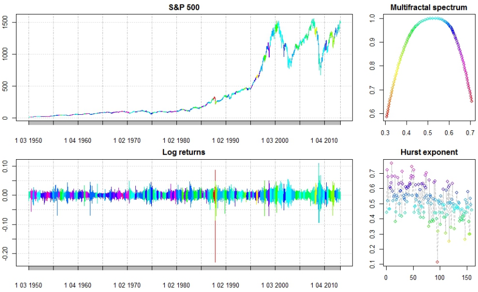
Techniques for multifractal spectrum estimation in financial time series



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Motivation

vs

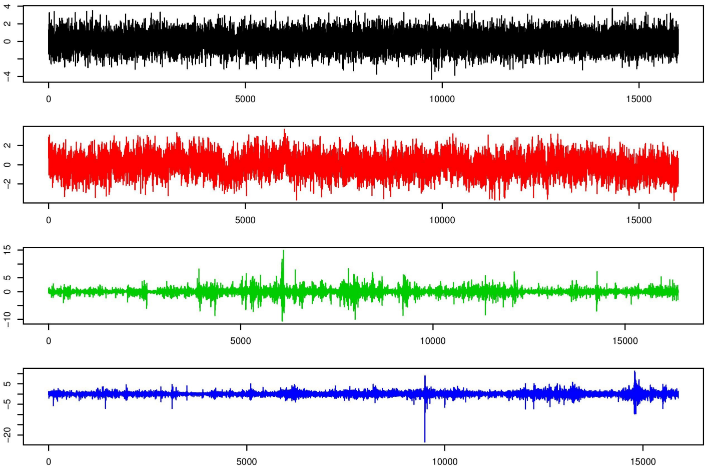
Many real systems have similar structure ⇒ similar description

Introduction

- Scaling properties are one of the most important quantifiers of complexity in many systems, e.g. financial time series
- Presence of scaling exponents can point to an inner fractal structure of the series
- Multiple scalings can be analyzed trough various techniques as Multifractal spectrum
- We examine different techniques of multifractality estimation, especially Detrended fluctuation analysis (DFA) and Diffusion entropy analysis (DEA)
- We introduce a procedure to proper estimation of Rényi entropy necessary in DEA algorithm
- We discuss both theoretical and practical properties of the techniques and compare them on various real financial time series

An example: real financial series

Which series is the real series of daily returns of S&P 500?



An example: real financial series

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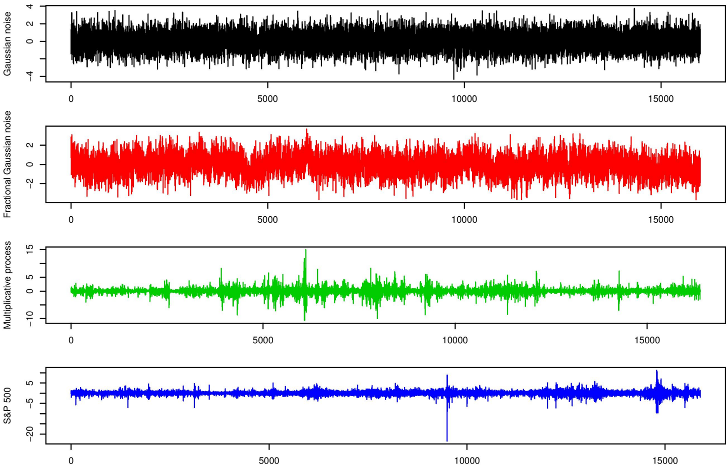


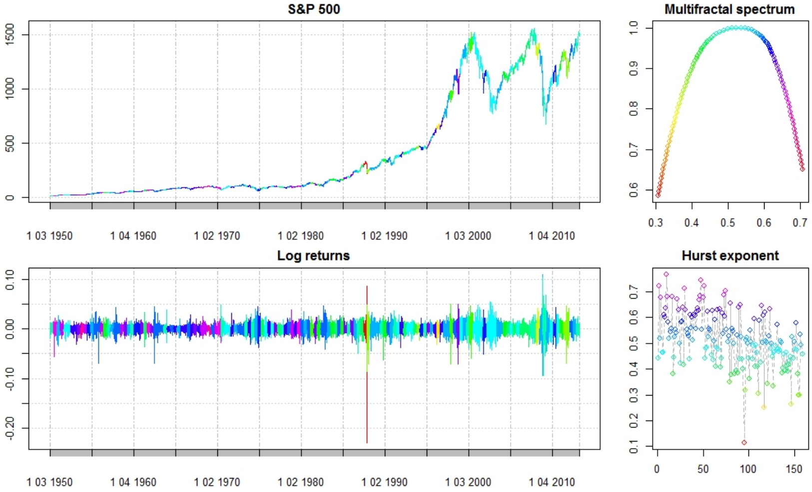
Multifractal spectrum

- Discrete time series $\{x_i\}_{i=1}^N$ in $\mathbb{R}^D$ with specific time lag s
- Empirical probability: $p_j = \frac{\#\{x_i \in K_j\}}{N}$
- Probabilities scale with the typical length as $p_j(s) \sim s^\alpha$
- Regions with different scalings are identified and distribution of scaling exponents has the form

$$\rho(\alpha, s)\mathrm{d}\alpha = c(\alpha)s^{-f(\alpha)}\mathrm{d}\alpha$$

- $f(\alpha)$ - Multifractal spectrum = fractal dimension of the subset with scaling exponent α

Multifractal spectrum of S&P 500



Scaling function and Rényi entropy

- Alternative approach - Partition function:

$$Z_q(s) = \sum_j p_j^q(s) \sim s^{\tau(q)}$$

- Relation to $f(\alpha)$ - *Legendre transform*: $f(\alpha) = \max_q(q\alpha - \tau(q))$
- $\tau(q)$ is related to Generalized dimension $D_q = \frac{\tau(q)}{q-1}$
  and Rényi entropy

$$S_q(s) = \frac{1}{q-1} \ln \sum_j p_j^q(s) = \frac{\ln Z_q(s)}{q-1}$$

- Multifractal exponents can be measured via estimation of Rényi entropy

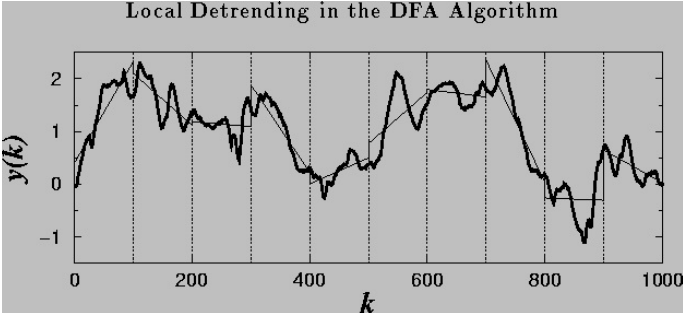
Estimation of Multifractal spectrum

- There exist several exponents of multifractal spectrum estimation
- Examples provide Generalized Hurst exponent or Wavelet analysis
- We focus on the most common, i.e. Detrended fluctuation analysis and Diffusion entropy analysis
- We introduce both methods and compare them on the real data

Detrended entropy analysis

- Method is based on measurement of fluctuations from local trends
- We divide the series into intervals of length s and calculate the aggregated deviation from local linear (quadratic,...) trends - Fluctuation function $F(s, \nu)$
- The total fluctuation function is calculated as a q-mean of local fluctuation functions $F_q(s) = \left(1/N \sum_\nu F(s,\nu)^q\right)^{1/q}$.
- Fluctuation function scales as $F_q(s) \propto s^{h(q)}$
- For stationary positive series is $\tau(q) = qh(q) - 1$

Estimation of local trends in DFA method



Diffusion entropy analysis

- Scaling exponents estimation: Diffusion entropy analysis - based on self-similarity property of PDF
- Monofractal case:

$$p(x,t)\mathrm{d}x = \frac{1}{t^\delta} F\left(\frac{x}{t^\delta}\right)\mathrm{d}x$$

- Shannon entropy identifies the exponent δ:

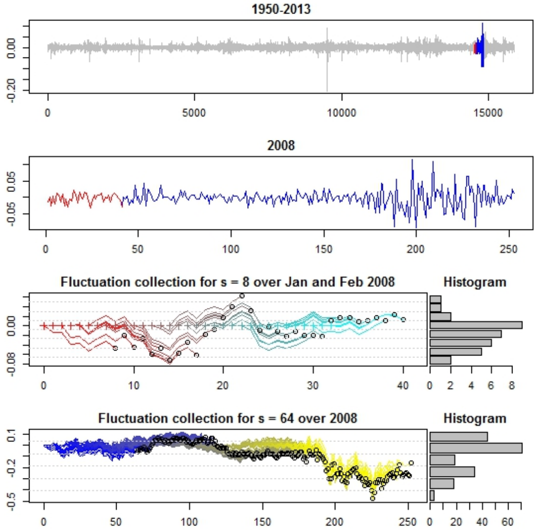
$$S(t) = -\int \mathrm{d}x \; p(x,t) \ln[p(x,t)] = A + \delta \ln t$$

- In multifractal case, whole class of Rényi entropies is calculated - class of scaling exponents $\delta(q) = \tau(q)/(q-1)$ is estimated from

$$S_q(t) = B_q + \delta(q) \ln t$$

- Fluctuation collection algorithm: all fluctuations over lag s are collected $\tilde{x}_s(t) = \sum_{i=1}^{s} x_{i+t}$, and PDF is estimated

Fluctuation collection algorithm of S&P 500



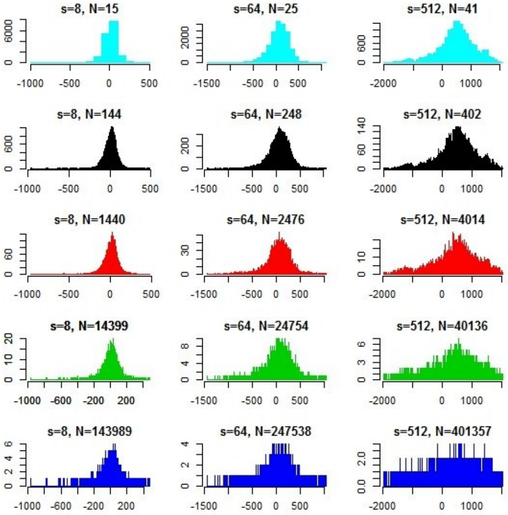
Basic properties of histograms

- Histogram: approximation of underlying PDF from data
- Equidistant boxes K_j of bin-width h; from frequency analysis:

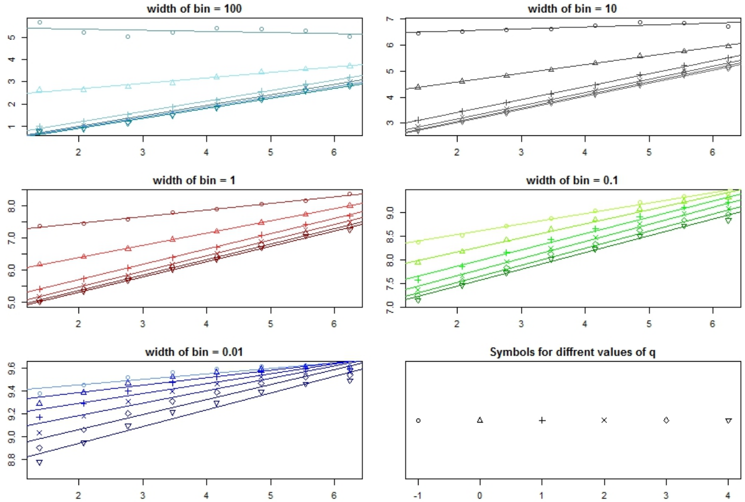
$$\hat{p}(x) = \frac{1}{Nh} \sum_{j=1}^{n_B} \nu_j \mathbb{1}_{K_j}(x)$$

  ν_j - # of $\{x_i\}_{i=1}^{N}$ in K_j, n_B - number of boxes
- The proper estimation of bin-width is crucial, because *underfitted* or *overfitted* histograms do not correspond to the underlying distribution
- Popular rules - Sturges: $n_B = 1 + log_2 N$
  Scott: $h = 3.5\hat{\sigma} N^{-1/3}$
  Freedman-Diaconis: $h = 2.6 \cdot IQR \cdot N^{-1/3}$

Histograms for different bin-widths



Entropy fits for different bin-widths



Optimal bin-width for Rényi entropy

- By minimizing the mean-squared integrated error we become an expression for optimal bin-width h_q^*

$$h_q^* = \sqrt[3]{\frac{6q^2}{N} \frac{\int p^{2q-1}(x)\mathrm{d}x}{\int (\mathrm{d}p^q(x)/\mathrm{d}x)^2}}$$

- We assume that $p(x)$ is normal distribution $\mathcal{N}(\mu, \sigma^2)$

$$h_q^* = \sigma N^{-1/3} \sqrt[3]{24\sqrt{\pi}} \frac{q^{1/2}}{\sqrt[6]{2q-1}} = h_{q=1}^* \rho_q$$

- For $q = 1$ we recover original Scott, resp. Freedman-Diaconis rules

Optimal bin-width for Rényi entropy

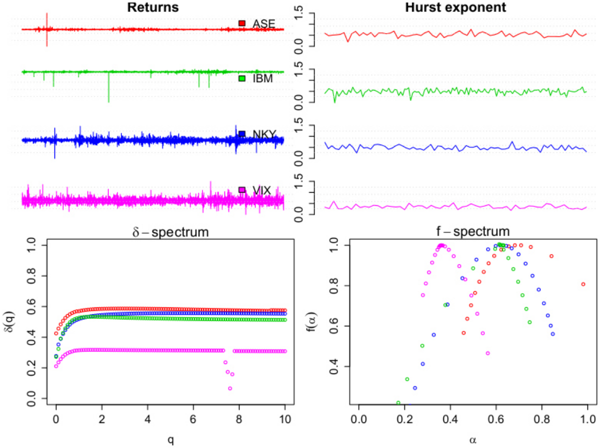
- In case of δ spectrum, we have to estimate several histograms on different specific lags $\{s_1, \ldots, s_m\}$ with the same bin-width
- We obtain the optimal bin-width by minimizing sum of particular errors

$$h_q^*(s_1, \ldots, s_m) = (24\sqrt{\pi})^{1/3} \rho_q \sqrt[3]{\frac{\sum_{i=1}^m \sigma_{s_i}^{2(1-q)} / N_{s_i}}{\sum_{i=1}^m \sigma_{s_i}^{-(1-2q)}}}$$

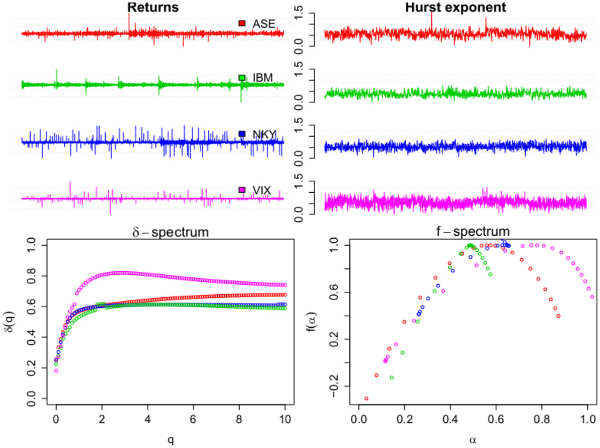
Comparison of multifractal methods on real data

- We have applied methods on the various financial data to test the stability of methods and to compare the spectra
- We used four financial series (Athens stock index, IBM stock price, Nikkei 225 stock index, Volatility index of S&P 500 VIX) recorded on minute and daily basis
- On the minute basis we observe some discontinuities due to the nature of the data (non-liquidity, heavy tails)
- Because each method has its own limitations, it is the best to use several multifractal methods to have the complete image of scaling exponents

Multifractal analysis for daily data



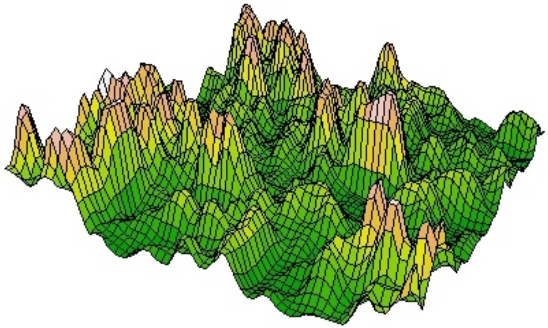
Multifractal analysis for minute data



Conclusions

- Many systems can be well described by multifractal scaling exponents
- Diffusion entropy analysis and Detrended fluctuation analysis represent possible ways how to estimate the exponents
- In case of DEA we have to properly estimate the histograms
- In case of DFA, the method does not work properly for heavy tailed distributions and long correlations
- Financial markets provide one example of complex system that can exhibit various kinds of multifractal spectra
- For high-frequency data is necessary to improve the method to be capable of dealing with heavy-tailed non-liquid time series

Back to the mountains!



Multifractal multiplicative cascade terrain model

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Thank you for your attention.