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NESTED BETHE ANSATZ FOR THE RTT ALGEBRA OF $sp(4)$ TYPE

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We show how to formulate the algebraic nested Bethe ansatz for RTT algebras with an R-matrix of the sp(4) type. We obtain the Bethe vectors and Bethe conditions for any highest-weight representation of these RTT algebras.

Keywords: RTT algebra, nested Bethe ansatz

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1. Introduction

The quantum inverse scattering method, or algebraic Bethe ansatz, formulated by the Leningrad school [1], provides eigenvectors and eigenvalues of the transfer matrix. The latter is the generating function of the conserved quantities of a large family of quantum integrable models. The transfer matrix eigenvectors are constructed in the framework of the representation theory of the RTT algebras. To obtain these eigenvectors, we should first construct Bethe vectors depending on a set of complex variables.

The Bethe vectors for $gl(n)$-invariant models were first constructed by Kulish and Reshetikhin in [2], where the nested algebraic Bethe ansatz was formulated. In the framework of this method, the Bethe vectors are given by a recursion on the rank of the algebra. The Bethe vectors corresponding to the algebra of higher rank are expressed in terms of the Bethe vectors of an algebra of lower rank using the embedding $gl(n-1) \subset gl(n)$, while the result for the case $gl(2)$ is well known. In [3], the construction was generalized to the cases of the RTT (super)algebras and deformed RTT (super)algebras with an R-matrix of $gl(m|n)$ type. We describe this scheme in detail for the known case of the RTT algebra of $gl(3)$ type. The crucial fact for this construction is that the RTT algebra of $gl(2)$ type is the RTT subalgebra of the RTT algebra of $gl(3)$ type.

In the case of R-matrices invariant under the action of the $SO(n)$ and $Sp(2n)$ groups, the Bethe vectors and Bethe conditions have been found only for special representations. The reason is that there is no obvious embedding of the RTT algebra of $sp(2n-2)$ type into the RTT algebra of $sp(2n)$ type. In [4] and [5], Reshetikhin studied representations for which the matrix Bethe ansatz can be used, and Martins and Ramos in [6] dealt with representations that are directly constructed using the R-matrix. Our aim here is to extend this scheme to the case of an arbitrary representation of the algebra $sp(4)$.

This paper is structured as follows. In Sec. 2, we give the basic definitions for RTT algebras and formulate our problem. In Sec. 3, we recall the known results regarding the algebraic Bethe ansatz for the RTT algebra of $gl(2)$ type and the main ideas of the nested Bethe ansatz for the RTT algebra of $gl(3)$ type. Our main result is in Sec. 4, where we show how to generalize the method of the nested Bethe ansatz to

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the RTT algebra of $sp(4)$ type. Using this generalization, we obtain eigenvectors and the Bethe conditions for the RTT algebra of $sp(4)$ type. In the conclusion, we compare our construction with the already known results for the Bethe vectors and Bethe conditions of the RTT algebra of $sp(4)$ type.

The proofs of many statements of this paper are simple but relatively long. We therefore do not include them in text here, but they can be found in [7].

2. RTT algebras

We let $\mathbf{E}_i^k$ denote a matrix that has all elements equal to zero except the element at the intersection of the ith row and kth column, which is unity. Then $\mathbf{I} = \sum_k \mathbf{E}_k^k$ is the unit matrix, and the relations $\mathbf{E}_i^k \mathbf{E}_r^s = \delta_r^k \mathbf{E}_i^s$ hold.

We consider an R-matrix

$$\mathbf{R}(x,y) = \sum_{i,k,r,s} R_{k,s}^{i,r}(x,y) \mathbf{E}_i^k \otimes \mathbf{E}_r^s,$$

which is invertible and satisfies the Yang–Baxter equation

$$\mathbf{R}_{1,2}(x,y)\mathbf{R}_{1,3}(x,z)\mathbf{R}_{2,3}(y,z) = \mathbf{R}_{2,3}(y,z)\mathbf{R}_{1,3}(x,z)\mathbf{R}_{1,2}(x,y). \tag{1}$$

The RTT algebra $\mathcal{A}$ is an associative algebra with the unit generated by the elements $T_k^i(x)$, where the monodromy operator

$$\mathbf{T}(x) = \sum_{i,k} \mathbf{E}_i^k \otimes T_k^i(x)$$

satisfies an RTT equation

$$\mathbf{R}_{1,2}(x,y)\mathbf{T}_1(x)\mathbf{T}_2(y) = \mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(x,y). \tag{2}$$

It follows from the invertibility of the R-matrix that the operator

$$H(x) = \mathrm{Tr}\, \mathbf{T}(x) = \sum_i T_i^i(x)$$

has the property $H(x)H(y) = H(y)H(x)$ for any x and y.

We consider the RTT algebras of the types $gl(2)$ and $gl(3)$, which are defined by the R-matrix of $gl(n)$ type

$$\mathbf{R}(x,y) = \frac{1}{f(x,y)} \left(\mathbf{I} \otimes \mathbf{I} + g(x,y) \sum_{i,k=1}^{n} \mathbf{E}_k^i \otimes \mathbf{E}_i^k \right),$$

where

$$g(x,y) = \frac{1}{x-y}, \qquad f(x,y) = \frac{x-y+1}{x-y},$$

and also the RTT algebra of $sp(4)$ type, which is associated with the R-matrix

$$\mathbf{R}(x,y) = \frac{1}{f(x,y)} \left(\mathbf{I} \otimes \mathbf{I} + g(x,y) \sum_{i,k=-2}^{2} \mathbf{E}_k^i \otimes \mathbf{E}_i^k - h(x,y) \sum_{i,k=-2}^{2} \epsilon_i \epsilon_k \mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i} \right),$$

where the indices i and k take the values ± 1 and ± 2 and the function $h(x,y)$ has the form

$$h(x,y) = \frac{1}{x-y+3}, \qquad \epsilon_i = \mathrm{sgn}(i).$$

All these R-matrices satisfy Yang–Baxter equation (1) and the unitarity relation

$$\mathbf{R}(x,y)\mathbf{R}(y,x) = \mathbf{I}\otimes\mathbf{I}.$$

We assume that in the vector space $\mathcal{W}$ of a representation of the RTT algebra $\mathcal{A}$, there exists an element $\omega \in \mathcal{W}$, called the vacuum vector, for which $\mathcal{W} = \mathcal{A}\omega$ and we have the relations

$$T^i_k(x)\omega = 0 \quad \text{for } i > k, \qquad T^i_i(x)\omega = \lambda_i(x)\omega. \tag{3}$$

The values of $\lambda_i(x)$ define a concrete representation of the RTT algebra.

We seek the eigenvectors of $H(x)$ in the vector space $\mathcal{W} = \mathcal{A}\omega$, i.e., nonzero vectors $w \in \mathcal{W}$ such that $H(x)w = E(x)w$ holds for any x.

3. Nested Bethe ansatz for RTT algebras of the $gl(2)$ and $gl(3)$ types

We define the vacuum vector ω by equalities (3). For the $gl(n)$ RTT algebra, RTT equation (2) is equivalent to the commutation relations

$$T^i_k(x)T^r_s(y) + g(x,y)T^r_k(x)T^i_s(y) = T^r_s(y)T^i_k(x) + g(x,y)T^r_k(y)T^i_s(x),$$

$$T^i_k(x)T^r_s(y) + g(y,x)T^i_s(x)T^r_k(y) = T^r_s(y)T^i_k(x) + g(y,x)T^i_s(y)T^r_k(x),$$

whence it follows that $T^i_k(x)T^i_k(y) = T^i_k(y)T^i_k(x)$ for any i, k and x, y.

3.1. Bethe ansatz for the RTT algebra of $gl(2)$ type. The Bethe ansatz for the RTT algebra of $gl(2)$ type means that we seek the eigenvectors of $H(x) = T^1_1(x) + T^2_2(x)$ in the form

$$|\bar u\rangle = T^1_2(\bar u)\omega, \quad \text{where } T^1_2(\bar u) = T^1_2(u_1)T^1_2(u_2)\cdots T^1_2(u_N).$$

In this paper, we use a special notation for sets of complex numbers

$$\bar u = \{u_1, u_2, \ldots, u_N\}, \qquad \bar u_k = \bar u \setminus \{u_k\}.$$

Lemma 1. *In the RTT algebra of $gl(2)$ type, we have the relations*

$$\begin{aligned} T^1_1(x)T^1_2(\bar u) &= F(\bar u, x)T^1_2(\bar u)T^1_1(x) - \\ &\quad - \sum_{u_k\in\bar u} g(u_k,x)F(\bar u_k,u_k)T^1_2(\{\bar u_k,x\})T^1_1(u_k), \\ T^2_2(x)T^1_2(\bar u) &= F(x,\bar u)T^1_2(\bar u)T^2_2(x) - \\ &\quad - \sum_{u_k\in\bar u} g(x,u_k)F(u_k,\bar u_k)T^1_2(\{\bar u_k,x\})T^2_2(u_k), \end{aligned} \tag{4}$$

where

$$F(\bar u,x) = \prod_{u_k\in\bar u} f(u_k,x), \qquad F(x,\bar u) = \prod_{u_k\in\bar u} f(x,u_k).$$

Applying Eq. (4) to the vacuum vector ω, we obtain the following result.

Theorem 1. *If the Bethe condition*

$$\lambda_1(u_k)F(\bar u_k,u_k) = \lambda_2(u_k)F(u_k,\bar u_k)$$

is satisfied for any $u_k \in \bar u$, then the vector $|\bar u\rangle$ is the eigenvector of the operator $H(x)$ for any x with the eigenvalue

$$E(x;\bar u) = \lambda_1(x)F(\bar u,x) + \lambda_2(x)F(x,\bar u).$$

This result can found in [1].

3.2. Nested Bethe ansatz for the RTT algebra of $gl(3)$ type. The nested Bethe ansatz for the RTT algebra of $gl(3)$ type (see [2]) is characterized by the fact that the eigenvectors of $H(x) = T_1^1(x) + T_2^2(x) + T_3^3(x)$ can be found in the form

$$\sum_{a_1,a_2,\ldots,a_M=1}^{2} T_3^{a_1}(v_1)T_3^{a_2}(v_2)\cdots T_3^{a_M}(v_M)\Phi_{a_1,a_2,\ldots,a_M}.$$

Here, $\Phi_{a_1,a_2,\ldots,a_M} \in \widetilde{\mathcal{W}} = \widetilde{\mathcal{A}}\omega$, $\widetilde{\mathcal{A}}$ is the RTT algebra generated by the elements $T_b^a(x)$, $a, b = 1, 2$, and

$$\widetilde{\mathbf{R}}(x,y) = \frac{1}{f(x,y)}\Bigl(\mathbf{I}^{(2)} \otimes \mathbf{I}^{(2)} + g(x,y)\sum_{a,b=1}^{2} \mathbf{E}_b^a \otimes \mathbf{E}_a^b\Bigr), \qquad \mathbf{I}^{(2)} = \sum_{a=1}^{2} \mathbf{E}_a^a,$$

is the R-matrix of $gl(2)$ type. Because the commutation relations between the elements $T_b^a(x)$ are the same as in the RTT algebra $\mathcal{A}$, the RTT algebra $\widetilde{\mathcal{A}}$ is the RTT subalgebra of $\mathcal{A}$.

It is essential for the Bethe ansatz that the subspace $\widetilde{\mathcal{W}}$ is invariant not only under the actions of the elements $T_1^1(x)$ and $T_2^2(x)$ but also under the action of $T_3^3(x)$. This follows because the commutation relations yield

$$\begin{aligned} T_3^3(x)T_b^a(y) &= T_b^a(y)T_3^3(x) + g(x,y)T_3^a(y)T_b^3(x) - g(x,y)T_3^a(x)T_b^3(y), \\ T_c^3(x)T_b^a(y) &= T_b^a(y)T_c^3(x) + g(x,y)T_c^a(y)T_b^3(x) - g(x,y)T_c^a(x)T_b^3(y), \end{aligned} \qquad a,b,c = 1,2.$$

Because $T_a^3(x)\omega = 0$ for $a = 1, 2$, we conclude that the equations

$$T_a^3(x)\mathbf{w} = 0, \qquad T_3^3(x)\mathbf{w} = \lambda_3(x)\mathbf{w}$$

hold for any element $\mathbf{w} \in \widetilde{\mathcal{W}}$. It is also easy to see that the commutation relations between $T_3^3(x)$ and $T_b^a(x)$, $a, b = 1, 2$, restricted to the subspace $\widetilde{\mathcal{W}}$ are

$$\begin{gathered} T_3^3(x)T_3^3(y) = T_3^3(y)T_3^3(x), \qquad T_3^3(x)\widetilde{\mathbf{T}}(y) = \widetilde{\mathbf{T}}(y)T_3^3(x), \\ \widetilde{\mathbf{R}}_{1,2}(x,y)\widetilde{\mathbf{T}}_1(x)\widetilde{\mathbf{T}}_2(y) = \widetilde{\mathbf{T}}_2(y)\widetilde{\mathbf{T}}_1(x)\widetilde{\mathbf{R}}_{1,2}(x,y), \end{gathered}$$

where $\widetilde{\mathbf{T}}(x) = \sum_{a,b=1}^{2} \mathbf{E}_a^b \otimes T_b^a(x)$. It follows from the commutation relations that the eigenvectors of $\widetilde{H}(x) = T_1^1(x) + T_2^2(x) \in \widetilde{\mathcal{A}}$ on the space $\widetilde{\mathcal{W}}$ are simultaneously eigenvectors of $H(x) \in \mathcal{A}$.

Remark. The original work [2] on the $gl(n)$ case used the fact that the RTT algebra $\widetilde{\mathcal{A}}$ with the monodromy operator $\widetilde{\mathbf{T}}(x) = \sum_{a,b=1}^{n-1} \mathbf{E}_a^b \otimes T_b^a(x)$ and the R-matrix $\widetilde{\mathbf{R}}(x,y)$ is the RTT subalgebra of the RTT algebra $\mathcal{A}$ and also the fact that $T_n^n(x)$ acts as a multiple of the unit operator on the vector space $\widetilde{\mathcal{W}}$. But another interpretation of the commutation relations on the vector space $\widetilde{\mathcal{W}}$ is also possible. The commutation relations can be written in the form of the RTT equation

$$\mathbf{R}_{1,2}^{(0)}(x,y)\mathbf{T}_1^{(0)}(x)\mathbf{T}_2^{(0)}(y) = \mathbf{T}_2^{(0)}(y)\mathbf{T}_1^{(0)}(x)\mathbf{R}_{1,2}^{(0)}(x,y),$$

where

$$\mathbf{R}^{(0)}(x,y) = \mathbf{E}_3^3 \otimes \mathbf{E}_3^3 + \mathbf{E}_3^3 \otimes \mathbf{I}^{(2)} + \mathbf{I}^{(2)} \otimes \mathbf{E}_3^3 + \widetilde{\mathbf{R}}(x,y)$$

and $\mathbf{T}^{(0)}(x) = \mathbf{E}_3^3 \otimes T_3^3(x) + \widetilde{\mathbf{T}}(x)$. Because the R-matrix $\mathbf{R}^{(0)}(x,y)$ satisfies the Yang–Baxter equation and the condition $\mathbf{R}^{(0)}(x,y)\mathbf{R}^{(0)}(y,x) = \mathbf{I} \otimes \mathbf{I}$, we can consider the RTT algebra $\mathcal{A}^{(0)}$ generated by the elements $T_3^3(x)$ and $T_b^a(x)$ with the R-matrix $\mathbf{R}^{(0)}(x,y)$. But the RTT algebra $\mathcal{A}^{(0)}$ is no longer an RTT subalgebra of $\mathcal{A}$. It is in fact the RTT algebra of $gl(2)$ type generated by the elements $T_b^a(x)$, where $a, b = 1, 2$, with the R-matrix $\widetilde{\mathbf{R}}(x,y)$ extended by the central element $T_3^3(x)$. This is precisely the interpretation that we use in considering the nested Bethe ansatz for the RTT algebra of type $sp(4)$.

We permute $T_1^1(x)$, $T_2^2(x)$, and $T_3^3(x)$ with $T_3^{a_1}(v_1)\cdots T_3^{a_M}(v_M)$. To avoid a long expression with indices, we introduce the notation

$$\mathbf{b}(v) = \sum_{a=1}^{2} \mathbf{e}_a \otimes T_3^a(v) \in \mathcal{V}\otimes\mathcal{A},$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ form a basis of the two-dimensional vector space $\mathcal{V}$. In general, for an ordered set of M pairwise distinct numbers $\vec{v} = (v_1, v_2, \dots, v_M)$, we define

$$\mathbf{b}_{1,2,\dots,M}(\vec{v}\,) = \mathbf{b}_1(v_1)\mathbf{b}_2(v_2)\cdots\mathbf{b}_M(v_M) =$$
$$= \sum_{a_1,a_2,\dots,a_M=1}^{2} \mathbf{e}_{a_1}\otimes\mathbf{e}_{a_2}\otimes\cdots\otimes\mathbf{e}_{a_M}\otimes T_3^{a_1}(v_1)T_3^{a_2}(v_2)\cdots T_3^{a_M}(v_M) \in \underbrace{\mathcal{V}\otimes\cdots\otimes\mathcal{V}}_{M\text{ times}}\otimes\mathcal{A}.$$

Let $\mathbf{f}^b$ be dual bases of the dual space $\mathcal{V}^*$. Hence, $\langle\mathbf{e}_a, \mathbf{f}^b\rangle = \delta_b^a$. We can write the assumed form of the eigenvector as

$$\sum_{a_1,a_2,\dots,a_M=1}^{2} T_3^{a_1}(v_1)T_3^{a_2}(v_2)\cdots T_3^{a_M}(v_M)\Phi_{a_1,a_2,\dots,a_M} = \langle\mathbf{b}_{1,2,\dots,M}(\vec{v}\,), \boldsymbol{\Phi}\rangle,$$

where

$$\boldsymbol{\Phi} = \sum_{b_1,b_2,\dots,b_M=1}^{2} \mathbf{f}^{b_1}\otimes\mathbf{f}^{b_2}\otimes\cdots\otimes\mathbf{f}^{b_M}\otimes\Phi_{b_1,b_2,\dots,b_M} \in \underbrace{\mathcal{V}^*\otimes\cdots\otimes\mathcal{V}^*}_{M\text{ times}}\otimes\widetilde{\mathcal{W}} = \widehat{\mathcal{W}}.$$

We regard the matrix $\mathbf{E}_b^a$ as a linear map on the space $\mathcal{V}$ defined by the formulas $\mathbf{E}_b^a\mathbf{e}_c = \delta_c^a\mathbf{e}_b$. The corresponding dual map is denoted by $\mathbf{F}_b^a$, and we hence have the equations

$$\langle\mathbf{E}_b^a\mathbf{e}_c, \mathbf{f}^d\rangle = \langle\mathbf{e}_c, \mathbf{F}_b^a\mathbf{f}^d\rangle, \quad \text{i.e.,} \quad \mathbf{F}_b^a\mathbf{f}^d = \delta_b^d\mathbf{f}^a.$$

Then $\mathbf{I}^* = \sum_{a=1}^{2}\mathbf{F}_a^a$ is an identical map on $\mathcal{V}^*$, and in contrast to similar relations for the map $\mathbf{E}_b^a$, we have the relation $\mathbf{F}_b^a\mathbf{F}_d^c = \delta_b^c\mathbf{F}_d^a$.

Using the above notation, we can write the interesting commutation relations like

$$T_3^3(x)\mathbf{b}(v) = f(x,v)\mathbf{b}(v)T_3^3(x) - g(x,v)\mathbf{b}(x)T_3^3(v),$$
$$\widetilde{\mathbf{T}}_0(x)\langle\mathbf{b}_1(v), \mathbf{I}\otimes\mathbf{f}^b\rangle = f(v,x)\langle\mathbf{b}_1(v), \widehat{\mathbf{T}}_{0,1^*}(x;v)(\mathbf{I}\otimes\mathbf{f}^b)\rangle - g(v,x)\langle\mathbf{b}_1(x), \widehat{\mathbb{T}}_{0,1^*}(v)(\mathbf{I}\otimes\mathbf{f}^b)\rangle,$$

where

$$\widehat{\mathbf{T}}_{0,1^*}(x;v) = \widehat{\mathbf{R}}_{0,1^*}(x,v)\widetilde{\mathbf{T}}_0(x), \qquad \widehat{\mathbb{T}}_{0,1^*}(v) = \widehat{\mathbf{T}}_{0,1}(v;v) = \widehat{\mathbb{R}}_{0,1^*}\widetilde{\mathbf{T}}_0(v),$$
$$\widehat{\mathbf{R}}_{0,1^*}(x;v) = \frac{1}{f(v,x)}\left(\mathbf{I}\otimes\mathbf{I}^* + g(v,x)\sum_{a,b=1}^{2}\mathbf{E}_b^a\otimes\mathbf{F}_a^b\right),$$
$$\widehat{\mathbb{R}}_{0,1^*} = \widehat{\mathbf{R}}_{0,1^*}(v,v) = \sum_{a,b=1}^{2}\mathbf{E}_b^a\otimes\mathbf{F}_a^b.$$

It can be shown that for any $\mathbf{\Phi} \in \widehat{\mathcal{W}}$, we have the relations

$$T_3^3(x)\langle \mathbf{b}_{1,2,\dots,M}(\vec{v}\,), \mathbf{\Phi}\rangle = \lambda_3(x)F(x,\bar{v})\langle \mathbf{b}_{1,2,\dots,M}(\vec{v}\,), \mathbf{\Phi}\rangle - \\ - \sum_{v_k\in\bar{v}} \lambda_3(v_k)g(x,v_k)F(v_k,\bar{v}_k)\langle \mathbf{b}_{k;1,2,\dots,M}(x,\vec{v}_k), \widetilde{\mathbf{R}}^*_{k;1,2,\dots,k}(\vec{v}\,)\mathbf{\Phi}\rangle,$$

$$\widetilde{\mathbf{T}}_0(x)\langle \mathbf{b}_{1,2,\dots,M}(\vec{v}\,), \mathbf{\Phi}\rangle = F(\bar{v},x)\langle \mathbf{b}_{1,2,\dots,M}(\vec{v}\,), \widehat{\mathbf{T}}_{0,1,\dots,M}(x;\vec{v}\,)\mathbf{\Phi}\rangle - \\ - \sum_{v_k\in\bar{v}} g(v_k,x)F(\bar{v}_k,v_k)\langle \mathbf{b}_{k;1,2,\dots,M}(x,\vec{v}_k), \widetilde{\mathbf{R}}^*_{k;1,2,\dots,k}(\vec{v}\,)\widehat{\mathbb{T}}_{k;0,1,\dots,M}(\vec{v}\,)\mathbf{\Phi}\rangle,$$

where

$$\mathbf{b}_{k;1,2,\dots,M}(x,\vec{v}_k) = \mathbf{b}_k(x)\mathbf{b}_1(v_1)\mathbf{b}_2(v_2)\cdots\mathbf{b}_{k-1}(v_{k-1})\mathbf{b}_{k+1}(v_{k+1})\cdots\mathbf{b}_N(v_M),$$

$$\widetilde{\mathbf{R}}^*_{2,1}(y,x) = \frac{1}{f(y,x)}\left(\mathbf{I}^*\otimes\mathbf{I}^* + g(y,x)\sum_{a,b=1}^{2}\mathbf{F}^a_b\otimes\mathbf{F}^b_a\right),$$

$$\widetilde{\mathbf{R}}^*_{k;1,2,\dots,k}(\vec{v}\,) = \widetilde{\mathbf{R}}^*_{k,1}(v_k,v_1)\widetilde{\mathbf{R}}^*_{k,2}(v_k,v_2)\cdots\widetilde{\mathbf{R}}^*_{k,k-1}(v_k,v_{k-1}), \qquad \widetilde{\mathbf{R}}_{1;1\dots,1}(\vec{v}\,) = \mathbf{I}^*,$$

$$\widehat{\mathbf{T}}_{0,1,\dots,M}(x;\vec{v}\,) = \widehat{\mathbf{R}}_{0,1^*}(x,v_1)\cdots\widehat{\mathbf{R}}_{0,M^*}(x,v_M)\widetilde{\mathbf{T}}_0(x),$$

$$\widehat{\mathbb{T}}_{k;0,1,\dots,M}(\vec{v}\,) = \widehat{\mathbf{T}}_{0,1,\dots,M}(v_k;\vec{v}\,).$$

If we set

$$\widehat{\mathbf{T}}_{0,1,\dots,M}(x,\vec{v}\,) = \sum_{i,k=1}^{2}\mathbf{E}^k_i\otimes\widehat{T}^i_k(x;\vec{v}\,), \qquad \widehat{T}^i_k(x;\vec{v}\,)\in\underbrace{\mathcal{V}^*\otimes\cdots\otimes\mathcal{V}^*}_{M\text{ times}}\otimes\widetilde{\mathcal{A}},$$

then we obtain the following proposition.

Proposition 1. *Let $\mathbf{\Phi}\in\widehat{\mathcal{W}}$ for any x be an eigenvector of the operator*

$$\widehat{H}_{1,2,\dots,M}(x;\vec{v}\,) = \widehat{T}^1_1(x;\vec{v}\,) + \widehat{T}^2_2(x;\vec{v}\,)$$

with the eigenvalue $\widehat{E}(x;\vec{v}\,)$. If the equations

$$\lambda_3(v_k)F(v_k,\bar{v}_k) = \widehat{E}(v_k;\vec{v}\,)F(\vec{v}_k,v_k)$$

hold for each $v_k\in\bar{v}$, then $\langle \mathbf{b}_{1,2,\dots,M}(\vec{v}\,), \mathbf{\Phi}\rangle$ is an eigenvector of the operator

$$H(x) = T^1_1(x) + T^2_2(x) + T^3_3(x)$$

with the eigenvalue

$$E(x;\vec{v},\mu) = \lambda_3(x)F(x,\bar{v}) + \widehat{E}(x;\vec{v}\,)F(\bar{v},x).$$

Hence, we must find eigenvectors of the operators $\widehat{H}_{1,2,\dots,M}(x;\vec{v}\,)$. It is easy to see that $\widehat{\mathbf{T}}_{0,1,\dots,M}(x;\vec{v}\,)$ satisfy the RTT equation

$$\widetilde{\mathbf{R}}_{0,0'}(x,y)\widehat{\mathbf{T}}_{0,1,\dots,M}(x;\vec{v}\,)\widehat{\mathbf{T}}_{0',1,\dots,M}(y;\vec{v}\,) = \widehat{\mathbf{T}}_{0',1,\dots,M}(y;\vec{v}\,)\widehat{\mathbf{T}}_{0,1,\dots,M}(x;\vec{v}\,)\widetilde{\mathbf{R}}_{0,0'}(x,y).$$

This means that they generate the RTT algebra of type $gl(2)$. Therefore, if we find a vacuum vector in the vector space $\widehat{\mathcal{W}}$, then we can use Theorem 1 to construct eigenvectors of $\widehat{H}_{1,2,\dots,M}(x;\vec{v}\,)$.

W can show that for the vector $\widehat{\Omega} = \underbrace{\mathbf{f}^2 \otimes \cdots \otimes \mathbf{f}^2}_{M \text{ times}} \otimes \omega$, we have

$$\widehat{T}_1^2(x;\vec{v}\,)\widehat{\Omega} = 0, \qquad \widehat{T}_1^1(x;\vec{v}\,)\widehat{\Omega} = \mu_1(x;\vec{v}\,)\widehat{\Omega}, \qquad \widehat{T}_2^2(x;\vec{v}\,)\widehat{\Omega} = \mu_2(x;\vec{v}\,)\widehat{\Omega},$$

where

$$\mu_1(x;\vec{v}\,) = \frac{\lambda_1(x)}{F(\bar{v},x)}, \qquad \mu_2(x;\vec{v}\,) = \lambda_2(x).$$

We obtain the following proposition from Theorem 1.

Proposition 2. *If the equality*

$$\lambda_1(u_k)F(\bar{u}_k,u_k) = \lambda_2(u_k)F(u_k,\bar{u}_k)F(\bar{v},u_k)$$

holds for every $u_k \in \bar{u}$, then the vector $\mathbf{\Phi}(\bar{u};\vec{v}\,) = \widehat{T}_1^2(\bar{u};\vec{v}\,)\widehat{\Omega}$ is the eigenvector of the operator $\widehat{H}_{1,2,\dots,M}(x;\vec{v}\,)$ with the eigenvalue

$$\widehat{E}(x;\bar{u};\vec{v}\,) = \lambda_1(x)\frac{F(\bar{u},x)}{F(\bar{v},x)} + \lambda_2(x)F(x,\bar{u}).$$

From Propositions 1 and 2, we then obtain the final result for the RTT algebra of $gl(3)$ type.

Theorem 2. *Let the Bethe conditions*

$$\lambda_1(u_k)F(\bar{u}_k,u_k) = \lambda_2(u_k)F(\bar{v},u_k)F(u_k,\bar{u}_k),$$

$$\lambda_3(v_k)F(v_k,\bar{v}_k) = \lambda_2(v_k)F(\bar{v}_k,v_k)F(v_k,\bar{u})$$

be satisfied for every $u_i \in \bar{u}$ and $v_k \in \bar{v}$. Then the vector $|\vec{v},\bar{u}\rangle = \langle \mathbf{b}_{1,2,\dots,M}(\vec{v}\,);\mathbf{\Phi}(\bar{u};\vec{v}\,)\rangle$ is the eigenvector of the operator $H(x)$ with the eigenvalue

$$E(x;\bar{u};\vec{v}\,) = \lambda_1(x)F(\bar{u},x) + \lambda_2(x)F(x,\bar{u})F(\bar{v},x) + \lambda_3(x)F(x,\bar{v}).$$

This theorem follows directly from the results in [2].

4. Nested Bethe ansatz for the RTT algebra of $sp(4)$ type

The vacuum vector ω for the RTT algebra $\mathcal{A}$ of $sp(4)$ type is defined by the relations

$$T_k^i(x)\omega = 0 \quad \text{for } i<k, \qquad T_i^i(x)\omega = \lambda_i(x)\omega \quad \text{for } i=\pm1,\pm2.$$

From the RTT equation, we obtain the commutation relations

$$\begin{aligned}
&T_k^i(x)T_s^r(y) + g(x,y)T_k^r(x)T_s^i(y) + \delta^{i,-r}h(x,y)\sum_{p=-2}^{2}\epsilon_p\epsilon_r T_k^p(x)T_s^{-p}(y) = \\
&\qquad = T_s^r(y)T_k^i(x) + g(x,y)T_k^r(y)T_s^i(x) + \delta_{k,-s}h(x,y)\sum_{p=-2}^{2}\epsilon_k\epsilon_p T_p^r(y)T_{-p}^i(x), \\
&T_k^i(x)T_s^r(y) + g(y,x)T_s^i(x)T_k^r(y) + \delta_{k,-s}h(y,x)\sum_{p=-2}^{2}\epsilon_s\epsilon_p T_p^i(x)T_{-p}^r(y) = \\
&\qquad = T_s^r(y)T_k^i(x) + g(y,x)T_s^i(y)T_k^r(x) + h(y,x)\delta^{i,-r}\sum_{p=-2}^{2}\epsilon_i\epsilon_p T_s^p(y)T_k^{-p}(x).
\end{aligned} \tag{5}$$

It follows from these formulas that $T_k^i(x)T_k^i(y) = T_k^i(y)T_k^i(x)$ for any x and y and for all i and k.

4.1. The RTT algebra $\widetilde{\mathcal{A}}$. There are two RTT subalgebras $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ in the RTT algebra $\mathcal{A}$, which are respectively generated by the elements $T^i_k(x)$ and $T^{-i}_{-k}(x)$, where $i,k=1,2$.

First, we consider the vector space

$$\mathcal{W}_0 = \mathcal{A}^{(+)}\mathcal{A}^{(-)}\omega \subset \mathcal{W} = \mathcal{A}\omega.$$

Using commutation relations (5), we can prove the following statements.

Lemma 2. *We have $T^{-i}_k(x)\mathbf{w}=0$ for all $i,k=1,2$ and any $\mathbf{w}\in\mathcal{W}_0$.*

We introduce the notation

$$\mathbf{T}^{(+)}(x) = \sum_{i,k=1}^{2} \mathbf{E}^k_i \otimes T^i_k(x), \qquad \mathbf{T}^{(-)}(x) = \sum_{i,k=1}^{2} \mathbf{E}^{-k}_{-i} \otimes T^{-i}_{-k}(x),$$

$$\mathbf{R}^{(+,+)}(x,y) = \frac{1}{f(x,y)}\Bigl(\mathbf{I}_+\otimes\mathbf{I}_+ + g(x,y)\sum_{i,k=1}^{2}\mathbf{E}^i_k\otimes\mathbf{E}^k_i\Bigr),$$

$$\mathbf{R}^{(-,-)}(x,y) = \frac{1}{f(x,y)}\Bigl(\mathbf{I}_-\otimes\mathbf{I}_- + g(x,y)\sum_{i,k=1}^{2}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^{-k}_{-i}\Bigr),$$

$$\mathbf{R}^{(+,-)}(x,y) = \mathbf{I}_+\otimes\mathbf{I}_- - k(x,y)\sum_{i,k=1}^{2}\mathbf{E}^i_k\otimes\mathbf{E}^{-i}_{-k},$$

$$\mathbf{R}^{(-,+)}(x,y) = \mathbf{I}_-\otimes\mathbf{I}_+ - h(x,y)\sum_{i,k=1}^{2}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k,$$

where

$$\mathbf{I}_+ = \sum_{i=1}^{2}\mathbf{E}^i_i, \qquad \mathbf{I}_- = \sum_{i=1}^{2}\mathbf{E}^{-i}_{-i}, \qquad k(x,y) = \frac{1}{x-y-1}.$$

Lemma 3. *On the vector space $\mathcal{W}_0$, the relations*

$$\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x,y)\mathbf{T}^{(\epsilon_1)}_1(x)\mathbf{T}^{(\epsilon_2)}_2(y) = \mathbf{T}^{(\epsilon_2)}_2(y)\mathbf{T}^{(\epsilon_1)}_1(x)\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x,y)$$

hold for any $\epsilon_1,\epsilon_2=\pm$.

Proposition 3. *If we set*

$$\widetilde{\mathbf{R}}(x,y) = \mathbf{R}^{(+,+)}(x,y) + \mathbf{R}^{(+,-)}(x,y) + \mathbf{R}^{(-,+)}(x,y) + \mathbf{R}^{(-,-)}(x,y),$$

$$\widetilde{\mathbf{T}}(x) = \mathbf{T}^{(+)}(x) + \mathbf{T}^{(-)}(x),$$

then the RTT equation

$$\widetilde{\mathbf{R}}_{1,2}(x,y)\widetilde{\mathbf{T}}_1(x)\widetilde{\mathbf{T}}_2(y) = \widetilde{\mathbf{T}}_2(y)\widetilde{\mathbf{T}}_1(x)\widetilde{\mathbf{R}}_{1,2}(x,y)$$

is satisfied on the space $\mathcal{W}_0$. Moreover, the R-matrix $\widetilde{\mathbf{R}}(x,y)$ is invertible and satisfies the Yang–Baxter equation

$$\widetilde{\mathbf{R}}_{1,2}(x,y)\widetilde{\mathbf{R}}_{1,3}(x,z)\widetilde{\mathbf{R}}_{2,3}(y,z) = \widetilde{\mathbf{R}}_{2,3}(y,z)\widetilde{\mathbf{R}}_{1,3}(x,z)\widetilde{\mathbf{R}}_{1,2}(x,y).$$

We let $\widetilde{\mathcal{A}}$ denote the RTT algebra defined by the R-matrix $\widetilde{\mathbf{R}}(x,y)$. For generators of the RTT algebra $\widetilde{\mathcal{A}}$, we use $\widetilde{T}^i_k(x)$ and $\widetilde{T}^{-i}_{-k}(x)$. If these operators act on the vector space $\mathcal{W}_0$, then we let $T^i_k(x)$ and $T^{-i}_{-k}(x)$ denote them. Using the commutation relations in the RTT algebra $\widetilde{\mathcal{A}}$, we can easily prove the following lemma.

Lemma 4. *In the RTT algebra $\widetilde{\mathcal{A}}$, we have the following relations for any x and y:*

1. *For any i and k,*
$$\widetilde{T}^i_k(x)\widetilde{T}^i_k(y)=\widetilde{T}^i_k(y)\widetilde{T}^i_k(x),\qquad \widetilde{T}^{-i}_{-k}(x)\widetilde{T}^{-i}_{-k}(y)=\widetilde{T}^{-i}_{-k}(y)\widetilde{T}^{-i}_{-k}(x).$$
2. *For any $i\neq k$,*
$$\widetilde{T}^i_k(x)\widetilde{T}^{-k}_{-i}(y)=\widetilde{T}^{-k}_{-i}(y)\widetilde{T}^i_k(x).$$
3. *For the operators $\widetilde{H}^{(+)}(x)=\widetilde{T}^1_1(x)+\widetilde{T}^2_2(x)$ and $\widetilde{H}^{(-)}(x)=\widetilde{T}^{-1}_{-1}(x)+\widetilde{T}^{-2}_{-2}(x)$, we have*
$$\widetilde{H}^{(+)}(x)\widetilde{H}^{(+)}(y)=\widetilde{H}^{(+)}(y)\widetilde{H}^{(+)}(x),\qquad \widetilde{H}^{(-)}(x)\widetilde{H}^{(-)}(y)=\widetilde{H}^{(-)}(y)\widetilde{H}^{(-)}(x),$$
$$\widetilde{H}^{(+)}(x)\widetilde{H}^{(-)}(y)=\widetilde{H}^{(-)}(y)\widetilde{H}^{(+)}(x).$$

4.2. General form of eigenvectors. We let $\vec{u}=(u_1,u_2,\ldots,u_N)$ denote an ordered set of pairwise distinct numbers. We seek the transfer matrix eigenvectors for the RTT algebra $\mathcal{A}$ in the form

$$\sum_{\substack{i_1,i_2,\ldots,i_N=1,\\ k_1,k_2,\ldots,k_N=1}}^{2} T^{i_1}_{-k_1}(u_1)T^{i_2}_{-k_2}(u_2)\cdots T^{i_N}_{-k_N}(u_N)\Phi^{-k_1,-k_2,\ldots,-k_N}_{i_1,i_2,\ldots,i_N},\quad \Phi^{-k_1,-k_2,\ldots,-k_N}_{i_1,i_2,\ldots,i_N}\in\mathcal{W}_0.$$

Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the basis of the two-dimensional space $\mathcal{V}_+\sim\mathbb{C}^2$ and $\mathbf{f}^1$ and $\mathbf{f}^2$ be its dual basis in $\mathcal{V}^*_+$. Similarly, let $\mathbf{e}_{-1}$ and $\mathbf{e}_{-2}$ be the basis of the vector space $\mathcal{V}_-\sim\mathbb{C}^2$ and $\mathbf{f}^{-1}$ and $\mathbf{f}^{-2}$ be its dual basis in $\mathcal{V}^*_-$. We set

$$\mathbf{B}_1(u)=\sum_{i,k=1}^{2}\mathbf{e}_i\otimes\mathbf{f}^{-k}\otimes T^i_{-k}(u)\in\mathcal{V}_{1_+}\otimes\mathcal{V}^*_{1_-}\otimes\mathcal{A}$$

and

$$\mathbf{B}_{1,2,\ldots,N}(\vec{u}\,)=\mathbf{B}_1(u_1)\mathbf{B}_2(u_2)\cdots\mathbf{B}_N(u_N)\in\mathcal{V}_+\otimes\mathcal{V}^*_-\otimes\mathcal{A},$$

where $\mathcal{V}_+=\mathcal{V}_{1_+}\otimes\mathcal{V}_{2_+}\otimes\cdots\otimes\mathcal{V}_{N_+}$ and $\mathcal{V}^*_-=\mathcal{V}^*_{1_-}\otimes\mathcal{V}^*_{2_-}\otimes\cdots\otimes\mathcal{V}^*_{N_-}$. Specifically,

$$\mathbf{B}_{1,2,\ldots,N}(\vec{u}\,)=\sum_{\vec{i},\vec{k}}\mathbf{e}_{\vec{i}}\otimes\mathbf{f}^{-\vec{k}}\otimes T^{\vec{i}}_{-\vec{k}}(\vec{u}\,),$$

where

$$\mathbf{e}_{\vec{i}}=\mathbf{e}_{i_1}\otimes\mathbf{e}_{i_2}\otimes\cdots\otimes\mathbf{e}_{i_N}\in\mathcal{V}_{1_+}\otimes\mathcal{V}_{2_+}\otimes\cdots\otimes\mathcal{V}_{N_+}=\mathcal{V}_+,$$

$$\mathbf{f}^{-\vec{k}}=\mathbf{f}^{-k_1}\otimes\mathbf{f}^{-k_2}\otimes\cdots\otimes\mathbf{f}^{-k_N}\in\mathcal{V}^*_{1_-}\otimes\mathcal{V}^*_{2_-}\otimes\cdots\otimes\mathcal{V}^*_{N_-}=\mathcal{V}^*_-,$$

$$T^{\vec{i}}_{-\vec{k}}(\vec{u}\,)=T^{i_1}_{-k_1}(u_1)T^{i_2}_{-k_2}(u_2)\cdots T^{i_N}_{-k_N}(u_N)\in\mathcal{A}.$$

We can write the general form of the eigenvector as $\langle\mathbf{B}_{1,2,\ldots,N}(\vec{u}\,),\boldsymbol{\Phi}\rangle$, where

$$\boldsymbol{\Phi}=\mathbf{f}^{(\vec{s}\,)}\otimes\mathbf{e}_{-\vec{r}}\otimes\Phi^{-\vec{r}}_{\vec{s}}\in\mathcal{V}^*_+\otimes\mathcal{V}_-\otimes\mathcal{W}_0=\widehat{\mathcal{W}}_0,\quad \Phi^{-\vec{r}}_{\vec{s}}=\Phi^{-r_1,r_2,\ldots,-r_N}_{s_1,s_2,\ldots,s_N}\in\mathcal{W}_0,$$

$$\mathbf{f}^{(\vec{s}\,)}=\mathbf{f}^{s_1}\otimes\mathbf{f}^{s_2}\otimes\cdots\otimes\mathbf{f}^{s_N}\in\mathcal{V}^*_{1_+}\otimes\mathcal{V}^*_{2_+}\otimes\cdots\otimes\mathcal{V}^*_{N_+}=\mathcal{V}^*_+,$$

$$\mathbf{e}_{-\vec{r}}=\mathbf{e}_{-r_1}\otimes\mathbf{e}_{-r_2}\otimes\cdots\otimes\mathbf{e}_{-r_N}\in\mathcal{V}_{1_-}\otimes\mathcal{V}_{2_-}\otimes\cdots\otimes\mathcal{V}_{N_-}=\mathcal{V}_-.$$

4.3. Some commutation relations and their consequences. We now find the operator action of $\mathbf{T}^{(+)}(x)$ and $\mathbf{T}^{(-)}(x)$ on the assumed eigenvector $\langle \mathbf{B}_{1,2,\ldots,N}(\vec{u}), \mathbf{\Phi}\rangle$.

Lemma 5. *In the RTT algebra of $sp(4)$ type, we have the relations*

$$\mathbf{T}_0^{(+)}(x)\langle \mathbf{B}_1(u), \mathbf{f}^r \otimes \mathbf{e}_{-s}\rangle = f(u,x)\langle \mathbf{B}_1(u), \widehat{\mathbf{T}}_{0,1}^{(+)}(x;u)(\mathbf{I}\otimes \mathbf{f}^r \otimes \mathbf{e}_{-s})\rangle -$$
$$- g(u,x)\langle \mathbf{B}_1(x), \widehat{\mathbb{T}}_{0,1}^{(+)}(u)(\mathbf{I}\otimes \mathbf{f}^r \otimes \mathbf{e}_{-s})\rangle,$$
$$\mathbf{T}_0^{(-)}(x)\langle \mathbf{B}_1(u), \mathbf{f}^r \otimes \mathbf{e}_{-s}\rangle = f(x,u)\langle \mathbf{B}_1(u), \widehat{\mathbf{T}}_{0,1}^{(-)}(x;u)(\mathbf{I}\otimes \mathbf{f}^r \otimes \mathbf{e}_{-s})\rangle -$$
$$- g(x,u)\langle \mathbf{B}_1(x), \widehat{\mathbb{T}}_{0,1}^{(-)}(u)(\mathbf{I}\otimes \mathbf{f}^r \otimes \mathbf{e}_{-s})\rangle,$$

where

$$\widehat{\mathbf{T}}_{0,1}^{(\epsilon)}(x;u) = \widehat{\mathbf{R}}_{0,1_+^*}^{(\epsilon,+)}(x,u)\mathbf{T}_0^{(\epsilon)}(x)\mathbf{R}_{0,1_-}^{(\epsilon,-)}(x,u),$$

$$\widehat{\mathbf{R}}_{0,1_+^*}^{(+,+)}(x,u) = \frac{1}{f(u,x)}\left(\mathbf{I}_{0_+}\otimes \mathbf{I}_{1_+}^* + g(u,x)\sum_{r,s=1}^{2}\mathbf{E}_s^r \otimes \mathbf{F}_r^s\right),$$

$$\widehat{\mathbf{R}}_{0,1_+^*}^{(-,+)}(x,u) = \mathbf{I}_{0_-}\otimes \mathbf{I}_{1_+}^* - k(u,x)\sum_{r,s=1}^{2}\mathbf{E}_{-s}^{-r}\otimes \mathbf{F}_s^r,$$

$$\widehat{\mathbb{T}}_{0,1}^{(\epsilon)}(u) = \widehat{\mathbb{R}}_{0,1_+^*}^{(\epsilon,+)}\mathbf{T}_0^{(\epsilon)}(u)\mathbb{R}_{0,1_-}^{(\epsilon,-)},$$

$$\widehat{\mathbb{R}}_{0,1_+^*}^{(+,+)} = \widehat{\mathbf{R}}_{0,1_+^*}^{(+,+)}(u,u) = \sum_{r,s=1}^{2}\mathbf{E}_s^r\otimes \mathbf{F}_r^s,$$

$$\widehat{\mathbb{R}}_{0,1_+^*}^{(-,+)} = \widehat{\mathbf{R}}_{0,1_+^*}^{(-,+)}(u,u) = \mathbf{I}_{0_-}\otimes \mathbf{I}_{1_+}^* + \sum_{r,s=1}^{2}\mathbf{E}_{-s}^{-r}\otimes \mathbf{F}_s^r,$$

$$\mathbb{R}_{0,1_-}^{(+,-)} = \mathbf{R}_{0,1_-}^{(+,-)}(u,u) = \mathbf{I}_{0_+}\otimes \mathbf{I}_{1_-} + \sum_{r,s=1}^{2}\mathbf{E}_s^r\otimes \mathbf{E}_{-s}^{-r},$$

$$\mathbb{R}_{0,1_-}^{(-,-)} = \mathbf{R}_{0,1_-}^{(-,-)}(u,u) = \sum_{r,s=1}^{2}\mathbf{E}_{-s}^{-r}\otimes \mathbf{E}_{-r}^{-s}.$$

To formulate and prove the following lemma, it is convenient to define a linear map $(\mathbf{R}^*)^{(+,+)}(x,y)$ by the relation

$$\langle \mathbf{R}^{(+,+)}(x,y)(\mathbf{e}_{s_1}\otimes \mathbf{e}_{s_2}), \mathbf{f}^{r_1}\otimes \mathbf{f}^{r_2}\rangle = \langle \mathbf{e}_{s_1}\otimes \mathbf{e}_{s_2}, (\mathbf{R}^*)^{(+,+)}(x,y)(\mathbf{f}^{r_1}\otimes \mathbf{f}^{r_2})\rangle,$$

i.e.,

$$(\mathbf{R}^*)^{(+,+)}(x,y) = \frac{1}{f(x,y)}\left(\mathbf{I}_+^*\otimes \mathbf{I}_+^* + g(x,y)\sum_{i,k=1}^{2}\mathbf{F}_k^i\otimes \mathbf{F}_i^k\right).$$

We let $\bar{u}$ denote the set of elements $\vec{u}$, i.e., $\bar{u} = \{u_1, u_2, \ldots, u_N\}$, and recall that $\bar{u}_k = \bar{u}\setminus\{u_k\}$.

Lemma 6. *We have the relations*

$$
\begin{aligned}
&\mathbf{T}_0^{(+)}(x)\langle \mathbf{B}_{1,2,\dots,N}(\vec{u}\,), \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}}\rangle = \\
&\quad = F(\bar{u},x)\langle \mathbf{B}_{1,2,\dots,N}(\vec{u}\,), \widehat{\mathbf{T}}^{(+)}_{0,1,\dots,,N}(x;\vec{u}\,)(\mathbf{I}\otimes \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}})\rangle - \sum_{u_k\in\bar{u}} g(u_k,x)F(\bar{u}_k,u_k)\,\times \\
&\qquad \times \langle \mathbf{B}_{k;1,2,\dots,N}(x,\vec{u}_k), (\mathbf{R}^*)^{(+,+)}_{1,2,\dots,k}(\vec{u}\,)\mathbf{R}^{(-,-)}_{1,2,\dots,k}(\vec{u}\,)\widehat{\mathbb{T}}^{(+)}_{k;0,1,\dots,N}(\vec{u}\,)(\mathbf{I}\otimes \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}})\rangle, \\
&\mathbf{T}_0^{(-)}(x)\langle \mathbf{B}_{1,2,\dots,N}(\vec{u}\,), \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}}\rangle = \\
&\quad = F(x,\bar{u})\langle \mathbf{B}_{1,2,\dots,N}(\vec{u}\,), \widehat{\mathbf{T}}^{(-)}_{0,1,\dots,,N}(x;\vec{u}\,)(\mathbf{I}\otimes \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}})\rangle - \sum_{u_k\in\bar{u}} g(x,u_k)F(u_k,\bar{u}_k)\,\times \\
&\qquad \times \langle \mathbf{B}_{k;1,2,\dots,N}(x,\vec{u}_k), (\mathbf{R}^*)^{(+,+)}_{1,2,\dots,k}(\vec{u}\,)\mathbf{R}^{(-,-)}_{1,2,\dots,k}(\vec{u}\,)\widehat{\mathbb{T}}^{(-)}_{k;0,1,\dots,N}(\vec{u}\,)(\mathbf{I}\otimes \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}})\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
&\widehat{\mathbf{T}}^{(\epsilon)}_{0,1,\dots,,N}(x;\vec{u}\,) = \widehat{\mathbf{R}}^{(\epsilon,+)}_{0;1^*_+,\dots,N^*_+}(x;\vec{u}\,)\mathbf{T}^{(\epsilon)}_0(x)\mathbf{R}^{(\epsilon,-)}_{0;1_-,\dots,N_-}(x;\vec{u}\,), \\
&\widehat{\mathbf{R}}^{(\epsilon,+)}_{0;1^*_+,\dots,N^*_+}(x;\vec{u}\,) = \widehat{\mathbf{R}}^{(\epsilon,+)}_{0,1^*_+}(x,u_1)\widehat{\mathbf{R}}^{(\epsilon,+)}_{0,2^*_+}(x,u_2)\cdots\widehat{\mathbf{R}}^{(\epsilon,+)}_{0,N^*_+}(x,u_N), \\
&\mathbf{R}^{(\epsilon,-)}_{0;1_-,\dots,N_-}(x;\vec{u}\,) = \mathbf{R}^{(\epsilon,-)}_{0,N_-}(x,u_N)\cdots\mathbf{R}^{(\epsilon,-)}_{0,2_-}(x,u_2)\mathbf{R}^{(\epsilon,-)}_{0,1_-}(x,u_1), \\
&(\mathbf{R}^*)^{(+,+)}_{1,2,\dots,k}(\vec{u}\,) = (\mathbf{R}^*)^{(+,+)}_{k^*_+,1^*_+}(u_k,u_1)(\mathbf{R}^*)^{(+,+)}_{k^*_+,2^*_+}(u_k,u_2)\cdots(\mathbf{R}^*)^{(+,+)}_{k^*_+,(k-1)^*_+}(u_k,u_{k-1}), \\
&\mathbf{R}^{(-,-)}_{1,2,\dots,k}(\vec{u}\,) = \mathbf{R}^{(-,-)}_{1_-,k_-}(u_1,u_k)\mathbf{R}^{(-,-)}_{2_-,k_-}(u_2,u_k)\cdots\mathbf{R}^{(-,-)}_{(k-1)_-,k_-}(u_{k-1},u_k), \\
&\mathbf{B}_{k;1,2,\dots,N}(x,\vec{u}_k) = \mathbf{B}_k(x)\mathbf{B}_1(u_1)\cdots\mathbf{B}_{k-1}(u_{k-1})\mathbf{B}_{k+1}(u_{k+1})\cdots\mathbf{B}_N(u_N), \\
&\widehat{\mathbb{T}}^{(\epsilon)}_{k;0,1,\dots,N}(\vec{u}\,) = \widehat{\mathbf{T}}^{(\epsilon)}_{0,1,\dots,,N}(u_k;\vec{u}\,) = \widehat{\mathbf{R}}^{(\epsilon,+)}_{0;1^*_+,\dots,N^*_+}(u_k;\vec{u}\,)\mathbf{T}^{(\epsilon)}_0(u_k)\mathbf{R}^{(\epsilon,-)}_{0;1_-,\dots,N_-}(u_k;\vec{u}\,),
\end{aligned}
$$

and the products over the empty sets $(\mathbf{R}^*)^{(+,+)}_{1,\dots,1}(\vec{u}\,)$ *and* $\mathbf{R}^{(-,-)}_{1,\dots,1}(\vec{u}\,)$ *are understood as identity maps.*

The following proposition follows directly from Lemma 6.

Proposition 4. *Let* $\mathbf{\Phi} \in \widehat{\mathcal{W}}_0$ *be the eigenvectors of the operators*

$$
\begin{aligned}
\widehat{H}^{(+)}_{1,2,\dots,N}(x;\vec{u}\,) &= (\widehat{\mathbf{T}}^{(+)}_{0,1,\dots,N}(x;\vec{u}\,))^1_1 + (\widehat{\mathbf{T}}^{(+)}_{0,1,\dots,N}(x;\vec{u}\,))^2_2, \\
\widehat{H}^{(-)}_{1,2,\dots,N}(x;\vec{u}\,) &= (\widehat{\mathbf{T}}^{(-)}_{0,1,\dots,N}(x;\vec{u}\,))^{-1}_{-1} + (\widehat{\mathbf{T}}^{(-)}_{0,1,\dots,N}(x;\vec{u}\,))^{-2}_{-2}
\end{aligned}
$$

with the eigenvalues $E^{(+)}(x;\vec{u}\,)$ *and* $E^{(-)}(x;\vec{u}\,)$. *If*

$$F(\bar{u}_k,u_k)E^{(+)}(u_k;\vec{u}\,) = F(u_k,\bar{u}_k)E^{(-)}(u_k;\vec{u}\,)$$

for any $u_k \in \bar{u}$, *then* $\langle \mathbf{B}_{1,2,\dots,N}(\vec{u}\,), \mathbf{\Phi}\rangle$ *are the eigenvectors of the operator* $H(x)$ *with the eigenvalue*

$$E(x;\vec{u}\,) = F(\bar{u},x)E^{(+)}(x;\vec{u}\,) + F(x,\bar{u})E^{(-)}(x;\vec{u}\,).$$

4.4. Operators $\widehat{\mathbf{T}}^{(\pm)}_{1,2,\dots,N}(x;\vec{v})$ and the RTT algebra $\widetilde{\mathcal{A}}$. Proposition 4 transforms the original problem into the problem of finding eigenvectors of $\widehat{H}^{(\pm)}(x;\vec{u}\,) = \widehat{H}^{(\pm)}_{1,2,\dots,N}(x;\vec{u}\,)$ on the vector space $\widehat{\mathcal{W}}_0$. We show that the operators $\widehat{T}^i_k(x;\vec{u}\,)$ and $\widehat{T}^{-i}_{-k}(x,\vec{u}\,)$, where $i,k=1,2$, and

$$\widehat{\mathbf{T}}^{(+)}_{0,1,\dots,,N}(x;\vec{u}\,) = \sum_{i,k=1}^{2} \mathbf{E}^k_i \otimes \widehat{T}^i_k(x;\vec{u}\,),$$

$$\widehat{\mathbf{T}}^{(-)}_{0,1,\dots,,N}(x;\vec{u}\,) = \sum_{i,k=1}^{2} \mathbf{E}^{-k}_{-i} \otimes \widehat{T}^{-i}_{-k}(x;\vec{u}\,),$$

restricted to the space $\widehat{\mathcal{W}}_0$ are generators of the RTT algebra $\widetilde{\mathcal{A}}$ and that the representation contains a vacuum vector.

Lemma 7. *For any $\mathbf{\Phi} \in \widehat{\mathcal{W}}_0$, we have the RTT relation*

$$\mathbf{R}^{(\epsilon_0,\epsilon_{0'})}_{0,0'}(x,y)\widehat{\mathbf{T}}^{(\epsilon_0)}_{0,1,\dots,N}(x,\vec{u}\,)\widehat{\mathbf{T}}^{(\epsilon_{0'})}_{0',1,\dots,N}(y;\vec{u}\,)\mathbf{\Phi} =$$
$$= \widehat{\mathbf{T}}^{(\epsilon_{0'})}_{0',1,\dots,N}(y;\vec{u}\,)\widehat{\mathbf{T}}^{(\epsilon_0)}_{0,1,\dots,N}(x,\vec{u}\,)\mathbf{R}^{(\epsilon_0,\epsilon_{0'})}_{0,0'}(x,y)\mathbf{\Phi}, \quad \epsilon_0,\epsilon_{0'} = \pm.$$

Lemma 8. *Let a vector $\widehat{\Omega}$ have the form*

$$\widehat{\Omega} = \underbrace{\mathbf{f}^1 \otimes \cdots \otimes \mathbf{f}^1}_{N\ times} \otimes \underbrace{\mathbf{e}_{-1} \otimes \cdots \otimes \mathbf{e}_{-1}}_{N\ times} \otimes\ \omega \in \widehat{\mathcal{W}}_0.$$

Then

$$\widehat{T}^1_2(x;\vec{u}\,)\widehat{\Omega} = 0, \qquad \widehat{T}^{-2}_{-1}(x;\vec{u}\,)\widehat{\Omega} = 0, \qquad \widehat{T}^i_i(x;\vec{u}\,)\widehat{\Omega} = \mu_i(x,\bar{u})\widehat{\Omega},$$

where

$$\mu_1(x;\bar{u}) = \lambda_1(x)F(\bar{u},x-1), \qquad \mu_2(x;\bar{u}) = \frac{\lambda_2(x)}{F(\bar{u},x)},$$

$$\mu_{-1}(x;\bar{u}) = \lambda_{-1}(x)F(x+1,\bar{u}), \qquad \mu_{-2}(x;\bar{u}) = \frac{\lambda_{-2}(x)}{F(x,\bar{u})}.$$

4.5. Nested Bethe ansatz for the RTT algebra $\widetilde{\mathcal{A}}$. In the space $\widehat{\mathcal{W}}_0$, we seek common eigenvectors of the operators

$$\widehat{H}^{(+)}_{1,2,\dots,N}(x;\vec{u}\,) = \widehat{T}^1_1(x;\vec{u}\,) + \widehat{T}^2_2(x;\vec{u}\,),$$

$$\widehat{H}^{(-)}_{1,2,\dots,N}(x;\vec{u}\,) = \widehat{T}^{-1}_{-1}(x;\vec{u}\,) + \widehat{T}^{-2}_{-2}(x;\vec{u}\,)$$

for any $\vec{u}$.

We study the representation of the RTT algebra $\widetilde{\mathcal{A}}$ with the generators $\widetilde{T}^i_k(x)$ and $\widetilde{T}^{-i}_{-k}(x)$ on $\widetilde{\mathcal{W}}_0 = \widetilde{\mathcal{A}}\tilde{\omega}$, where $\tilde{\omega}$ is the vacuum vector for which we have the relations

$$\widetilde{T}^1_2(x)\tilde{\omega} = \widetilde{T}^{-2}_{-1}(x)\tilde{\omega} = 0, \qquad \widetilde{T}^i_i(x)\tilde{\omega} = \mu_i(x)\tilde{\omega}, \qquad \widetilde{T}^{-i}_{-i}(x)\tilde{\omega} = \mu_{-i}(x)\tilde{\omega}.$$

We seek the common eigenvectors of

$$\widetilde{H}^{(+)}(x) = \widetilde{T}^1_1(x) + \widetilde{T}^2_2(x), \qquad \widetilde{H}^{(-)}(x) = \widetilde{T}^{-1}_{-1}(x) + \widetilde{T}^{-2}_{-2}(x)$$

in the form

$$|\bar{v},\bar{w}\rangle = \widetilde{T}_1^2(\bar{v})\widetilde{T}_{-2}^{-1}(\bar{w})\tilde{\omega} = \widetilde{T}_1^2(w_1)\widetilde{T}_1^2(w_2)\cdots\widetilde{T}_1^2(v_P)\widetilde{T}_{-2}^{-1}(w_1)\widetilde{T}_{-2}^{-1}(w_2)\cdots\widetilde{T}_{-2}^{-1}(w_Q)\tilde{\omega},$$

where $\bar{v} = \{v_1, v_2, \ldots, w_P\}$ and $\bar{w} = \{w_1, w_2, \ldots, w_Q\}$.

Lemma 9. *In the RTT algebra $\widetilde{\mathcal{A}}$, we have the relations*

$$\widetilde{T}_1^1(x)\widetilde{T}_1^2(\bar{v}) = F(x,\bar{v})\widetilde{T}_1^2(\bar{v})\widetilde{T}_1^1(x) - \sum_{v_r\in\bar{v}} g(x,v_r)F(v_r,\bar{v}_r)\widetilde{T}_1^2(\bar{v}_r,x)\widetilde{T}_1^1(v_r),$$

$$\widetilde{T}_2^2(x)\widetilde{T}_1^2(\bar{v}) = F(\bar{v},x)\widetilde{T}_1^2(\bar{v})\widetilde{T}_2^2(x) - \sum_{v_r\in\bar{v}} g(v_r,x)F(\bar{v}_r,v_r)\widetilde{T}_1^2(\bar{v}_r,x)\widetilde{T}_2^2(v_r),$$

$$\widetilde{T}_{-1}^{-1}(x)\widetilde{T}_{-2}^{-1}(\bar{w}) = F(\bar{w},x)\widetilde{T}_{-2}^{-1}(\bar{w})\widetilde{T}_{-1}^{-1}(x) - \sum_{w_s\in\bar{w}} g(w_s,x)F(\bar{w}_s,w_s)\widetilde{T}_{-2}^{-1}(\bar{w}_s,x)\widetilde{T}_{-1}^{-1}(w_s),$$

$$\widetilde{T}_{-2}^{-2}(x)\widetilde{T}_{-2}^{-1}(\bar{w}) = F(x,\bar{w})\widetilde{T}_{-2}^{-1}(\bar{w})\widetilde{T}_{-2}^{-2}(x) - \sum_{w_s\in\bar{w}} g(x,w_s)F(w_s,\bar{w}_s)\widetilde{T}_{-2}^{-1}(\bar{w}_s,x)\widetilde{T}_{-2}^{-2}(w_s),$$

$$\widetilde{T}_1^1(x)\widetilde{T}_{-2}^{-1}(\bar{w}) = F(x-2,\bar{w})\widetilde{T}_{-2}^{-1}(\bar{w})\widetilde{T}_1^1(x) +$$
$$+ \sum_{w_s\in\bar{w}} g(x-2,w_s)F(w_s,\bar{w}_s)\widetilde{T}_1^2(x)\widetilde{T}_{-2}^{-1}(\bar{w}_s)\widetilde{T}_{-2}^{-2}(w_s),$$

$$\widetilde{T}_2^2(x)\widetilde{T}_{-2}^{-1}(\bar{w}) = F(\bar{w},x-2)\widetilde{T}_{-2}^{-1}(\bar{w})\widetilde{T}_2^2(x) +$$
$$+ \sum_{w_s\in\bar{w}} g(w_s,x-2)F(\bar{w}_s,w_s)\widetilde{T}_1^2(x)\widetilde{T}_{-2}^{-1}(\bar{w}_s)\widetilde{T}_{-1}^{-1}(w_s),$$

$$\widetilde{T}_{-1}^{-1}(x)\widetilde{T}_1^2(\bar{v}) = F(\bar{v},x+2)\widetilde{T}_1^2(\bar{v})\widetilde{T}_{-1}^{-1}(x) +$$
$$+ \sum_{v_r\in\bar{v}} g(v_r,x+2)F(\bar{v}_r,v_r)\widetilde{T}_1^2(\bar{v}_r)\widetilde{T}_{-2}^{-1}(x)\widetilde{T}_2^2(v_r),$$

$$\widetilde{T}_{-2}^{-2}(x)\widetilde{T}_1^2(\bar{v}) = F(x+2,\bar{v})\widetilde{T}_1^2(\bar{v})\widetilde{T}_{-2}^{-2}(x) +$$
$$+ \sum_{v_r\in\bar{v}} g(x+2,v_r)F(v_r,\bar{v}_r)\widetilde{T}_1^2(\bar{v}_r)\widetilde{T}_{-2}^{-1}(x)\widetilde{T}_1^1(v_r).$$

Applying the equations of Lemma 9 to the assumed form of the eigenvector, we obtain the following statement.

Lemma 10. *For a representation of the RTT algebra $\widetilde{\mathcal{A}}$ with the highest weight $\mu_i(x)$, we obtain*

$$\widetilde{T}_1^1(x)|\bar{v};\bar{w}\rangle = \mu_1(x)F(x,\bar{v})F(x-2,\bar{w})|\bar{v};\bar{w}\rangle -$$
$$- \sum_{v_r\in\bar{v}} \mu_1(v_r)g(x,v_r)F(v_r,\bar{v}_r)F(v_r-2,\bar{w})|\{\bar{v}_r,x\};\bar{w}\rangle +$$
$$+ \sum_{w_s\in\bar{w}} \mu_{-2}(w_s)g(x-2,w_s)F(w_s+2,\bar{v})F(w_s,\bar{w}_s)|\{\bar{v},x\};\bar{w}_s\rangle,$$

$$\widetilde{T}_2^2(x)|\bar v;\bar w\rangle = \mu_2(x)F(\bar v,x)F(\bar w,x-2)|\bar v;\bar w\rangle -$$
$$-\sum_{v_r\in\bar v}\mu_2(v_r)g(v_r,x)F(\bar v_r,v_r)F(\bar w,v_r-2)|\{\bar v_r,x\};\bar w\rangle +$$
$$+\sum_{w_s\in\bar w}\mu_{-1}(w_s)g(w_s,x-2)F(\bar v,w_s+2)F(\bar w_s,w_s)|\{\bar v,x\};\bar w_s\rangle,$$
$$\widetilde{T}_{-1}^{-1}(x)|\bar v;\bar w\rangle = \mu_{-1}(x)F(\bar v,x+2)F(\bar w,x)|\bar v;\bar w\rangle +$$
$$+\sum_{v_r\in\bar v}\mu_2(v_r)g(v_r,x+2)F(\bar v_r,v_r)F(\bar w,v_r-2)|\bar v_r;\{\bar w,x\}\rangle -$$
$$-\sum_{w_s\in\bar w}\mu_{-1}(w_s)g(w_s,x)F(\bar v,w_s+2)F(\bar w_s,w_s)|\bar v;\{\bar w_s,x\}\rangle,$$
$$\widetilde{T}_{-2}^{-2}(x)|\bar v;\bar w\rangle = \mu_{-2}(x)F(x+2,\bar v)F(x,\bar w)|\bar v;\bar w\rangle +$$
$$+\sum_{v_r\in\bar v}\mu_1(v_r)g(x+2,v_r)F(v_r,\bar v_r)F(v_r-2,\bar w)|\bar v_r;\{\bar w,x\}\rangle -$$
$$-\sum_{w_s\in\bar w}\mu_{-2}(w_s)g(x,w_s)F(w_s+2,\bar v)F(w_s,\bar w_s)|\bar v;\{\bar w_s,x\}\rangle.$$

From Lemma 10, we almost immediately obtain the following theorem, which gives Bethe conditions for the eigenvectors in the RTT algebra $\widetilde{\mathcal{A}}$.

Theorem 3. *If the conditions*

$$\begin{aligned}
&\mu_1(v_r)F(v_r,\bar v_r)F(v_r-2,\bar w) = \mu_2(v_r)F(\bar v_r,v_r)F(\bar w,v_r-2),\\
&\mu_{-1}(w_s)F(\bar v,w_s+2)F(\bar w_s,w_s) = \mu_{-2}(w_s)F(w_s+2,\bar v)F(w_s,\bar w_s)
\end{aligned} \tag{6}$$

are satisfied for any $v_r\in\bar v$ and $w_s\in\bar w$, then the vector $|\bar v;\bar w\rangle = \widetilde{T}_1^2(\bar v)\widetilde{T}_{-2}^{-1}(\bar w)\omega$ is the eigenvector $\widetilde{H}^{(+)}(x)$ and $\widetilde{H}^{(-)}(x)$ with the eigenvalues

$$\begin{aligned}
&\mu^{(+)}(x;\bar v,\bar w) = \mu_1(x)F(x,\bar v)F(x-2,\bar w)+\mu_2(x)F(\bar v,x)F(\bar w,x-2),\\
&\mu^{(-)}(x;\bar v,\bar w) = \mu_{-1}(x)F(\bar v,x+2)F(\bar w,x)+\mu_{-2}(x)F(x+2,\bar v)F(x,\bar w).
\end{aligned}$$

4.6. Eigenvectors for the RTT algebra of *sp*(4) type. The following statement is a corollary of Theorem 3.

Proposition 5. *If the conditions*

$$\begin{aligned}
&\lambda_1(v_r)F(\bar u,v_r)F(\bar u,v_r-1)F(v_r,\bar v_r)F(v_r-2,\bar w) = \lambda_2(v_r)F(\bar v_r,v_r)F(\bar w,v_r-2),\\
&\lambda_{-1}(w_s)F(w_s,\bar u)F(w_s+1,\bar u)F(\bar v,w_s+2)F(\bar w_s,w_s) = \lambda_{-2}(w_s)F(w_s+2,\bar v)F(w_s,\bar w_s)
\end{aligned}$$

are satisfied for any $v_r\in\bar v$ and $w_s\in\bar w$, then the vectors $\mathbf{\Phi}(\vec u;\bar v;\bar w) = \widehat{T}_1^2(\bar v;\vec u)\widehat{T}_{-2}^{-1}(\bar w;\vec u)\widehat{\Omega}\in\widehat{\mathcal{W}}_0$ are *common eigenvectors of the operators* $\widehat{H}^{(+)}_{1,2,\ldots,N}(x;\vec{u}\,)$ *and* $\widehat{H}^{(-)}_{1,2,\ldots,N}(x;\vec{u}\,)$ *with the eigenvalues*

$$E^{(+)}(x;\vec{u},\bar{v},\bar{w}) = \lambda_1(x)F(\bar{u},x-1)F(x,\bar{v})F(x-2,\bar{w}) + \lambda_2(x)\,\frac{F(\bar{v},x)F(\bar{w},x-2)}{F(\bar{u},x)},$$

$$E^{(-)}(x;\vec{u},\bar{v},\bar{w}) = \lambda_{-1}(x)F(x+1,\bar{u})F(\bar{v},x+2)F(\bar{w},x) + \lambda_{-2}(x)\frac{F(x+2,\bar{v})F(x,\bar{w})}{F(x,\bar{u})}.$$

Because of the identity

$$\frac{F(\bar{u}_k,u_k)}{F(\bar{u},u_k)} = \frac{F(u_k,\bar{u}_k)}{F(u_k,\bar{u})} = \frac{1}{f(u_k,u_k)} = 0,$$

from Propositions 4 and 5, we immediately derive our main result giving the Bethe vectors and Bethe conditions for the RTT algebra of $sp(4)$ type.

Theorem 4. *Let the conditions*

$$\begin{aligned}
&\lambda_1(u_k)F(\bar{u}_k,u_k-1)F(\bar{u}_k,u_k)F(u_k,\bar{v})F(u_k-2,\bar{w}) = \\
&\qquad\qquad = \lambda_{-1}(u_k)F(u_k+1,\bar{u}_k)F(u_k,\bar{u}_k)F(\bar{v},u_k+2)F(\bar{w},u_k),\\
&\lambda_1(v_r)F(\bar{u},v_r-1)F(\bar{u},v_r)F(v_r,\bar{v}_r)F(v_r-2,\bar{w}) = \lambda_2(v_r)F(\bar{v}_r,v_r)F(\bar{w},v_r-2),\\
&\lambda_{-1}(w_s)F(w_s+1,\bar{u})F(w_s,\bar{u})F(\bar{v},w_s+2)F(\bar{w}_s,w_s) = \lambda_{-2}(w_s)F(w_s+2,\bar{v})F(w_s,\bar{w}_s)
\end{aligned}$$

be satisfied for any $u_k\in\bar{u}$, $v_r\in\bar{v}$, *and* $w_s\in\bar{w}$. *Then the vector* $|\vec{u};\bar{v};\bar{w}\rangle = \langle \mathbf{B}_{1,2,\ldots,N}(\vec{u}\,),\mathbf{\Phi}(\vec{u};\bar{v};\bar{w})\rangle$ *is the eigenvector of* $H(x)$ *with the eigenvalue*

$$\begin{aligned}
E(x;\vec{u},\bar{v},\bar{w}) = {} & \lambda_1(x)F(\bar{u},x)F(\bar{u},x-1)F(x,\bar{v})F(x-2,\bar{w}) + \\
& + \lambda_{-1}(x)F(x,\bar{u})F(x+1,\bar{u})F(\bar{v},x+2)F(\bar{w},x) + \\
& + \lambda_2(x)F(\bar{v},x)F(\bar{w},x-2) + \lambda_{-2}(x)F(x+2,\bar{v})F(x,\bar{w}).
\end{aligned}$$

5. Conclusion

We have formulated the nested Bethe ansatz for the RTT algebra of $sp(4)$ type. We showed that the auxiliary RTT algebra $\widetilde{\mathcal{A}}$, which is not an RTT subalgebra of the RTT algebra of $sp(4)$ type, can be used to construct the Bethe vectors and the Bethe conditions. The first outline of this idea can be found in [4].

We note that we here assumed that a highest-weight representation of the RTT algebra of $sp(4)$ type is known. This representation must be given to obtain concrete physical models. At present, we explicitly know only two nontrivial representations. One of these representations was used in [4]. In that case, the representation of the RTT algebra $\widetilde{\mathcal{A}}$ is one-dimensional. The second representation is constructed using the R-matrix. The construction of the Bethe vectors and the Bethe conditions for this representation can be found in [6]. This construction does not use the auxiliary RTT algebra $\widetilde{\mathcal{A}}$, and we are convinced that this is possible only because the representation is relatively simple.

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