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# BRST–BFV and BRST–BV descriptions for bosonic fields with continuous spin on $R^{1,d-1}$

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*The paper is dedicated to the loving memory of the father of A.A.R.*
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Gauge-invariant descriptions for a free bosonic scalar field of continuous spin in a $d$-dimensional Minkowski space–time using a metric-like formulation are constructed on the basis of a constrained BRST–BFV approach we propose. The resulting BRST–BFV equations of motion for a scalar field augmented by ghost operators contain different sets of auxiliary fields, depending on the manner of a partial gauge-fixing and a resolution of some of the equations of motion for a BRST-unfolded first-stage reducible gauge theory. To achieve an equivalence of the resulting BRST-unfolded constrained equations of motion with the initial irreducible Poincaré group conditions of a Bargmann–Wigner type, it is demonstrated that one should replace the field in these conditions by a class of gauge-equivalent configurations. Triplet-like, doublet-like constrained descriptions, as well as an unconstrained quartet-like non-Lagrangian and Lagrangian formulations, are derived using both Fronsdal-like and new tensor fields. In particular, the BRST–BV equations of motion and Lagrangian using an appropriate set of Lagrangian multipliers

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in the minimal sector of the respective field and antifield configurations are constructed in a manifest way.

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## 1. Introduction

The Poincaré group is a cornerstone of relativistic quantum field theories. For the first time, its representations in $\mathbb{R}^{1,3}$ were studied by Wigner.[1] The number of group representations describes the quantum states found in a local field theory, being some massless particles of fixed helicity (photon) and massive particles of integer (for vector and Higgs bosons) and half-integer (for quarks and leptons) spin. In higher space–time dimensions, the Poincaré group $ISO(1, d-1)$ is shown to be useful in (super-)string theories.[2–4] Until now, no examples have been found to realize any other representations that exist in the Nature. So, a tachyon representation of imaginary mass, which appears to be an excitation of the lowest energy in the spectrum of bosonic string theories, is used as an indicator of instabilities, for instance, in spontaneous symmetry breaking. The other representations are known as *continuous spin representations* (CSR) which describe a massless object with an infinite number of helicities for which eigenstates of various helicities are mixed under the Lorentz transformations, in a way similar to the set of massive particles, leading to an infinite heat capacity of the vacuum, due to Wigner's argumentation.[5]

Numerous attempts have been undertaken to associate CSR with physical systems. It appears that the actual discovery of this procedure is yet to come. At the same time, the single-valued (bosonic) and double-valued (fermionic) CSR with an infinite number of degrees of freedom (see e.g. Ref. 6) have not yet been observed with confidence[7] in the respective spectra of second-quantized bosonic strings and superstrings, as compared to the massless higher-spin (HS) fields of all the integer $(0, 1, 2, \ldots)$ and half-integer $(1/2, 3/2, 5/2, \ldots)$ helicities (each having a finite number of degrees of freedom), so as to be extracted using the (super-)string tensionless limit.[8,9] However, there are ways to construct a special tensionless string limit[10,11] in which CSR in the truncated string field may be found.

The above property of CS particles is quite attractive nowadays due to an intense development of higher-spin theory;[12–17] see the reviews in Refs. 18 and 19, the discussion in the string-theory context in Ref. 20 and references therein.

Unitary irreducible representations (UIR) using CS for the Poincaré and super-Poincaré groups in a $d$-dimensional Minkowski space–time with $d > 4$ were first studied by the team of Brink and Ramond,[21] and in further detail, by Bekaert and Boulanger.[22] It was shown by Khan and Ramond[23] that it is possible to consider CSR with CS $\Xi$ as a special limit for an HS particle of mass $m$ and spin $s$, when $\lim_{m\to 0; s\to\infty} ms = \Xi$, used to derive the Fronsdal- and Fronsdal–Fang-like equations,[24,25] albeit having CSR in the limit corresponding to massive HS particles,[26] and shown to be equivalent to the Wigner and Wigner–Bargmann equations[27,28] (for a review, see e.g. Ref. 29).

In turn, a search for Lagrangian formulations (LFs) and forms of relativistic field equations, not necessarily Lagrangian ones, which are to equivalently reproduce the conditions selecting massless UIR with CS, has been variously developed for $\mathbb{R}^{1,d-1}$, in both $d = 4$ and higher dimensions. So, a local covariant action for bosonic CS particle formulated using an auxiliary Lorentz vector $\eta_m$ and localized to the unit hyperboloid $\eta^2 = -1$ has been presented by an integral over $d^4xd^4\eta$ in Ref. 30 (see also Ref. 31). An LF for a scalar bosonic CSR field in terms of an infinite set of (double-)traceless totally-symmetric tensor fields of any rank in constant-curvature $d$-dimensional spaces has been realized using an oscillator formalism (in accordance with tensor representation[30]) by Metsaev,[32] which was used in Ref. 33 to construct a quantum action for CSR field in $\mathbb{R}^{1,d-1}$, whereas a twistor description for massless particles with CS has been suggested in Ref. 34 (for relationship between the Fronsdal-like and Fang–Fronsdal-like equations[35] and ones obtained in Ref. 32 and for interactions, see as well, Refs. 36–39).

Some of the most efficient tools to reconstruct a local gauge-invariant LF from the initial UIR of the Poincaré or anti-de Sitter groups previously used merely for particles of discrete spin on a basis of the BRST–BFV approach originating from the BFV method,[40–43] invented to quantize dynamical constrained systems, and applied, nevertheless, to a solution of the inverse problem, in fact, to formulate an LF in terms of Hamiltonian-like objects using an auxiliary Hilbert space whose vectors consists of HS (spin-)tensor fields. It is not surprising that a first application in this way of the BRST–BFV method to CS fields in $R^{1,3}$ has been recently proposed by Bengtsson,[44] one of the inventors of the constrained BRST–BFV approach to lower-spin fields.[45–48] An inclusion of holonomic (traceless and mixed symmetry) constraints, together with differential ones, into a total system of constraints which is to be closed with respect to Hermitian conjugation with an appropriate conversion procedure for a subsystem with second-class constraints, has resulted in augmenting the original method by an unconstrained BRST–BFV method, with no restrictions imposed on the entire set of initial and auxiliary HS fields. The application of this method have been initiated by Pashnev and Tsulaia,[49] followed by Buchbinder, Krykhtin and Reshetnyak,[50–58] for totally symmetric HS fields and mixed (anti-)symmetric HS fields in $R^{1,d-1}$ and $\mathrm{AdS}_d$, see Refs. 59–67 (for a review and the interaction problem, see Ref. 68). A detailed correspondence between constrained and unconstrained BRST–BFV methods for arbitrary massless and massive IR of the $ISO(1, d-1)$ group with a generalized discrete spin has been recently studied in Ref. 69, where a constrained BRST–BFV LF for fermionic HS fields subject to an arbitrary Young tableaux $Y(s_1, \ldots, s_k)$, $k \leq [d/2]$ was first suggested and an equivalence between the underlying constrained and unconstrained LF was established. A development of this topic has resulted in an (un-)constrained BRST–BV method of finding minimal BV actions necessary to construct a quantum action within the BV quantization[70–72] presented in Ref. 73 (for bosonic HS fields, also see Refs. 74–77). An application of the BRST–BFV method to a scalar bosonic CS field in $\mathbb{R}^{1,3}$ on the basis of a so-called *four-constraint formalism*[44] was recently proposed using the Weyl spinor notation in Ref. 78 (for recent developments, see also Refs. 79–82, 85). A prescription for a four-constraint formalism to derive an unconstrained BRST–BFV LF for a CS field[44] is different from the one applied to HS fields of any discrete helicity, because the set of conditions extracting a massless bosonic UIR of any integer spin and the one having CS[27,28] contains the respective 2 and 4 equations, so that the "naive" numbers of the respective constraints being linear in the ghost approximations of Hermitian BRST operators should be 3 and 7.

Having in mind the equivalence between unconstrained and constrained BRST–BFV LF for one and the same HS field of a generalized discrete spin in $\mathbb{R}^{1,d-1}$,[69] we shall assume that the same property is to be valid for BRST–BFV unconstrained and constrained descriptions for the equations of motion (EoM), we intend to construct an BRST–BFV descriptions for free massless CSR particles propagating in $\mathbb{R}^{1,d-1}$. The paper is devoted to the following problems:

(1) Derivation of a constrained BRST–BFV approach to constrained gauge-invariant both non-Lagrangian description for EoM (and Lagrangians) for a scalar CS field in $\mathbb{R}^{1,d-1}$ in the Bargmann–Wigner form, with a compatible set of off-shell BRST-extended constraints in the metric formulation.
(2) Study of an equivalence between the resulting BRST-EoM for a scalar CS field in $\mathbb{R}^{1,d-1}$ with initial conditions extracting UIR of the $ISO(1,d-1)$ group with CS and making a comparison with a Fronsdal-like representation.
(3) Construction of constrained BRST–BV descriptions for EoM (and action) in the minimal sector of the field–antifield formalism on a basis of the suggested gauge-invariant constrained EoM (and action) for a scalar CS field in $\mathbb{R}^{1,d-1}$ in the Bargmann–Wigner form.
(4) Construction of an unconstrained gauge-invariant EoM (and action) from a constrained BRST–BFV description for EoM (and action) on a basis of additional compensating field.

The paper is organized as follows. In Sec. 2, we find an higher continuous spin (HCS) symmetry algebra for a massless bosonic field with a given CS in $\mathbb{R}^{1,d-1}$ in Bargmann–Wigner form and suggest (in Sec. 3) a constrained BRST–BFV formulation for EoM. In the latter point, we construct a constrained BRST operator with an off-shell holonomic constraint, obtain a properly gauge-invariant EoM and action with help of the Lagrangian multipliers, find its representations in terms of Fronsdal-like fields and resolve the problem 2 concerning an equivalence with the initial set of UIR CS conditions. BRST–BV minimal formulations for EoM (and action) are derived in Sec. 4 and an unconstrained quartet-like EoM and action are presented in Sec. 5. The short-list of the results is presented in the Conclusion. Finally, in App. A we construct a representation for higher continuous spin symmetry algebra with two pairs of oscillators, then demonstrate in App. B the problem of Fock space realization and, thus, LF for the single pair of oscillators within Wigner–Bargmann form of CSR equations as well as suggest a new way to find CSR in the special tensionless limit for open bosonic string in App. C.

The convention $\eta_{mn} = \mathrm{diag}(+,-,\ldots,-)$ for the metric tensor, with the Lorentz indices $m,n = 0,1,\ldots,d-1$, and the notation $\epsilon(A)$, $[gh_H, gh_L, gh_{\rm tot}](A)$ for the respective values of Grassmann parity, BFV, $gh_H$, BV, $gh_L$ and total, $gh_{\rm tot} = gh_H + gh_L$, ghost numbers of a quantity $A$ are used. The totally symmetric in indices $m_1,\ldots,m_k$ quantities $\Phi^{m_1\cdots m_k}$ and $A^{m_1}\cdots A^{m_k}$ are denoted, respectively as $\Phi^{(m)_k}$ and $\left(\prod A\right)^{(m)_k}$. The supercommutator $[A,B\}$ of quantities $A$, $B$ with definite values of Grassmann parity is given by $[A,B\} = AB - (-1)^{\epsilon(A)\epsilon(B)}BA$. The Heaviside $\theta$-symbol determined as $\theta_{k,l} = 1(0)$ for $k > l (k \le l)$.

## 2. Higher Continuous Spin Symmetry Algebra $\mathcal{A}(\Xi;\mathbb{R}^{1,d-1})$

The irreducible Poincaré group massless bosonic representation with CS in $\mathbb{R}^{1,d-1}$ is described by the $\mathbb{R}$-valued function $\Phi(x,\omega)$ of two independent variables $x^m,\omega^m$ (being by scalar CS field[21,30]) on which the quadratic $C_2 = P^mP_m$ and quartic, $C_4 = W_{m_1\cdots m_{d-3}}W^{m_1\cdots m_{d-3}}$ Casimir operators take the values

$$C_2\Phi(x,\omega) = 0\,, \quad C_4\Phi(x,\omega) = \nu\Xi^2\Phi(x,\omega) \quad \text{with} \quad W^{m_1\cdots m_{d-3}} = \epsilon^{m_1\cdots m_d}P_{m_{d-2}}M_{m_{d-1}m_d}\,. \tag{2.1}$$

$W^{m_1\cdots m_{d-3}}$ is the generalized Pauli–Lubanski $(d-3)$-rank tensor[a] with Levi-Civita tensor $\epsilon^{m_1\cdots m_d}$, momentum $P_m = -\imath\frac{\partial}{\partial x^m}$, angular momentum $M_{mn} = \hat{M}_{mn} + S_{mn}$, for orbital and spin parts:

$$\hat{M}_{mn} = \imath x_m\frac{\partial}{\partial x^n} - \imath x_n\frac{\partial}{\partial x^m}\,, \quad S_{mn} = \imath\omega_m\frac{\partial}{\partial\omega^n} - \imath\omega_n\frac{\partial}{\partial\omega^m}\,, \tag{2.2}$$

and with the real positive constant $\Xi$, enumerating the value of CS in $\mathbb{R}^{1,d-1}$ when $\nu = 1$. Explicitly, the field $\Phi(x,\omega)$ should satisfy to the 4 relations $\big[$as it was suggested for $d = 4$ case by Wigner and Bargmann[27,28] when $\nu = 1$ for the field $\tilde{\Phi}(p,\xi)$ in momentum representation, being Fourier transform of $\Phi(x,\omega)$: $\tilde{\Phi}(p,\xi) = (2\pi)^{-d/2}\int d^dx\, d^d\omega \exp\{ip_mx^m + i\xi_m\omega^m\}\Phi(x,\omega)\big]$. In terms of $\tilde{\Phi}(p,\xi)$ and $\Phi(x,\omega)$ the equations read:

$$\left(\eta^{mn}p_mp_n, \eta^{mn}\xi_mp_n, \imath\eta^{mn}\frac{\partial}{\partial\xi^m}p_n - \Xi, \eta^{mn}\xi_m\xi_n + \nu\right)\tilde{\Phi}(p,\xi) = (0,0,0,0)\,, \tag{2.3}$$

$$\eta^{mn}\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n}\Phi(x,\omega) = 0\,, \quad \eta^{mn}\frac{\partial}{\partial\omega^n}\frac{\partial}{\partial x^m}\Phi(x,\omega) = 0\,, \tag{2.4}$$

[a]For $d > 4$ there exist additional Pauli–Lubanski tensors

$$W^{m_1\cdots m_e} = \epsilon^{m_1\cdots m_d}P_{m_{e+1}}M_{m_{e+2}m_{e+3}} \times \cdots \times M_{m_{d-1}m_d}\,,$$

such that $[P_m, W^{m_1\cdots m_e}] = 0$, thus providing for the operators $C_{2e} = W_{m_1\cdots m_e}W^{m_1\cdots m_e}$, $e = 1,3,\ldots,d-3$ for $d = 2N$, ($e = 0,2,\ldots,d-3$ for $d = 2N-1$) to be by Casimir operators[21] which are characterized by the parameters $\nu,\Xi$ and integer spin-like parameter $s_1,\ldots,s_k$ for $k = [(d-4)/2]$.

$$-\imath\omega^m \frac{\partial}{\partial x^m}\Phi(x,\omega) = \Xi\Phi(x,\omega)\,, \quad \eta^{mn}\frac{\partial}{\partial\omega^m}\frac{\partial}{\partial\omega^n}\Phi(x,\omega) = \nu\Phi(x,\omega)\,, \tag{2.5}$$

with some dimensionless parameter $\nu \in \mathbb{R}$ (being the squared length for the space-like internal vector $\xi^m$, $\xi^2 = -\nu$), expressing the fact of ambiguity in definition of internal variables $\omega^m$ and determining the value of the quartic Casimir operator $C_4$ on the elements of IR space of the Poincaré algebra $iso(1, d-1)$ as $\nu\Xi^2$.[b] Equations (2.4) and (2.5) are non-Lagrangian.

Because of the absence of nontrivial solutions, with except for, $\Phi(x,\omega) = 0$, due to the first equation in (2.5), when expanding $\Phi(x,\omega)$ in powers of only non-negative degrees in $\omega^m$, we consider the representation of $\Phi(x,\omega)$ in the form of series both in powers of $\omega^m$ and in powers of its inverse degrees, $(\omega^m/\omega^2)$ in terms of independent usual in HS field theory tensor fields $\Phi^0_{(m)_k,(n)_0}(x) \equiv \Phi_{(m)_k}(x)$ and new ones for $l > 0$: $\Phi^l_{(m)_k,(n)_l}(x)$:

$$\Phi(x,\omega) = \sum_{l\geq 0}\sum_{k\geq 0}\frac{1}{k!l!}\Phi^l_{(m)_k,(n)_l}(x)\prod_{i=1}^{k}\omega^{m_i}\prod_{j=1}^{l}\frac{\omega^{n_j}}{\omega^2} \equiv \left(\Phi^0 + \sum_{l\geq 1}\Phi^l\right)(x,\omega)\,, \tag{2.6}$$

(for $\omega^2 = \omega^m\omega_m$) to provide the completeness property when resolving the respective BRST complex in the corresponding vector (or Hilbert with endowing by an appropriate finite scalar product) space. Because of the monomial $\omega^{(m)_k}\frac{\omega^{(n)_l}}{\omega^{2l}}$ does not uniquely determine the positive and negative degrees of $\omega^m$ its dual element $\Phi^l_{(m)_k,(n)_l}$ should have the respective dual property for $\forall l \geq 0$:

$$\left(\omega^{(m)_k}\frac{\omega^{(n)_l}}{\omega^{2l}} = \omega^{(m)_k}\left\{\prod_{j=1}^{i}\omega^{m_{k+j}}\right\}\frac{\omega^{(n)_l}\left\{\prod_{j=1}^{i}\omega^{n_{l+j}}\right\}}{\omega^{2(l+i)}}\prod_{j=1}^{i}\eta_{m_{k+j}n_{l+j}}\right)$$
$$\Rightarrow \left(\Phi^l_{(m)_k,(n)_l} \to \tilde{\Phi}^l_{(m)_k,(n)_l} = \sum_{i\geq 0} d^{-i}\Phi^{l+i}_{(m)_{k+i},(n)_{l+i}}\prod_{j=1}^{i}\eta^{m_{k+j}n_{l+j}}\right), \tag{2.7}$$

where for the latter row a decomposition for the monomial $\omega^{(m)_k}\frac{\omega^{(n)_l}}{\omega^{2l}}$ on trace and traceless parts was used (with account of the notation for totally-symmetric sets of indices $(m)_k \equiv m_1\cdots m_k$ and $(n)_l \equiv n_1\cdots n_l$ and $\frac{\partial}{\partial x^n} \equiv \partial_n$)

$$\left(\omega^m\frac{\omega^n}{\omega^2} = d^{-1}\eta^{mn} + \left(\delta^m_\rho\delta^n_\sigma - d^{-1}\eta^{mn}\eta_{\rho\sigma}\right)\omega^\rho\frac{\omega^\sigma}{\omega^2}\right)$$
$$\Rightarrow \left(\omega^{(m)_k}\frac{\omega^{(n)_l}}{\omega^{2l}} = \prod_{i=1}^{\min(k,l)}\left\{d^{-1}\eta^{m_in_i} + \left(\delta^{m_i}_{\rho_i}\delta^{n_i}_{\sigma_i} - d^{-1}\eta^{m_in_i}\eta_{\rho_i\sigma_i}\right)\omega^{\rho_i}\frac{\omega^{\sigma_i}}{\omega^2}\right\}\right.$$
$$\left.\times\left\{\theta_{k,l}\omega^{(m)_{k-l}} + \theta_{l,k-1}\frac{\omega^{(n)_{l-k}}}{\omega^{2(l-k)}}\right\}\right) \tag{2.8}$$

[b]For $\Xi = 1$, $\nu = \mu^2$ from above equations the relations given by (1.1)–(1.4) in Ref. 34 are obtained, whereas for $\nu = 1$ the Wigner and Bargmann equations[27,28] hold.

(for $\omega^{(n)_0} \equiv 1$). So, the trace part of $\omega^m \frac{\omega^n}{\omega^2}$ corresponds to scalar $d^{-1}\eta^{mn}\Phi^1_{m,n}$ which can be added to pure scalar $\Phi^0$. The component fields $\Phi^l_{(m)_k,(n)_l}$ are totally symmetric not only with respect to separate group of indices $(m)_k$ and $(n)_l$, but with respect to whole set of indices $(m)_k, (n)_l$ due to the relation: $\omega^{m_1}\frac{\omega^{n_1}}{\omega^2} = \omega^{n_1}\frac{\omega^{m_1}}{\omega^2}$. In particular, the property holds

$$\Phi^l_{(m)_k,(n)_l} = \begin{cases} \Phi^l_{(n)_l m_{l+1}\cdots m_k,(m)_l} & \text{for } k \geq l\,, \\ \Phi^l_{(n)_k,(m)_k n_{k+1}\cdots n_l} & \text{for } k < l \end{cases} \tag{2.9}$$

true. The symmetry properties (2.9) mean that the different traces of the functions $\Phi^l_{(m)_k,(n)_l}$ are equal:

$$\begin{aligned} \Phi^l_{(m)_k,(n)_l}\eta^{m_k n_l} &= \Phi^l_{(m)_{k-2}m_k m_{k+1},m_{k-1}(n)_{l-1}}\eta^{m_k m_{k+1}} \\ &= \Phi^l_{(m)_{k-1}n_{l-1},(n)_{l-2}n_l n_{l+1}}\eta^{n_l n_{l+1}}\,. \end{aligned} \tag{2.10}$$

Indeed, we have the sequence of relations, e.g. for the first equality,

$$\begin{aligned} \Phi^l_{(m)_k,(n)_l}\eta^{m_k n_l} &= \Phi^l_{(m)_{k-2}m_k n_l,m_{k-1}(n)_{l-1}}\eta^{m_k n_l} \\ &= \Phi^l_{(m)_{k-2}m_k m_{k+1},m_{k-1}(n)_{l-1}}\eta^{m_k m_{k+1}}\,. \end{aligned}$$

In particular, for $k-1 = l = 1$ $(l-1 = k = 1)$ we have,

$$\Phi^1_{(m)_2,n}\eta^{m_2 n} = \Phi^1_{nm_2,m_1}\eta^{m_2 n} \quad \left(\Phi^2_{m,(n)_2}\eta^{n_1 n_2} = \Phi^2_{n_1,mn_2}\eta^{n_1 n_2}\right). \tag{2.11}$$

The relations (2.4), (2.5) take equivalent, [but ambiguous(!) due to freedom in the definition of the monomials and therefore component functions (2.7), (2.8)] representation in powers of $\omega^m$ (2.12) and of $\prod_{i=1}^k \omega^{m_i} \times \prod_{j=1}^l \omega^{n_j}/\omega^2$ (2.13):

$$\omega^{(m)_k}: \begin{cases} \eta^{mn}\partial_m\partial_n\Phi^0_{(m)_k}(x) = 0\,, \\ \partial^{m_{k+1}}\Phi^0_{(m)_{k+1}}(x) = 0\,, \\ -\imath\partial_{\{m_k}\Phi^0_{(m)_{k-1}\}}(x) = \Xi\Phi^0_{(m)_k}(x)\,, \\ \eta^{m_{k+1}m_{k+2}}\Phi^0_{(m)_{k+2}}(x) = \nu\Phi^0_{(m)_k}(x)\,; \end{cases} \tag{2.12}$$

$$\omega^{(m)_k}\frac{\omega^{(n)_l}}{\omega^{2l}}: \begin{cases} \eta^{mn}\partial_m\partial_n\Phi^l_{(m)_k,(n)_l} = 0\,, \\ \partial^{m_{k+1}}\Phi^l_{(m)_{k+1},(n)_l} + \eta_{\{n_{l-1}n_l}\partial^n\Phi^{l-1}_{(m)_k,(n)_{l-2}\}n} \\ \qquad - 2(l-1)\partial_{\{n_l}\Phi^{l-1}_{(m)_k,(n)_{l-1}\}} = 0\,, \\ \imath\partial_{\{m_k}\Phi^l_{(m)_{k-1}\},(n)_l} + \Xi\eta_{\{m_k\{n_l}\Phi^{l-1}_{(m)_{k-1}\},(n)_{l-1}\}} = 0\,, \\ \Phi^l_{(m)_k m}{}^{m}{}_{,(n)_l} - \nu\Phi^l_{(m)_k,(n)_l} + 2\Phi^{l-1}_{(m)_k n,\{(n)_{l-2}}{}^{n}\eta_{n_{l-1}n_l\}} \\ \qquad - 4(l-1)\Phi^{l-1}_{(m)_k\{n_l,(n)_{l-1}\}} \\ \qquad + \Phi^{l-2}_{(m)_k,\{\{(n)_{l-4}n}{}^{n}\eta_{n_{l-3}n_{l-2}\}}\eta_{n_{l-1}n_l\}} \\ \qquad + 2(l-2)(2-d)\Phi^{l-2}_{(m)_k,\{(n)_{l-2}}\eta_{n_{l-1}n_l\}} = 0 \end{cases} \tag{2.13}$$

(for $\Phi^l \equiv 0$ when $l < 0$; $k \in \mathbb{N}_0$) being respectively for each systems by D'Alambert, divergentless, gradient and generalized traceless equations. Note, first, that all the component tensor functions $\Phi^0_{(m)_k}$ in (2.12) and $\Phi^l_{(m)_k,(n)_l}$ in (2.13) are determined with accuracy up to the transformations (2.7), i.e. may be changed on the respective functions $\tilde{\Phi}^0_{(m)_k}$ and $\tilde{\Phi}^l_{(m)_k,(n)_l}$. Second, we have used the symmetrization in indices $m_k$, $(m)_{k-1}$: $\{m_k, (m)_{k-1}\}$; in $n_{l-1}n_l$, $(n)_{l-2}$: $\{n_{l-1}n_l, (n)_{l-2}\}$ and in 4 indices $n_{l-3}n_{l-2}, \dots, n_l$ with $(n)_{l-4}$ in (2.13) without numerical factor. Third, the left-hand side of the last equation in (2.12) may be equivalently written as, $\eta^{m_{k+1}m_{k+2}}\Phi^0_{(m)_{k+2}}(x) = \Phi^0_{(m)_k m}{}^m(x)$ as it was done in the similar traceless equations in (2.13). The representation (2.6) leads to nonempty set of nontrivial solutions for the systems (2.12), (2.13) with account for the first note. Fourth, the set of Eqs. (2.12) appears by the subsystem of the set (2.13) for $l = 0$ with allowance made for ambiguity (2.7), (2.8). For instance, from the third equations in (2.13) for $l = 1$, $k = 1$ (with $\eta_{\{m_1\{n_1}\Phi^0_{(m)_0\},(n)_0\}} \equiv 2\eta_{m_1n_1}\Phi^0$) it follows:

$$\begin{aligned} &\imath\partial_{\{m_1}\tilde{\Phi}^1_{n_1\}}(x) + 2\eta_{m_1n_1}\Xi\tilde{\Phi}^0(x) = 0 \\ &\Leftrightarrow \begin{cases} \imath d^{-1}\partial^n\tilde{\Phi}^1_n(x) + \Xi\tilde{\Phi}^0(x) = 0\,, \\ \imath\left(\delta^\rho_{\{m}\delta^\sigma_{n\}} - d^{-1}\eta_{mn}\eta^{\rho\sigma}\right)\partial_\rho\tilde{\Phi}^1_\sigma(x) = 0\,. \end{cases} \end{aligned} \tag{2.14}$$

Note, that the similar equivalent equations take place for the third equations in (2.13) for $l > 1$.[c]

To describe the dynamics of the fields $\Phi^l$ jointly, we may follow by two ways both being based on BRST–BFV approach. First variant consists in applying the algorithm[50–66] for construction of the Lagrangian formulation, second variant is concentrated within scope of non-Lagrangian equations of motion, as it was realized for UIR with discrete spin, see e.g. Ref. 76 and for CSR.[82] Whereas, the former case requires an introduction of more complicated set of oscillators to provide string-inspired Fock space structure presented in App. A, the latter one may be realized without using of finite scalar product.

In spite of these problems, Eqs. (2.12) and (2.13) can be derived from the Lagrangian actions both in unconstrained with help of four sets of Lagrangian multipliers $\lambda_i^{l|(m)_k,(n)_l}$, $i = 1, 2, 3, 4$, $k, l \in \mathbb{N}_0$ and in constrained form with three sets of

[c]Another variant of solutions for (2.4), (2.5) due to the first equation in (2.5) can be chosen without poles in $\omega^m$ as its explicit solution:

$$\Phi(x,\omega) = \delta(\omega p - \Xi)\varphi(x,\omega)\,,$$
$$\varphi(x,\omega) = \sum_{k\geq 0}\frac{1}{k!}\varphi_{(m)_k}(x)\omega^{m_1}\cdots\omega^{m_k}\,,$$
$$p_m = -\imath\frac{\partial}{\partial x^m}\,.$$

To get the UIR with CS for the field $\varphi(x,\omega)$ one should to modify the rest equations in order to include the value of CS $\Xi$ in it that should provide the fulfillment of the equations on the Casimir operators (2.1) that was done, e.g. in Refs. 32 and 78. We develop the above procedure in Ref. 97.

Lagrangian multipliers without $\lambda_4^{l|(m)_k,(n)_l}$ as follows:

$$S_{C|\Xi} = \int d^dx \sum_{k,l\geq 0} \Big\{ \lambda_1^{l|(m)_k,(n)_l} \partial^2 \Phi^l_{(m)_k,(n)_l} + \lambda_2^{l|(m)_k,(n)_l} \Big( \partial^{m_{k+1}} \Phi^l_{(m)_{k+1},(n)_l} + \eta_{\{n_{l-1}n_l} \partial^n \Phi^{l-1}_{(m)_k,(n)_{l-2}\}n} - 2(l-1)\partial_{\{n_l} \Phi^{l-1}_{(m)_k,(n)_{l-1}\}} \Big) + \lambda_3^{l|(m)_k,(n)_l} \Big( \imath \partial_{\{m_k} \Phi^l_{(m)_{k-1}\},(n)_l} + \Xi \eta_{\{m_k\{n_l} \Phi^{l-1}_{(m)_{k-1}\},(n)_{l-1}\}} \Big) \Big\}, \tag{2.15}$$

$$S_\Xi = S_{C|\Xi} + \int d^dx \sum_{k,l\geq 0} \lambda_4^{l|(m)_k,(n)_l} \Big( \Phi^l_{(m)_k m}{}^m{}_{,(n)_l} - \nu \Phi^l_{(m)_k,(n)_l} + 2\Phi^{l-1}_{(m)_k n,\{(n)_{l-2}}{}^n \eta_{n_{l-1}n_l\}} - 4(l-1)\Phi^{l-1}_{(m)_k\{n_l,(n)_{l-1}\}} + \Phi^{l-2}_{(m)_k,\{\{(n)_{l-4}n}{}^n \eta_{n_{l-3}n_{l-2}\}} \eta_{n_{l-1}n_l\}} + 2(l-2)(2-d)\Phi^{l-2}_{(m)_k,\{(n)_{l-2}} \eta_{n_{l-1}n_l\}} \Big). \tag{2.16}$$

In this respect, note that since the Bargmann–Wigner equations is linear in $\Phi$ the respective actions (2.15), (2.16) have a quadratic form which leads, of course, to a nondiagonal form of the respective propagator and to a wider set of the degrees of freedom due to Lagrangian multipliers. However, the EoM for $\Phi(x,\omega)$ and those for $\lambda_i^{l|(m)_k,(n)_l}(x)$ are completely decoupled from each other, and thereby form independent systems. By selecting the appropriate initial and boundary conditions for $\lambda_i^{l|(m)_k,(n)_l}(x)$ one can always fix the unwanted degrees of freedom completely. We will use this form of free actions as auxiliary ones in order to deduce the EoM.[d]

[d]The actions (2.15), (2.16) reflect the points of a general procedure known as the augmentation method for the classical and quantum descriptions of (non-)Lagrangian systems,[83] which, owing to an idea also suggested in Ref. 84, may be used to obtain the respective actions entirely in terms of the field $\Phi(x,\omega)$. To do so, one should consider the actions $S^A_{C|\Xi}$, $S^A_\Xi$ augmented by the terms quadratic in $\lambda_i^{l|(m)_k,(n)_l}(x) \equiv \lambda_i^{M_i}(x)$:

$$S^A_{(C|)\Xi} = S_{(C|)\Xi} + \int d^dx \frac{1}{2} \lambda_i^{M_i}(x) G_{M_iN_i} \lambda_i^{N_i}(x) : \frac{\delta S^A_{(C|)\Xi}}{\delta \lambda_i^{N_i}(x)} = \lambda_i^{M_i}(x) G_{M_iN_i} + F(\Phi(x))_{N_i i} = 0\,,$$

for $F(\Phi)_{N_i i} \equiv F^{M_i}_{N_i i}\Phi_{M_i i}$ being EoM (2.13) for constrained at $i = 1,2,3$ and unconstrained $i = 1,2,3,4$ cases. For a nondegenerate matrix $\|G_{M_iN_i}\|$, we have actions being quadratic at the extremals $F(\Phi)_{N_i i}$:

$$\mathcal{S}^A_{(C|)\Xi} = S^A_{(C|)\Xi}|_{\left(\lambda_{M_i i} = -F(\Phi)_i^{N_i} G^{-1}_{N_iM_i}\right)} = -\frac{1}{2} \int d^dx\, F(\Phi)_i^{M_i} G^{-1}_{M_iN_i} F(\Phi)_i^{N_i} \equiv -\frac{1}{2} \int d^dx \sum_{k,l,k',l'\geq 0} \Phi^{l|(m)_k,(n)_l} \mathcal{G}^{ll'}_{(m)_k,(n)_l;(m_1)_{k'},(n_1)_{l'}} \Phi^{l'|(m_1)_{k'},(n_1)_{l'}}\,.$$

The Poincaré group IR relations (2.4), (2.5) take the equivalent form in terms of the operators,

$$\big(l_0, l_1, m_1^+, m_{11}\big)\Phi(x,\omega)^{\mathrm{e}} = 0\,, \tag{2.17}$$

$$\big(l_0, l_1, m_1^+, m_{11}\big) = \bigg(\eta^{mn}\partial_m\partial_n, -\frac{\partial}{\partial\omega^m}\partial^m, -\omega^m\partial_m + \imath\Xi, -\eta^{mn}\frac{\partial}{\partial\omega^m}\frac{\partial}{\partial\omega^n} + \nu\bigg)\,. \tag{2.18}$$

It is impossible as it is shown in App. B to realize a Fock space structure on a set of the generating functions $\Phi(x,\omega)$ given by (2.6) and its dual with finite scalar product with standard Hermitian conjugation property, when using the one set of oscillators: $(a_m, a^{+n}) \equiv -\imath(\partial/\partial\omega^m, \omega^n)$, $\big[a^m, a^{+n}\big] = -\eta^{mn}$ for $a^m|0\rangle$ when acting on a vacuum vector $|0\rangle$. However, we enlarge the set of the operators (2.18) by its (formal) duals $\big(l_1^+, m_1, m_{11}^+\big)$:

$$\big(l_1^+, m_1, m_{11}^+\big) = \bigg(-\omega^m\partial_m, -\frac{\partial}{\partial\omega^m}\partial^m - \imath\Xi, -\omega^m\omega_m + \nu\bigg)\,, \tag{2.19}$$

and by the "number particle" operator,

$$g_0 = \frac{1}{4}\big[m_{11}, m_{11}^+\big\} : g_0 = \omega_m\frac{\partial}{\partial\omega_m} + \frac{d}{2} \equiv \frac{1}{2}\bigg\{\omega_m, \frac{\partial}{\partial\omega_m}\bigg\}\,, \tag{2.20}$$

being characterized by its action on $\Phi(x,\omega) = \sum_s \big\{\Phi^{s,0}(x,\omega) + \sum_{l>0}\Phi^{s,l}(x,\omega)\big\}$, for the values of the map degree $\big(\deg_\omega, \deg_{\omega/\omega^2}\big)\Phi^{s,l} = (s,l)$:

$$(g_0 - d/2)\Phi(x,\omega) = \sum_s\bigg\{s\Phi^{s,0}(x,\omega) + \sum_{l>0}(s-l)\Phi^{s,l}(x,\omega)\bigg\}\,, \tag{2.21}$$

so that the component tensors $\Phi^{s,l}(x,\omega)$ from $\Phi(x,\omega)$, for $s = l$, belong to the kernel of the operator $(g_0 - d/2)$. From the commutators:

$$\big[g_0, m_{11}^+\big\} = 2\big(m_{11}^+ - \nu\big)\,, \quad \big[g_0, m_{11}\big\} = -2\big(m_{11} - \nu\big) \tag{2.22}$$

it follows, that the nonzero number $\nu$ should be considered as a noncentral charge.

footnote d (*Continued*)

In the case of a gauge presence for $\Phi(x,\omega)$: $\delta\Phi(x,\omega) = \mathcal{R}(x,\partial_x;\omega,\partial_\omega)\varsigma(x,\omega)$, we have the determinant $\det\|G_{M_iN_i}\| = 0$, so that there exist (local) generators $R_\alpha^{N_i}$ of gauge transformations (dual to the (local) generators $\mathcal{R}(x,\partial_x;\omega,\partial_\omega)$): $\delta\lambda_i^{M_i} = R_\alpha^{N_i}\sigma^\alpha$ with their own gauge parameters $\sigma^\alpha$. The quantities $R_\alpha^{N_i}$ are proper eigenvectors of $\|G_{M_iN_i}\|$, so that we should find an invertible supermatrix $\|G_{\tilde{M}_i\tilde{N}_i}\|$ in a larger configuration space $M_\lambda = \big\{\lambda_i^{M_i}, C_{\lambda}{}_i^{\alpha_i}, \bar{C}_{\lambda}{}_i^{\alpha_i}, b_{\lambda}{}_i^{\alpha_i}\big\}$ having its own ghost $C_\lambda$, antighost $\bar{C}_\lambda$ and Nakanishi–Lautrup $b_\lambda$ fields in addition to the ones for the EoM $F(\Phi)_{N_i i}$. The only problem here which may be controlled by an appropriate choice of the initial conditions is that the maximal order of the EoM following from the $\mathcal{S}^A_{(C|)\Xi}$ is greater than the one for EoM implied by the respective actions $S_{(C|)\Xi}$.

[e]For quartic Casimir operator $C_4 = (M_{mn}P^n)^2$ evaluated for massless case on $\Phi(x,\omega)$ we have after explicit calculation with allowance made for Eqs. (2.4), (2.5) that: $C_4\Phi(x,\omega) = (l_1^+)^2(m_{11} - \nu)\Phi(x,\omega) = \Xi^2\nu\Phi(x,\omega)$, so that the relations (2.1) hold.

Table 1. Higher continuous spin symmetry algebra $\mathcal{A}(\Xi;\mathbb{R}^{1,d-1})$.

| $[\downarrow,\rightarrow\}$ | $l_0$ | $m_1$ | $m_1^+$ | $l_1$ | $l_1^+$ | $m_{11}$ | $m_{11}^+$ | $g_0$ |
|---|---|---|---|---|---|---|---|---|
| $l_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $m_1$ | 0 | 0 | $l_0$ | 0 | $l_0$ | 0 | $-2l_1^+$ | $l_1$ |
| $m_1^+$ | 0 | $-l_0$ | 0 | $-l_0$ | 0 | $2l_1$ | 0 | $-l_1^+$ |
| $l_1$ | 0 | 0 | $l_0$ | 0 | $l_0$ | 0 | $-2l_1^+$ | $l_1$ |
| $l_1^+$ | 0 | $-l_0$ | 0 | $-l_0$ | 0 | $2l_1$ | 0 | $-l_1^+$ |
| $m_{11}$ | 0 | 0 | $-2l_1$ | 0 | $-2l_1$ | 0 | $4g_0$ | $2(m_{11}-\nu)$ |
| $m_{11}^+$ | 0 | $2l_1^+$ | 0 | $2l_1^+$ | 0 | $-4g_0$ | 0 | $-2(m_{11}^+-\nu)$ |
| $g_0$ | 0 | $-l_1$ | $l_1^+$ | $-l_1$ | $l_1^+$ | $-2(m_{11}-\nu)$ | $2(m_{11}^+-\nu)$ | 0 |

Because of any linear combination of the constraints $o_I = \bigl(o_\alpha, o_\alpha^+\bigr)$ should be constraint, we have, that

$$m_1^+ - l_1^+ = \imath\Xi\,, \quad m_1 - l_1 = -\imath\Xi\,, \tag{2.23}$$

and $\Xi$ should be considered as the noncentral charge too, because of not extending the zero-mode constraint, $l_0$. Note, because of the operators $l_1^+, m_1$ cannot be imposed as the constraints on $\Phi(x,\omega)$ we could ignore the reducibility above.

Being combined, the total set of bosonic operators $o_I = \bigl\{o_A, o_a, o_a^+; g_0, \Xi, \nu\bigr\}$, for $\{o_A\} = \bigl\{l_0, l_1, l_1^+, m_1, m_1^+\bigr\}$, $\bigl\{o_a^{(+)}\bigr\} = \bigl\{m_{11}^{(+)}\bigr\}$ can be interpreted within the Hamiltonian analysis of the dynamical systems as the respective operator-valued five first-class and two second-class constraints subsystems among $\{o_I\}$ for a topological gauge system, with additional operators $g_0$, $\Xi$, $\nu$, which are not the constraints due to (2.20), and because of the commutation relations for the operators $o_I$ (forming a Lie algebra)

$$[o_I, o_J\} = f_{IJ}^K o_K\,, \quad f_{IJ}^K = -f_{JI}^K\,, \tag{2.24}$$

the following subsets can be extracted:

$$[o_a, o_b^+\} = f_{ab}^c o_c + \Delta_{ab}(g_0)\,, \quad [o_A,\ o_B\} = f_{AB}^C o_C\,, \quad [o_a,\ o_B\} = f_{aB}^C o_C\,. \tag{2.25}$$

Here, the constants $f_{ab}^c$, $f_{AB}^C$, $f_{aB}^C$ are determined by the Multiplication Table 1 and possess the antisymmetry property with respect to permutations of lower indices, whereas the quantities $\Delta_{ab}(g_0)$ form a nondegenerate $(2\times 2)$ matrix: $\|\Delta\| = \mathrm{antidiag}(-4g_0, 4g_0)$, in the space $\mathcal{V}$ of vectors $\{\Phi(x,\omega)\}$ (2.6) on the surface $\Sigma\subset\mathcal{V}$: $\|\Delta\|_{|\Sigma}\neq 0$, which is determined by the equations: $(o_A, o_a)\Phi(x,\omega) = (0,0)$.

We call the algebra of the operators $o_I$ as the *higher continuous spin symmetry algebra in Minkowski space* with notation $\mathcal{A}(\Xi;\mathbb{R}^{1,d-1})$ (shortly HCS symmetry algebra).[e] Note, first, that we omitted in Table 1 the center of $\mathcal{A}(\Xi;\mathbb{R}^{1,d-1})$

[e]One should not identify the term "*higher continuous spin symmetry algebra*" using here for free HS formulation starting from Ref. 56 with the algebraic structure known as "*higher-spin algebra*" (see, for instance Refs. 15–17) arising to describe the (half-)integer HS interactions.

consisting from $\Xi, \nu$. Second, the linear dependence of $o_k = (m_1, l_1, \Xi)$ and $o_k^+ = \left(m_1^+, l_1^+, \Xi\right)$ for $k = 1, 2, 3$ means the existence of independent bosonic proper zero eigenvectors $Z^k$; $Z^{+k}$:

$$\begin{aligned} o_k Z^k = 0\,, \quad o_k^+ Z^{+k} = 0\,, \quad \text{for} \quad Z^k = \beta(1, -1, \imath)\,, \\ Z^{+k} = \beta(1, -1, -\imath)\,, \quad \forall \beta \in \mathbb{R} \setminus \{0\}\,, \end{aligned} \tag{2.26}$$

whose set is linear independent. Third, because of the elements $\partial^\omega_m$, $\omega_m$ transfer the fields $\Phi^{s,0}(x, \omega)$, $s \geq 0$ (see (2.21)) into the fields $\theta_{s,0}\Phi^{s-1,0}_m(x, \omega)$, $\Phi^{s+1,0}_m$, the elements $o_I$ from $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$ have the same property and for their actions on the fields $\Phi^{s,l}$, $l > 0$ both operators $\partial^\omega_m$, $\omega_m$ obey by the similar property:

$$\partial^\omega_m \Phi^{s,l} = s\Phi^{s-1,l}_m - l\Phi^{s,l+1}_m\,, \quad \omega_m \Phi^{s,l} = \Phi^{s+1,l}_m\,. \tag{2.27}$$

Thus, all operators $o_I$ when acting on $\Phi(x, \omega)$, preserve the grading in $\mathcal{V}$, induced by the decomposition of $\Phi(x, \omega)$ by $g_0$: (2.21).

The algebra $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$ contains the subalgebra $\mathcal{A}_{BW}(\Xi; \mathbb{R}^{1,d-1}) = \{l_0, l_1, m_1^+, m_{11}\}$ being closed with respect to the $[\,,]$-multiplication. This algebra does not closed with respect to the formal Hermitian conjugation in $\mathcal{V}$, but may be effectively used to formulate (un-)constrained BRST–BFV description of dynamics for the CS field $\Phi(x, \omega)$.

## 3. Constrained BRST–BFV Descriptions

To construct constrained formulations we extend the results of general research,[69] realized there for HS fields with generalized integer and half-integer spins on $\mathbb{R}^{1,d-1}$, for CS case, however on the level of the equations of motion.

### 3.1. *Constrained BRST operators, BRST-extended constraints*

There are two ways to develop constrained BRST–BFV descriptions for CS field, based respectively on the HS symmetry algebras $\mathcal{A}\big(\Xi; \mathbb{R}^{1,d-1}\big)$ and $\mathcal{A}_{BW}\big(\Xi; \mathbb{R}^{1,d-1}\big)$.

#### 3.1.1. *Case of* $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$

We consider the set of the first-class constraints $\{o_A\}$ as the dynamical one with the element $\Xi$, and the off-shell algebraic constraint (one from the second-class constraints) $m_{11}$. Due to the fact that the operator $g_0$ does not now relate to CS value $\Xi$, as it was for the case of discrete spin,[69] we introduce *generating equation* for superalgebra of the Grassmann-odd constrained BRST operator, $Q_C$, and extended in the space $\mathcal{V}_C$: $\mathcal{V}_C = \mathcal{V} \otimes \mathcal{V}^{o_A}_{gh}$, off-shell constraint $\hat{M}_{11}$ in the form:

$$[Q_C, Q_C\} = 0\,, \quad [Q_C, \hat{M}_{11}\} = 0 \quad \text{for} \quad gh_H(Q_C, \hat{M}_{11}) = \epsilon(Q_C, \hat{M}_{11}) = (1, 0)\,, \tag{3.1}$$

with boundary conditions for $Q_C$, $\hat{M}_{11}$:

$$\left(\frac{\vec{\delta}}{\delta\mathcal{C}^A}, \frac{\vec{\delta}}{\delta\eta_\Xi}, \frac{\vec{\delta}}{\delta\eta_Z^+}, \frac{\vec{\delta}}{\delta\eta_Z}\right) Q_C\Big|_{\mathcal{C}=0} = \left(o_A, \Xi, \sum_k Z^k\mathcal{P}_k, \sum_k Z^{+k}\mathcal{P}_k^+\right), \qquad (3.2)$$
$$\hat{M}_{11}\big|_{\mathcal{C}=\mathcal{P}=0} = m_{11}\,,$$

when vanishing ghost coordinates, momenta $(\mathcal{C}^A, \mathcal{P}_A; \eta_\Xi, \mathcal{P}_\Xi)$ for constraints $(o_A, \Xi)$ and ones for eigenvectors $Z^k$, $Z^{+k}$: $(\eta_Z^{(+)}, \mathcal{P}_Z^{(+)})$, being by the generating elements for the space $\mathcal{V}_{gh}^{o_A}$.

The solution for the system (3.1) is sought in powers series in ghost operators with choice of some $(\mathcal{CP})$-ordering for $[C^A, \mathcal{P}_B\} = \delta^A_B$, which satisfy to the Grassmann, ghost number distributions and respective nonvanishing (anti-)commutator relations:

| | $\mathcal{C}^A$ | $\mathcal{P}_A$ | $\eta_\Xi$ | $\mathcal{P}_\Xi$ | $\eta_Z^{(+)}$ | $\mathcal{P}_Z^{(+)}$ |
|---|---|---|---|---|---|---|
| $\epsilon$ | 1 | 1 | 1 | 1 | 0 | 0 |
| $gh_H$ | 1 | −1 | 1 | −1 | 2 | −2 |

, (3.3)

$$\{\eta_1, \mathcal{P}_1^+\} = \{\eta_1^+, \mathcal{P}_1\} = 1\,, \quad \{\eta_1^m, \mathcal{P}_1^{m+}\} = \{\eta_1^{m+}, \mathcal{P}_1^m\} = 1\,,$$
$$\{\eta_0, \mathcal{P}_0\} = \{\eta_\Xi, \mathcal{P}_\Xi\} = \imath\,, \qquad [\eta_Z, \mathcal{P}_Z^+] = [\mathcal{P}_Z, \eta_Z^+] = 1\,. \qquad (3.4)$$

In (3.3) and (3.4) the formal Hermitian conjugation for the zero-mode ghosts is determined by the rule: $(\eta_0, \mathcal{P}_0, \eta_\Xi, \mathcal{P}_\Xi)^+ = (\eta_0, -\mathcal{P}_0, \eta_\Xi, -\mathcal{P}_\Xi)$ for formal Hermitian operators $l_0$, $\Xi$ from the center of $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$ with the rest ghost operators, which form the Wick pairs.

The BRST operator, $Q'$, for the Lie algebra $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$ of the constraints $o_I$ (2.24), whose linear dependence means the presence of proper zero eigenvectors $Z^I_{I_1}$: $o_I Z^I_{I_1} = 0$, $\epsilon(Z^I_{I_1}) = 0$, such that they supercommute with $o_I$: $[o_I, Z^J_{I_1}\} = 0$, should be found from the equation: $[Q', Q'\} = 2(Q')^2 = 0$, and has the form:

$$Q' = \mathcal{C}^I o_I + \frac{1}{2}\mathcal{C}^I\mathcal{C}^J f^K_{JI}\mathcal{P}_K + \mathcal{C}^{I_1} Z^I_{I_1}\mathcal{P}_I\,. \qquad (3.5)$$

Here the set of fermionic ghost operators $(\mathcal{C}^I, \mathcal{P}_J)$ for the bosonic constraints $o_I$ and bosonic ghost coordinates and momenta $(\mathcal{C}^{I_1}, \mathcal{P}_{J_1})$ for $Z^J_{I_1}$ corresponds to the minimal sector of BRST–BFV method[43] for the topological (i.e. without Hamiltonian) first-stage reducible dynamical system with the first-class constraints.

For the case of the constraints $o_A$ (2.25) whose algebra is subject to Table 1 and constant proper zero eigenvectors $Z^I_{I_1} = (Z^k, Z^{+k})$, the solution for $Q_C$ in (3.1) follows from the general anzatz (3.5) in the form:

$$\tilde{Q}_C = \mathcal{C}^A\left(o_A + \frac{1}{2}\mathcal{C}^B f^D_{BA}\mathcal{P}_D\right) + \eta_\Xi\Xi + \eta_Z\sum_k Z^{+k}\mathcal{P}_k^+ + \eta_Z^+\sum_k Z^k\mathcal{P}_k\,, \qquad (3.6)$$

for $\mathcal{P}_k = \big(\mathcal{P}^m_1, \mathcal{P}_1, \mathcal{P}_\Xi\big)$ and $\mathcal{P}^+_k = \big(\mathcal{P}^{m+}_1, \mathcal{P}^+_1, -\mathcal{P}_\Xi\big)$. Explicitly, we have

$$\begin{aligned}\tilde{Q}_C &= \eta_0 l_0 + \eta^+_1 l_1 + l^+_1\eta_1 + m_1\eta^{m+}_1 + \eta^m_1 m^+_1 \\ &\quad + \imath\big(\eta^+_1\eta_1 + \eta^{m+}_1\eta^m_1 + \eta^+_1\eta^m_1 + \eta^{m+}_1\eta_1\big)\mathcal{P}_0 \\ &\quad + \eta_\Xi\Xi + \eta_Z\big(\mathcal{P}^{m+}_1 - \mathcal{P}^+_1 - \mathcal{P}_\Xi\big) + \eta^+_Z\big(\mathcal{P}^m_1 - \mathcal{P}_1 + \mathcal{P}_\Xi\big)\,. \end{aligned} \tag{3.7}$$

Because of, first, the physical space (as the set of states being equivalent to one described by Eqs. (2.4) and the first from (2.5)), in fact, should be extracted by imposing of linear in ghost $\mathcal{C}^A, \Xi$ terms from $\tilde{Q}_C$ (see, e.g. Statement 2 in Ref. 69), second, the operator $\Xi$ cannot be imposed as the constraint on the vectors from $\mathcal{V}$, we instead consider another variant of inclusion of the term, $\eta_\Xi\Xi$, when calculating of zero ghost number cohomology of $\tilde{Q}_C$ in $\mathcal{V}_C$.

To do so, we define the representation in $\mathcal{V}_C$:

$$\begin{aligned}&\big(\eta_1, \eta^{m+}_1, \mathcal{P}_1, \mathcal{P}^{m+}_1, \mathcal{P}_0, \eta_\Xi, \eta_Z, \mathcal{P}_Z\big) \\ &\quad = \left(\frac{\partial}{\partial\mathcal{P}^+_1}, \frac{\partial}{\partial\mathcal{P}^m_1}, \frac{\partial}{\partial\eta^+_1}, \frac{\partial}{\partial\eta^m_1}, \imath\frac{\partial}{\partial\eta_0}, \imath\frac{\partial}{\partial\mathcal{P}_\Xi}, \frac{\partial}{\partial\mathcal{P}^+_Z}, \frac{\partial}{\partial\eta^+_Z}\right), \end{aligned} \tag{3.8}$$

such that the requirement $(\eta_\Xi)\Xi\chi_C = 0$, for arbitrary physical vector $\chi_C\colon \chi_C \in \mathcal{V}_C$, $gh_H(\chi_C) = 0$ to be not depending on $\mathcal{P}_\Xi$ means that we, in fact, extract only linear independent constraints, when acting on arbitrary $\tilde{\chi}_C = \tilde{\chi}_C(x, \omega, \mathcal{C}, \mathcal{P})$:

$$\begin{aligned}\tilde{\chi}_C &= \sum_n \eta_0^{n_{f0}}\big(\eta^+_1\big)^{n_{f1}}\big(\eta^m_1\big)^{n_{fm}}\big(\mathcal{P}^+_1\big)^{n_{p1}}\big(\mathcal{P}^m_1\big)^{n_{pm}}\big(\eta^+_Z\big)^{n_{fz}}\big(\mathcal{P}^+_Z\big)^{n_{pz}}\big(\mathcal{P}_\Xi\big)^{n_{p\Xi}} \\ &\quad \times \Phi(x,\omega)_{n_{f0};n_{f1},n_{fm},n_{p1},n_{pm},n_{fz},n_{pz},n_{p\Xi}}\,, \end{aligned} \tag{3.9}$$

where $n_{fz}$, $n_{pz}$ are running from 0 to $\infty$, whereas the rest $n$'s from 0 to 1. The ghost-independent vectors $\Phi(\omega)_{n_{f0};\ldots}$ have the dependence in $\omega$ according to (2.6). Thus, we resolved the linear dependence problem for the sets: $\{l_1, m_1, \Xi\}$ and $\big\{l^+_1, m^+_1, \Xi\big\}$ on the space of $\mathcal{P}_\Xi$-independent vectors (3.9) and should remove dependence on proper zero eigenvectors and respective ghosts $\eta^{(+)}_Z$, $\mathcal{P}^{(+)}_Z$ in $\tilde{Q}_C$ and $\tilde{\chi}_C$ turning to

$$\big(Q_C, \chi_C\big) = \big(\tilde{Q}_C, \tilde{\chi}_C\big)\Big|_{\left(\eta^{(+)}_Z = \mathcal{P}^{(+)}_Z = \mathcal{P}_\Xi = 0\right)}\,. \tag{3.10}$$

It provides that from $Q_C\chi_C = 0$ it follows, due to the choice of (3.8), the equations in power of ghosts: $\big(l_0 + O\big(\mathcal{C}^A\big), l_1 + O\big(\mathcal{C}^A\big), m^+_1 + O\big(\mathcal{C}^A\big)\big)\chi_C = 0$, being compatible with (2.17). It is in the agreement with the observation that the operators $l^+_1$, $m_1$ cannot be imposed as the constraints on $\Phi(x, \omega)$, so that among the operators $l^+_1$, $m^+_1$ $(l_1, m_1)$ the only $m^+_1$ $(l_1)$ is the constraint. The solution for the second equation in (3.1) can be found in the form

$$\hat{M}_{11} = m_{11} + 2\eta_1\mathcal{P}_1 + 2\eta^m_1\mathcal{P}_1\,. \tag{3.11}$$

The respective BRST-extended number particle operator $\hat{\sigma}_C(g)$, $\epsilon\big(\hat{\sigma}_C(g)\big) = 0$ (known for the discrete spin as the spin operator[69]), which should satisfy to the additional equations

$$\big[Q_C, \hat{\sigma}_C(g)\big\} = 0\,, \quad \big[\hat{M}_{11}, \hat{\sigma}_C(g)\big\} = 2\big(\hat{M}_{11} - \nu\big)\,, \tag{3.12}$$

is uniquely determined in the form[f]

$$\hat{\sigma}_C(g) = g_0 + \eta_1^+\mathcal{P}_1 - \eta_1\mathcal{P}_1^+ + \eta_1^{m+}\mathcal{P}_1 - \eta_1^m\mathcal{P}_1^+ \,. \tag{3.13}$$

3.1.2. *Case of $\mathcal{A}_{BW}(\Xi;\mathbb{R}^{1,d-1})$*

The nilpotent unconstrained $Q'_{BW}$ and constrained $Q_{BW|C}$ BRST operators for the algebra $\mathcal{A}_{BW}(\Xi;\mathbb{R}^{1,d-1})$, which contain only linear independent first-class constraints subsystems by the respective numbers 4 and 3 look as (for the Grassmann-odd ghost coordinate and momenta $\eta_{11}^+$, $\mathcal{P}_{11}$: $\{\eta_{11}^+, \mathcal{P}_{11}\} = 1$ corresponding to $m_{11}$)

$$Q'_{BW} = Q_{BW|C} + \eta_{11}^+\hat{M}_{11}^{BW} \,, \quad Q_{BW|C} = \eta_0 l_0 + \eta_1^+ l_1 + \eta_1^m m_1^+ + \imath\eta_1^+\eta_1^m\mathcal{P}_0 \,, \tag{3.14}$$

$$Q_{BW|C} = \tilde{Q}_C\big|_{\left(\eta_1^{m+}=\eta_1=\mathcal{P}_1^+=\mathcal{P}_1^m=\eta_\Xi=\mathcal{P}_\Xi=\eta_Z^{(+)}=\mathcal{P}_Z^{(+)}=0\right)} \,. \tag{3.15}$$

The BRST-extended constraint $\hat{M}_{11}^{BW}$ is determined from the generating equation, $\big[Q_{BW|C}, \hat{M}_{11}^{BW}\big\} = 0$, by the expression:

$$\hat{M}_{11}^{BW} = m_{11} + 2\eta_1^m\mathcal{P}_1 \,, \quad \hat{M}_{11}^{BW} = \hat{M}_{11} - 2\eta_1\mathcal{P}_1 \,. \tag{3.16}$$

The extended vector $\chi_{BW|C}$ from the space $\mathcal{V}_{BW|C}$: $\mathcal{V}_{BW|C} = \mathcal{V}\otimes\mathcal{V}_{gh}^{o_{BW}}$ for the primary constraints $o_{BW} = \big\{l_0, l_1, m_1^+\big\}$ has the representation

$$\chi_{BW|C} = \sum_n \eta_0^{n_{f0}}\big(\eta_1^+\big)^{n_{f1}}\big(\eta_1^m\big)^{n_{fm}}\big(\mathcal{P}_1^+\big)^{n_{p1}}\big(\mathcal{P}_1^m\big)^{n_{pm}}\Phi(x,\omega)_{n_{f0};n_{f1},n_{fm},n_{p1},n_{pm}} \,, \tag{3.17}$$

that implies $\chi_{BW|C} = \tilde{\chi}_C\big|_{(\eta_\Xi=\mathcal{P}_\Xi=\eta_Z^+=\mathcal{P}_Z^+=0)}$.

### 3.2. *Constrained dynamics*

To derive BRST constrained dynamics we should solve spectral problem for the vectors $\chi_C^l \in \mathcal{V}_C^l$ due to existence of $\mathbb{Z}$-grading in $\mathcal{V}_C$: $\mathcal{V}_C = \bigoplus_k \mathcal{V}_C^k$ for $gh_H(\chi_C^k) = -k$, $k\in\mathbb{N}_0$ for the case of $\mathcal{A}\big(\Xi;\mathbb{R}^{1,d-1}\big)$ algebra:

$$Q_C\chi_C^0 = 0 \,, \qquad \hat{M}_{11}\chi_C^0 = 0 \,, \quad (\epsilon, gh_H)\big(\chi_C^0\big) = (0,0) \,, \tag{3.18}$$

$$\delta\chi_C^0 = Q_C\chi_C^1 \,, \quad \hat{M}_{11}\chi_C^1 = 0 \,, \quad (\epsilon, gh_H)\big(\chi_C^1\big) = (1,-1) \,, \tag{3.19}$$

$$\delta\chi_C^1 = Q_C\chi_C^2 \,, \quad \hat{M}_{11}\chi_C^2 = 0 \,, \quad (\epsilon, gh_H)\big(\chi_C^2\big) = (0,-2) \,. \tag{3.20}$$

The closedness of the superalgebra of $Q_C$, $\hat{M}_{11}$ guarantees, the joint set of solution for the system (3.18)–(3.20). Thus, the physical state $\chi_C \equiv \chi_C^0$ for the vanishing of all ghost variables $\eta_0$, $\eta_1^+$, $\eta_1^m$, $\mathcal{P}_1^+$, $\mathcal{P}_1^m$, contains only the physical vector $\Phi = \Phi(x,\omega)_{0_{f0};0_{f1},0_{fm},0_{p1},0_{pm},0_{fz},0_{pz}}$ (2.6), so that

$$\chi_C^0 = \Phi + \Phi_{\text{aux}} \,, \quad \Phi_{\text{aux}}\big|_{(\eta_0,\eta_1^+,\eta_1^m,\mathcal{P}_1^+,\mathcal{P}_1^m)=0} = 0 \,. \tag{3.21}$$

[f]The operator $\hat{\sigma}_C(g)$ (3.13) is differed from the standard spin operator by the latter two terms due to $[g_0, m_1^+] = l_1^+ \neq m_1^+$.

The vectors $\chi_C^k$ inherit by the construction the decomposition (2.6): $\chi_C^k = \chi_C^{(0)k} + \sum_{l\geq 1}\chi_C^{(l)k}$, in the sum of the vectors with only positive and mixed degrees in $\omega_m$. The equations of motion: $Q_C\chi_C = 0$ $\left(\chi_C \equiv \chi_C^0\right)$ in (3.18) obtained at independent degrees in powers of the ghost oscillators are yet non-Lagrangian (due to absence of finite scalar product definition in $\mathcal{V}_C$) to be invariant with respect to reducible gauge transformations with off-shell constraints

$$Q_C\chi_C^0 = 0\,, \quad \delta\chi_C^k = \theta_{2,k}Q_C\chi_C^{k+1}\,, \quad \hat{M}_{11}\chi_C^k = 0\,, \quad k = 0,1,2\,. \tag{3.22}$$

The vanishing of all $\chi_C^l$, for $l \geq 3$ is due to the possible maximal ghost momenta degree: $\mathcal{P}_1^+\mathcal{P}_1^m$ to be realized for only $\chi_C^2$:

$$\chi_C^2 = \mathcal{P}_1^+\mathcal{P}_1^m\varpi(x,\omega) \quad \text{for} \quad \varpi(x,\omega) \equiv \Phi(x,\omega)_{0_{f0};0_{f1},0_{fm},1_{p1},1_{pm}}\,. \tag{3.23}$$

Thus, we constructed the *constrained gauge-invariant non-Lagrangian formulation of the first-stage reducibility* for the massless scalar bosonic field with CS $\Xi$ for $\nu = 1$.

Having the decomposition in ghost oscillators for the field and first-level gauge parameters $\chi_C^l$, $l = 0,1$ with $\mathbb{R}$-valued coefficient functions (as well as for $\varpi(x,\omega)$):

$$\begin{aligned}\chi_C^0 &= \Phi(\omega) + \eta_1^+\left(\mathcal{P}_1^+\chi_1(\omega) + \mathcal{P}_1^m\chi_1^m(\omega)\right) + \eta_1^m\left(\mathcal{P}_1^+\chi_2(\omega) + \mathcal{P}_1^m\chi_2^m(\omega)\right)\\ &\quad + \eta_0\left(\mathcal{P}_1^+\chi_0(\omega) + \mathcal{P}_1^m\chi_0^m(\omega) + \eta_1^+\mathcal{P}_1^+\mathcal{P}_1^m\chi_{01}(\omega) + \eta_1^m\mathcal{P}_1^+\mathcal{P}_1^m\chi_{01}^m(\omega)\right)\\ &\quad + \eta_1^+\eta_1^m\mathcal{P}_1^+\mathcal{P}_1^m\chi_{11}^m(\omega)\,,\end{aligned} \tag{3.24}$$

$$\chi_C^1 = \mathcal{P}_1^+\varsigma(\omega) + \mathcal{P}_1^m\varsigma^m(\omega) + \mathcal{P}_1^+\mathcal{P}_1^m\left(\eta_1^+\varsigma_{01}(\omega) + \eta_1^m\varsigma_{11}(\omega) + \eta_0\varsigma_0(\omega)\right)\,, \tag{3.25}$$

from the BRST-extended constraints (3.22) and structure of operator $\hat{M}_{11}$ (3.11) it follows the constraints in powers of independent ghost monomials for the gauge parameters and field vectors:

$$l = 2: \quad m_{11}\varpi = 0\,, \tag{3.26}$$

$$\begin{aligned} l = 1: \quad & m_{11}(\varsigma,\varsigma_0,\varsigma_{01}) = 0\,, \quad m_{11}\varsigma^m + 2\varsigma_{01} = 0\,,\\ & m_{11}\varsigma_{11} + 2\varsigma_{01} = 0\,,\end{aligned} \tag{3.27}$$

$$\begin{aligned} l = 0: \quad & m_{11}(\chi_0,\chi_{01}) = 0\,, \quad m_{11}\left(\chi_1,\chi_1^m,\chi_{11}^m\right) = 0\,,\\ & m_{11}\chi_0^m + 2\chi_{01} = 0\,,\end{aligned} \tag{3.28}$$

$$\begin{aligned} & m_{11}\chi_{01}^m + 2\chi_{01} = 0\,, \quad m_{11}\chi_2^m - 2\chi_{11}^m + 2\chi_1^m = 0\,,\\ & m_{11}\chi_2 + 2\chi_1 = 0\,,\end{aligned} \tag{3.29}$$

$$m_{11}\Phi + 2\chi_1 = 0\,. \tag{3.30}$$

The $\eta_0$-independent equivalent representation for the equations of motion and gauge transformations (3.22) in the supermatrix form look:

$$\begin{pmatrix} l_0 & -\Delta Q_C \\ -\Delta Q_C & (\eta_1^+ + \eta_1^{m+})(\eta_1 + \eta_1^m) \end{pmatrix} \begin{pmatrix} S_C^0(\omega) \\ B_C^0(\omega) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.31}$$

$$\delta \begin{pmatrix} S_C^l(\omega) \\ B_C^l(\omega) \end{pmatrix} = \begin{pmatrix} \Delta Q_C & -(\eta_1^+ + \eta_1^{m+})(\eta_1 + \eta_1^m) \\ l_0 & -\Delta Q_C \end{pmatrix} \times \begin{pmatrix} S_C^{l+1}(\omega) \\ B_C^{l+1}(\omega) \end{pmatrix} \theta_{2,l}, \quad l = 0, 1, 2, \tag{3.32}$$

$$\Delta Q_C = \eta_1^+ l_1 + \eta_1^m m_1^+ + l_1^+ \eta_1 + m_1 \eta_1^{m+} \tag{3.33}$$

for $\chi_C^l(\omega) = S_C^l(\omega) + \eta_0 B_C^l(\omega)$, $\chi_C^{-1}(\omega) = B_C^2(\omega) \equiv 0$. The respective gauge transformations in the ghost-independent form follow from (3.32) for the zeroth-level gauge parameters:

$$\delta(\varsigma, \varsigma^m, \varsigma_{01}, \varsigma_{11}, \varsigma_0)(\omega) = (-m_1, l_1^+, l_1, m_1^+, l_0)\varpi(\omega), \tag{3.34}$$

and for the field vectors (omitting explicit $\omega$-dependence)

$$\begin{aligned} \delta\Phi &= l_1^+ \varsigma + m_1 \varsigma^m - \varsigma_0, \\ \delta\chi_1 &= l_1 \varsigma + m_1 \varsigma_{01}, \\ \delta\chi_1^m &= l_1 \varsigma^m - l_1^+ \varsigma_{01} - \varsigma_0, \end{aligned} \tag{3.35}$$

$$\begin{aligned} \delta\chi_2 &= m_1^+ \varsigma + m_1 \varsigma_{11} - \varsigma_0, \\ \delta\chi_2^m &= m_1^+ \varsigma^m - l_1^+ \varsigma_{11}, \\ \delta\chi_{11}^m &= l_1 \varsigma_{11} - m_1^+ \varsigma_{01} - \varsigma_0, \end{aligned} \tag{3.36}$$

$$\delta\chi_0 = m_1 \varsigma_0 + l_0 \varsigma, \quad \delta\chi_0^m = -l_1^+ \varsigma_0 + l_0 \varsigma^m, \tag{3.37}$$

$$\delta\chi_{01} = -l_1 \varsigma_0 + l_0 \varsigma_{01}, \quad \delta\chi_{01}^m = -m_1^+ \varsigma_0 + l_0 \varsigma_{11}. \tag{3.38}$$

The respective ghost-independent equations of motion from (3.31) take the form in powers of ghost monomials $C(C\mathcal{P})^k$ for $k = 0$:

$$\eta_0 : \quad l_0 \Phi - l_1^+ \chi_0 - m_1 \chi_0^m = 0, \tag{3.39}$$

$$\eta_1^+ : \quad l_1 \Phi - l_1^+ \chi_1 - m_1 \chi_1^m - \chi_0 - \chi_{01} = 0, \tag{3.40}$$

$$\eta_1^m : \quad m_1^+ \Phi - l_1^+ \chi_2 - m_1 \chi_2^m + \chi_0^m - \chi_{01}^m = 0, \tag{3.41}$$

$$m_{11} \Phi + 2\chi_1 = 0, \quad m_{11} \chi_1 = 0, \tag{3.42}$$

as well as for $k=1,2$:

$$\eta_0\eta_1^+\mathcal{P}_1^+ : \quad l_0\chi_1 - l_1\chi_0 - m_1\chi_{01} = 0\,, \tag{3.43}$$

$$\eta_0\eta_1^+\mathcal{P}_1^m : \quad l_0\chi_1^m - l_1\chi_0^m + l_1^+\chi_{01} = 0\,, \tag{3.44}$$

$$\eta_0\eta_1^m\mathcal{P}_1^+ : \quad l_0\chi_2 - m_1^+\chi_0 - m_1\chi_{01}^m = 0\,, \tag{3.45}$$

$$\eta_0\eta_1^m\mathcal{P}_1^m : \quad l_0\chi_2^m - m_1^+\chi_0^m + l_1^+\chi_{01}^m = 0\,, \tag{3.46}$$

$$\eta_0\eta_1^+\eta_1^m\mathcal{P}_1^+\mathcal{P}_1^m : \quad l_0\chi_{11}^m + m_1^+\chi_{01} - l_1\chi_{01}^m = 0\,, \tag{3.47}$$

$$\eta_1^+\eta_1^m\mathcal{P}_1^+ : \quad -m_1^+\chi_1 + l_1\chi_2 - m_1\chi_{11}^m - \chi_0 - \chi_{01} = 0\,, \tag{3.48}$$

$$\eta_1^+\eta_1^m\mathcal{P}_1^m : \quad -m_1^+\chi_1^m + l_1\chi_2^m + l_1^+\chi_{11}^m - \chi_0^m + \chi_{01}^m = 0\,. \tag{3.49}$$

Thus, the relations (3.34)–(3.49) determine the constrained gauge theory of the first-stage reducibility for the massless free field $\Phi(x,\omega)$ of CS $\Xi$ in $\mathbb{R}^{1,d-1}$ subject to the constraints (3.26)–(3.30), (3.42) with 9 auxiliary tensor fields. This theory is non-Lagrangian with off-shell holonomic constraint (3.42) and corresponds to the algebra $\mathcal{A}(\Xi;\mathbb{R}^{1,d-1})$.

The special structure of the constraints permits to make gauge-fixing procedure starting from the lowest gauge parameter $\varpi$ which together with the linear combination of the gauge parameters $(\varsigma^m - \varsigma_{11})$ (that follows from (3.27)) belongs to the set of $\ker m_{11}$. After invertible change of the basis of the zeroth-level gauge parameters:

$$\big\{\varsigma,\varsigma^m,\varsigma_{01},\varsigma_{11},\varsigma_0\big\} \to \big\{\varsigma,\tilde{\varsigma}^m,\varsigma_{01},\tilde{\varsigma}_{11},\varsigma_0\big\} \quad \text{for} \quad \big(\tilde{\varsigma}^m,\tilde{\varsigma}_{11}\big) = \frac{1}{2}\big(\varsigma^m \mp \varsigma_{11}\big) \tag{3.50}$$

we may gauge away the parameter $\tilde{\varsigma}^m$ from $\delta\tilde{\varsigma}^m = -\frac{\imath}{2}\Xi\varpi$ by means of complete using of $\varpi$. Now, the theory becomes by irreducible gauge theory with independent gauge-invariant parameters $\varsigma$, $\varsigma_{01}$, $\tilde{\varsigma}_{11}$, $\varsigma_0$ for $m_{11}^2\tilde{\varsigma}_{11} = 0$ and with the rest parameters satisfying to the first constraints in (3.27).

Turning to the field vectors we replace the parameters $\varsigma^m$, $\varsigma_{11}$ in (3.35)–(3.38) on $\tilde{\varsigma}_{11}$ and see, that two pairs of the fields $\chi_0^m$, $\chi_{01}^m$ and $\chi_1^m$, $\chi_{11}^m$ obey to similar constraints in (3.28)–(3.30) as the parameters $\big(\varsigma^m - \varsigma_{11}\big) = 2\tilde{\varsigma}_{11}$. Making invertible change of the basis of the fields:

$$\big\{\chi_0^m,\chi_{01}^m,\chi_1^m,\chi_{11}^m\big\} \to \big\{\tilde{\chi}_0^m,\tilde{\chi}_{01}^m,\tilde{\chi}_1^m,\tilde{\chi}_{11}^m\big\}$$
$$\text{for} \quad \big(\tilde{\chi}_0^m,\tilde{\chi}_{01}^m;\tilde{\chi}_1^m,\tilde{\chi}_{11}^m\big) = \frac{1}{2}\big(\chi_0^m \mp \chi_{01}^m;\chi_1^m \mp \chi_{11}^m\big)\,, \tag{3.51}$$

with untouched rest fields: $\Phi$, $\chi_0$, $\chi_{01}$, $\chi_1$, $\chi_2$, $\chi_2^m$, we may gauge away the fields $\tilde{\chi}_0^m$ from $\delta\tilde{\chi}_0^m = \frac{\imath}{2}\Xi\varsigma_0$ and $\tilde{\chi}_1^m$ from $\delta\tilde{\chi}_1^m = \frac{\imath}{2}\Xi\varsigma_{01}$ by means of complete using of $\varsigma_0$ and $\varsigma_{01}$, respectively, in view of theirs satisfaction to the same constraint. Then, from the gauge transformation, $\delta\chi_2^m = \imath\Xi\tilde{\varsigma}_{11}$ we gauge away the field $\chi_2^m$, which now obeys, together with $\tilde{\varsigma}_{11}$, to the constraints: $m_{11}\big|\chi_2^m\rangle = m_{11}\tilde{\varsigma}_{11} = 0$ (3.29) after using $\varsigma_{01}$. Thus, the following 7 fields with gauge transformations survive after partial gauge-fixing with unique gauge parameter $\varsigma$, $m_{11}\varsigma = 0$:

$$\delta\big(\Phi, \chi_1, \chi_2, \tilde{\chi}^m_{11}, \chi_0, \chi_{01}, \tilde{\chi}^m_{01}\big) = \big(l_1^+, l_1, m_1^+, 0, l_0, 0, 0\big)\varsigma\,. \tag{3.52}$$

From the transformed equations of motion (3.46), (3.49): $\imath\Xi\big(\tilde{\chi}^m_{01}, \tilde{\chi}^m_{11}\big) = 0$ follow vanishing of the fields $\tilde{\chi}^m_{01}$, $\tilde{\chi}^m_{11}$. Therefore, the remaining equations (3.39)–(3.49) transform as follows except for (3.43):

$$l_0\Phi - l_1^+\chi_0 = 0\,, \quad l_1\Phi - l_1^+\chi_1 - \chi_0 - \chi_{01} = 0\,, \quad m_1^+\Phi - l_1^+\chi_2 = 0\,, \tag{3.53}$$

$$l_1^+\chi_{01} = 0\,, \quad m_1^+\chi_{01} = 0\,, \quad l_0\chi_2 - m_1^+\chi_0 = 0\,, \tag{3.54}$$

$$-m_1^+\chi_1 + l_1\chi_2 - \chi_0 - \chi_{01} = 0\,. \tag{3.55}$$

The first two equations in (3.54) have unique general solution: $\chi_{01} = 0$. The resulting equations of motion for the rest 4 fields take the form

$$l_0\Phi - l_1^+\chi_0 = 0\,, \qquad l_1\Phi - l_1^+\chi_1 - \chi_0 = 0\,, \qquad l_0\chi_1 - l_1\chi_0 = 0\,, \tag{3.56}$$

$$l_0\chi_2 - m_1^+\chi_0 = 0\,, \quad l_1\chi_2 - m_1^+\chi_1 - \chi_0 = 0\,, \quad m_1^+\Phi - l_1^+\chi_2 = 0\,, \tag{3.57}$$

which may be considered with account of algebraic traceless constraints (3.28)–(3.30) as the *triplet-like non-Lagrangian formulation* for scalar bosonic field with CS in the Bargmann–Wigner form, due to presence of the field $\chi_0(\omega)$ by analogy with case of HS fields with integer spin.[86–88] Indeed, Eqs. (3.56) do not contain the field $\chi_2(\omega)$ and coincide with the conditions which determine the triplet formulation for the case of integer spin, but with more involved structure of the fields. Almost the same triplet interpretation (if makes the change $l_1^{(+)} \to m_1^{(+)}$) valid for the latter equation in (3.56) and two first equations in (3.57) for the triplet: $\chi_2(\omega)$, $\chi_0(\omega)$, $\chi_1(\omega)$. The latter equation in (3.57) entangles the basic fields: $\Phi$, $\chi_2$ from both triplet equations.

After resolution of the second algebraic equation of motion in (3.56) with respect to $\chi_0$ we get the system:

$$\big(l_0 - l_1^+l_1\big)\Phi + \big(l_1^+\big)^2\chi_1 = 0\,, \quad \big(l_0 + l_1l_1^+\big)\chi_1 - (l_1)^2\Phi = 0\,, \tag{3.58}$$

$$\begin{aligned} l_0\chi_2 - m_1^+l_1\Phi + m_1^+l_1^+\chi_1 &= 0\,,\\ l_1(\chi_2 - \Phi) - \imath\Xi\chi_1 &= 0\,,\\ l_1^+(\chi_2 - \Phi) - \imath\Xi\Phi &= 0\,, \end{aligned} \tag{3.59}$$

given in the configuration space $\mathcal{M}_{cl}$ parametrized by $\Phi(\omega)$, $\chi_1(\omega)$, $\chi_2(\omega)$ subject to the gauge transformations (3.52) and to the independent holonomic constraints

$$\begin{aligned} m_{11}\chi_1(\omega) = m_{11}\varsigma(\omega) &= 0\,,\\ m_{11}\chi_2(\omega) + 2\chi_1(\omega) &= 0\,,\\ m_{11}\Phi(\omega) + 2\chi_1(\omega) &= 0\,. \end{aligned} \tag{3.60}$$

Equations (3.58) and (3.59) may be interpreted as the *duplet-like non-Lagrangian formulation* for scalar bosonic field with CS in the Bargmann–Wigner form by analogy with case of HS fields with integer spin.[86–88] Equations (3.58) coincide by the form with the conditions which determine the duplet formulation for discrete spin.

If we gauge away the field $\chi_1(\omega)$ by using the part of the gauge parameter $\varsigma(\omega)$, which therefore should be divergentless, $l_1\varsigma(\omega) = 0$, (for details, see Subsec. 3.4), the same gauge-fixing procedure may be applied for now $m_{11}$-traceless field $\chi_2$ so that the field $\chi_2$ may be removed completely by means of using of the gauge transformations with remaining degrees of freedom in $\varsigma(\omega)$ so that the remaining from the systems (3.58)–(3.60) equations on the initial field $\Phi(\omega)$ coincide with the IR conditions (2.4), (2.5).

Expressing the field $\chi_1(\omega)$ as generalized trace of the basic field $\Phi(\omega)$: $\chi_1 = -\frac{1}{2}m_{11}\Phi$, according to (3.60), we derive from the duplet-like formulation (3.56), (3.59) and (3.60) the system:

$$\begin{aligned} &\Big\{l_0 - l_1^+ l_1 - \frac{1}{2}(l_1^+)^2 m_{11}\Big\}\Phi \equiv \mathcal{L}_0\Phi = 0\,, \\ &\Big\{\frac{1}{2}(l_0 + l_1 l_1^+)m_{11} + (l_1)^2\Big\}\Phi = \hat{\mathcal{L}}_0\Phi = 0\,, \end{aligned} \tag{3.61}$$

$$\begin{aligned} &l_0\chi_2 - m_1^+\Big(l_1 + \frac{1}{2}l_1^+ m_{11}\Big)\Phi \equiv \mathcal{F}(\Phi,\chi_2) = 0\,, \\ &l_1^+(\chi_2 - \Phi) - \imath\Xi\Phi \equiv \mathcal{L}_1^+(\Phi,\chi_2) = 0\,, \end{aligned} \tag{3.62}$$

$$l_1(\chi_2 - \Phi) + \frac{1}{2}\imath\Xi m_{11}\Phi = \mathcal{L}_1(\Phi,\chi_2) = 0\,, \quad m_{11}\big(\chi_2 - \Phi\big) = m_{11}^2\Phi = 0\,. \tag{3.63}$$

The second equation in (3.61) and the first one in (3.63) are the respective algebraic consequences of the first equation in (3.61) and the second one (3.62), when applying to the latter ones of the trace operator $m_{11}$:

$$\hat{\mathcal{L}}_0\Phi = m_{11}\mathcal{L}_0\Phi\,, \quad \mathcal{L}_1(\Phi,\chi_2) = m_{11}\mathcal{L}_1^+(\Phi,\chi_2)\,. \tag{3.64}$$

Thus, the independent system of equations consists from 3 differential equations and 2 $m_{11}$-traceless (holonomic) constraints to be invariant with respect to gauge transformations, $\delta(\Phi(\omega),\chi_2(\omega)) = \big(l_1^+, m_1^+\big)\varsigma(\omega)$ for $m_{11}\varsigma(\omega) = 0$. Note, the imposing of the above described gauge-fixing procedure on the auxiliary field $\chi_2$ turns the system (3.61)–(3.63) to the Bargmann–Wigner equations (2.4), (2.5).

Presenting the independent gauge-invariant equations (3.58)–(3.63) in powers of $\omega^{(m)_k} \times \frac{\omega^{(n)_l}}{\omega^{2l}}$, $k,l \in \mathbb{N}_0$ similar to Eqs. (2.12), (2.13):

$$\begin{aligned} &\omega^{(m)_k}\frac{\omega^{(n)_l}}{\omega^{2l}} : \big(\mathcal{L}_0\Phi\big)^l_{l|(m)_k,(n)_l} = 0\,, \\ &\big(\mathcal{L}_1^+(\Phi,\chi_2)\big)^l_{l|(m)_k,(n)_l} = 0\,, \quad \big(\mathcal{F}(\Phi,\chi_2)\big)^l_{l|(m)_k,(n)_l} = 0\,, \end{aligned} \tag{3.65}$$

we get with help of the real-valued Lagrangian multipliers $\hat{\lambda}_i^{l|(m)_k,(n)_l}$, $i = 1, 2, 3$, $k$, $l \in \mathbb{N}_0$ (being different with ones used in (2.15)) the constrained gauge-invariant LF for CSR scalar (real-valued) field $\Phi$ of CS $\Xi$ in the tensor form

$$\begin{aligned}\mathcal{S}_{C|\Xi}(\Phi, \chi_2, \hat{\lambda}_i) = \int d^dx \sum_{k,l\geq 0} \Big\{ &\hat{\lambda}_1^{l|(m)_k,(n)_l} (\mathcal{L}_0\Phi)^l_{(m)_k,(n)_l} \\ &+ \imath \hat{\lambda}_2^{l|(m)_k,(n)_l} \big(\mathcal{L}_1^+(\Phi, \chi_2)\big)^l_{l|(m)_k,(n)_l} \\ &+ \hat{\lambda}_3^{l|(m)_k,(n)_l} \big(\mathcal{F}(\Phi, \chi_2)\big)^l_{l|(m)_k,(n)_l} \Big\}, \end{aligned} \tag{3.66}$$

$$\begin{aligned}&\delta\big(\Phi^l, \chi_2^l\big)_{(m)_k,(n)_l} \\ &\quad = -\Big(\partial_{\{m_k}\varsigma^l_{(m)_{k-1}\},(n)_l}, \partial_{\{m_k}\varsigma^l_{(m)_{k-1}\},(n)_l} - \imath\Xi\eta_{\{m_k\{n_l}\varsigma^{l-1}_{(m)_{k-1}\},(n)_{l-1}\}}\Big), \end{aligned} \tag{3.67}$$

$$\delta\hat{\lambda}_i^{l(m)_k,(n)_l}(x) = \sum_{k',l'} \hat{R}^{l(m)_k,(n)_l}_{il'(m_1)_{k'}(n_1)_{l'}}(x, \partial_x)\sigma^{l'(m_1)_{k'}(n_1)_{l'}}(x) \Leftrightarrow \delta\hat{\lambda}_i^{M_i} = \hat{R}^{N_i}_{\alpha}\sigma^{\alpha} \tag{3.68}$$

with certain generators $\hat{R}^{l(m)_k,(n)_l}_{il'(m_1)_{k'}(n_1)_{l'}}$ of gauge transformations for $\hat{\lambda}_i^{l(m)_k,(n)_l}$, $\delta_\sigma\mathcal{S}_{C|\Xi} = 0$, being dual (see footnote d) to those for the fields, whose specific form may be determined explicitly. Here, we should impose the double (single) $m_{11}$-traceless holonomic constraints on the auxiliary fields $\hat{\lambda}_i^{l|(m)_k,(n)_l}$ (on dual gauge parameters: $(m_{11}\sigma)^\alpha = 0$) due to the structure of the action and the constraints (3.63) on the fields $\Phi^{l|(m)_k,(n)_l}, \chi_2^{l|(m)_k,(n)_l}$ and gauge parameters $\varsigma^l_{p(m)_k,(n)_l}$ according to the last representation in (2.13):

$$\begin{aligned}\big(m_{11}^2\hat{\lambda}_i\big)^{l|(m)_k,(n)_l} &= \big(m_{11}^2\Phi\big)^l_{(m)_k,(n)_l} = \big(m_{11}\big(\chi_2 - \Phi\big)\big)^l_{(m)_k,(n)_l} \\ &= \big(m_{11}\varsigma\big)^l_{(m)_k,(n)_l} = 0\,. \end{aligned} \tag{3.69}$$

The analogous forms of the constrained gauge-invariant LFs may be formulated with help of respective sets of the Lagrangian multipliers for the triplet-like non-Lagrangian formulation (3.56), (3.57) (see, Eq. (5.4) in Sec. 5) and for duplet-like non-Lagrangian formulation (3.58), (3.59) within Bargmann–Wigner form of the CSR equations.

One should stress that an analysis of the form for the nonscalar field $\Psi(x,\omega)_{(\mu^1)_{s_1}\cdots(\mu^k)_{s_k}}$ with CS, $\Xi$, and integer generalized spin, $\mathbf{s} = (s_1, \ldots, s_k)$, $k \leq [(d-4)/2]$,[21] should be necessary by a gauge-invariant theory with reducible gauge symmetry, and based on the respective HCS symmetry algebra.

### 3.2.1. *On higher continuous spin symmetry algebra* $\mathcal{A}(\Xi; Y(k), \mathbb{R}^{1,d-1})$

The most general massless nonscalar CS one-valued irreducible representation of Poincaré group in a Minkowski space $\mathbb{R}^{1,d-1}$ is described by a tensor field

$\Psi(\omega)_{(\mu^1)_{s_1}\cdots(\mu^k)_{s_k}} \equiv \Psi(x,\omega)_{\mu^1_1\cdots\ \cdots\mu^1_{s_1},\mu^2_1\cdots\mu^2_{s_2},\ldots,\mu^k_1\cdots\mu^k_{s_k}}$ of rank $\sum_{i\geq 1}^{k} s_i$ to be corresponding to a Young tableaux with $k$ rows of length $s_1, s_2, \ldots, s_k$, respectively

$$\Psi(\omega)_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}} \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|c|}\hline \mu^1_1 & \mu^1_2 & \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot & \mu^1_{s_1} \\ \hline \mu^2_1 & \mu^2_2 & \cdot\ \cdot\ \cdot & \cdot & \cdot & \mu^2_{s_2} \\ \cline{1-6} \cdots \\ \cline{1-1} \mu^k_1 & \mu^k_2 & \cdot\ \cdot\ \cdot & \cdot & \mu^k_{s_k} \\ \cline{1-5} \end{array} \,. \quad (3.70)$$

This field is symmetric with respect to the permutations of each type of Lorentz indices $\mu^i$, and obeys in addition to (2.4)–(2.5) to the Klein–Gordon, divergentless (3.71), traceless (3.72) and mixed-symmetry equations (3.73) (for $i,j = 1,\ldots,k$; $l_i$, $m_i = 1,\ldots,s_i$):

$$\begin{aligned} \partial^\mu\partial_\mu \Psi(\omega)_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}} &= 0\,, \\ \partial^{\mu^i_{l_i}} \Psi(\omega)_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}} &= 0\,, \end{aligned} \quad (3.71)$$

$$\begin{aligned} &\eta^{\mu^i_{l_i}\mu^i_{m_i}} \Psi(\omega)_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}} \\ &\quad = \eta^{\mu^i_{l_i}\mu^j_{m_j}} \Psi(\omega)_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}} = 0\,, \quad l_i < m_i\,, \end{aligned} \quad (3.72)$$

$$\Psi(\omega)_{(\mu^1)_{s_1},\ldots,\{(\mu^i)_{s_i}\underbrace{,\ldots,\mu^j_1\cdots}\mu^j_{l_j}\}\cdots\mu^j_{s_j},\ldots,(\mu^k)_{s_k}} = 0\,, \quad i<j\,, \quad 1 \leq l_j \leq s_j\,, \quad (3.73)$$

where the bracket below denote that the indices in it do not include in symmetrization, i.e. the symmetrization concerns only indices $(\mu^i)_{s_i}, \mu^j_{l_j}$ in $\{(\mu^i)_{s_i}\underbrace{,\ldots,\mu^j_1\cdots}\mu^j_{l_j}\}$.

For a joint description of all the CSRs with given CS, but different spin $\mathbf{s}$ we following to the case of HS fields with only integer spin[65] introduce an auxiliary vector space $\mathcal{V}_k$, ($\mathcal{V}_0 \equiv \mathcal{V}$) generated in addition to $\omega$ by $k$ sets of bosonic variables $\vec{\omega} = (\omega^1_{\mu^1}, \ldots, \omega^k_{\mu^k})$, $i$, $j = 1,\ldots,k$; $\mu^i$, $\nu^j = 0,1,\ldots,d-1$, and a set of constraints for an arbitrary string-like vector $\Psi(x,\omega,\vec{\omega}) \in \mathcal{V}_k$,

$$\Psi(x,\omega,\vec{\omega}) = \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{s_1}\cdots\sum_{s_k=0}^{s_{k-1}} \Psi(x,\omega)_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}} \prod_{i=1}^{k}\prod_{l_i=1}^{s_i} \omega_i^{\mu^i_{l_i}}\,, \quad (3.74)$$

$$\begin{aligned} &\big(l_0, l^i, l^{ij}, t^{i_1 j_1}\big)\Psi(x,\omega,\vec{\omega}) \\ &\quad = \left(\partial^2, -\frac{\partial}{\partial\omega_i^\mu}\partial^\mu, -\frac{1}{2}\eta^{mn}\frac{\partial}{\partial\omega_i^m}\frac{\partial}{\partial\omega_j^n}, \omega_\mu^{i_1}\frac{\partial}{\partial\omega_{j_1\mu}}\right)\Psi(x,\omega,\vec{\omega}) = 0\,, \end{aligned} \quad (3.75)$$

(for $i \leq j; i_1 < j_1$). The set of $(k(k+1)+1)$ primary constraints (3.75) with $\{o^{\mathbf{s}}_\alpha\} = \{l_0, l^i, l^{ij}, t^{i_1 j_1}\}$ describes all CSR with given CS $\Xi$, if, in addition, the constraints $o_{\hat{\alpha}}$ hold

$$\big(l_0, l_1, m_1^+, m_{11}\big)\Psi(x,\omega,\vec{\omega}) = 0\,. \quad (3.76)$$

In turn, if we impose to Eqs. (3.75) the additional constraints with number particles operators, $g_0^i$,

$$g_0^i\Psi(x,\omega,\vec{\omega}) = \big(s_i + d/2\big)\Psi(x,\omega,\vec{\omega})\,, \quad g_0^i = \frac{1}{2}\Big\{\omega^i_\mu, \frac{\partial}{\partial\omega^i_\mu}\Big\}\,, \tag{3.77}$$

then these combined conditions [which are reduced to ones (2.17) for scalar CS field, $\Phi(x,\omega) = \Psi(x,\omega,\vec{0})$] are equivalent to Eqs. (3.71)–(3.73) for the field $\Psi(\omega)_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}}$ with given spin $(\Xi,\mathbf{s})$.

The procedure of LF implies the property of BRST–BFV operator $Q$, $Q = C^\alpha o^{\mathbf{s}}_\alpha + C^{\hat\alpha} o_{\hat\alpha}$+more, to be Hermitian, that is equivalent to the formal requirements: $\{o^{\mathbf{s}}_\alpha\}^+ = \{o^{\mathbf{s}}_\alpha\}$, $\{o_{\hat\alpha}\}^+ = \{o_{\hat\alpha}\}$ and closedness for $\{o_{\hat\alpha}, o^{\mathbf{s}}_\alpha\}$ with respect to the commutator multiplication $[\,,\,]$. To provide these conditions we consider an quasi-scalar product, $(\,,\,)$, on $\mathcal{V}_k$:

$$\begin{aligned}&\left(\Omega\Big(\omega,\frac{\partial}{\partial\vec{\omega}}\Big),\Psi(\omega,\vec{\omega})\right)\\ &\quad = \int d^dx \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{s_1}\cdots\sum_{s_k=0}^{s_{k-1}}\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{p_1}\cdots\sum_{p_k=0}^{p_{k-1}}\Omega^*(x,\omega)_{(\nu^1)_{p_1},(\nu^2)_{p_2},\ldots,(\nu^k)_{p_k}}\\ &\qquad\times\prod_{j=1}^{k}\prod_{m_j=1}^{p_j}\frac{\partial}{\partial\omega_j^{\nu^j_{m_j}}}\Psi(x,\omega)_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}}\prod_{i=1}^{k}\prod_{l_i=1}^{s_i}\omega_i^{\mu^i_{l_i}}\,.\end{aligned} \tag{3.78}$$

The quasi-scalar product $(\,,\,)$ presents the $\omega$-dependent bilinear operation on $\mathcal{V}_k^*\otimes\mathcal{V}_k$ being differed from the standard scalar product on the Fock space $\mathcal{H}$ used for HS fields with only integer spin $\mathbf{s}$, see Ref. 65 [under identification of the oscillators $\big(a^i_{\mu^i}, a^{j+}_{\nu^j}\big) \equiv -\imath\big(\partial/\partial\omega_i^{\mu^i}, \omega^{j+}_{\nu^j}\big)$ with commutation relations, $\big[a^i_{\mu^i}, a^{j+}_{\nu^j}\big] = -\eta_{\mu^i\nu^j}\delta^{ij}\big]$. As the result, the set of $\{o_{\hat\alpha}\}$ and $\big\{o^{\mathbf{s}}_\alpha\big\}$ are extended respectively, formally to $o_I$ (2.24) and to $\{o^{\mathbf{s}}_P\} = \big\{o^{\mathbf{s}}_\alpha, (o^{\mathbf{s}}_\alpha)^+, g^i_0\big\}$ extended by means of the operators $\big(o^{\mathbf{s}}_\alpha\big)^+$,

$$\big(l^{i+}, l^{ij+}, t^{i_1j_1+}\big) = \left(-\omega^i_\mu\partial^\mu, -\frac{1}{2}\omega^i_\mu\omega^{j\mu}, \omega^{j_1}_\mu\frac{\partial}{\partial\omega_{i_1\mu}}\right), \quad i\le j\,, \quad i_1 < j_1\,, \tag{3.79}$$

with taken into account of formally self-conjugated operators, $\big(l_0^+, {g_0^i}^+\big) = \big(l_0, g_0^i\big)$.

The set of all operators $o_W$ has therefore the structure,

$$\{o_W\} = \big\{o_I; o^{\mathbf{s}}_P\big\} = \big\{o_{\hat\alpha}, o^+_{\hat\alpha}, g_0, \Xi, \nu; o^{\mathbf{s}}_\alpha, o^{\mathbf{s}+}_\alpha; g^i_0\big\}\,, \quad \big[o_I, o^{\mathbf{s}}_P\big] = 0\,. \tag{3.80}$$

Explicitly, operators $o_W$ satisfy to the Lie-algebra commutation relations,

$$[o_W, o_X] = f^Y_{WX}o_Y\,, \quad f^Y_{WX} = \big(f^K_{IJ}, f^Q_{PR}\big) = -f^Y_{XW}\,, \tag{3.81}$$

where the structure constants $f^Y_{WX}$ with nonvanishing components $\big(f^K_{IJ}, f^Q_{PR}\big)$ are determined from the multiplication Tables 1 and 2.

Table 2. Higher continuous spin symmetry algebra $\mathcal{A}(\Xi; Y(k), \mathbb{R}^{1,d-1})$.

| $[\downarrow, \rightarrow\}$ | $t^{i_1j_1}$ | $t^+_{i_1j_1}$ | $l_0$ | $l^i$ | $l^{i+}$ | $l^{i_1j_1}$ | $l^{i_1j_1+}$ | $g_0^i$ |
|---|---|---|---|---|---|---|---|---|
| $t^{i_2j_2}$ | $A^{i_2j_2,i_1j_1}$ | $B^{i_2j_2}{}_{i_1j_1}$ | 0 | $l^{j_2}\delta^{i_2i}$ | $-l^{i_2+}\delta^{j_2i}$ | $l^{\{j_1j_2}\delta^{i_1\}i_2}$ | $-l^{i_2\{i_1+}\delta^{j_1\}j_2}$ | $F^{i_2j_2,i}$ |
| $t^+_{i_2j_2}$ | $-B^{i_1j_1}{}_{i_2j_2}$ | $A^+_{i_1j_1,i_2j_2}$ | 0 | $l_{i_2}\delta^i_{j_2}$ | $-l^+_{j_2}\delta^i_{i_2}$ | $l_{i_2}{}^{\{j_1}\delta^{i_1\}}_{j_2}$ | $-l_{j_2}{}^{\{j_1+}\delta^{i_1\}}_{i_2}$ | $-F_{i_2j_2}{}^{i+}$ |
| $l_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $l^j$ | $-l^{j_1}\delta^{i_1j}$ | $-l_{i_1}\delta^j_{j_1}$ | 0 | 0 | $l_0\delta^{ji}$ | 0 | $-\frac{1}{2}l^{\{i_1+}\delta^{j_1\}j}$ | $l^j\delta^{ij}$ |
| $l^{j+}$ | $l^{i_1+}\delta^{j_1j}$ | $l^+_{j_1}\delta^j_{i_1}$ | 0 | $-l_0\delta^{ji}$ | 0 | $\frac{1}{2}l^{\{i_1}\delta^{j_1\}j}$ | 0 | $-l^{j+}\delta^{ij}$ |
| $l^{i_2j_2}$ | $-l^{j_1\{j_2}\delta^{i_2\}i_1}$ | $-l_{i_1}{}^{\{i_2+}\delta^{j_2\}}_{j_1}$ | 0 | 0 | $-\frac{1}{2}l^{\{i_2}\delta^{j_2\}i}$ | 0 | $L^{i_2j_2,i_1j_1}$ | $l^{i\{i_2}\delta^{j_2\}i}$ |
| $l^{i_2j_2+}$ | $l^{i_1\{i_2+}\delta^{j_2\}j_1}$ | $l_{j_1}{}^{\{j_2+}\delta^{i_2\}}_{i_1}$ | 0 | $\frac{1}{2}l^{\{i_2+}\delta^{ij_2\}}$ | 0 | $-L^{i_1j_1,i_2j_2}$ | 0 | $-l^{i\{i_2+}\delta^{j_2\}i}$ |
| $g_0^j$ | $-F^{i_1j_1,j}$ | $F_{i_1j_1}{}^{j+}$ | 0 | $-l^i\delta^{ij}$ | $l^{i+}\delta^{ij}$ | $-l^{j\{i_1}\delta^{j_1\}j}$ | $l^{j\{i_1+}\delta^{j_1\}j}$ | 0 |

Note that in Table 2, the operators $t^{i_2j_2}$, $t^+_{i_2j_2}$ satisfy the properties

$$\big(t^{i_2j_2}, t^+_{i_2j_2}\big) \equiv \big(t^{i_2j_2}, t^+_{i_2j_2}\big)\theta^{j_2i_2}\,, \tag{3.82}$$

the products $B^{i_2j_2}_{i_1j_1}$, $A^{i_2j_2,i_1j_1}$, $F^{i_1j_1,i}$, $L^{i_2j_2,i_1j_1}$ are determined by the explicit relations,[65]

$$\begin{aligned}
B^{i_2j_2}{}_{i_1j_1} &= \big(g_0^{i_2} - g_0^{j_2}\big)\delta^{i_2}_{i_1}\delta^{j_2}_{j_1} + \big(t_{j_1}{}^{j_2}\theta^{j_2}{}_{j_1} + t^{j_2}{}^+_{j_1}\theta_{j_1}{}^{j_2}\big)\delta^{i_2}_{i_1}\\
&\quad - \big(t^{+\,i_2}_{i_1}\theta^{i_2}{}_{i_1} + t^{i_2}{}_{i_1}\theta_{i_1}{}^{i_2}\big)\delta^{j_2}_{j_1}\,,\\
A^{i_2j_2,i_1j_1} &= t^{i_1j_2}\delta^{i_2j_1} - t^{i_2j_1}\delta^{i_1j_2}\,, \quad F^{i_2j_2,i} = t^{i_2j_2}\big(\delta^{j_2i} - \delta^{i_2i}\big)\,,\\
L^{i_2j_2,i_1j_1} &= \frac{1}{4}\big\{\delta^{i_2i_1}\delta^{j_2j_1}\big[2g_0^{i_2}\delta^{i_2j_2} + g_0^{i_2} + g_0^{j_2}\big]\\
&\quad - \delta^{j_2\{i_1}\big[t^{j_1\}i_2}\theta^{i_2j_1\}} + t^{i_2j_1\}+}\theta^{j_1\}i_2}\big]\\
&\quad - \delta^{i_2\{i_1}\big[t^{j_1\}j_2}\theta^{j_2j_1\}} + t^{j_2j_1\}+}\theta^{j_1\}j_2}\big]\big\}\,,
\end{aligned} \tag{3.83}$$

with known properties of theirs antisymmetry and Hermitian conjugation.[65]

We call the algebra of the operators $o_W$ the *higher continuous spin symmetry algebra in Minkowski space with a Young tableaux having k rows* with notation $\mathcal{A}(\Xi; Y(k), \mathbb{R}^{1,d-1})$. Note that $\mathcal{A}(\Xi; Y(0), \mathbb{R}^{1,d-1}) = \mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$. The subalgebra of $\{o^{\mathrm{s}}_P\}$ without space–time derivatives, i.e. $\big(l^{ij}, t^{i_1j_1}, l^{ij+}, t^{i_1j_1+}, g^i_0\big)$ is isomorphic to the symplectic algebra $sp(2k)$. The Howe dual algebra[89,90] to the algebra $so(1, d-1)$ is $sp(2k)$ if $k = [(d-4)/2]$.

The algebra $\mathcal{A}(\Xi; Y(k), \mathbb{R}^{1,d-1})$ provides a basis to construct BRST–BFV gauge-invariant descriptions for the equations of motion and for the (un-)constrained LFs following to the proposed above receipt for scalar CSR in the Bargmann–Wigner form.

The situation with $ISO(1, d-1)$ representations with integer spin looks another.[49,68,69]

### 3.3. *Comparison with dynamics for higher integer spin fields*

First, let us present the result for the BRST–BFV descriptions for scalar CS field obtained in Subsec. 3.2 in terms of the bosonic fields being subject to the usual traceless or double traceless condition, generated by the operator $l_{11} = m_{11}|_{\nu=0}$, following to Fronsdal proposal for (half-)integer HS fields.[24,25] To this end we will use the representation (2.6), (2.21) for the fields $\Phi(x,\omega)$ and for $\chi^k_C$:

$$\chi^k_C = \sum_s \Big\{\chi^{(s,0)k} + \sum_{l>0}\chi^{(s,l)k}\Big\}\,, \quad \chi^{(s,l)k}(x,\omega,\mathcal{C},\mathcal{P}) = \mathcal{C}^p\mathcal{P}^{p+k}\varphi^{(s,l)k}(x,\omega)\,, \tag{3.84}$$

$$gh_H(\chi^k_C) = gh_H(\chi^{(s,l)k}) = -k\,, \quad \big(\deg_\omega, \deg_{\omega/\omega^2}\big)\chi^{(s,l)k} = (s,l)\,, \tag{3.85}$$

so that the generalized-traceless constraints for the field and gauge parameter satisfying to:

$$m_{11}^2\Phi(\omega) = 0\,, \quad m_{11}\varsigma(\omega) = 0 \tag{3.86}$$

should be rewritten in terms of Fronsdal-like double traceless $\phi^{s,0}(\omega)$ and traceless $\psi^{s,0}(\omega)$ fields and new fields double traceless $\phi^{s,l}(\omega)$ and traceless $\psi^{s,l}(\omega)$, $\forall s$, $(l+1) \in \mathbb{N}_0$, (for fixed degree in powers of $\omega^m$ according to (3.84) and due to relation: $\Phi^{s,l}(\omega) = (-1)^l\tilde{\Phi}^{s+l,0}(\omega)\big(l_{11}^+\big)^{-l}$) as:[g]

$$\Phi(\omega) = \sum_{n\geq 0}\sum_{k\geq 0}^{[n/2]}\left(\gamma_{k,n}\big(l_{11}^+\big)^k\phi^{n-2k,0} + \sum_{l>1}\tilde{\gamma}^l_{k,n}\big(l_{11}^+\big)^k\phi^{n-2k,l}\right), \tag{3.87}$$

$$\varsigma(\omega) = \sum_{n\geq 0}\sum_{k\geq 0}^{[n/2]}\left(\delta_{k,n}\big(l_{11}^+\big)^k\psi^{n-2k,0} + \sum_{l>1}\tilde{\delta}^l_{k,n}\big(l_{11}^+\big)^k\psi^{n-2k,l}\right), \tag{3.88}$$

with untouched negative-like part of $\Phi(\omega)$ (as for the standard Fronsdal-like fields) and in more general form

$$\Phi(\omega) = \sum_{n\geq 0}\sum_{k\geq 0}^{[n/2]}\sum_{s\geq 0}^{[n/2]-k}\left(\gamma^s_{k,n}\big(l_{11}^+\big)^{k+s}\phi^{n-2k-s,s} + \sum_{l>1}\tilde{\gamma}^{l,s}_{k,n}\big(l_{11}^+\big)^{k+s}\phi^{n-2k-s,l+s}\right), \tag{3.89}$$

$$\varsigma(\omega) = \sum_{n\geq 0}\sum_{k\geq 0}^{[n/2]}\sum_{s\geq 0}^{[n/2]-k}\left(\delta^s_{k,n}\big(l_{11}^+\big)^{k+s}\psi^{n-2k-s,s} + \sum_{l>1}\tilde{\delta}^{l,s}_{k,n}\big(l_{11}^+\big)^{k+s}\psi^{n-2k-s,l+s}\right), \tag{3.90}$$

$$\vartheta(\omega) = \left(\vartheta^0 + \sum_{l\geq 1}\vartheta^l\right)(\omega) \quad \text{for} \quad \vartheta \in \{\phi,\psi\}\,, \quad l_{11}^2\phi^{n,l} = 0\,, \quad l_{11}\psi^{n,l} = 0\,, \tag{3.91}$$

where the fields $\phi(\omega)$, $\psi(\omega)$ have the decomposition (2.6) according to (2.21). Here, the unknown rational coefficients $\tilde{\gamma}^l_{k,n}$, $\tilde{\delta}^l_{k,n}$ for the decomposition (3.87), (3.88) are determined from (3.86) as the solutions of the system of recursive equations at each fixed monomial $\big(l_{11}^+\big)^k\vartheta^{n-2k,l}$, $\big(l_{11}^+\big)^{k+1}l_{11}\phi^{n-2k,l}$ and $\big(l_{11}^+\big)^k\psi^{n-2k,l}$ for $k = 0,\ldots,[n/2]$; $n,l\in\mathbb{N}_0$:

$$\left\{\nu^2\tilde{\gamma}^l_{k,n}\big(l_{11}^+\big)^k + 2\nu\tilde{\gamma}^l_{k+1,n+2}l_{11}\big(l_{11}^+\big)^{k+1} + \tilde{\gamma}^l_{k+2,n+4}\left[l_{11}^2, \big(l_{11}^+\big)^{k+2}\right]\right\}\phi^{n-2k,l} = 0\,, \tag{3.92}$$

$$\left\{\nu\tilde{\delta}^l_{k,n}\big(l_{11}^+\big)^k + \tilde{\delta}^l_{k+1,n+2}\left[l_{11}, \big(l_{11}^+\big)^{k+1}\right]\right\}\psi^{n-2k,l} = 0 \tag{3.93}$$

(for $\gamma_{k,n} \equiv \tilde{\gamma}^0_{k,n}$, $\delta_{k,n} \equiv \tilde{\delta}^0_{k,n}$). In case of decomposition (3.89), (3.90) the coefficients $\tilde{\gamma}^{l,s}_{k,n}$, $\tilde{\delta}^{l,s}_{k,n}$ should be determined from more general system at each fixed

[g]The presentation of (double) traceless condition, $l_{11}^{(2)}\psi^{s,l}(\omega) = 0$ is as usual for $l = 0$, whereas for $l > 0$ should be understood according to the latter algebraic equations, respectively in (2.12), (2.13), but for $\nu = 0$.

monomial $\big(l^+_{11}\big)^{k+s}\vartheta^{n-2k-s,l+s}$, $\big(l^+_{11}\big)^{k+s}l_{11}\phi^{n-2k-s,l+s}$ and $\big(l^+_{11}\big)^{k+s}\psi^{n-2k-s,l+s}$ for $k=0,\dots,[n/2]$, $s=0,\dots,[n/2]-k$; $n$, $l\in\mathbb{N}_0$:

$$\Big\{\nu^2\tilde{\gamma}^{l,s}_{k,n}\big(l^+_{11}\big)^{k+s}+2\nu\tilde{\gamma}^{l,s}_{k+1,n+2}l_{11}\big(l^+_{11}\big)^{k+s+1} + \tilde{\gamma}^{l,s}_{k+2,n+4}\Big[l^2_{11},\big(l^+_{11}\big)^{k+s+2}\Big]\Big\}\phi^{n-2k-s,l+s}=0\,, \tag{3.94}$$

$$\Big\{\nu\tilde{\delta}^{l,s}_{k,n}\big(l^+_{11}\big)^{k+s}+\tilde{\delta}^{l,s}_{k+1,n+2}\Big[l_{11},\big(l^+_{11}\big)^{k+s+1}\Big]\Big\}\psi^{n-2k-s,l+s}=0\,, \tag{3.95}$$

(for $\gamma^s_{k,n}\equiv\tilde{\gamma}^{0,s}_{k,n}$, $\delta^s_{k,n}\equiv\tilde{\delta}^{0,s}_{k,n}$). The solution for the system (3.92), (3.93) is found in the form

$$\tilde{\gamma}^l_{k,n}=\frac{(-\nu)^k\tilde{\gamma}^l_{0,n-2k}}{4^kk!\prod_{i=1}^k\big[\big(n-2k-l+d/2-1\big)+i-1\big]} = \frac{(-\nu)^k\tilde{\gamma}^l_{0,n-2k}}{4^kk!\big(n-2k-l+d/2-1\big)_k}\,, \tag{3.96}$$

$$\tilde{\delta}^l_{k,n}=\frac{(-\nu)^k\tilde{\delta}^l_{0,n-2k}}{4^kk!\prod_{i=1}^k\big[\big(n-2k-l+d/2\big)+i-1\big]} = \frac{(-\nu)^k\tilde{\delta}^l_{0,n-2k}}{4^kk!\big(n-2k-l+d/2\big)_k} \tag{3.97}$$

with arbitrary constants $\tilde{\delta}^l_{0,n}$, $\tilde{\gamma}^l_{0,n}$ concrete choice of which depends on $n$, $l$, $d$ and with $(x)_n$ being by the Pochhammer symbol. The coefficients related as $\big(\tilde{\delta}^l_{k,n-1}/\tilde{\delta}^l_{0,n-1-2k}\big)=\big(\tilde{\gamma}^l_{k,n}/\tilde{\gamma}^l_{0,n-2k}\big)$. The solution (3.96) for (3.92) follows from the recursive relations

$$\nu^2\tilde{\gamma}^l_{k,n}+8(k+1)\nu\tilde{\gamma}^l_{k+1,n+2}[n-k-l+d/2] + 4^2\tilde{\gamma}^l_{k+2,n+4}\prod_{i=k}^{k+1}(i+1)[i+n-2k-l+d/2]=0\,, \tag{3.98}$$

$$\big\{2\nu\tilde{\gamma}^l_{k+1,n+2}+8(k+2)\tilde{\gamma}^l_{k+2,n+4}[k+n-2k-l+d/2]\big\}l_{11}=0\,, \tag{3.99}$$

with account for

$$g_0\vartheta^{n-2k,l}=((n-2k-l)+d/2)\vartheta^{n-2k,l}\,,\quad \vartheta\in\{\phi,\psi\}\,. \tag{3.100}$$

Substituting $\tilde{\gamma}^l_{k+2,n+4}$ expressed from (3.99) in terms of $\tilde{\gamma}^l_{k+1,n+2}$ in (3.98) we get (3.96).

Note, as to the general systems (3.94), (3.95), that due to the ambiguity in the definition of the monomial

$$\big(l^+_{11}\big)^{k+s}\vartheta^{n-2k-s,l+s} = \big(l^+_{11}\big)^{(k+m)+(s-m)}\vartheta^{(n+m)-2(k+m)-(s-m),(l+m)+(s-m)}\,,\quad m=1,\dots,s\,, \tag{3.101}$$

the respective coefficients $\tilde{\gamma}^{l,s}_{k,n}$, $\tilde{\delta}^{l,s}_{k,n}$ should satisfy to the relations:

$$\tilde{\gamma}^{l,s}_{k,n} = \tilde{\gamma}^{l+m,s-m}_{k+m,n+m}\,, \quad \tilde{\delta}^{l,s}_{k,n} = \tilde{\delta}^{l+m,s-m}_{k+m,n+m}\,, \quad m = 1,\ldots,s\,. \tag{3.102}$$

One can easily see that the solutions for (3.94), (3.95) can be found analogously to (3.96), (3.97) in the form:

$$\left(\tilde{\gamma}^{l,s}_{k,n}, \tilde{\delta}^{l,s}_{k,n}\right) = \frac{(-\nu)^k}{4^k k!}\left(\frac{\tilde{\gamma}^{l,s}_{0,n-2k}}{(n-2(k+s)-l+d/2-1)_k}, \frac{\tilde{\delta}^{l,s}_{0,n-2k}}{(n-2(k+s)-l+d/2)_k}\right), \tag{3.103}$$

so that $\tilde{\gamma}^{l,0}_{k,n} \equiv \tilde{\gamma}^{l}_{k,n}$, $\tilde{\delta}^{l,0}_{k,n} \equiv \tilde{\delta}^{l}_{k,n}$.

Now, substituting, instead of $\Phi$, $\chi_j$, $\varsigma$, for $j = 1, 2$ theirs presentations in terms of series of respective traceless: $\chi^{n,l}_{F|1}$, $n$, $l \in \mathbb{N}_0$ and double traceless tensor fields: $w^{n,l}_i$, $w_i \in \{\phi, \chi_{F|2}\}$, $i = 1, 2$, as well as the gauge parameters $\epsilon^{n,l}$:

$$W_i(\omega) = \sum_{n\geq 0}\sum_{k\geq 0}^{[n/2]}\sum_{l\geq 0} \frac{(-\nu)^k \tilde{\gamma}^{l}_{0,n-2k}}{4^k k!(n-2k-l+d/2-1)_k}\left(l^+_{11}\right)^k w^{n-2k,l}_i\,,$$

$$l^2_{11} w^{n,l}_i = 0\,, \tag{3.104}$$

$$\chi_1(\omega) = \sum_{n\geq 0}\sum_{k\geq 0}^{[n/2]}\sum_{l\geq 0} \frac{(-\nu)^k \tilde{\delta}^{l}_{0,n-2k}}{4^k k!(n-2k-l+d/2)_k}\left(l^+_{11}\right)^k \chi^{n-2k,l}_{F|1}\,,$$

$$l_{11}\chi^{n-2k,l}_{F|1} = 0\,, \tag{3.105}$$

$$\varsigma(\omega) = \sum_{n\geq 0}\sum_{k\geq 0}^{[n/2]}\sum_{l\geq 0} \frac{(-\nu)^k \tilde{\delta}^{l}_{0,n-2k}}{4^k k!(n-2k-l+d/2)_k}\left(l^+_{11}\right)^k \epsilon^{n-2k,l}\,,$$

$$l_{11}\epsilon^{n,l} = 0\,, \tag{3.106}$$

(for $W_i \in \{\Phi, \chi_2\}$) we get gauge-invariant non-Lagrangian duplet-like (3.58), (3.59) and with expressed field $\chi_1$, $\chi_1 = 1/2m_{11}\Phi$, (3.61)–(3.63) (with the fields $\Phi$, $\chi_2$ and $\phi^{n,l}$, $\chi^{n,l}_{F|2}$) formulations in terms of Fronsdal-like totally-symmetric standard (for $l = 0$) and new (for $l > 0$) fields. Representing the expression (3.104) in powers of $\omega^{(m)_k}\frac{\omega^{(n)_l}}{\omega^{2l}}$ we may derive the latter gauge-invariant EoM from the action $\mathcal{S}_{C|\Xi}(\Phi, \chi_2)$ (3.66) with help of the Lagrangian multipliers $\hat{\lambda}^{l|(m)_k,(n)_l}_i$, $i = 1, 2, 3$, $k$, $l \in \mathbb{N}_0$. Because of the multipliers are double $m_{11}$-traceless as well (3.69) they should be expressed according to the above receipt (3.104) in terms of double $l_{11}$-traceless Fronsdal-like multipliers $\hat{\lambda}^{l|(m)_k,(n)_l}_{F|i}$. The constrained gauge-invariant action (3.66) for CSR scalar field of CS $\Xi$ presenting in terms of Fronsdal-like fields in the tensor form (but in Bargmann–Wigner approach) in this case looks as

$$\mathcal{S}_{C|\Xi}\left(\phi^{n,l}, \chi^{n,l}_{F|2}, \hat{\lambda}^{n,l}_{F|i}\right)$$

$$= \mathcal{S}_{C|\Xi}(\Phi, \chi_2, \hat{\lambda}_i)\big|_{\left(\Phi,\chi_2,\hat{\lambda}^{l|(m)_k,(n)_l}_i\right)=\left(\Phi(\phi^{n,l}),\chi_2(\chi^{n,l}_{F|2}),\hat{\lambda}^{n,l}_i(\hat{\lambda}^{n,l}_{F|i})\right)}\,. \tag{3.107}$$

To compare these results with triplet, duplet and Fronsdal LFs for the fields with all integer spins we remind that the latters are encoded by only the D'Alambert, divergentless and usual traceless (for $\nu = 0$) Eqs. (2.12) for the basic field $\phi^s \equiv \phi^{s,0}$ of any integer spin $s$, $s \in \mathbb{N}_0$: $g_0\phi^s = (s + \frac{d}{2})\phi^s$, without the presence of new fields $\phi^{\bullet,l}$, $l > 0$.

The respective constrained gauge-invariant LFs for the massless field, $\phi^s$ of integer spin $s$ in terms, first, of triplet: $\phi^s$, $\chi_0^{s-1}$, $\chi_1^{s-2}$, second, of duplet $\phi^s$, $\chi_1^{s-2}$ (having expressed of $\chi_0^{s-1}$, being similar to $\chi_0(\omega)$ (3.24), from triplet formulation through algebraic EoM) with indices $s$, $s-1$, $s-2$ meaning the rank of the component Lorentz tensors, i.e. $\deg_\omega \phi(\chi)^s = s$ (3.87), and third, in terms of unique field $\phi^s$ look as

$$\mathcal{S}_{C|s}(\phi, \chi_0, \chi_1) = \big(\phi^s(\partial_\omega)\chi_0^{s-1}(\partial_\omega)\chi_1^{s-2}(\partial_\omega)\big) \times \begin{pmatrix} l_0 & -l_1^+ & 0 \\ -l_1 & 1 & l_1^+ \\ 0 & l_1 & -l_0 \end{pmatrix} \begin{pmatrix} \phi^s(\omega) \\ {\chi_0}^{s-1}(\omega) \\ {\chi_1}^{s-2}(\omega) \end{pmatrix}, \tag{3.108}$$

$$\begin{aligned} &\delta(\phi^s(\omega), \chi_0^{s-1}(\omega), \chi_1^{s-2}(\omega)) = (l_1^+, l_0, l_1)\epsilon^{s-1}(\omega)\,, \\ &l_{11}(\phi(\omega), \chi_0(\omega), \chi_1(\omega), \epsilon(\omega)) = (-2\chi_1(\omega), 0, 0, 0)\,, \end{aligned} \tag{3.109}$$

$$\hat{\mathcal{S}}_{C|s}(\phi, \chi_1) = \big(\phi^s(\partial_\omega)\chi_1^{s-2}(\partial_\omega)\big) \begin{pmatrix} l_0 - l_1^+ l_1 & (l_1^+)^2 \\ l_1^2 & -l_0 - l_1 l_1^+ \end{pmatrix} \begin{pmatrix} \phi^s(\omega) \\ \chi_1^{s-2}(\omega) \end{pmatrix}, \tag{3.110}$$

$$\hat{\mathcal{S}}^0_{C|s}(\phi) = \phi^s(\partial_\omega)\Big(l_0 - l_1^+ l_1 - \frac{1}{2}\big(l_1^+\big)^2 l_{11} - \frac{1}{2} l_{11}^+ l_1^2 - \frac{1}{4} l_{11}^+\big(l_0 + l_1 l_1^+\big) l_{11}\Big)\phi^s(\omega)\,, \tag{3.111}$$

$$\delta\phi^s(\omega) = l_1^+\epsilon^{s-1}(\omega) \quad \text{and} \quad l_{11}^2\phi^s(\omega) = l_{11}\epsilon^{s-1}(\omega) = 0\,, \tag{3.112}$$

for $\hat{\mathcal{S}}_{C|s} = \mathcal{S}_{C|s}|_{\chi_0 = \chi_0(\phi,\chi_1)}$ and $\hat{\mathcal{S}}^0_{C|s} = \hat{\mathcal{S}}_{C|s}|_{\chi_1 = -(1/2)l_{11}\phi}$. Thus, the gauge-invariant actions

$$\Big(\mathcal{S}_{C|\infty}\big(\phi^0, \chi_0^0, \chi_1^0\big), \hat{\mathcal{S}}_{C|\infty}\big(\phi^0, \chi_1^0\big), \hat{\mathcal{S}}^0_{C|\infty}\big(\phi^0\big)\Big) = \sum_{s\geq 0}\big(\mathcal{S}_{C|s}, \hat{\mathcal{S}}_{C|s}, \hat{\mathcal{S}}^0_{C|s}\big)\,, \tag{3.113}$$

with

$$\big(\phi^0, \chi_0^0, \chi_1^0\big)(\omega) = \Bigg(\sum_{s\geq 0}\phi^s, \sum_{s\geq 1}\chi_0^{s-1}, \sum_{s\geq 2}\chi_1^{s-2}\Bigg)(\omega)$$

according to the rules (2.6), (3.91) for massless fields of all spins $s = 0, 1, 2, \ldots$ take in the ghost-independent vector-like notations the respective forms: (3.108), (3.110), (3.111) with allowance made for the changes $\big(\phi^s, \chi_0^{s-1}, \chi_1^{s-2}\big) \to \big(\phi^0, \chi_0^0, \chi_1^0\big)$. The corresponding gauge transformations (3.109), (3.112) are now written for the fields of all integer spins with the gauge parameter: $\epsilon^0 = \sum_{s\geq 1}\epsilon^{s-1}$, with the same forms of the traceless constraints.

In the tensor notations the latter duplet and Fronsdal LFs read (up to the common factor (1/2)):

$$\hat{\mathcal{S}}^0_{C|\infty}(\phi^0, \chi^0_1) = \sum_{s\geq 0}\frac{(-1)^s}{s!}\int d^dx\Big\{\phi_{(m)_s}\Big(\partial^2\phi^{(m)_s} - s\partial^{m_s}\partial^n\phi^{(m)_{s-1}}{}_n + s(s-1)\partial^{m_{s-1}}\partial^{m_s}\chi_1^{(m)_{s-2}}\Big) - s(s-1)\chi_{1(m)_{s-2}} \times\Big(2\partial^2\chi_1^{(m)_{s-2}} + (s-2)\partial^{m_{s-2}}\partial^m\chi_1^{(m)_{s-3}}{}_m - \partial_{m_{s-1}}\partial_{m_s}\phi^{(m)_s}\Big)\Big\}, \tag{3.114}$$

$$\hat{\mathcal{S}}^0_{C|\infty}(\phi^0) = \sum_{s\geq 0}\frac{(-1)^s}{s!}\int d^dx\Big\{\phi_{(m)_s}\Big(\partial^2\phi^{(m)_s} - s\partial^{m_s}\partial^n\phi^{(m)_{s-1}}{}_n + s(s-1)\partial^{m_{s-1}}\partial^{m_s}\phi^{(m)_{s-2}m}{}_m\Big) - \frac{1}{2}s(s-1)\phi_{(m)_{s-2}m}{}^m \times\Big(\partial^2\phi^{(m)_{s-2}n}{}_n + \frac{1}{2}(s-2)\partial^{m_{s-2}}\partial^m\phi^{(m)_{s-3}}{}_{mn}{}^n\Big)\Big\}, \tag{3.115}$$

with traceless field: $\sum_{s\geq 2}\chi_1{}^{(m)_{s-2}}$: $\chi_1{}^{(m)_{s-4}m}{}_m = 0$, gauge parameter $\sum_{s\geq 1}\epsilon^{(m)_{s-1}}$: $\epsilon^{(m)_{s-3}m}{}_m = 0$, and double traceless basic field: $\sum_{s\geq 0}\phi^{(m)_s}$: $\phi^{(m)_{s-2}m}{}_m = 2\chi_1{}^{(m)_{s-2}}$, providing the standard form of the gauge transformations:

$$\delta\Bigg(\sum_{s\geq 0}\phi_{(m)_s}, \sum_{s\geq 2}\chi_{1(m)_{s-2}}\Bigg) = -\sum_{s\geq 0}\big(\partial_{\{m_s}\epsilon_{(m)_{s-1}\}}, \partial^{m_{s-1}}\epsilon_{(m)_{s-1}}\big), \tag{3.116}$$

from (3.109), (3.112). To be complete, note the constrained BRST–BFV LF for HS field, $\phi^s$, of integer spin $s$ are given by the relations:

$$\mathcal{S}_{C|s}(\phi^0, \chi^0_0, \chi^0_1) = \int d\eta_0\, \chi^{0s}_C(\partial_\omega)Q_{C|\text{int}}\chi^{0s}_C(\omega), \qquad \delta\big(\chi^{0s}_C, \chi^{1s}_C\big)(\omega) = \big(Q_{C|\text{int}}\chi^{1s}_C(\omega), 0\big), \tag{3.117}$$

$$\hat{L}_{11}\chi^{ks}_C = (l_{11} + 2\eta_1\mathcal{P}_1)\chi^{ks}_C = 0, \qquad \hat{\sigma}_{C|\text{int}}(g)\chi^{ks}_C = \left(s + \frac{d}{2}\right)\chi^{ks}_C, \quad k = 0, 1, \tag{3.118}$$

which are related with ones (3.22) for CS field $\Phi(\omega)$ as follows:

$$\Bigg(Q_{C|\text{int}}, \sum_{s\geq 0}\chi^{ks}_C, \hat{L}_{11} + \nu, \hat{\sigma}_{C|\text{int}}(g)\Bigg) = \Big(Q_C, \chi^{0k^0}_C, \hat{M}_{11}, \hat{\sigma}_C(g)\Big)\Big|_{(\eta_1^m = \eta_1^{m+} = \mathcal{P}_1^{m+} = \mathcal{P}_1^m = 0)}, \tag{3.119}$$

where $\chi^{0k}_C = \chi^{0k^0}_C + \sum_{l>0}\chi^{0k^l}_C$ and $\chi^0_C(\partial_\omega)$ is dual for $\chi^0_C(\omega)$ with respect to the natural scalar product in the respective Hilbert space $\mathcal{H}_C$. Explicit comparison of the duplet LF (3.114), (3.116) for HS fields of all integer spins and duplet-like LF (3.66), (3.67), (3.69) for CS field shows its difference both by the contents of the configuration spaces, due to presence, first, of "negative spin value" fields: $\left(\Phi^{\bullet,l},\chi_1^{\bullet,l},\chi_2^{\bullet,l}\right)$, second, by the field $\left(\sum_{s,l\geq 0}\chi^{s,l}(\omega)\right)$ in the latter, third, by the Lagrangian multipliers presence for CS field, by the structure of the constraints among the fields. The only EoM (3.58) for the standard fields $\Phi^0(\omega),\chi^0_1(\omega)$ have the Lagrangian form (without using Lagrangian multipliers) and when rewritten in terms of Fronsdal-like fields $\phi^0(\omega)$, $\chi^0_{F|1}(\omega)$ may be derived from the functional $\mathcal{S}^0_{C|\Xi}\left(\Phi^0,\chi^0_1\right)$:

$$\mathcal{S}^0_{C|\Xi}\left(\Phi^0,\chi^0_1\right)|_{\left(\Phi^0,\chi^0_1\right)=\left(\Phi^0(\phi^{n,0}),\chi^0_1(\chi^{n,0}_{F|1})\right)}=\hat{\mathcal{S}}^0_{C|\infty}\left(\phi^0,\chi^0_1\right). \tag{3.120}$$

We stress that the main difference concerns the presence of infinite number of new tensor fields with "negative spin values," that in turn, follows from the Bargmann–Wigner and Fronsdal forms of the equations selecting respectively the CSR and the integer spin representations. In case of Fronsdal-like form of the equations (suggested by Schuster and Toro[30]) they can be derived from the Lagrangian BRST–BFV EoM (without new fields presence) with the constrained Lagrangian formulation closely related with one for massless totally-symmetric fields for all integer spins (see footnote c).

### 3.4. *Equivalence to initial irreducible relations*

Let us preliminarily consider the problem of establishing of the equivalence of the Lagrangian EoM for massless totally-symmetric field $\phi_{(m)_s}(x)$ with integer spin $s$, in the triplet formulation (3.108), (3.109), which have the form, when expanding the EoM: $Q_{C|\mathrm{int}}\chi^{0s}_C(\omega)=0$, in powers of ghost coordinates (together with respective traceless constraints (3.109)):

$$\begin{aligned}\eta_0 &: l_0\phi^{s,0}-l_1^+\chi_0^{s-1,0}=0\,,\\ \eta_1^+ &: l_1\phi^{s,0}-l_1^+\chi_1^{s-2,0}-\chi_0^{s-1,0}=0\,,\end{aligned} \tag{3.121}$$

$$\begin{aligned}&\eta_0\eta_1^+\mathcal{P}_1^+ : l_0\chi_1^{s-2,0}-l_1\chi_0^{s-1,0}=0\,,\\ &l_{11}\left(\phi^{s,0},\chi_0^{s-1,0},\chi_1^{s-2,0},\epsilon^{s-1,0}\right)=\left(-2\chi_1^{s-2,0},0,0,0\right)\end{aligned} \tag{3.122}$$

with non-Lagrangian conditions which should extract the massless UIR of the Poincaré group $ISO(1,d-1)$ with discrete spin $s$ in terms of tensor fields:

$$(l_0,l_1,l_{11})\phi^{s,0}=(0,0,0)\,. \tag{3.123}$$

The conditions (3.123) do not fix completely an ambiguity in the definition of $\phi^{s,0}$ as a representative of the UIR space of $ISO(1,d-1)$ group due to existence of a residual gauge symmetry, which we intend to determine. First, we use part of the degrees of freedom from the gauge parameter $\epsilon^{s-1,0}$ to gauge away the field $\chi_1^{s-2,0}$ by means of the gauge transformations (3.109). For $s=0$ the equivalence is trivial, whereas for $s=1$ there is no field $\chi_1^{-1,0}\equiv 0$. To do so, we expand $\epsilon^{s-1,0}$ into sum of longitudinal, $\epsilon_L^{s-1,0}$, and transverse, $\epsilon_\perp^{s-1,0}$, components:

$$\epsilon^{s-1,0}=\epsilon_L^{s-1,0}+\epsilon_\perp^{s-1,0}\equiv\sum_{k=1}^{s-1}(-1)^{k-1}\frac{(l_1^+)^k l_1^k}{k!(l_0)^k}\epsilon^{s-1,0}+\left(1+\sum_{k=1}^{s-1}(-1)^k\frac{(l_1^+)^k l_1^k}{k!(l_0)^k}\right)\epsilon^{s-1,0}\,, \tag{3.124}$$

so that $l_1\epsilon_\perp^{s-1,0}\equiv 0$ and both of the components are traceless: $l_{11}\epsilon_L=l_{11}\epsilon_\perp=0$. Thus, first, we use only part: $\epsilon_L^\chi$ from the parameter $\epsilon_L^{s-1,0}=\epsilon_L^\chi+\epsilon_L^\phi$ for $s\geq 2$ to gauge away the field $\chi_1^{s-2,0}$ completely. So we have, from the stability of the solution $\chi_1^{s-2,0}=0$ under the gauge symmetry that $\delta\chi_1^{s-2,0}=0\Leftrightarrow l_1\epsilon_L^{s-1,0}=0\Rightarrow l_1\epsilon_L^{\phi s-1,0}=-l_1\epsilon_L^{\chi s-1,0}$.[h]

Second, from the first equation in (3.122) we observe that the field $\chi_0^{s-1,0}$ is the transverse one and we may therefore use the unused parameter $\epsilon_\perp^{s-1,0}$ choosing $\epsilon_\perp^{s-1,0}=-(l_0)^{-1}\chi_0^{s-1,0}$ to gauge away this field completely, so that the stability of the solution $\chi_0^{s-1,0}=0$ under the gauge transformations means that $\delta\chi_0^{s-1,0}=0\Leftrightarrow l_0\epsilon^{s-1,0}=0$.

As the result, from Eqs. (3.121), (3.122) it follows the validity of the system (3.123) with residual gauge transformations determined by the longitudinal gauge parameter, $\epsilon_L^{\phi s-1,0}$, which satisfy to the same restrictions as the field $\phi^{s,0}$ in (3.123). Therefore, the conditions which should select the (tensor) field of any spin $s\in\mathbb{N}_0$ as the element of irreducible massless unitary representation must be determined as:

$$(l_0,l_1,l_{11})\phi^{s,0}=(0,0,0)\,,\quad \delta\phi^{s,0}=l_1^+\epsilon^{s-1,0}\,,\quad (l_0,l_1,l_{11})\epsilon^{s-1,0}=(0,0,0)\,. \tag{3.125}$$

The latter equations on $\epsilon^{s-1,0}$ means that the parameter may be considered as the element of massless UIR of $ISO(1,d-1)$ of spin $s-1$, but without own gauge symmetry.[i] Note, first that the dimensional reduction procedure being applied to massless

[h] The realization of the first step allows one to get Maxwell-like LF with traceless field $\phi^{s,0}$, when having substituted $\chi_0^{s-1,0}$ being expressed from the second equation in (3.121) into the first one, as follows: $(l_0-l_1^+l_1)\phi^{s,0}=0$, so that $\mathcal{S}^0_{C|s}(\phi^0)=\phi^{s,0}(\partial_\omega)(l_0-l_1^+l_1)\phi^{s,0}$ with $\delta\phi^{s,0}=l_1^+\epsilon^{s-1,0}$, $l_1\epsilon^{s-1,0}=0$. The equivalent reducible respective LF for the latter with elimination of the differential constraint on $\epsilon$ were considered in Refs. 92 and 93 among them for AdS space.

[i] For the case of mixed-symmetric massless HS field with generalized integer spin $\mathbf{s}=(s_1,\ldots,s_k)$ given on $\mathbb{R}^{1,d-1}$ the conditions of extraction of only UIR of Poincaré group $ISO(1,d-1)$ in the space of tensor fields $\phi_{(m^1)_{s_1}\cdots(m^k)_k}(x)\in Y(s_1,\ldots,s_k)$, $k\leq[d/2]$: $(l_0,l_i,l_{ij},t_{rs})|\phi\rangle_{(s)_k}=0$ being initial in Ref. 65 (with use of the Fock space notations) should be augmented according to (3.125) by adding the reducible gauge symmetry: $\delta\phi_{(m^1)_{s_1}\cdots(m^k)_k}=l_1^+\epsilon_{(m^1)_{s_1-1}\cdots(m^k)_k},\ldots$, subject to the same requirements as for the tensor field itself. For the half-integer totally- and mixed-symmetric massless HS fields the situation with the exact formulation of the UIR is the same, e.g. one can show that for totally-symmetric case it is necessary to add the gauge transformations of the same form with gauge spin-tensor of rank $(n-1)$, but for basic spin-tensor field $\psi_{(m)_n}$ of spin, $n+1/2$, with suppressed Dirac indice and being subject to the same conditions: Dirac and $\gamma$-traceless constraints. Thus, the theorem in Ref. 69 concerning the equivalence of the solutions of the equations of motion from the respective constrained BRST–BFV LF and ones for UIR conditions will be guaranteed, because of the latter solutions contains some gauge identities due to residual gauge symmetry presence.

UIR conditions (3.125) of $ISO(1,d)$ in $\mathbb{R}^{1,d}$ space–time permits one explicitly derive the massive UIR conditions of $ISO(1,d-1)$ in $\mathbb{R}^{1,d-1}$ with the same spin, as follows: $\big(l_0+m^2, l_1, l_{11}\big)\phi^{s,0} = (0,0,0)$ without any gauge symmetry. Second, the independent counting of the numbers of the physical degrees of freedom being extracted by (3.125) and by Eqs. (3.121), (3.122) with the gauge symmetry transformations (3.109) shows their coincidence.

Having in mind, the above analysis for HS field with integer spin, let us consider non-Lagrangian EoM for the basic field $\Phi(\omega)$ with CS, which follow from BRST–BFV equation (3.18) (or from (3.32)), as well as the holonomic constraints.

Again it may turn out that the conditions (2.17) do not fix completely an ambiguity in the definition of $\Phi(\omega)$ as a representative of the CSR space of $ISO(1,d-1)$ group, due to existence of a residual gauge symmetry, which we should to determine. We will call Eqs. (3.39)–(3.49) as the *BRST-unfolded equations*, due to the appearance of any field variable there with a coefficient being, at most the first degree in powers of the symmetry algebra $\mathcal{A}(\Xi;\mathbb{R}^{1,d-1})$ elements $o_I$.[j]

First, we repeat the procedure from Subsec. 3.2 of gauge fixing up to surviving of only the fields $(\Phi,\chi_1,\chi_2,\chi_0)(\omega)$ with equivalent transforming of Eqs. (3.39)–(3.49) into *triplet-like non-Lagrangian formulation* (3.56), (3.57) with the gauge transformations (3.52) with unique independent gauge parameter $\varsigma(\omega)$ with account of algebraic traceless constraints (3.28)–(3.30). Second, we expand $\varsigma$ into sum of longitudinal, $\varsigma_L(\omega)$, and transverse, $\varsigma_\perp(\omega)$, components:

$$\varsigma = \varsigma_L + \varsigma_\perp \equiv \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(l_1^+)^k l_1^k}{k!(l_0)^k}\varsigma + \sum_{k=0}^{\infty}(-1)^{k}\frac{(l_1^+)^k l_1^k}{k!(l_0)^k}\varsigma\,, \qquad (3.126)$$

so that $l_1\varsigma_\perp \equiv 0$ and both of the components are generalized traceless: $m_{11}\varsigma_L = m_{11}\varsigma_\perp = 0$. Then, we use a part $\varsigma_L^{\chi_1}(\omega)$ of the longitudinal gauge parameter:

$$\varsigma_L = \varsigma_L^{\chi_1} + \varsigma_L^{\chi_2} + \varsigma_L^{\Phi} \qquad (3.127)$$

to gauge away the field $\chi_1(\omega)$ completely. From the stability: $\delta\chi_1(\omega) = 0$, of the solution $\chi_1(\omega) = 0$ under the gauge symmetry, it follows the relations:

$$\delta\chi_1 = 0 \Leftrightarrow l_1\varsigma_L = l_1\varsigma = 0 \Rightarrow l_1\varsigma_L^{\Phi} + l_1\varsigma_L^{\chi_2} = -l_1\varsigma_L^{\chi_1} \quad \text{and} \quad m_{11}\Phi = 0\,. \qquad (3.128)$$

Third, from the second equation in (3.57) we obtain that: $l_1\chi_2 = \chi_0$ and thus the field $\chi_2$ is double transverse, due to $l_1\chi_0 = 0$ from the third equation in (3.56). Then, we use the remaining degrees of freedom from the parameter $\varsigma$ (both $\varsigma_L^{\chi_2}$ and $\varsigma_\perp$) to gauge away the field $\chi_2$ completely. Then, the requirement $\delta\chi_2 = 0$ leads to:

$$m_1^+\big(\varsigma_L^{\chi_2} + \varsigma_L^{\Phi} + \varsigma_\perp\big) = m_1^+\varsigma = 0 \Rightarrow m_1^+\varsigma_L^{\Phi} = -m_1^+\big(\varsigma_L^{\chi_2} + \varsigma_\perp\big) \quad \text{and} \quad \chi_0 = 0\,. \qquad (3.129)$$

[j]The analogous type of the BRST-unfolded equations were written in (3.121), (3.122) for totally-symmetric integer spin case.

As the result, the only initial field $\Phi$ survives after the procedure above, satisfying to the relations (2.17) without any residual gauge transformations due to using of the parameter $\varsigma$ completely. Indeed, from a possible expression $\delta\Phi = l_1^+\varsigma$ and $\big(l_0, l_1, m_1^+, m_{11}\big)\varsigma = (0,0,0,0)$ it should be that $\delta\Phi = -\imath\Xi\varsigma$ and it would mean that the field $\Phi$ does not contain any physical degrees of freedom. It means, that in opposite to the case of integer massless UIR of $ISO(1,d-1)$ one-valued CSR conditions do not include residual gauge transformations. Again, we suppose, that the dimensional reduction when applied to massless CSR conditions (2.17) in $\mathbb{R}^{1,d}$ can be used to derive massive-like CSR relations in $\mathbb{R}^{1,d-1}$ for the same value of CS $\Xi$.

Thus, we show, that the CSR equations (2.17) [or, equivalently, (2.4), (2.5)], can be achieved by using the BRST–BFV equations (3.22) after gauge-fixing and removing the auxiliary fields by using a total set of the equations of motion.

## 4. BRST–BV Minimal Descriptions

To construct a quantum action being sufficient for determination of the nondegenerate path integral within conventional BV quantization method,[70–72] one necessary to derive preliminarily the so-called BV action in the minimal sector of field and antifield variables organized in terms of respective vectors on a space $\mathcal{V}_g$, when considering instead of the field vector $\chi^0_C \in \mathcal{V}_C$ the *generalized field–antifield vector* $\chi_{g|C} \in \mathcal{V}_{g|C}$:

$$\mathcal{V}_{g|C} := \mathcal{V}_g \otimes \mathcal{V}_{gh}^{o_A} \quad \text{with } \mathbb{Z}\text{-grading} \quad \mathcal{V}_{g|C} = \lim_{M\to\infty} \bigoplus_{l=-M}^{M} \mathcal{V}^l_{g|C} \tag{4.1}$$

for $gh_{\rm tot}\big(\chi^l_{g|C}\big) = -l$, $\chi^l_{g|C} \in \mathcal{V}^l_{g|C}$. The total configuration space for initial first-stage reducible gauge constrained LF in the minimal sector, $\mathcal{M}_{\min} = \big\{\Phi^A_{\min}(x,\omega)\big\}$, contains, in addition to the field $\chi^0_C$, the zeroth-level ghost field vector, $C^0_C$, and first-level ghost field one, $C^1_C$, introduced by the rule according to (2.6):

$$\varpi(x,\omega) = C^1(x,\omega)\mu_0\mu_1 \Rightarrow \chi^2_C = C^1_C\mu_0\mu_1\,, \tag{4.2}$$

$$\varsigma_k(x,\omega) = C^0_k(x,\omega)\mu_0 \Rightarrow \chi^1_C = C^0_C\mu_0\,, \quad C^i_C = C^{0|i}_C + \sum_{l\geq 1} C^{l|i}_C\,, \tag{4.3}$$

| | $\mathcal{C}^A$ | $\mathcal{P}_A$ | $C^1(x,\omega)$ | $C^0_k(x,\omega)$ | $C^i_C$ | $\mu_i$ |
|---|---|---|---|---|---|---|
| $\epsilon$ | 1 | 1 | 0 | 1 | 0 | 1 |
| $gh_H$ | 1 | $-1$ | 0 | 0 | $-1-i$ | 0 |
| $gh_L$ | 0 | 0 | 2 | 1 | $i+1$ | $-1$ |
| $gh_{\rm tot}$ | 1 | $-1$ | 2 | 2 | 0 | $-1$ |

$, \quad i = 0, 1\,, \tag{4.4}$

(where under $\varsigma_k$ and $C^0_k$ we mean all component fields in $\chi^1_C$ (3.25) and with constant $\mu_i$: $\{\mu_i, \mu_j\} = 0$, $i, j = 0, 1$), which due to the vanishing of the total ghost number and Grassmann parity may be combined with $\chi^0_C$ into *generalized field vector*:

$$\chi^0_{\mathrm{gen}|C} = \chi^{0|0}_{\mathrm{gen}|C} + \sum_{l\geq 1} \chi^{l|0}_{\mathrm{gen}|C} = \chi^0_C + \sum_{i=0}^{1} C^i_C\,, \quad (\epsilon, gh_{\mathrm{tot}})\chi^0_{\mathrm{gen}|C} = (0,0)\,. \tag{4.5}$$

The corresponding (according to (3.9)) antifields

$$\Phi^*_{A\,\min}(x,\omega) = \big(\Phi^*_{n_{f0};n_{f1},n_{fm},n_{p1},n_{pm},0,0,0}; C^{*0}_k, C^{*1}\big)(x,\omega)$$

and respective space vectors from $\mathcal{V}^0_{g|C}$ with the $\mathbb{Z}_2$-, and $\mathbb{Z}$-gradings

| | $\Phi^*_{n_{f0};\ldots}(x,\omega)$ | $C^{*0}_k(x,\omega)$ | $C^{*1}(x,\omega)$ | $\chi^{*0}_C$ | $C^{*i}_C$ |
|---|---|---|---|---|---|
| $\epsilon$ | 1 | 0 | 1 | 0 | 0 |
| $gh_H$ | 0 | 0 | 0 | 1 | $2+i$ |
| $gh_L$ | $-1$ | $-2$ | $-3$ | $-1$ | $-2-i$ |
| $gh_{\mathrm{tot}}$ | $-1$ | $-2$ | $-3$ | 0 | 0 |

$, \quad i = 0, 1\,; \qquad (4.6)$

are combined into *generalized antifield vector* as follows:

$$\chi^{*0}_{\mathrm{gen}|C} = \chi^{*0}_C + \sum_{i=0}^{1} C^{*i}_C = \big\{B^{*0}_C + B^{*0}_{c|C}\big\} + \eta_0\Big\{S^{*0}_C + \sum_{i\geq 0} S^{*i}_{c|C}\Big\}\,, \tag{4.7}$$

$$\begin{aligned}\chi^{*0}_C(\mathcal{C},\mathcal{P},\omega) &= \eta^+_1\chi^*_0(\omega) + \eta^m_1\chi^{*m}_0(\omega) + \eta^+_1\mathcal{P}^+_1\eta^m_1\chi^*_{01}(\omega) + \eta^m_1\eta^+_1\mathcal{P}^m_1\chi^{*m}_{01}(\omega)\\ &\quad + \eta_0\big[\Phi^* + \mathcal{P}^+_1\big(\eta^+_1\chi^*_1(\omega) + \eta^m_1\chi^{*m}_1(\omega)\big)\\ &\quad + \mathcal{P}^m_1\big(\eta^+_1\chi^*_2(\omega) + \eta^m_1\chi^{*m}_2(\omega)\big) + \eta^+_1\eta^m_1\mathcal{P}^+_1\mathcal{P}^m_1\chi^{*m}_{11}(\omega)\big]\,, \end{aligned}\tag{4.8}$$

$$\begin{aligned}C^{*0}_C(\mathcal{C},\mathcal{P},\omega) &= \eta_0\big(\eta^+_1C^*_\varsigma(\omega) + \eta^m_1C^{*m}_\varsigma(\omega) + \eta^+_1\mathcal{P}^+_1\eta^m_1C^*_{\varsigma|01}(\omega)\\ &\quad + \mathcal{P}^m_1\eta^+_1\eta^m_1C^*_{\varsigma|11}(\omega)\big) + \eta^+_1\eta^m_1C^*_{\varsigma|0}(\omega)\,, \end{aligned}\tag{4.9}$$

$$C^{*1}_C(\mathcal{C},\mathcal{P},\omega) = \eta_0\eta^+_1\eta^m_1C^*_\varpi(\omega)\,, \tag{4.10}$$

for $B^{*1}_{c|C} \equiv 0$ and $\chi^{*0}_{\mathrm{gen}|C} = \chi^{0*0}_{\mathrm{gen}|C} + \sum_{l\geq 1}\chi^{l*0}_{\mathrm{gen}|C}$. The ghost-independent antifield vectors have the decompositions in powers of $\omega^m$ similar to (2.6) and (2.21) as for the respective field vectors. The generalized field (4.5) and antifield (4.7) vectors are uniquely written in terms of the generalized field–antifield vector:

$$\chi^0_{\mathrm{g}|C} = \chi^0_{\mathrm{gen}|C} + \chi^{*0}_{\mathrm{gen}|C} = \chi^{0|0}_{\mathrm{g}|C} + \sum_{l\geq 1}\chi^{l|0}_{\mathrm{g}|C}\,, \quad (\epsilon, gh_{\mathrm{tot}})\chi^0_{\mathrm{g}|C} = (0,0)\,. \tag{4.11}$$

The presentation of the constrained minimal BV actions are different for the case of Lagrangian form of BRST–BFV EoM, $Q_{C|\mathrm{int}}\chi^{0\infty}_C(\omega) = 0$ as for the field

$\chi^{0\infty}_C = \sum_{s\geq 0}\chi^{0s}_C$, of all integer spin $s = 0, 1, \ldots$ (3.117) and for the non-Lagrangian BRST–BFV EoM (3.22) as for the free massless field $\Phi(x,\omega)$ of CS $\Xi$ (for $\nu = 1$) in $\mathbb{R}^{1,d-1}$. In the former case the minimal action $S_{\min} \equiv S_{C|\infty}$ is given according to the general prescription: $S_{\min} = \mathcal{S}_0 + \Phi^*_{A\,\min}\overrightarrow{s}\Phi^A_{\min}$,[70–72] with account for specific of the vector space $\mathcal{V}^0_{g|C}$, now being endowed by Fock space structure, irreducibility of the gauge theory and reality of $S_{C|\infty}$:

$$S_{C|\infty} = \mathcal{S}_{C|\infty} + \int d\eta_0\big\{\chi^{*\infty}_C(\partial_\omega)\overrightarrow{s}_\infty C^{0\infty}_C + C^{0\infty}_C(\partial_\omega)\overleftarrow{s}_\infty\chi^{*\infty}_C(\omega)\big\}\,, \tag{4.12}$$

$$\chi^{*\infty}_C(\omega) = \eta_0\big[\Phi^{*\infty} + \mathcal{P}^+_1\eta^+_1\chi^{*\infty}_1(\omega)\big] + \eta^+_1\chi^{*\infty}_0(\omega)\,, \tag{4.13}$$

with right $\overleftarrow{s}_\infty$ (left $\overrightarrow{s}_\infty$) generator of *Lagrangian BRST-like transformations* in the minimal sector of the fields combined within the generalized field $\chi^{0\infty}_{\mathrm{gen}|C}(\omega)$:

$$\chi^{0\infty}_{\mathrm{gen}|C} = \chi^{0\infty}_C + C^{\infty}_C\,, \quad \delta_B\chi^{0\infty}_{\mathrm{gen}|C} = \mu\overrightarrow{s}_\infty\chi^{0\infty}_{\mathrm{gen}|C} = \mu Q_{C|\mathrm{int}}\chi^{0\infty}_{\mathrm{gen}|C}\,. \tag{4.14}$$

For dual vector, $\chi^{0\infty}_{\mathrm{gen}|C}(\partial_\omega) = \big(\chi^{0\infty}_C + C^{\infty}_C\big)(\partial_\omega)$, the transformation (4.14) with account of hermiticity $Q_{C|\mathrm{int}}$, $\mu$ looks as:

$$\begin{aligned}\delta_B\chi^{0\infty}_{\mathrm{gen}|C}(\partial_\omega) &= \big(\delta_B\chi^{0\infty}_{\mathrm{gen}|C}(\omega)\big)^+ : \delta_B\chi^{0\infty}_{\mathrm{gen}|C}(\partial_\omega)\\ &= \chi^{0\infty}_{\mathrm{gen}|C}(\partial_\omega)\overleftarrow{s}_\infty\mu = \chi^{0\infty}_{\mathrm{gen}|C}(\partial_\omega)Q_{C|\mathrm{int}}\,\mu\,.\end{aligned} \tag{4.15}$$

Explicitly, the action $S_{C|\infty}$ and its BRST-like invariance transformations can be given in the form

$$\begin{aligned}S_{C|\infty} &= \int d\eta_0\,\chi^{0\infty}_{\mathrm{g}|C}(\partial_\omega)Q_{C|\mathrm{int}}\chi^{0\infty}_{\mathrm{g}|C}(\omega)\,,\\ \chi^{0\infty}_{\mathrm{g}|C}(\omega) &= \big(\chi^{0\infty}_{\mathrm{gen}|C} + \chi^{*0\infty}_{\mathrm{gen}|C}\big)(\omega)\,,\\ \delta_B S_{C|\infty} &= 0\,.\end{aligned} \tag{4.16}$$

Here, both the generalized field, $\chi^{0\infty}_{\mathrm{gen}|C}$, and antifield, $\chi^{*0\infty}_{\mathrm{gen}|C}$, vectors are subject to the off-shell BRST extended constraints $\hat{L}_{11}$ (3.118):

$$\hat{L}_{11}\chi^{0\infty}_{\mathrm{g}|C} = 0 \Leftrightarrow \hat{L}_{11}\big(\chi^{0\infty}_{\mathrm{gen}|C}, \chi^{*0\infty}_{\mathrm{gen}|C}\big) = (0,0)\,. \tag{4.17}$$

However for the non-Lagrangian BRST–BFV description for free massless CSR field $\Phi(x,\omega)$ in the Bargmann–Wigner representation the minimal BV action, $S_{\min\Xi} = S_{C|\Xi}$, may be found with help of only Lagrangian multipliers procedure. We apply it for the duplet-like non-Lagrangian formulation with expressed field $\chi_1(\omega)$: $\chi_1(\omega) = -\frac{1}{2}m_{11}\Phi(\omega)$ (3.61)–(3.63) with only independent gauge-invariant equations of motion derived from the action $\mathcal{S}_{C|\Xi}(\Phi, \chi_2, \hat{\lambda}_i)$ (3.66). The functional $S_{C|\Xi} = S_{C|\Xi}\big(\Phi, \chi_2, \hat{\lambda}_i, \Phi^*, \chi^*_2, \hat{\lambda}^*_i, C_\varsigma, C_\sigma\big)$ and its nonvanishing global invariance transformation (following from (3.67), (3.68))

$$S_{C|\Xi} = \mathcal{S}_{C|\Xi} - \int d^d x \sum_{k,l\geq 0} \Big\{ \Phi^{*(m)_k,(n)_l} \partial_{\{m_k} C^{l}_{(m)_{k-1}\},(n)_l} + \chi_2^{*(m)_k,(n)_l} \Big( \partial_{\{m_k} C^{l}_{(m)_{k-1}\},(n)_l} - \imath \Xi \eta_{\{m_k\{n_l} C_{\varsigma}{}^{l-1}_{(m)_{k-1}\},(n)_{l-1}\}} \Big) + \sum_{k',l'} \hat{\lambda}^{*l}_{i(m)_k,(n)_l} \hat{R}^{l(m)_k,(n)_l}_{il'(m_1)_{k'}(n_1)_{l'}} (x, \partial_x) C_{\sigma}^{l'(m_1)_{k'}(n_1)_{l'}}(x) \Big\} , \tag{4.18}$$

$$\delta_B \big(\Phi^l, \chi_2^l\big)_{(m)_k,(n)_l} = -\Big(\partial_{\{m_k} C_{\varsigma}{}^{l}_{(m)_{k-1}\},(n)_l}, \partial_{\{m_k} C_{\varsigma}{}^{l}_{(m)_{k-1}\},(n)_l} - \imath \Xi \eta_{\{m_k\{n_l} C_{\varsigma}{}^{l-1}_{(m)_{k-1}\},(n)_{l-1}\}}\Big)\mu , \tag{4.19}$$

$$\delta_B \hat{\lambda}_i^{l(m)_k,(n)_l} = \sum_{k',l'} \hat{R}^{l(m)_k,(n)_l}_{il'(m_1)_{k'}(n_1)_{l'}} (x, \partial_x) C_{\sigma}^{l'(m_1)_{k'}(n_1)_{l'}} \mu , \tag{4.20}$$

solve the problem in the tensor form. Thus, we have derived the constrained BRST–BV minimal action for an irreducible form of constrained BRST–BFV LF (3.22) for free CSR of the $ISO(1, d-1)$ group described by the field $\Phi(x,\omega)$ and auxiliary classical $\chi_2(x,\omega)$, the ghost $C_\varsigma(x,\omega)$, the Lagrangian multipliers $\hat{\lambda}_i(x)$, $i = 1, 2, 3$, the ghost $C_\sigma$ fields and theirs antifields subject to the generalized traceless constraints (3.69) where the substitution $\varsigma \leftrightarrow C_\varsigma$ should be made.

The difference of the BRST–BFV descriptions for the fields describing all integer spin representations and the fields for CSR presented in the Subsec. 3.3 is inherited for BRST–BV descriptions as well. Note, the constrained BRST–BV actions for the CSR field in case of Fronsdal-like form of the equations (following to Schuster and Toro[30]) can be derived from the constrained Lagrangian BRST–BFV LF (without the fields with "negative spin values") which should be closely related with the minimal BV action $S_{C|\infty}$ (4.16) for massless totally-symmetric fields for all integer spins.

Different BRST–BV minimal actions may be used as the starting points to construct a LF for the CS field, being interacting both with itself, or with another scalar CS fields and with HS fields with integer spin in $R^{1,d-1}$ on a base of preservation underlying master equation.

## 5. Generalized Quartet-like Unconstrained Descriptions

To solve the problem, beyond of the extension of the constrained BRST–BFV approach to unconstrained one, it is sufficient to start from the triplet-like non-Lagrangian formulation (3.56), (3.57). We may obtain unconstrained quartet-like non-Lagrangian formulation (following, in part, to idea of Ref. 91 for the case of integer spin) by introducing a compensator field $\vartheta(\omega)$: $\delta\vartheta(\omega) = m_{11}\varsigma(\omega)$. Then we should enlarge the constraints (3.28)–(3.30) on the fields $(\Phi, \chi_1, \chi_2, \chi_0)$ with nontrivial gauge transformations (3.52) leaving by invariant the EoM (3.56), (3.57) up to the gauge-invariant equations as follows:

$$m_{11}\chi_0 - l_0\vartheta = 0\,, \qquad m_{11}\chi_1 - l_1\vartheta = 0\,, \tag{5.1}$$

$$m_{11}\chi_2 + 2\chi_1 - m_1^+\vartheta = 0\,, \quad m_{11}\Phi + 2\chi_1 - l_1^+\vartheta = 0\,. \tag{5.2}$$

Introducing four new sets of bosonic (real one-valued) fields $\lambda_j$, $j = 1, \ldots, 4$, playing the role of the Lagrangian multipliers for the modified constraints (5.1), (5.2) in addition to ones $\tilde{\lambda}_p$, $p = 1, \ldots, 6$ for the equations of motion (3.56), (3.57), we get an unconstrained LF with the action in the tensor form:

$$\begin{aligned}\mathcal{S}_\Xi = {} & \mathcal{S}_{C|\Xi}\big(\Phi, \chi_0, \chi_1, \chi_2; \tilde{\lambda}_p\big) \\ & + \int d^dx \sum_{k,l\geq 0}\Big[\lambda_1^{l|(m)_k,(n)_l}\big(m_{11}\Phi + 2\chi_1 - l_1^+\vartheta\big)^l_{(m)_k,(n)_l} \\ & + \lambda_2^{l|(m)_k,(n)_l}\big(m_{11}\chi_2 + 2\chi_1 - m_1^+\vartheta\big)^l_{(m)_k,(n)_l} \\ & + \lambda_3^{l|(m)_k,(n)_l}\big(m_{11}\chi_1 - l_1\vartheta\big)^l_{(m)_k,(n)_l} \\ & + \lambda_4^{l|(m)_k,(n)_l}\big(m_{11}\chi_0 - l_0\vartheta\big)^l_{(m)_k,(n)_l}\Big]\,,\end{aligned} \tag{5.3}$$

$$\begin{aligned}& \mathcal{S}_{C|\Xi}\big(\Phi, \chi_0, \chi_1, \chi_2; \tilde{\lambda}_p\big) \\ & \quad = \int d^dx \sum_{k,l\geq 0}\Big[\tilde{\lambda}_1^{l|(m)_k,(n)_l}\big(l_0\Phi - l_1^+\chi_0\big)^l_{(m)_k,(n)_l} \\ & \qquad + \tilde{\lambda}_2^{l|(m)_k,(n)_l}\big(l_1\Phi - l_1^+\chi_1 - \chi_0\big)^l_{(m)_k,(n)_l} \\ & \qquad + \tilde{\lambda}_3^{l|(m)_k,(n)_l}\big(l_0\chi_1 - l_1\chi_0\big)^l_{(m)_k,(n)_l} \\ & \qquad + \tilde{\lambda}_4^{l|(m)_k,(n)_l}\big(l_0\chi_2 - m_1^+\chi_0\big)^l_{(m)_k,(n)_l} \\ & \qquad + \tilde{\lambda}_5^{l|(m)_k,(n)_l}\big(l_1\chi_2 - m_1^+\chi_1 - \chi_0\big)^l_{(m)_k,(n)_l} \\ & \qquad + \tilde{\lambda}_6^{l|(m)_k,(n)_l}\big(m_1^+\Phi - l_1^+\chi_2\big)^l_{(m)_k,(n)_l}\Big]\,,\end{aligned} \tag{5.4}$$

which is invariant with respect to the gauge transformations with the unconstrained gauge parameter $\varsigma(\omega)$ for the fields

$$\delta(\Phi, \chi_1, \chi_2, \chi_0, \vartheta)(\omega) = \big(l_1^+, l_1, m_1^+, l_0, m_{11}\big)\varsigma(\omega)\,, \tag{5.5}$$

and with respect to the dual gauge transformations for the Lagrangian multipliers $\tilde{\lambda}_p$, $\lambda_j$ with additional unconstrained gauge parameters $\sigma^{l|(m)_k,(n)_l}$:

$$\delta\big(\tilde{\lambda}^l_p, \lambda^l_j\big)^{(m)_k,(n)_l}(x) = \sum_{k',l'}(\tilde{R}_p, R_j)^{l(m)_k,(n)_l}_{l'(m_1)_{k'}(n_1)_{l'}}(x, \partial_x)\sigma^{l'(m_1)_{k'}(n_1)_{l'}}(x) \tag{5.6}$$

with some local field-independent generators $\tilde{R}_{p}{}^{l(m)_k,(n)_l}_{l'(m_1)_{k'}(n_1)_{l'}}$, $R_{j}{}^{l(m)_k,(n)_l}_{l'(m_1)_{k'}(n_1)_{l'}}$ whose specific form should be derived from the gauge invariance for the EoM for the

Lagrangian multipliers:

$$\begin{aligned}
\frac{\delta \mathcal{S}_\Xi}{\delta \Phi} &= l_0\tilde{\lambda}_1 + l_1^d\tilde{\lambda}_2 + m_1^{+d}\tilde{\lambda}_6 + m_{11}^d\lambda_1 = 0\,,\\
\frac{\delta \mathcal{S}_\Xi}{\delta \chi_0} &= -l_1^{+d}\tilde{\lambda}_1 - \tilde{\lambda}_2 - l_1^d\tilde{\lambda}_3 - m_1^{+d}\tilde{\lambda}_4 - \tilde{\lambda}_5 + m_{11}^d\lambda_4 = 0\,,\\
\frac{\delta \mathcal{S}_\Xi}{\delta \chi_2} &= l_0\tilde{\lambda}_4 + l_1^d\tilde{\lambda}_5 - l_1^{+d}\tilde{\lambda}_6 + m_{11}^d\lambda_2 = 0\,, \qquad (5.7)\\
\frac{\delta \mathcal{S}_\Xi}{\delta \chi_1} &= l_0\tilde{\lambda}_3 - l_1^{+d}\tilde{\lambda}_2 - m_1^{+d}\tilde{\lambda}_5 + m_{11}^d\lambda_3 + 2(\lambda_1+\lambda_2) = 0\,,\\
\frac{\delta \mathcal{S}_\Xi}{\delta \vartheta} &= -l_0\lambda_4 - l_1^d\lambda_1 - m_1^{+d}\lambda_2 - l_1^d\lambda_3 = 0\,.
\end{aligned}$$

Here, the form of the dual operators $l_1^d$, $m_1^{+d}$, $l_1^{+d}$, $m_{11}^d$ are determined with help of (2.13), (2.18) according the rule for any tensors $G^l_{(m)_k,(n)_l}(x)$, $F^l_{(m)_k,(n)_l}(x)$ with a compact support

$$\begin{aligned}
\int d^dx\, G^l_{(m)_k,(n)_l}(\hat{A}F)^{l|(m)_k,(n)_l} &= \int d^dx(\hat{A}^dG)^l_{(m)_k,(n)_l}F^{l|(m)_k,(n)_l}\,,\\
&\qquad \hat{A}\in\left\{l_1, m_1^{(+)}, l_1^+\right\}, \qquad (5.8)\\
G^l_{(m)_k,(n)_l}(m_{11}F)^{l|(m)_k,(n)_l} &= \left(m_{11}^dG\right)^l_{(m)_k,(n)_l}F^{l|(m)_k,(n)_l}\,.
\end{aligned}$$

Again, by the choice of appropriate initial and boundary conditions for $\tilde{\lambda}_p^{l|(m)_k,(n)_l}$, $\lambda_j^{l|(m)_k,(n)_l}$ we always able to fix their unwanted degrees of freedom completely.[83] We use again such form of the free actions as the auxiliary ones to derive preferably the EoM for the fields $\Phi$, $\chi_1$, $\chi_2$, $\chi_0$, $\vartheta$.

Applying the terminology from the HS fields with discrete spin we will call the obtained irreducible gauge-invariant LF as the *quartet-like unconstrained formulation* for scalar bosonic field with CS $\Xi$ on $\mathbb{R}^{1,d-1}$ within Bargmann–Wigner representation. In turn, the functional $\mathcal{S}_{C|\Xi}\big(\Phi,\,\chi_0,\chi_1,\chi_2;\tilde{\lambda}_p\big)$ (5.4) should describe the constrained irreducible gauge-invariant LF in the so-called *triplet-like formulation* with constrained fields, Lagrangian multipliers and gauge parameters $\sigma^{l'(m_1)_{k'}(n_1)_{l'}}$.

The unconstrained LF given by the relations (5.3)–(5.6) presents the basic result of the section.

## 6. Conclusion

In this paper, we have developed a constrained BRST–BFV approach to a gauge-invariant description of EoM and action of free scalar CSR for the Poincaré group, with a fixed arbitrary CS $\Xi$ (when parameter $\nu = 1$) in Minkowski space–time $\mathbb{R}^{1,d-1}$ of an arbitrary dimension in a “metric-like” formulation within Bargmann–Wigner representation. The final constrained BRST–BFV representation for EoM, given by (3.22), in fact, determined by Wigner fields of two space–time variables

$x^m, \omega^m$, represents a first-stage reducible gauge theory and contains an auxiliary set of fields providing a BRST-unfolded form (in a ghost-independent representation), of the field equations (3.39)–(3.49) and gauge transformations (3.34), (3.35)–(3.38).

To present a constrained BRST–BFV gauge-invariant description of EoM, by transforming the Bargmann–Wigner equations (2.3) into four constraints (2.4), (2.5) (equivalently, (2.17)) imposed on the CS real-valued field $\Phi(x, \omega)$ in the coordinate form. The decomposition (2.6) for the field $\Phi(x, \omega)$ presents an original ansatz for the nontrivial solution for (2.4), (2.5) in powers of direct and inverse degrees in the variables $\omega^m$ (2.6), using an infinite set of conventional independent $\Phi^0_{(m)_k}(x)$ and additional $\Phi^l_{(m)_k,(n)_l}(x)$ tensor fields. The vector $\Phi(x, \omega)$ (in the space $\mathcal{V}$ having no scalar product structure) contains a standard contribution with the usual $\Phi^0_{(m)_k}(x)$ massless tensor fields of rank $k = 0, 1, 2, \ldots$ and a new one, $\Phi^l_{(m)_k,(n)_l}(x)$ from which the number particle operator extracts some vectors with "mixed positive and negative spin values" $(k, l)$: $k, l - 1 = 0, 1, 2, \ldots$. CSR realizations on (2.12) and (2.13) ones are different but not independent, due to "*coupling*" *equation* for $l = 0$ in the second, third and fourth lines of (2.13), due to an ambiguity in the definitions of $\Phi^l_{(m)_k,(n)_l}(x)$ (2.7), (2.8). The closure of the constraint algebra (2.17) under the commutator multiplication and a formal Hermitian conjugation generates a higher continuous spin symmetry algebra $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$ given in Table 1 with two center elements: the parameter $\nu$ and the value of CS, $\Xi$ for $\nu = 1$, since any linear combination of constraints should also be a constraint. Extracting a second-class constraint subsystem: the generalized trace, $m_{11}$, its dual, $m^+_{11}$, and the particle number, $g_0$, operators from the remaining $(4 + 1)$ first-class differential constraints, i.e. the divergence, $l_1$, the generalized divergence, $m_1$, their formal duals, $l^+_1$, $m^+_1$, and the D'Alambert operator, we construct, with respect to a reducible set of first-class constraints (considering $m_1 - l_1 = -i\Xi$ as a constraint), a constrained BRST operator, $\tilde{Q}_C$ (3.7), and a BRST-extended off-shell constraint, $\hat{M}_{11} = m_{11} + \cdots$, in an enlarged vector space, $\mathcal{V}_C$. They are found as a solution of the generating equations (3.1) with the boundary conditions (3.2). Calculating the $Q_C$-cohomology in the ghost number zero subspace of $\mathcal{V}_C$, which should lead to the Bargmann–Wigner equations fixes in a unique way, the representation (3.8) in $\mathcal{V}_C$, which allows one to select an independent set of constraints and then to reduce $\tilde{Q}_C$ to the constrained BRST operator $Q_C$ (3.10), without first-stage reducible ghost operators, and to determine the off-shell constraint $\hat{M}_{11}$ (3.11). The familiar application of the spectral problem, with a BRST equation $Q_C \chi^0_C = 0$, (3.18)–(3.20), however with no spin condition, as in the case of HS fields with discrete spin,[69] leads to the constrained BRST–BFV description of the first-stage reducible EoM (3.22). In the ghost-independent form the latter problem is realized with EoM (3.39)–(3.49) for one initial and nine auxiliary fields (in Bargmann–Wigner form with two sets of variables $x^m, \omega^m$) invariant with respect to reducible gauge transformations (3.35)–(3.38) with five gauge parameters, invariant under the transformations (3.34) with an independent gauge for gauge parameter $\varpi(x, \omega)$ and off-shell holonomic constraints (3.26)–(3.30).

A specific structure of the constraints and gauge transformations has permitted one to realize a partial gauge-fixing, jointly with a resolution of some of the non-Lagrangian EoM to obtain from the constrained BRST–BFV description the *triplet-like* (3.56), (3.57) and then *duplet-like* (3.58)–(3.60) formulations of EoM for scalar CSR fields in Bargmann–Wigner representation. These formulations are classified as irreducible gauge theories, respectively, with constrained three and two additional auxiliary fields, by analogy with the triplet and doublet descriptions for an HS field of an integer spin $s$.[86–88] Expressing the field $\chi_1(x,\omega)$ as a generalized trace of the basic CS field $\Phi(x,\omega)$, the non-Lagrangian gauge-invariant EoM (3.61)–(3.63) has also been derived with the help of an additional field $\chi_2(x,\omega)$. The respective constrained gauge-invariant LF for triplet-like and latter formulations with the actions $\mathcal{S}_{C|\Xi}(\Phi,\chi_0,\chi_1,\chi_2;\tilde{\lambda}_p)$, $\mathcal{S}_{C|\Xi}(\Phi,\chi_2,\hat{\lambda}_i)$ have been obtained with the help of some appropriate sets of real-valued gauge Lagrangian multipliers (following Ref. 83) for CSR scalar (real-valued) field $\Phi$ of CS $\Xi$ in the tensor form, respectively, in (3.66)–(3.69) and (5.4)–(5.6) for only gauge $\tilde{\lambda}_p$. The fields and gauge parameter $\varsigma(\omega)$ satisfy the generalized traceless (simply, $m_{11}$-traceless) conditions (3.60). We stress that the dynamics of the fields and Lagrangian multipliers is completely decoupled in the presented LFs for free CS field and the unwanted degrees of freedom for the Lagrangian multipliers can be accurately treated, e.g. it can be removed by the appropriate choice of the initial conditions for the respective EoM.

The characteristic feature of the constrained BRST–BFV descriptions of EoM and theirs derivative descriptions is the presence of respective sets of new infinite set of tensor fields with the so-called "negative spin values."

We have found, first, the interrelations of the resulting BRST–BFV description of EoM for a scalar CSR field (in the Bargmann–Wigner form) given in the basis of $m_{11}$-traceless fields with those for totally-symmetric HS fields with any integer spin $s=0,1,2,\dots$ in terms of Fronsdal-like (traceless) standard and new fields. Second, we have found the correspondence of the (double) $m_{11}$-traceless fields with the usual and new Fronsdal-like (double) traceless fields in (3.104)–(3.106). The latter allows one to present the parts of all the constrained LFs which contain the usual tensor and auxiliary fields for an CSR field entirely in terms of Fronsdal-like fields. We have shown that the constrained LFs for all integer spins do not coincide with respective ones for a scalar CSR field. However, for vanishing CS $\Xi=0$ the EoM (3.61)–(3.63) under the identification $\Phi=\chi_2$ without new (negative spin values) tensors they coincide with ones for HS field $\phi^0(x,\omega)$ (3.113) with all integer spins. We stress that, in case of the Fronsdal-like form of equations (suggested by Schuster and Toro[30]) which select the CSR field, they can be derived from the Lagrangian BRST–BFV EoM (without new fields presence) with the constrained LF closely related with that for massless totally-symmetric fields for all integer spins. We intend to solve this problem in a separate work based on the result of App. C, which justifies the presence of a CSR field in the spectrum of an open bosonic string within a special tensionless limit.

We have established an equivalence of non-Lagrangian EoM in the BRST unfolded form (3.39)–(3.49) of the suggested constrained BRST–BFV description with the irreducible CSR relations (2.4), (2.5). Incidentally, we have clarified the form of conditions necessary to select UIR of $ISO(1, d-1)$ with integer spin (3.125) and with residual gauge transformations, thus determining a class of gauge equivalent configurations instead of its unique representative. Note that the constraints in the respective conditions that select massless UIR both with CS and with integer spin are sufficient (without using the residual gauge transformations) to construct the constrained BRST operators and to derive the respective BRST–BFV description of EoM and LFs.

We have developed a BRST–BV approach to the suggested constrained BRST–BFV gauge-invariant description of EoM for a CSR field in a $R^{1,d-1}$ space–time and explicitly constructed the BRST–BV action (4.18) for the classical action $\mathcal{S}_{C|\Xi}(\Phi, \chi_2, \hat{\lambda}_i)$ with 2 fields and with a corresponding BRST-like invariance (4.19), (4.20) in the minimal set of constrained field–antifield configurations both for the fields and for the Lagrangian multipliers. The crucial point here is that all the fields, ghost fields and their antifields are combined within a unique generalized field–antifield vector (4.11) and contain new auxiliary (anti)field tensors as well. The actions serve, first, to construct quantum actions under an appropriate choice of gauge conditions, and second, to develop a construction of theories interacting with the CS field with accurate elaboration of the degrees of freedom for the set of Lagrangian multipliers. We stress that the construction of the minimal BRST–BV actions is differed from the procedure of finding BRST–BV minimal and quantum actions developed in Ref. 33 for the scalar CS field.

An unconstrained quartet-like LF (similar to the one for the integer spin case[91]) has also been found in (5.3) by including a compensator field to remove the $m_{11}$-tracelessness of the gauge parameter and by adding to the action for a triplet-like LF (5.4) of the augmented gauge-invariant constraint conditions (5.1), (5.2) with $(4+6)$ unconstrained Lagrangian multipliers. These multipliers are subject to the attributed gauge transformations (5.6) being dual for the gauge transformations (5.5) for the fields.

The higher continuous spin symmetry algebra $\mathcal{A}(\Xi; Y(k), \mathbb{R}^{1,d-1})$ which corresponds to the most general massless nonscalar CS one-valued irreducible representation of Poincaré group in Minkowski space $\mathbb{R}^{1,d-1}$ for $k = 0, 1, \ldots, k = [(d-4)/2]$ of the Bargmann–Wigner form is suggested in Subsec. 3.2.1 as well to be different from one in Ref. 82.

We have presented in App. A another way of higher continuous spin symmetry algebra $\mathcal{A}\big(\Xi; \mathbb{R}^{1,d-1}\big)$ realization by means of two sets of oscillator pairs corresponding to direct and inverse degrees in variables $\omega^m$ with endowing the Fock space $\mathcal{V} \equiv \mathcal{H}$ with a new scalar product. It should serve for further study of the algebra $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$ and its application for BRST LFs for CSR field.

It has been shown in App. B that there is no possibility to endow the vector space $\mathcal{V}$ with a Hilbert space structure with finite scalar product when explicitly working with the inverse degrees in powers of oscillators. This point proves an impossibility to use the latters for the purpose of BRST–BFV Lagrangian formulation of the form $S_{\Xi} \sim \langle\Phi|Q|\Phi\rangle'$.

There are numerous ways to elaborate the suggested constrained BRST–BFV and BRST–BV approaches for CSR in the Bargmann–Wigner representation, so as to study the Lagrangian dynamics of CSR in $\mathbb{R}^{1,d-1}$ in the case of arbitrary one-valued mixed-symmetric UIR with CS as well as to adapt the formalism to accommodate two-valued CSR in $\mathbb{R}^{1,d-1}$.

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## Appendix A. Higher Continuous Spin Symmetry Algebra $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$ with Two Sets of Oscillators

In this appendix, we describe another way to present the algebra $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$ in the sector of new tensor fields $\Phi^{l}_{(m)_k,(n)_l}(x)$. To this end, we endow $\mathcal{V}$ by the Fock space structure $\mathcal{V} \to \mathcal{H}$ with a new scalar product by presenting $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}^0 + \sum_{l>0} \mathcal{H}^l$, which is generated by two pairs of the Grassmann-even bosonic (dependent) oscillators with help of translational invariant vacuum vector: $|0\rangle$: $\partial^m|0\rangle = 0$:

$$\begin{gathered}\big(a_m, a^{+n}\big) \equiv -\imath\big(\partial^m_\omega, \omega^n\big)\,,\\ \big(b_m, b^{+n}\big) \equiv -\left(\frac{\partial}{\partial b^{+m}}, \frac{\imath\omega^n}{\omega^2}\right), \quad (a_m, b_m)|0\rangle = (0,0)\,,\end{gathered} \tag{A.1}$$

which are subject to the commutation relations:

$$\begin{aligned}\big[a^m, a^{+n}\big] &= -\eta^{mn}\,, \quad \big[b^m, b^{+n}\big] = -\eta^{mn}\,,\\ \big[a^+_m, b^+_n\big] &= 0\,, \qquad \big[a_m, b^+_n\big] = \eta_{mn}(b^+)^2 - 2b^+_m b^+_n\,.\end{aligned} \tag{A.2}$$

The validity of the latter commutator in (A.2) follows from (A.1) and the explicit calculation of $\big[\partial^m_\omega, \imath\omega^n/\omega^2\big]$, whereas the commutators $[a_m, b_n]$, $\big[a^+_m, b_n\big]$ are still remained undetermined.

The field $\Phi(x,\omega)$ (2.6) is presented as the vector from $\mathcal{H}$

$$|\Phi\rangle = \sum_{k,l\geq 0} \frac{\imath^{k+l}}{k!l!} \Phi^l_{(m)_k,(n)_l}(x) \prod_{i=1}^{k} a^{+m_i} \prod_{j=1}^{l} b^{+n_j}|0\rangle$$

$$\equiv \left[\Phi^0(x,\imath a^+) + \sum_{l\geq 1}\Phi^l\big(x,\imath a^+,\imath b^+\big)\right]|0\rangle \equiv \big|\Phi^0\big\rangle + \big|\Phi^-\big\rangle\,, \qquad \text{(A.3)}$$

for square integrable component functions in $\Phi^0\big(x,\imath a^+\big)$ and $\Phi^l\big(x,\imath a^+,\imath b^+\big)$ obtained from the decomposition (2.6). The different pairs of the oscillators are not independent, in view of $\omega^{m_1}\frac{\omega^{n_1}}{\omega^2} = \omega^{n_1}\frac{\omega^{m_1}}{\omega^2}$, (2.9) and (A.1):

$$\big(a^{+m}b^{+n} = a^{+n}b^{+m} \text{ and } a^{+m}b^+_m = -1\big) \Rightarrow \qquad \text{(A.4)}$$

$$a^{+2}b^{+2} = 1 \Rightarrow C^{+m} = -D^{+m}/D^{+2} = -D^{+m}C^{+2} \text{ for } C, D \in \{b, a\}\,, \qquad \text{(A.5)}$$

because of, $a^{+2}b^{+2} = (-\omega^m\omega_m)\cdot\big(-\omega^n/\omega^2\big)\big(\omega_n/\omega^2\big) = \big(-\omega^n\{\omega_n/\omega^2\}\big)\cdot\big(-\omega_m\{\omega^m/\omega^2\}\big) = (a^{+n}b^+_n)\cdot(a^+_m b^{+m})$, so that they both look as the inverse-like operators for each other, creating "particle" and "antiparticle," respectively.

An idea to consider the oscillators $a^{+m}$, $b^{+n}$ as independent ones but with additional constraints imposed, explicitly on the vector $\big|\Phi\rangle$:

$$\big[F(a^m b_m), G(a^2b^2)\big]|\Phi\rangle = 0\,, \quad \text{for} \quad [F(0), G(0)] = 0$$

$$\text{and} \quad \left[\frac{dF(y)}{dy}, \frac{dG(y)}{dy}\right]\bigg|_{y=0} = [C_F, C_G]\,, \qquad \text{(A.6)}$$

(for unknown analytical functions $F$, $G$) with some real constants $C_F$, $C_G$ leads to highly nonlinear expressions for the operators $F(a^m b_m)$ and $G(a^2b^2)$ generating the mixed traces for the component tensors in $|\Phi\rangle$ as it was shown in Sec. 2. Instead, we will explicitly resolve the oscillator constraints (A.4) thus reducing the ambiguity (2.7) in the choice of the component fields in $|\Phi\rangle$. To this end, we introduce system of projectors generated by the decomposition (2.8): $P_0$, $P_1$; $P_0 + P_1 = 1$, such that $P_iP_j = \delta_{ij}P_i$, $i,j = 0,1$, which are associated with the decomposition of any product $a^{+m}b^{+n}$ on trace and traceless parts (according the rules (2.7), (2.8)) when decompose of any product $a^{+m}b^{+n}$ in each monomial $\big(\prod a^+\big)^{(k)_l}\big(\prod b^+\big)^{(n)_l}$ for $k$, $l > 0$ and the same for the product $a^{+m_1}a^{+m_2}b^{+n_1}b^{+n_2}$, but for $k$, $l > 1$ as follows:

$$a^{+m}b^{+n} = d^{-1}\eta^{mn}\eta_{k_1k_2}a^{+k_1}b^{+k_2} + \big(\delta^m_{k_1}\delta^n_{k_2} - d^{-1}\eta^{mn}\eta_{k_1k_2}\big)a^{+k_1}b^{+k_2}$$

$$\equiv \big(P_0{}^{mn}_{k_1k_2} + P_1{}^{mn}_{k_1k_2}\big)a^{+k_1}b^{+k_2}\,, \qquad \text{(A.7)}$$

$$\Big(\prod a^+\Big)^{(m)_2}\Big(\prod b^+\Big)^{(n)_2}$$

$$= d^{-2}\eta^{m_1m_2}\,\eta^{n_1n_2}\eta_{k_1k_2}\eta_{l_1l_2}\Big(\prod a^+\Big)^{(k)_2}\Big(\prod b^+\Big)^{(l)_2}$$

$$+ d^{-1}\eta^{m_1m_2}\big(\delta^{n_1}_{l_1}\delta^{n_2}_{l_2} - d^{-1}\eta^{n_1n_2}\eta_{l_1l_2}\big)a^{+2}\Big(\prod b^+\Big)^{(l)_2}$$

$$
\begin{aligned}
&+ d^{-1}\eta^{n_1n_2}\big(\delta^{m_1}_{k_1}\delta^{m_2}_{k_2} - d^{-1}\eta^{m_1m_2}\eta_{k_1k_2}\big)\Big(\prod a^+\Big)^{(k)_2} b^{+2}\\
&+ \big(\delta^{m_1}_{k_1}\delta^{m_2}_{k_2}\delta^{n_1}_{l_1}\delta^{n_2}_{l_2} - d^{-2}\eta^{m_1m_2}\eta^{n_1n_2}\eta_{k_1k_2}\eta_{l_1l_2}\big)\Big(\prod a^+\Big)^{(k)_2}\Big(\prod b^+\Big)^{(l)_2}\\
\equiv{}& \sum_{i=0}^{1}\sum_{j=0}^{1} P_{i}{}^{m_1m_2}_{k_1k_2} P_{j}{}^{n_1n_2}_{l_1l_2}\Big(\prod a^+\Big)^{(k)_2}\Big(\prod b^+\Big)^{(l)_2}. \qquad \text{(A.8)}
\end{aligned}
$$

Due to the properties (A.4) the first summands in (A.7), (A.8) are equal respectively to

$$
\begin{aligned}
P_{0}{}^{mn}_{k_1k_2} a^{+k_1} b^{+k_2} &= -d^{-1}\eta^{mn}\,,\\
P_{0}{}^{m_1m_2}_{k_1k_2} P_{0}{}^{n_1n_2}_{l_1l_2}\Big(\prod a^+\Big)^{(k)_2}\Big(\prod b^+\Big)^{(l)_2} &= d^{-2}\eta^{m_1m_2}\eta^{n_1n_2}\,.
\end{aligned} \qquad \text{(A.9)}
$$

The projectors $P_{i}{}^{m_1m_2}_{k_1k_2}$ (because of the total symmetry of all component tensor fields $\Phi^l_{(m)_k,(n)_l}$, and thus due to $a^{+m}b^{+n} = a^{+n}b^{+m}$ inside $|\Phi\rangle$) satisfy to the symmetry properties:

$$
P_{i}{}^{m_1m_2}_{k_1k_2} = P_{i}{}^{m_2m_1}_{k_1k_2} = P_{i}{}^{m_1m_2}_{k_2k_1}\,, \quad i = 0, 1\,. \qquad \text{(A.10)}
$$

The last properties permits to find that the decomposition of the quartic term $\big(\prod a^+\big)^{(m)_2} \times \big(\prod b^+\big)^{(n)_2} \equiv \big(a^+b^+\big)^{(mn)_2}$ into components generated by the relations (A.7), (A.9):

$$
\begin{aligned}
\big(a^+b^+\big)^{(mn)_2} ={}& d^{-2}\eta^{m_1\{n_1}\eta^{m_2\}n_2}\\
&+ \Big(P_{1}{}^{m_1\{n_1}_{k_1l_1} P_{0}{}^{m_2\}n_2}_{k_2l_2} + \delta^{m_1}_{k_1}\delta^{\{n_1}_{l_1} P_{1}{}^{m_2\}n_2}_{k_2l_2}\Big)(a^+b^+)^{(kl)_2} \qquad \text{(A.11)}
\end{aligned}
$$

coincides with the decomposition (A.8) with account for Eqs. (A.4), (A.9):

$$
\begin{aligned}
\big(a^+b^+\big)^{(mn)_2} ={}& d^{-2}\eta^{m_1\{m_2}\eta^{n_1\}n_2}\\
&+ \Big(P_{1}{}^{m_1\{m_2}_{k_1k_2} P_{0}{}^{n_1\}n_2}_{l_1l_2} + \delta^{m_1}_{k_1}\delta^{\{m_2}_{k_2} P_{1}{}^{n_1\}n_2}_{l_1l_2}\Big)(a^+b^+)^{(kl)_2}\,. \qquad \text{(A.12)}
\end{aligned}
$$

Therefore, the decomposition (A.7) is sufficient to reduce an ambiguity in the choice of the component tensors in $\big|\Phi\rangle$ (A.3), which should now be determined as

$$
\begin{aligned}
\big|\Phi\rangle ={}& \sum_{k\geq 0}\Bigg\{\sum_{l\geq 0}^{k}\frac{\imath^{k+l}}{k!l!}\Bigg(\prod_{h=1}^{l} P_{1}{}^{\sigma_h\rho_h}_{m_hn_h}\Bigg)\Phi^l_{(\sigma)_l m_{l+1}\cdots m_k,(\rho)_l}\\
&+ \sum_{l>k}\frac{\imath^{k+l}}{k!l!}\Bigg(\prod_{h=1}^{k} P_{1}{}^{\sigma_h\rho_h}_{m_hn_h}\Bigg)\Phi^l_{(\sigma)_k,(\rho)_k n_{k+1}\cdots n_l}\Bigg\}\prod_{i=1}^{k} a^{+m_i}\prod_{j=1}^{l} b^{+n_j}|0\rangle\\
\equiv{}& \Bigg(\Phi^0\big(x,\imath a^+\big) + \sum_{l\geq 1}\hat{\Phi}^l\big(x,\imath a^+,\imath b^+\big)\Bigg)|0\rangle \equiv \big|\Phi^0\big\rangle + \big|\hat{\Phi}^-\big\rangle\,, \qquad \text{(A.13)}
\end{aligned}
$$

$$\hat{\Phi}^l_{(m)_k,(n)_l} \equiv \left(\prod_{h=1}^{l} P_1{}^{\sigma_h\rho_h}_{m_h n_h}\right)\Phi^l_{(\sigma)_l m_{l+1}\cdots m_k,(\rho)_l}\theta_{k,l} + \left(\prod_{h=1}^{k} P_1{}^{\sigma_h\rho_h}_{m_h n_h}\right)\Phi^l_{(\sigma)_k,(\rho)_k n_{k+1}\cdots n_l}\theta_{l,k-1}\,, \tag{A.14}$$

where the operators $a^{+m}$, $b^{+n}$ are already considered on the set of such vectors as independent oscillators. Note, the traceless projection concerned only new tensor fields $\Phi^l_{(m)_k,(n)_l}(x)$, $l>0$, whereas the standard fields, $\Phi^0_{(m)_k}(x)$, have not been touched when resolving the operator identities. Because of $P_0{}^{m_i n_i}_{\sigma_i\rho_i}\hat{\Phi}^l_{(m)_k,(n)_l}(x)=0$ for $i=1,\ldots,\min(k,l)$ the field $\tilde{\Phi}^l_{(m)_k,(n)_l}$ given by Eq. (2.7) contains the field $\hat{\Phi}^l_{(m)_k,(n)_l}$ as its traceless part without summands proportional to $P_0{}^{\sigma_h\rho_h}_{m_h n_h}$ projector according to (A.14), (2.10). It means, in fact that all new tensor fields $\Phi^l_{(m)_k,(n)_l}(x)$, $l>0$ should be traceless when calculating of any traces:

$$\eta^{m_{k-i}m_k}\Phi^l_{(m)_k,(n)_l} = \eta^{m_{k-i}n_{l-j}}\Phi^l_{(m)_k,(n)_l} = \eta^{n_{l-j}n_l}\Phi^l_{(m)_k,(n)_l} = 0\,, \tag{A.15}$$

for $i=1,\ldots,k-1$, $j=1,\ldots,l-1$, when $l,k>0$. Therefore, instead of the tensors $\hat{\Phi}^l_{(m)_k,(n)_l}$ we may equivalently write $\Phi^l_{(m)_k,(n)_l}$ in the decomposition (A.13) implying validity of the traceless condition (A.15).

The Poincaré group IR relations (2.4), (2.5) in the tensor form (2.12), (2.13) take the equivalent representation in terms of the operators [confer with Eqs. (2.17), (2.18)]

$$\left(l_0, l_1, m_1^+, m_{11}\right)|\Phi\rangle = 0\,, \tag{A.16}$$

$$l_1 = -\imath a^m\partial_m - \imath b^{+n}\left[2b^{+m}b_n - b^+_n b^m\right]\partial_m\,, \quad m_1^+ = -\imath a^{+m}\partial_m + \imath\Xi\,, \tag{A.17}$$

$$\tilde{m}_{11} = a^2 + \left\{a^m, b^{+n}\left[2b^+_m b_n - b^+_n b_m\right]\right\} + b^{+2}\left(b^{+2}b^2 - 2(d-2)b^{+k}b_k\right) + \nu\,, \tag{A.18}$$

where, first, the sign "{ , }" in (A.18) is the anticommutator, second, we have used the rule to express the derivative $\partial/\partial\omega^m$ in terms of the oscillators without their negative degrees:

$$\frac{\partial}{\partial\omega^m} = \frac{\partial}{\partial\omega^m}\Big|_{-\imath\frac{\omega^n}{\omega^2}=b^{+n}=\text{const}} + \frac{\partial}{\partial\omega^m}\Big|_{-\imath\omega^n=a^{+n}=\text{const}} \tag{A.19}$$

$$\begin{aligned} &= \imath a_m + \frac{\partial b^{+n}}{\partial\omega^m}\frac{\partial}{\partial b^{+n}} = \imath a_m - \imath\frac{\partial\left(\omega^n/\omega^2\right)}{\partial\omega^m}\frac{\partial}{\partial b^{+n}} \\ &= \imath a_m + \imath\left(\frac{\delta^n_m}{\omega^2} - 2\frac{\omega^n\omega_m}{\omega^4}\right)b_n = \imath a_m + \imath\left(\frac{\delta^n_m\omega^k\omega_k}{\omega^2\cdot\omega^2} - 2\frac{\omega^n\omega_m}{\omega^4}\right)b_n \\ &= \imath a_m + \imath b^{+n}\left[2b^+_m b_n - b^+_n b_m\right]\,, \end{aligned} \tag{A.20}$$

To get Lagrangian form of the equations (A.16) (without Lagrangian multipliers) we need $\mathbb{R}$-valued Lagrangian action within BRST–BFV approach. Therefore, the set of initial constraints $\{o_\alpha\} = \left(l_0, l_1, m_1^+, m_{11}\right)$ (A.17), (A.18) should be closed with respect to $[\,,]$-multiplication and Hermitian conjugation in $\mathcal{H}$. To do so we determine the scalar product on the space of the vectors (A.13) and its dual as follows:

$$\begin{aligned}\langle\Psi|\Phi\rangle &= \big\langle\Psi^0\big|\Phi^0\big\rangle + \big\langle\Psi^-\big|\Phi^-\big\rangle \\ &= \int d^dx \Bigg\{ \sum_{k_1,k_2=0}^{\infty} \frac{\imath^{k_1}(-\imath)^{k_2}}{k_1!k_2!} \langle 0| \prod_{j=1}^{k_2} a^{m_j} \Psi^*_{(m)_{k_2}} \Phi_{(n)_{k_1}} \prod_{i=1}^{k_1} a^{+n_i}|0\rangle \\ &\quad + \sum_{k_1,k_2,l_1,l_2>0}^{\infty} \frac{(-\imath)^{l_2+k_2}\imath^{l_1+k_1}}{k_1!l_1!k_2!l_2!} \langle 0| \prod_{i_2,j_2=1}^{k_2,l_2} b^{n'_{j_2}} a^{m'_{i_2}} \Psi^{l_2 *}_{(m')_{k_2},(n')_{l_2}} \\ &\quad \times \Phi^{l_1}_{(m)_{k_1},(n)_{l_1}} \prod_{i_1,j_1=1}^{k_1,l_1} a^{+m_{i_1}} b^{+n_{j_1}} |0\rangle \Bigg\} \\ &= \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l}}{k!l!} \int d^dx\, \Psi^{l*}_{(m)_k,(n)_l} \Phi^{l(m)_k,(n)_l}\,,\end{aligned} \tag{A.21}$$

where the nondiagonal terms proportional to

$$\left(\sum_{p=1} z_{p,l} \Psi^{2p+l*}_{(m)_p(m')_k,,(n')_l(n)_p}{}^{(n)_p}\right) \Phi^{p+l(m')_k,(m)_p(n')_l} + \text{c.c.} \tag{A.22}$$

(for some rationals $z_{p,l}$) arising in (A.21) from the noncommutativity of $a^m$ and $b^+_n$ (A.2) and its (usual) Hermitian conjugated for $a^{+m}$ and $b_n$ should vanish due to the traceless condition (A.15). For instance, for $p = 1$; $l, k = 0$, we have,

$$\begin{aligned}\langle 0|\Psi^{2*}_{m,(n)_2} b^{n_1} b^{n_2} a^m \Phi^1_k b^{+k}|0\rangle &= 2\Psi^{2*}_{m,n}{}^n \Phi^{1m} - 4\Psi^{2*}_{m,}{}^m{}_n \Phi^{1n} \\ &= -2\Psi^{2*}_{m,n}{}^n \Phi^{1m} = 0\,.\end{aligned} \tag{A.23}$$

The Hermitian conjugated for (A.2) mixed oscillator's commutators take the form

$$[a_m, b_n] = 0\,, \quad \big[a^+_m, b_n\big] = -\eta_{mn} b^2 + 2b_m b_n\,. \tag{A.24}$$

From the Hermitian conjugation of the identity (A.4), $\big(a^{+m}b^+_m\big)^+ = b_m a^m = -1$, the operator $\tilde{m}_{11}$ in (A.18) is simplified to

$$\begin{aligned} m_{11} &= a^2 + 2\big\{a^m, b^{+n}b^+_m b_n\big\} - a^m b^{+2} b_m \\ &\quad + b^{+2}\big(1 + b^{+2}b^2 - 2(d-2)b^{+k}b_k\big) + \nu\,. \end{aligned} \tag{A.25}$$

The closedness of the set of operators (A.17), (A.25), first, with respect to the Hermitian conjugation in $\mathcal{H}$ with a scalar product (A.21) leads to its augmentation by the operators

$$m_1 = -\imath a^m \partial_m - \imath\Xi\,, \tag{A.26}$$

$$l_1^+ = -\imath a^{+m}\partial_m - \imath\big[2b_n^+ b^m - b^{+m} b_n\big] b^n \partial_m\,, \tag{A.27}$$

$$\begin{aligned} m_{11}^+ &= a^{+^2} + 2\big\{a^{+m}, b^{+n} b_m b_n\big\} - b_m^+ b^2 a^{+m} \\ &\quad + \big(1 + b^{+^2} b^2 - 2(d-2) b^{+l} b_l\big) b^2 + \nu\,. \end{aligned} \tag{A.28}$$

Second, its closedness with respect to the $[\ ,\ ]$-multiplication for $l_1^+$, $l_1$, $m_1$, $m_1^+$:

$$\begin{aligned} \big[l_1, l_1^+\big] &= \big(1 + 3b^{+^2} b^2 - 4(b^{+l} b_l)^2\big) l_0 \\ &\quad - 2\left\{b^{+^2} b_m b_k + b_m^+ b_k^+ b^2 - 2b_m^+ \left(b^{+l} b_l + \frac{1}{2}(d-2)\right) b_k\right\}\partial^m \partial^k \\ &= \big(1 + 3b^{+^2} b^2 - 4(b^{+l} b_l)^2\big) l_0 - 2 l_1^{b+}\big(l_1^+ - m_1^+\big) \\ &\quad - 2(l_1 - m_1) l_1^b - 4 l_1^{b+}\left\{\frac{1}{2}(d-2) - b^{+l} b_l\right\} l_1^b - 2\imath\Xi\big(l_1^{b+} - l_1^b\big)\,, \end{aligned} \tag{A.29}$$

$$\begin{aligned} \big[l_1, m_1^+\big] &= \big(1 + b^{+^2} b^2\big) l_0 - 2\big(b^{+^2} b_m b_k + b_m^+ b_k^+ b^2\big)\partial^m \partial^k \\ &\quad + 4 b_m^+ b^{+l} b_l b_k \partial^m \partial^k \\ &= \big(1 + b^{+^2} b^2\big) l_0 - 2 l_1^{b+}\big(l_1^+ - m_1^+\big) - 2(l_1 - m_1) l_1^b \\ &\quad - 4 l_1^{b+} b^{+l} b_l l_1^b - 2\imath\Xi\big(l_1^{b+} - l_1^b\big)\,, \end{aligned} \tag{A.30}$$

$$\big[m_1, l_1^+\big] = \big[l_1, m_1^+\big] \equiv \big(\big[l_1, m_1^+\big]\big)^+\,, \tag{A.31}$$

$$\big[l_1, m_1\big] = 2\big(b^{+^2} l_0 + 4b^{+^2} l_1^{b+^2}\big) b^{+l} b_l - 4b^{+^2} l_1^{b+} l_1^b\,, \tag{A.32}$$

(with account of $\big[l_1^+, m_1^+\big] = -\big(\big[l_1, m_1\big]\big)^+$) implies an inclusion in itself of the operators of divergence and gradient with respect to the second group indices $(n)_l$ in the tensors $\Phi^l_{(m)_k,(n)_l}$,

$$\big(l_1^b, l_1^{b+}\big) \overset{\text{def}}{=} -\imath\big(b_m, b_m^+\big)\partial^m \text{ so that } \big[l_1^b, l_1^{b+}\big] = l_0\,. \tag{A.33}$$

This fact requires a further careful study of the nonlinear HCS symmetry algebra, in question, in the representation with two pairs of dependent oscillators.

## Appendix B. On Problems with Lagrangian Formulations with Single Set of Oscillators

In this appendix, we present the algebra $\mathcal{A}(\Xi;\mathbb{R}^{1,d-1})$ with use of the inverse degrees in $a^{+m} = -\imath\omega^m$ oscillators, which however leads to the problem with finiteness of the scalar product. To this end, we endow $\mathcal{V}$ with a new scalar product, $\langle\ |\ \rangle'$, by presenting $\mathcal{V}$ as $\mathcal{V} = \mathcal{V}^0 + \sum_{l>0}\mathcal{V}^l = \mathcal{V}^0 + \mathcal{V}^-$ permitting the representation for the vector $\big|\Phi\big\rangle$ as in (2.6) or (A.3) but for $b^{+n} = -\imath\omega^n/\omega^2 = -a^{+n}/a^{+2}$:

$$
\begin{aligned}
\langle\Psi|\Phi\rangle' &= \big\langle\Psi^0\big|\Phi^0\big\rangle' + \langle\Psi^-|\Phi^-\rangle' \\
&= \int d^dx \Bigg\{ \sum_{k_1,k_2=0}^{\infty} \frac{\imath^{k_1}(-\imath)^{k_2}}{k_1!k_2!} \langle 0| \prod_{j=1}^{k_2} a^{m_j}\,\Psi^*_{(m)_{k_2}}\,\Phi_{(n)_{k_1}} \prod_{i=1}^{k_1} a^{+n_i}|0\rangle \\
&\quad + \sum_{k_1,k_2,l_1,l_2>0}^{\infty} \frac{(-\imath)^{l_2+k_2}\imath^{l_1+k_1}}{k_1!l_1!k_2!l_2!} \langle 0| \prod_{i_2,j_2=1}^{k_2,l_2} \frac{-a^{n'_{j_2}}}{a^2} a^{m'_{i_2}}\,\Psi^{l_2*}_{(m')_{k_2},(n')_{l_2}} \\
&\quad \times \Phi^{l_1}_{(m)_{k_1},(n)_{l_1}} \prod_{i_1,j_1=1}^{k_1,l_1} a^{+m_{i_1}} \frac{-a^{+n_{j_1}}}{a^{+2}}|0\rangle \Bigg\} \\
&= \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l}}{k!l!} \int d^dx\, \Psi^{l*}_{(m)_k,(n)_l}\,\Phi^{l(m)_k,(n)_l} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int d^dx \Bigg\{ \Psi^{0*}_{(n)_k}\Phi^{0(n)_k} + \sum_{l>0} \frac{(-1)^l}{l!} \big[\Psi^{l*}_{(n)_l} K_{l,l}\Phi^{l(n)_l} + \mathit{more}\big] \Bigg\},
\end{aligned}
\tag{B.1}
$$

with some real numbers $K_{l,l}$. Here, the term "*more*" denotes the summands proportional to a respective product $\Psi^{l*}_{(m)_k,(n)_l}\Phi^{l(m)_k,(n)_l}$ for $k > 0$ and some possible others.

Indeed, the orthogonality properties among the vectors $\langle 0|a^m(a^{+m}|0\rangle)$ and $\frac{a^{+m}}{a^{+2}}|0\rangle\big(\langle 0|\frac{a^m}{a^2}\big)$ take the form:

$$
\begin{aligned}
\langle 0| \prod_{j=1}^{p} a^{m_j} \prod_{i=1}^{q} a^+_{n_i}|0\rangle &= \delta_{pq}(-1)^p p!\, S^{(m)_p}_{(n)_p}\,, \\
\langle 0| \prod_{j=1}^{p} a^{m'_j} \prod_{i_1,j_1=1}^{k_1,l_1} a^{+m_{i_1}} \frac{a^{+n_{j_1}}}{a^{+2}}|0\rangle &= 0\,,
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
&\langle 0| \prod_{j=1}^{p} \frac{a_{m_j}}{a^2} \prod_{i=1}^{k} \frac{a^{+n_i}}{(a^+)^2}|0\rangle \\
&\quad = \begin{cases} \delta_{p,k+2l}(-1)^p p!\, S^{(n)_p}_{(m)_p} K_{k+2l,k}\,\eta_{n_{k+1}n_{k+2}}\cdots\eta_{n_{k+2l-1}n_{k+2l}} & p > k\,, \\ \delta_{p+2l,k}(-1)^k k!\, S^{(n)_k}_{(m)_k} K_{p,p+2l}\,\eta^{m_{p+1}m_{p+2}}\ldots\eta^{m_{p+2l-1}m_{p+2l}} & p \le k \end{cases}
\end{aligned}
\tag{B.3}
$$

$\big($and with more complicated form for $\langle 0|\prod_{j_1,i_1=1}^{p_1,k_1}\frac{a_{m'_{i_1}}}{a^2}a^{n'_{j_1}}\prod_{i_2=1,j_2=1}^{p_2,k_2}a^{+n_{j_2}}\times\frac{a^{+n_{i_2}}}{(a^+)^2}|0\rangle\big)$, where $K_{p,p+2l}=K_{p+2l,p}$[k] and explicitly for $l=0$:

$$K_{p,p}=\int_0^\infty dt_1\,dt_2\sum_{k=0}\frac{(t_1t_2)^k}{(k!)^2}\prod_{j=1}^{pk}2j[d+2(j+p-1)]\,, \tag{B.4}$$

$\forall p,\,k,\,q\in\mathbb{N}$ with the symmetrizer $S^{(m)_k}_{(n)_k}$.

The first products in (B.2) are standard, whereas to prove the validity of the second ones, we apply the induction. For $p=1$, $\forall k\in\mathbb{N}$ we have

$$\begin{aligned}\langle 0|a^{m_1}\prod_{i=1}^k\frac{a^{+n_i}}{(a^+)^2}|0\rangle &= \langle 0|\left(\left[a^{m_1},a^{+n_1}\right]\frac{1}{(a^+)^2}-a^{+n_1}\left[a^{m_1},a^{+2}\right]\frac{1}{(a^+)^4}\right)\prod_{i=2}^k\frac{a^{+n_i}}{(a^+)^2}|0\rangle\\ &= -\langle 0|\big(\eta^{m_1n_1}(a^+)^2-2a^{+n_1}a^{+m_1}\big)\frac{1}{(a^+)^4}\prod_{i=2}^k\frac{a^{+n_i}}{(a^+)^2}|0\rangle=0\,,\end{aligned} \tag{B.5}$$

due to $\langle 0|a^{+n_1}=0$. Let for $\forall p\le p_0\in\mathbb{N}$ the same equations as one (B.5) hold. Then, for $p=p_0+1$, $\forall k\in\mathbb{N}$ it follows, with account of the relation above:

$$\begin{aligned}&\langle 0|\left(\prod_{j=1}^{p_0}a^{m_j}\right)a^{m_{p_0+1}}\prod_{i=1}^k\frac{a^{+n_i}}{(a^+)^2}|0\rangle\\ &=-\langle 0|\left\{\left(\prod_{j=1}^{p_0}a^{m_j}\right)\big(\eta^{m_{p_0+1}n_1}(a^+)^2-2a^{+n_1}a^{+m_{p_0+1}}\big)\frac{1}{(a^+)^4}-\frac{a^{+n_1}}{(a^+)^2}a^{m_{p_0+1}}\right\}\prod_{i=2}^k\frac{a^{+n_i}}{(a^+)^2}|0\rangle\\ &=\langle 0|\left(\prod_{j=1}^{p_0}a^{m_j}\right)\frac{a^{+n_1}}{(a^+)^2}a^{m_{p_0+1}}\prod_{i=2}^k\frac{a^{+n_i}}{(a^+)^2}|0\rangle\\ &\quad-\langle 0|\left(\prod_{j=1}^{p_0}a^{m_j}\right)\left(\eta^{m_{p_0+1}n_1}\eta_{m_{k+1}m_{k+2}}-\delta^{m_{p_0+1}}_{\{m_{k+1}}\delta^{n_1}_{m_{k+2}\}}\right)\prod_{i=2}^{k+2}\frac{a^{+n_i}}{(a^+)^2}|0\rangle\end{aligned}$$

[k]Again, the nondiagonal terms in the scalar product (B.1) are proportional to

$$\left(\sum_{p=1}Z_{p,l}\Psi^{2p+l*}_{(m)_p(m')_k,,(n')_l(n)_p}{}^{(n)_p}\right)\Phi^{p+l(m')_k,(m)_p(n')_l}+\text{c.c.}$$

as in (A.22) (for some rationals $Z_{p,l}\sim K_{p,p+2l}$) which may vanish due to symmetry (2.10) of the new tensor fields in case choosing only traceless components from it (A.15) following to (A.14) with projectors $\tilde{P}1^{\sigma_h\rho_h}_{m_hn_h}(a^+)=P1^{\sigma_h\rho_h}_{m_hn_h}(a^+,b^+=-a^+/a^{+2})$, so that the only diagonal terms with $K_{p,p}$ survive in (B.1). The last argument provides the correct spectral property and nondegeneracy for $\langle\Psi|\Phi\rangle'$, in which for any tensor $\Phi^l_{(m)_k,(n)_l}$ the only one tensor $\Psi^{l*}_{(m)_k,(n)_l}$ exists which gives input in the scalar product.

$$= -\langle 0|\left(\prod_{j=1}^{p_0} a^{m_j}\right)\left\{\left(\eta^{m_{p_0+1}n_2}\eta_{m_{k+3}m_{k+4}} - \delta^{m_{p_0+1}}_{\{m_{k+3}}\delta^{n_2}_{m_{k+4}\}}\right)\right.$$

$$\left.\times \frac{a^{+n_1}}{(a^+)^2}\prod_{i=3}^{k+4}\frac{a^{+n_i}}{(a^+)^2} - \frac{a^{+n_1}}{(a^+)^2}\frac{a^{+n_2}}{(a^+)^2}a^{m_{p_0+1}}\prod_{i=3}^{k}\frac{a^{+n_i}}{(a^+)^2}\right\}|0\rangle$$

$$= \cdots = \langle 0|\left(\prod_{j=1}^{p_0} a^{m_j}\right)\prod_{i=1}^{k}\frac{a^{+n_i}}{(a^+)^2}a^{m_{p_0+1}}|0\rangle = 0\,, \tag{B.6}$$

due to the repeated applying of the induction hypothesis, e.g. for the first summand in the relation before last (and for the second term in the previous relation), as well as with commutating of $a^{m_{p_0+1}}$ with $\frac{a^{+n_i}}{(a^+)^2}$ for $i = 3, \ldots, k$. The Hermitian conjugated quantities for ones in (B.2): $\langle 0|\prod_{i=1}^{k}\frac{a^{n_i}}{a^2}\prod_{j=1}^{p}a^{+m_j}|0\rangle$, vanish as well.

To establish validity of (B.3) we should commute $\frac{1}{(a^+)^{2p}}$ through $\frac{1}{a^{2k}}$, which may be done with help of the integral representation for $\frac{1}{(a^{(+)})^{2k}} = \int_0^\infty dt\,\exp\{-ta^{(+)2k}\}$, starting from the case, $p = k$, by means of the auxiliary relation:

$$\langle 0|\frac{1}{a^{2k}}\frac{1}{a^{+2k}}|0\rangle = \langle 0|\int_0^\infty dt_1\,dt_2\sum_{e\geq 0}\frac{(-t_1)^e(a^2)^{ke}}{e!}\sum_{g\geq 0}\frac{(-t_2)^g(a^{+2})^{kg}}{g!}|0\rangle \tag{B.7}$$

$$= \int_0^\infty dt_1\,dt_2\sum_{e,g\geq 0}\frac{(-t_1)^e(-t_2)^g\langle 0|\{(a^2)^{ke}(a^{+2})^{kg}\}|0\rangle}{e!g!}$$

$$= \int_0^\infty dt_1\,dt_2\sum_{e\geq 0}\frac{(t_1t_2)^e}{(e!)^2}\langle 0|\prod_{j=1}^{ke}4j(g_0+j-1)|0\rangle$$

$$= \int_0^\infty dt_1\,dt_2\sum_{e\geq 0}\frac{(t_1t_2)^e}{(e!)^2}\prod_{j=1}^{ke}2j[d+2(j-1)] \tag{B.8}$$

with using of the expansion above in Taylor series for $\exp\{-ta^{(+)2k}\}$, spectral properties: $\langle 0|(a^2)^{ke}(a^{+2})^{kg}|0\rangle \sim \delta_{ge}\cdots$, and that $\forall k \in \mathbb{N}_0$:

$$\langle 0|(a^2)^k(a^{+2})^k|0\rangle = \prod_{j=1}^{k}\langle 0|4j(g_0+j-1)|0\rangle \quad \text{and}$$

$$\langle 0|(g_0+j-1)|0\rangle = (d/2+j-1)\,. \tag{B.9}$$

Therefore, we have respectively for $p = k = 1$ and $p = 2$, $k = 1$

$$\langle 0|\frac{a^m}{a^2}\frac{a^+_n}{(a^+)^2}|0\rangle = \langle 0|\int_0^\infty dt_1\,dt_2\sum_{e\geq 0}\frac{(t_1t_2)^e}{(e!)^2}a^m\left(a^{2e}a^{+2e}\right)a^+_n|0\rangle$$

$$= -\delta^m_n\int_0^\infty dt_1\,dt_2\sum_{e\geq 0}\frac{(t_1t_2)^e}{(e!)^2}\prod_{j=1}^{e}\langle 0|4j(g_0+j)|0\rangle$$

$$= -\delta^m_n \int_0^\infty dt_1\, dt_2 \sum_{e\geq 0} \frac{(t_1t_2)^e}{(e!)^2} \prod_{j=1}^{e} 2j[d+2j]\,, \tag{B.10}$$

$$\langle 0| \frac{a^{m_1}a^{m_2}}{a^4} \frac{a^+_n}{(a^+)^2}|0\rangle = \int_0^\infty dt_1\, dt_2 \sum_{e\geq 0} (-1)^e \frac{(t_1t_2^2)^e}{e!(2e)!} \langle 0|a^{m_1}a^{m_2}\left(a^{4e}a^{+4e}\right)a^+_n|0\rangle = 0\,, \tag{B.11}$$

so that, for any $p = k+1$, $p$, $k \in \mathbb{N}$ the presentation (B.3) is valid. Whereas for $p = k$, $\forall p \in \mathbb{N}$ the average values in (B.3) calculated with account of (B.7), (B.8):

$$\langle 0| \prod_{j=1}^{p} \frac{a^{m_j}}{a^2} \prod_{i=1}^{p} \frac{a^+_{n_i}}{(a^+)^2}|0\rangle$$

$$= \int_0^\infty dt_1 dt_2 \sum_{e,g\geq 0} \frac{(-t_1)^e(-t_2)^g}{e!g!} \langle 0| \prod_{j=1}^{p} a^{m_j} \left\{(a^2)^{pe}(a^{+2})^{pg}\right\} \prod_{i=1}^{p} a^+_{n_i}|0\rangle$$

$$= \int_0^\infty dt_1\, dt_2 \sum_{e\geq 0} \frac{(t_1t_2)^e}{(e!)^2} \left\{ \langle 0| \prod_{j=1}^{p} a^{m_j} \prod_{j=1}^{pe}[4j(g_0+j-1)] \prod_{i=1}^{p} a^+_{n_i}|0\rangle \right\}$$

$$= \int_0^\infty dt_1\, dt_2 \sum_{e\geq 0} \frac{(t_1t_2)^e}{(e!)^2} \left\{ \langle 0| \prod_{j=1}^{pe}[4j(g_0+j+p-1)] \prod_{j=1}^{p} a^{m_j} \prod_{i=1}^{p} a^+_{n_i}|0\rangle \right\}$$

$$= (-1)^p p! S^{(m)_p}_{(n)_p} \int_0^\infty dt_1\, dt_2 \sum_{e\geq 0} \frac{(t_1t_2)^e}{(e!)^2} \prod_{j=1}^{pe} 2j(d+2(j+p-1))\,, \tag{B.12}$$

that proves the validity of (B.3) with $K_{p,p}$ in (B.4). The case $p \neq k$ in (B.3) due to the argument from footnote i is not essential for the evaluation of the scalar product (B.1).

Let us evaluate the finiteness of the quantities $K_{p,p}$ (B.4). It is enough to check it for simplest case of $K_{1,1}$:

$$K_{1,1} = \int_0^\infty dt_1\, dt_2 \sum_{k=0} \frac{(t_1t_2)^k}{(k!)^2} \prod_{j=1}^{k} 2j[d+2j]$$

$$> \int_0^\infty dt_1\, dt_2 \sum_{k=0} \frac{(t_1t_2)^k}{(k!)^2} \prod_{j=1}^{k} 4j^2$$

$$= \int_0^\infty dt_1\, dt_2 \sum_{k=0} (4t_1t_2)^k > \int_0^\infty dt_1\, dt_2 \exp\{4t_1t_2\} = \infty\,. \tag{B.13}$$

Thus, the operation $\langle\ |\ \rangle'$ (B.1) cannot be consider as the scalar product with finite norm, therefore not endowing the vector space $\mathcal{V}$ with the Hilbert space structure. This point proves impossibility to use inverse degrees in powers of oscillators for the purpose of BRST–BFV Lagrangian formulation of the form $S_\Xi \sim \langle\Phi|Q|\Phi\rangle'$.

## Appendix C. Towards Tensionless Limit in Open Bosonic String with CSR

In this appendix we will show the way to find the CSR fields in the spectrum of open bosonic string within a special tensionless limit.

Let us recall some standard properties of open bosonic string oscillators that satisfy the commutation relations

$$\left[\alpha_k^m, \alpha_l^n\right] = -k\delta_{k,-l}\eta^{mn}, \quad k,\ l \in \mathbb{Z}. \tag{C.1}$$

The Virasoro generators $L_k$, and the Virasoro algebra for their commutators take the form

$$L_k = -\frac{1}{2}\sum_{l=-\infty}^{\infty}\alpha_{k-l}^m\alpha_{ml}, \quad [L_k, L_l] = (k-l)L_{k+l} + \frac{d}{12}k(k^2-1), \tag{C.2}$$

with the zero mode rescaling as

$$\alpha_0^m = -i\sqrt{2\alpha'}\partial^m = \sqrt{2\alpha'}p^m. \tag{C.3}$$

We define the reduced generators

$$\begin{aligned} l_0 = -p^2 = -\frac{1}{2\alpha'}\alpha_0^m\alpha_{m0}, &\quad l_{\pm 1} = p^m\alpha_{m\pm 1} \mp \imath\Xi, \\ \Xi \equiv -\left(\frac{1}{\imath\sqrt{2\alpha'}}\right)&\alpha_{-2}^m\alpha_{m3}, \end{aligned} \tag{C.4}$$

$$l_k = p^m\alpha_{mk} = \frac{1}{\sqrt{2\alpha'}}\alpha_0^m\alpha_{mk}, \quad |k| > 1, \tag{C.5}$$

where the real-valued dimensional parameter $\Xi$, $\Xi = \Xi^+$ should satisfy the property

$$\Xi^+ = \left(\frac{1}{\imath\sqrt{2\alpha'}}\right)\alpha_{-3}^m\alpha_{m2} \Rightarrow \alpha_{-3}^m\alpha_{m2} = -\alpha_{-2}^m\alpha_{m3} \tag{C.6}$$

with a strong operator constraint on the values of the oscillators $\alpha_{\pm 3}^m$, $\alpha_{m\pm 2}$ (for comparison, see, e.g. Ref. 9, where the reducible massless (half-)integer representation of $ISO(1, d-1)$ was deduced for $\Xi = 0$).

The algebra of the constraints $l_0$, $l_{\pm 1}$, $l_l$ for $|l| > 1$ satisfies to the simpler algebra, related to the algebra for continuous spin fields in $\mathbb{R}^{1,d-1}$ in the Schuster–Toro-like form:[30]

$$\begin{aligned} [l_k, l_l] = k\delta_{k+l,0}&\left(l_0 + \frac{1}{2\alpha'}\delta_{|k|,1}\left\{2\alpha_{-2}^\mu\alpha_{\mu 2} - 3\alpha_{-3}^\mu\alpha_{\mu 3}\right\}\right) \\ &+ \frac{1}{\sqrt{2\alpha'}}l \cdot l_{k+l}\left(\delta_{k,1}\{\delta_{l,2} + \delta_{l,-3}\} + \delta_{k,-1}\{\delta_{l,-2} + \delta_{l,3}\}\right). \end{aligned} \tag{C.7}$$

The nondiagonal nonvanishing commutators above are

$$[l_{\pm 2}, l_{\pm 1}] = \mp\left(\frac{2}{\sqrt{2\alpha'}}\right)l_{\pm 3}, \quad [l_{\pm 3}, l_{\mp 1}] = \mp\left(\frac{3}{\sqrt{2\alpha'}}\right)l_{\pm 2}, \tag{C.8}$$

which in the naive tensionless limit $\alpha' \to \infty$ vanish as well as the terms, $(1/\alpha')\{2\alpha^{\mu}_{-2}\alpha_{\mu 2} - 3\alpha^{\mu}_{-3}\alpha_{\mu 3}\}$, in the first commutator.

The Grassmann-odd BRST charge $\mathcal{Q}$ subject to the ghost number $gh_H(\mathcal{Q}) = 1$ with the Grassmann-odd operators of ghost coordinates $C_k$ and momenta $P_k$, $gh_H(C_k) = -gh_H(P_k) = 1$, satisfying the anticommutator relations

$$\{C_k, P_l\} = \delta_{k,-l} \tag{C.9}$$

are written in the known form[94–96]

$$\mathcal{Q} = \sum_{-\infty}^{\infty}\left(C_{-k}L_k - \frac{1}{2}(k-l):C_{-k}C_{-l}P_{k+l}:\right) - C_0\,. \tag{C.10}$$

Rescaling the ghost operators $(C,P) \to (c,p)$ without changing the commutation relations (C.9):

$$\begin{gathered} c_k = -\sqrt{2\alpha'}C_k\,, \quad p_k = -\left(\frac{1}{\sqrt{2\alpha'}}\right)P_k\,, \\ k \neq 0\,, \quad (c_0, p_0) = \left(\alpha' C_0, \left(\frac{1}{\alpha'}\right)P_0\right) \end{gathered} \tag{C.11}$$

and make the tensionless limit in $\mathcal{Q}$:

$$\begin{aligned} \lim_{\alpha'\to\infty}\mathcal{Q} &= \lim_{\alpha'\to\infty}\Bigg[\left(\frac{1}{\alpha'}\right)c_0\{\alpha' l_0 + \tilde{L}_0\} - \sum_{k\neq 0}^{\infty}\left(\frac{1}{\sqrt{2\alpha'}}\right)c_{-k}\{-\sqrt{2\alpha'}l_k + \tilde{L}_k\} \\ &\quad + \sum_{k,l}\left(\frac{1}{2}(k-l)\left(\frac{1}{2\alpha'}\right)\right. \\ &\quad \left.\times :c_{-k}c_{-l}\{\sqrt{2\alpha'}p_{k+l}(1-\delta_{k,-l}) - \alpha'\delta_{k,-l}p_0\}:\right) + \left(\frac{1}{\alpha'}\right)c_0\Bigg] \\ &= c_0 l_0 + \sum_{k\neq 0}^{\infty}\left(c_{-k}l_k - \frac{k}{2}c_{-k}c_k p_0\right), \end{aligned} \tag{C.12}$$

where the operators $(\tilde{L}_0, \tilde{L}_k) = \left(L_0 - \alpha' l_0, \tilde{L}_k + \sqrt{2\alpha'}l_k\right)$ do not contain the terms with $\alpha'$-dependence and the algebra of the operators $l_k$ ((C.4), (C.5)) has the form (C.7) for $\alpha' \to \infty$:

$$[l_k, l_l] = k\delta_{k+l,0}l_0\,, \tag{C.13}$$

which encoded by the nilpotent BRST operator $Q$ for any $d$

$$Q = \sum_{-\infty}^{\infty}\left(c_{-k}l_k - \frac{k}{2}c_{-k}c_k p_0\right) = \lim_{\alpha'\to\infty}\mathcal{Q}\,, \tag{C.14}$$

which coincides with $Q_C$ (3.10) for $|k| \leq 1$, for vanishing $\eta_1$, $\eta_1^+$ and for

$$\begin{aligned} &\left(\alpha^m_{-1}, \alpha_{m1}; l_{-1}, l_1; c_0, c_{-1}, c_1, p_{-1}, p_1, p_0\right) \\ &\qquad \equiv \left(-\imath\omega^m, -\imath\partial^m_\omega; m_1^+, m_1; \eta_0, \eta_1^{m+}, \eta_1^m, \mathcal{P}_1^{m+}, \mathcal{P}_1^m, \imath\mathcal{P}_0\right). \end{aligned} \tag{C.15}$$

The operator $Q$ still contains the dependent oscillators $\alpha^m_{\pm 2}$, $\alpha^n_{\pm 3}$ due to identity on right-hand side of (C.6). In order to get the truncated BRST operator $\tilde{Q}$ from $\mathcal{Q}$ without its presence, i.e. without the constraints $l_{\pm 2}$, $l_{\pm 3}$ as well as without the ghost variables $c_{\pm k}$, $p_{\pm k}$, $k = 2, 3$ we modify the rescaling only for the latter ghosts as it was done for the zero mode ones:

$$(c_k, p_k) = -\left(2\alpha' C_k, \left(\frac{1}{2\alpha'}\right) P_k\right), \quad k = \pm 2, \pm 3\,. \tag{C.16}$$

As a result, the operator $\tilde{Q} = \lim_{\alpha' \to \infty} \mathcal{Q}$ has the form

$$\lim_{\alpha' \to \infty} \mathcal{Q} = \tilde{Q} = Q|_{(c_{\pm 2}, p_{\pm 2}; c_{\pm 3}, p_{\pm 3}) = 0} \tag{C.17}$$

and is nilpotent for any space–time dimension $d$. After rescaling for the oscillators $\alpha^m_{k_1} = \sqrt{|k_1|}\big(a^m_{k_1}\theta_{k_1,0}, a^{m+}_{-k_1}\theta_{0,k_1}\big)$ for $|k_1| > 0$ the relations (C.1) are transformed to nonvanishing commutators

$$\big[a^m_k, a^{n+}_l\big] = -\delta_{k,l}\eta^{mn}\,, \quad k, l \in \mathbb{N}\,. \tag{C.18}$$

The operators $Q$ and $\tilde{Q}$ coincide when acting on the Hilbert subspace $\tilde{\mathcal{H}}$ from the total Hilbert space $\mathcal{H}$ ($\tilde{\mathcal{H}} \subset \mathcal{H}$), whose vectors do not depend on $a^m_{-2}$, $a^n_{-3}$

$$Q|_{\tilde{\mathcal{H}}} = \tilde{Q} = c_0 l_0 + \sum_{k>0, k\neq 2,3}^{\infty} \big(c_k l^+_k + c^+_k l_k - c_k c^+_k p_0\big)\,, \tag{C.19}$$

whereas the algebra takes the form

$$\big[l_k, l^+_l\big] = \delta_{k,l} l_0\,, \quad l_k = p^m a_{mk} - \imath\Xi\delta_{1,k}\,, \quad l^+_k = p^m a^+_{mk} + \imath\Xi\delta_{1,k}\,. \tag{C.20}$$

From the nilpotency of $\mathcal{Q}$ in $d = 26$ and the standard string BRST-complex with free string equations and infinite chain of reducible gauge symmetries

$$\mathcal{Q}|\Phi\rangle = 0\,, \quad \delta|\Phi\rangle = \mathcal{Q}|\Lambda\rangle\,, \quad \delta|\Lambda\rangle = \mathcal{Q}|\Lambda^1\rangle, \ldots, \delta|\Lambda^{p-1}\rangle = \mathcal{Q}|\Lambda^p\rangle\,, \quad p \in \mathbb{N} \tag{C.21}$$

[for $gh_H(|\Phi\rangle, |\Lambda^p\rangle) = (0, -p-1)$] it follows the same BRST-complex with nilpotent $\tilde{Q}$ in the tensionless limit for any $d$.

Recalling the representation in $\tilde{\mathcal{H}}$ for the vacuum vector $|0\rangle$ has the form $\big(a^m_k, c_k, p_k, p_0\big)|0\rangle = 0$, $k > 0$. Extracting the zero-mode ghosts in $\tilde{Q}$ and in $|\Phi\rangle$, $|\Lambda^p\rangle$

$$\tilde{Q} = c_0 l_0 - M p_0 + \Delta Q\,, \quad (|\Phi\rangle, |\Lambda^p\rangle) = (|\phi\rangle, |\Lambda^p_0\rangle) + c_0(|\phi_1\rangle, |\Lambda^p_1\rangle) \tag{C.22}$$

for

$$M = \sum_{k\neq 0,2,3}^{\infty} c_k c^+_k\,, \quad \Delta Q = \sum_{k\neq 0,2,3}^{\infty} \big(c_k l^+_k + c^+_k l_k\big) \tag{C.23}$$

we get the $c_0$-independent sequence

$$\begin{pmatrix} l_0 & -\Delta Q \\ \Delta Q & -M \end{pmatrix} \begin{pmatrix} |\phi\rangle \\ |\phi_1\rangle \end{pmatrix} = 0, \quad \delta \begin{pmatrix} |\phi\rangle \\ |\phi_1\rangle \end{pmatrix} = \begin{pmatrix} \Delta Q & -M \\ l_0 & -\Delta Q \end{pmatrix} \begin{pmatrix} |\Lambda_0\rangle \\ |\Lambda_1\rangle \end{pmatrix}, \ldots\,. \tag{C.24}$$

In case of scalar CSR ($k = 1$) the fields $|\phi\rangle$, $|\phi_1\rangle$, gauge parameter $|\Lambda_0\rangle$ (for $|\Lambda_1\rangle \equiv 0$ and $|\Lambda^p\rangle \equiv 0$, when $p > 0$ due to $gh_H$ distribution) can be presented in powers of oscillators

$$\begin{aligned} |\phi\rangle &= \left( \sum_{l\geq 0} \frac{1}{l!} \varphi^{(m)_l}(x) + c_1^+ p_1^+ \sum_{l\geq 0} \frac{1}{l!} D^{(m)_l}(x) \right) a^+_{m_1} \cdots a^+_{m_l} |0\rangle \\ &\equiv |\varphi\rangle + c_1^+ p_1^+ |D\rangle \,, \end{aligned} \tag{C.25}$$

$$\begin{aligned} (|\phi_1\rangle, |\Lambda_0\rangle) &= p_1^+ \sum_{l\geq 0} \frac{\imath}{l!} \big(-C^{(m)_l}(x), \Lambda^{(m)_l}(x)\big) a^+_{m_1} \cdots a^+_{m_l} |0\rangle \\ &\equiv p_1^+ (|C\rangle, |\lambda\rangle) \,. \end{aligned} \tag{C.26}$$

Because the number particle operator, $g_0 = -\frac{1}{2}\{a^+_m, a^m\}$, (usually associated with the spin value of basic field in case of integer HS field) no longer commute with enlarged divergence (gradient) $l_1^{(+)}$:

$$[g_0, l_1] = -(l_1 + \imath\Xi) \,, \quad [g_0, l_1^+] = (l_1^+ - \imath\Xi) \,, \tag{C.27}$$

the ghost-independent equations and gauge transformations

$$l_0|\varphi\rangle - l_1^+|C\rangle = 0 \,, \quad l_1|\varphi\rangle - l_1^+|D\rangle = -|C\rangle \,, \quad l_0|D\rangle - l_1|C\rangle = 0 \,, \tag{C.28}$$

$$\delta(|\varphi\rangle, |C\rangle, |D\rangle) = \big(l_1^+, l_0, l_1\big)|\lambda\rangle \tag{C.29}$$

contain all tensor fields $\varphi^{(m)_k}(x)$, $C^{(m)_k}(x)$, $D^{(m)_k}(x)$ starting from the scalar fields.

Equations (C.28) are Lagrangian, follow from the action, $\mathcal{S}(\varphi, C, D) = \int dc_0 \langle\Phi|\tilde{Q}|\Phi\rangle$ and represent the triplet analogue of EoM for the fields from scalar reducible CSR. The irreducible CSR should be selected by means of the specified trace conditions imposed on $|\varphi\rangle$, $|C\rangle$, $|D\rangle$, $|\lambda\rangle$ which are realized by the operators

$$\mathcal{L}_{11} = l_{11} + \mathcal{O}(c_1, p_1) \quad \text{for} \quad l_{11} = a^m a_m \,, \tag{C.30}$$

which should commute with BRST operator $\tilde{Q}$: $[\tilde{Q}, \mathcal{L}_{11}] = 0$ to get consistent dynamics.

However, within the Fock space $\mathcal{H}$ generated by $a^+_m$, $c_1^+$, $p_1^+$ it seems impossible to realize such operator due to

$$\big[l_1^+, l_{11}\big] = 2(l_1 + \imath\Xi) \quad \big(\Rightarrow \big[l_1, l_{11}^+\big] = -2(l_1^+ - \imath\Xi),\ l_{11}^+ = a^+_m a^{+m}\big) \,. \tag{C.31}$$

The problem of nonclosing for the commutators (C.31) may be effectively resolved within a special conversion procedure in a larger Hilbert space $\tilde{\mathcal{H}} \otimes \mathcal{H}'$ (we develop the respective study in Ref. 97). One can show, that the evaluation of the Casimir operators (2.1) (see, as well the footnotes a and e) on the field $|\phi\rangle$: $C_2|\phi\rangle = P^2|\phi\rangle = 0$ and $C_4|\phi\rangle = (M^{mn}P_m)^2|\phi\rangle = \Xi^2|\phi\rangle + \delta_\lambda|\mathcal{F}\rangle$, with accuracy up to the gauge transformations of some vector $|\mathcal{F}\rangle$ can be done following the recipe of Ref. 30 with allowance for the appropriate traceless conditions.

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