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On representations of Higher Spin symmetry algebras for mixed-symmetry HS fields on AdS-spaces. Lagrangian formulation

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Abstract. We derive non-linear commutator HS symmetry algebra, which encode unitary irreducible representations of AdS group subject to Young tableaux $Y(s_1,\ldots,s_k)$ with $k\geq 2$ rows on d-dimensional anti-de-Sitter space. Auxiliary representations for specially deformed non-linear HS symmetry algebra in terms of generalized Verma module in order to additively convert a subsystem of second-class constraints in the HS symmetry algebra into one with first-class constraints are found explicitly for the case of HS fields for $k=2$ Young tableaux. The oscillator realization over Heisenberg algebra for obtained Verma module is constructed. The results generalize the method of auxiliary representations construction for symplectic $sp(2k)$ algebra used for mixed-symmetry HS fields on a flat spaces and can be extended on a case of arbitrary HS fields in AdS-space. Gauge-invariant unconstrained reducible Lagrangian formulation for free bosonic HS fields with generalized spin (s_1,s_2) is derived.

1. Introduction

Increased interest to higher-spin field theory is mainly conditioned by expected output of LHC on the planned capacity. It suspects not only the proof of supersymmetry display, the answer on the question on existence of Higgs boson, and possibly a new insight on origin of Dark Matter ([1], [2]) but permits one to reconsider the problems of an unique description of variety of elementary particles and all known interactions. If the above hopes are true the development of higher-spin (HS) field theory in view of its close relation to superstring theory on constant curvature spaces, which operates with an infinite set of massive and massless bosonic and fermionic HS fields subject to multi-row Young tableaux (YT) $Y(s_1,...,s_k)$, $k\geq 1$ (see for a review, [3]–[7]) seems by actual one. Corresponding description of such theories, having as the final aim the Lagrangian form, requires quite modern and complicated group-theoretic tools which are connected with a construction of different representations of algebras and superalgebras underlying above theories. Whereas for Lie (super)algebra case relevant for HS fields on flat spaces the finding and structure of mentioned objects like Verma modules and generalized Verma modules [8], [9] are rather understandable, an analogous situation with non-linear algebraic and superalgebraic structures which corresponds to HS fields on AdS spaces has not been classified to present with except for the case of totally-symmetric bosonic [10]–[12] and fermionic [13], [14] HS fields. The paper propose the results of regular method for Verma module constructing and Fock space realization for quadratic algebras, whose negative (equivalently positive) root

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vectors in Cartan-like triangular decomposition are entangled due to presence of the special parameter r being by the inverse square of AdS-radius vanishing in the flat space limit. The obtained objects permit to find Lagrangian formulations (LFs) for free integer HS fields on d-dimensional AdS-space with $Y(s_1,...,s_k)$ in Fronsdal [15] metric-like formalism within BFV-BRST procedure [16], [17] (known at present as BRST construction) as a starting point for an interacting HS field theory in the framework of conventional Quantum Field Theory. The application of the BRST construction to free HS field theory on AdS spaces consists in 4 steps and presents a solution of the *problem inverse* to that of the method [16] (as in the case of string field theory [18])and reflects one more side of the BV–BFV duality concept [19]–[21]. First, the conditions that determine the representations with given mass and spin are regarded as a topological (i.e. without Hamiltonian) gauge system of mixed-class operator constraints o_I, $I = 1, 2, ...$, in an auxiliary Fock space $\mathcal{H}$. Second, the whole system of o_I which form quadratic commutator algebra should be additively converted (see [22], [23] for the development of conversion methods) in deformed (in power of parameter r) algebra of O_I: $O_I = o_I + o'_I$ determined on wider Fock space, $\mathcal{H} \bigotimes \mathcal{H}'$ with first-class constraints O_α, $O_\alpha \subset O_I$. Third, the, Hermitian nilpotent BFV-BRST operator Q' for non-linear algebra of converted operators O_I which contains the BFV-BRST operator Q for only subsystem of O_α should be found. Fourth, the Lagrangian $\mathcal{L}$ for given HS field through a scalar product $\langle\ |\ \rangle$ in total Hilbert space $\mathcal{H}_{tot}$ like $\mathcal{L} \sim \langle\phi|Q|\phi\rangle$, to be invariant with respect to (reducible) gauge transformations $\delta|\phi\rangle = Q|\Lambda\rangle$ with $|\phi\rangle$ containing initial and auxiliary HS fields is constructed in such a way that the corresponding equations of motion reproduce the initial constraints.

The application of above algorithm for bosonic [24], [25] and fermionic [26]–[28] HS fields on flat spaces did not meet the problems in rather complicated second and third steps due to linear Lie structure of initial constraints o_I algebra, $[o_I, o_J\} = f^K_{IJ} o_K$ for structure constants f^K_{IJ}. Indeed, for the same algebra of additional parts o'_I it is sufficient to use the construction of Verma module (VM) for integer HS symmetry algebra $sp(2k)$ [29] which is in one-to-one correspondence with Lorentz $so(1, d-1)$ algebra unitary irreducible representations subject to $Y(s_1,...,s_k)$, $k \leq \left[\frac{d}{2}\right]$ due to Howe duality [30]. Then an oscillator realization of the symplectic algebra $sp(2k)$ in Fock space $\mathcal{H}'$ [29] represents a polynomial form as compared to totally-symmetric HS fields on AdS space [11]–[14] where, (super)algebras of o_I are non-linear, not coinciding with ones for o'_I. The problem of the same complication arises in constructing of BFV-BRST operator Q' for (super)algebra of converted constraints O_I which in transiting to AdS-space does not have the form to be quadratic in ghost coordinates $\mathcal{C}$ and requires the full study of algebraic relations starting from Jacobi identities resolution (see [31] for details of finding BRST operator Q' in question and ones for classical quadratic algebras [32]).

To be complete, note the details of Lagrangian description of mixed-symmetry HS tensors on (A)dS backgrounds were studied in "frame-like" formulation in [33]–[36] whereas the LF for the mixed-symmetry bosonic fields with off-shell traceless constraints in the case of (anti)-de-Sitter case are recently known for the Young tableaux with two rows [37], [38]. At last, the various aspects of mixed-symmetry HS fields Lagrangian dynamics on Minkowski space were discussed in [39], [40] and recently for interacting mixed-symmetry HS fields on AdS-spaces in [41], [42].

Present paper is devoted to the solution of the following problems:

(i) derivation of HS symmetry algebra for bosonic HS fields in d-dimensional AdS space subject to arbitrary YT $Y(s_1,...,s_k)$;

(ii) development of a method of Verma module construction for a non-linear HS symmetry algebra for YT with two rows $Y(s_1,s_2)$ and to the oscillator realization for given non-linear algebra as formal power series in creation and annihilation operators of corresponding Heisenberg algebra;

(iii) construction of unconstrained LF for free bosonic HS fields on AdS-space with $Y(s_1,s_2)$.

The paper is organized as follows. In Section 2, we examine the bosonic HS fields that includes, first, the derivation of HS symmetry algebra $\mathcal{A}(Y(k), AdS_d)$ for HS fields subject to Young tableaux with arbitrary number of rows k in Subsection 2.1, second, an auxiliary proposition that permits one to find the forms of deformation of general commutator polynomial algebras under additive conversion procedure in Subsection 2.2. In Section 3 we derive explicitly HS symmetry algebra of additional parts for HS fields with 2 families of indices, formulate and solve the problem of Verma module construction for algebra $\mathcal{A}'(Y(2), AdS_d)$ of additional parts o'_I in Subsection 3.1. We find Fock space realization for the algebra $\mathcal{A}'(Y(2), AdS_d)$ in Subsection 3.2. In Section 4, we derive an explicit form for non-linear algebra $\mathcal{A}_c(Y(2), AdS_d)$ of converted operators O_I, and present for it an expression for BFV–BRST operator in Subsection4.2. and develop the unconstrained Lagrangian formulation for bosonic HS fields with two-rows Young tableaux $Y(s_1, s_2)$ in Subsection 4.3. In Conclusion, we summarize the results of the work and discuss some open problems. Finally, in Appendix A we prove the proposition on additive conversion for polynomial algebras.

2. HS fields in AdS spaces with integer spin

In the section we derive numbers of special HS symmetry non-linear algebras which encode mixed-symmetry tensor fields as the elements of AdS group irreducible representations with generalized spin $\mathbf{s} = (s_1, \ldots, s_k)$ and mass m on Ads$_d$-space-time. We consider the problem of Verma module construction for one of them and solve it explicitly for non-linear algebra with two-rows Young tableaux. The construction of the Fock space representation for the non-linear algebra with found Verma module finishes the solution of the problem there.

2.1. HS symmetry algebra $\mathcal{A}(Y(k), AdS_d)$ for mixed-symmetry tensor fields with $Y(s_1, ..., s_k)$

A massive generalized integer spin $\mathbf{s} = (s_1, ..., s_k)$, $(s_1 \geq s_2 \geq ... \geq s_k > 0,\, k \leq [d/2])$, AdS group irreducible representation in an AdS$_d$ space is realized in a space of mixed-symmetry tensors, $\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},...,(\mu^k)_{s_k}} \equiv \Phi_{\mu^1_1...\mu^1_{s_1},\mu^2_1...\mu^2_{s_2},...,\mu^k_1...\mu^k_{s_k}}(x)$ to be corresponding to a Young tableaux

$$\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},...,(\mu^k)_{s_k}} \longleftrightarrow \begin{array}{|c|c|c|c|c|c|c|c|}\hline \mu^1_1 & \mu^1_2 & \cdot\ \ \cdot\ \ \cdot & \cdot & \cdot & \cdot & \cdot & \mu^1_{s_1} \\ \hline \mu^2_1 & \mu^2_2 & \cdot\ \ \cdot\ \ \cdot & \cdot & \cdot & \mu^2_{s_2} \\ \cline{1-7} \cdot & \cdot & \cdot\ \ \cdot\ \ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cline{1-7} \mu^k_1 & \mu^k_2 & \cdot\ \ \cdot\ \ \cdot & \cdot & \mu^k_{s_k} \\ \cline{1-6} \end{array}\ , \tag{1}$$

subject to the Klein-Gordon (2), divergentless (3), traceless (4) and mixed-symmetry equations (5) [for $\beta = (2; 3; ...; k+1) \Longleftrightarrow (s_1 > s_2; s_1 = s_2 > s_3; ...; s_1 = s_2 = ... = s_k)$] [43]:

$$[\nabla^2 + r[(s_1 - \beta - 1 + d)(s_1 - \beta) - \sum_{i=1}^{k} s_i] + m^2]\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},...,(\mu^k)_{s_k}} = 0, \tag{2}$$

$$\nabla^{\mu^i_{l_i}}\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},...,(\mu^k)_{s_k}} = 0, \quad i, j = 1, ..., k;\ l_i, m_i = 1, ..., s_i\,, \tag{3}$$

$$g^{\mu^i_{l_i}\mu^i_{m_i}}\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},...,(\mu^k)_{s_k}} = g^{\mu^i_{l_i}\mu^j_{m_j}}\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},...,(\mu^k)_{s_k}} = 0, \quad l_i < m_i,\,, \tag{4}$$

$$\Phi_{(\mu^1)_{s_1},...,\{(\mu^i)_{s_i}\underbrace{,...,\mu^j_1...}\mu^j_{l_j}\}...\mu^j_{s_j},...(\mu^k)_{s_k}} = 0, \quad i < j,\ 1 \leq l_j \leq s_j, \tag{5}$$

where the brackets below denote that the indices inside it do not include in symmetrization, i.e. the symmetrization concerns only indices $(\mu^i)_{s_i}, \mu^j_{l_j}$ in $\{(\mu^i)_{s_i}\underbrace{,...,\mu^j_1...}\mu^j_{l_j}\}$.

To obtain HS symmetry algebra (of o_I) for a description of all integer spin HS fields, we in a standard manner introduce a Fock space $\mathcal{H}$, generated by k pairs of bosonic creation $a^i_{\mu^i}(x)$

and annihilation $a^{j+}_{\nu^j}(x)$ operators, $i,j=1,...,k, \mu^i, \nu^j = 0,1...,d-1$[1]

$$[a^i_{\mu^i}, a^{j+}_{\nu^j}] = -g_{\mu^i\nu^j}\delta^{ij}\,, \qquad \delta^{ij} = diag(1,1,\ldots 1)\,, \tag{6}$$

and a set of constraints for an arbitrary string-like vector $|\Phi\rangle \in \mathcal{H}$ which we call as the basic vector,

$$|\Phi\rangle = \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{s_1}\cdots\sum_{s_k=0}^{s_{k-1}} \Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},...,(\mu^k)_{s_k}}(x)\prod_{i=1}^{k}\prod_{l_i=1}^{s_i} a_i^{+\mu^i_{l_i}}|0\rangle, \tag{7}$$

$$\tilde{l}_0|\Phi\rangle = (l_0 + \tilde{m}_b^2 + r((g_0^1 - 2\beta - 2)g_0^1 - \sum_{i=2}^{k} g_0^i))|\Phi\rangle = 0, \quad l_0 = [D^2 - r\tfrac{d(d-2(k+1))}{4}], \tag{8}$$

$$(l^i, l^{ij}, t^{i_1 j_1})|\Phi\rangle = (-ia^i_\mu D^\mu, \tfrac{1}{2}a^i_\mu a^{j\mu}, a^{i_1+}_\mu a^{j_1\mu})|\Phi\rangle = 0, \quad i \le j;\, i_1 < j_1, \tag{9}$$

with number particles operators, central charge, covariant derivative in $\mathcal{H}$ respectively,

$$g_0^i = -\frac{1}{2}\{a^{i+}_\mu, a^{\mu i}\}, \qquad \tilde{m}_b^2 = m^2 + r\beta(\beta+1), \tag{10}$$

$$D_\mu = \partial_\mu - \omega^{ab}_\mu(x)\Big(\sum_i a^+_{ia}a_{ib}\Big), \qquad a_i^{\mu(+)}(x) = e^\mu_a(x)a_i^{a(+)2}, \tag{11}$$

where e^μ_a, ω^{ab}_μ are vielbein and spin connection for tangent indices $a,b = 0,1...,d-1$. Operator D_μ is equivalent in its action in $\mathcal{H}$ to the covariant derivative ∇_μ [with d'Alambertian $D^2 = (D_a + \omega^b{}_{ba})D^a$]. The set of $k(k+1)$ primary constraints (8), (9) with $\{o_\alpha\} = \{\tilde{l}_0, l^i, l^{ij}, t^{i_1j_1}\}$ are equivalent to Eqs. (2)–(5) for all admissible values of spins and for the field $\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},...,(\mu^k)_{s_k}}$ with fixed spin $\mathbf{s} = (s_1, s_2, \ldots, s_k)$ if in addition to Eqs. (8), (9) we add k constraints with g_0^i,

$$g_0^i|\Phi\rangle = (s_i + \tfrac{d}{2})|\Phi\rangle. \tag{12}$$

The requirement of closedness of the algebra with o_α with respect to [,]-multiplication leads to enlargement of o_α by adding the operators g_0^i and hermitian conjugated operators o^+_α,

$$(l^{i+},\ l^{ij+},\ t^{i_1j_1+}) = (-ia^{i+}_\mu D^\mu,\ \tfrac{1}{2}a^{i+}_\mu a^{j\mu+},\ a^{i_1}_\mu a^{j_1\mu+}),\ i \le j;\ i_1 < j_1, \tag{13}$$

with respect to scalar product on $\mathcal{H}$,

$$\begin{aligned}\langle\Psi|\Phi\rangle \ = \ & \int d^dx\sqrt{|g|}\sum_{i=1}^{k}\sum_{s_i=0}^{s_{i-1}}\sum_{j=1}^{k}\sum_{p_j=0}^{s_{j-1}}\langle 0|\prod_{j=1}^{k}\prod_{m_j=1}^{p_j} a_j^{\nu^j_{m_j}}\Psi^*_{(\nu^1)_{p_1},(\nu^2)_{p_2},...,(\nu^k)_{p_k}}(x) \\ & \times\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},...,(\mu^k)_{s_k}}(x)\prod_{i=1}^{k}\prod_{l_i=1}^{s_i} a_i^{+\mu^i_{l_i}}|0\rangle,\ \texttt{for}\ s_{-1}, p_{-1} = \infty.\end{aligned} \tag{14}$$

This fact will guarantee the Hermiticity of corresponding BFV-BRST operator with taken into account of self-conjugated operators, $(l_0^+,\ g_0^{i\,+}) = (l_0,\ g_0^i)$ (therefore the reality of Lagrangian

[1] such choice of the oscillators corresponds to the case of symmetric basis, whereas there exists another realization of auxiliary Fock space generated by the fermionic oscillators (antisymmetric basis) $\hat{a}^m_{\mu^m}(x)$, $\hat{a}^{n+}_{\nu^n}(x)$ with anticommutation relations, $\{\hat{a}^m_{\mu^m}, \hat{a}^{n+}_{\nu^n}\} = -g_{\mu^m\nu^m}\delta^{mn}$, for $m,n = 1,...,s_1$, and develop the procedure below following to the lines of Ref. [44] for totally antisymmetric tensors for $s_1 = s_2 = ... = s_k = 1$.

[2] operators $a_i^{a_i}$, $a_j^{b_j+}$ satisfy to usual for $R^{1,d-1}$-space commutation relations $[a_i^{a_i}, a_j^{b_j+}] = -\eta^{a_ib_i}\delta_{ij}$ for $\eta^{ab} = \mathrm{diag}(+,-,\ldots,-)$

Table 1. HS symmetry non-linear algebra $\mathcal{A}_m(Y(k), AdS_d)$.

$[\downarrow, \rightarrow\}$	$t^{i_1 j_1}$	$t^+_{i_1 j_1}$	l_0	l^i	l^{i+}	$l^{i_1 j_1}$	$l^{i_1 j_1 +}$
$t^{i_2 j_2}$	$A^{i_2 j_2, i_1 j_1}$	$B^{i_2 j_2}{}_{i_1 j_1}$	0	$l^{j_2}\delta^{i_2 i}$	$-l^{i_2+}\delta^{j_2 i}$	$l^{\{j_1 j_2}\delta^{i_1\}i_2}$	$-l^{i_2\{i_1+}\delta^{j_1\}j_2}$
$t^+_{i_2 j_2}$	$-B^{i_1 j_1}{}_{i_2 j_2}$	$A^+_{i_1 j_1, i_2 j_2}$	0	$l_{i_2}\delta^i_{j_2}$	$-l^+_{j_2}\delta^i_{i_2}$	$l_{i_2}{}^{\{j_1}\delta^{i_1\}}_{j_2}$	$-l_{j_2}{}^{\{j_1+}\delta^{i_1\}}_{i_2}$
l_0	0	0	0	$-r\mathcal{K}^{i+}_1$	$r\mathcal{K}^i_1$	0	0
l^j	$-l^{j_1}\delta^{i_1 j}$	$-l^{i_1}\delta^{j_1 j}$	$r\mathcal{K}^{j+}_1$	W^{ji}_b	X^{ji}_b	0	$-\frac{1}{2}l^{\{i_1+}\delta^{j_1\}j}$
l^{j+}	$l^{i_1+}\delta^{j_1 j}$	$l^{j_1+}\delta^{i_1 j}$	$-r\mathcal{K}^j_1$	$-X^{ij}_b$	$-W^{ji+}_b$	$\frac{1}{2}l^{\{i_1}\delta^{j_1\}j}$	0
$l^{i_2 j_2}$	$-l^{j_1\{j_2}\delta^{i_2\}i_1}$	$-l^{i_1\{i_2}\delta^{j_2\}j_1}$	0	0	$-\frac{1}{2}l^{\{i_2}\delta^{j_2\}i}$	0	$L^{i_2 j_2, i_1 j_1}$
$l^+_{i_2 j_2}$	$l^{i_1+}_{\{i_2}\delta^{j_1}_{j_2\}}$	$l^+_{j_1\{j_2}\delta_{i_2\}i_1}$	0	$\frac{1}{2}l^+_{\{i_2}\delta^i_{j_2\}}$	0	$-L^{i_1 j_1, i_2 j_2}$	0
g^j_0	$-F^{i_1 j_1, j}$	$F^{i_1 j_1, j+}$	0	$-l^i\delta^{ij}$	$l^{i+}\delta^{ij}$	$-l^{j\{i_1}\delta^{j_1\}j}$	$l^{j\{i_1+}\delta^{j_1\}j}$

$\mathcal{L}$) for the system of all operators $\{o_\alpha, o^+_\alpha, g^i_0\}$. We call the algebra of these operators the *integer higher-spin symmetry algebra in AdS space with a Young tableaux having k rows*[3] and denote it as $\mathcal{A}(Y(k), AdS_d)$).

The maximal Lie subalgebra of operators $l^{ij}, t^{i_1 j_1}, g^i_0, l^{ij+},\ t^{i_1 j_1 +}$ is isomorphic to symplectic algebra $sp(2k)$ (see, [29] for details and we will refer on it later as $sp(2k)$) whereas the only nontrivial quadratic commutators in $\mathcal{A}(Y(k), AdS_d)$) are due to operators with D_μ: $l^i, \tilde{l}_0, l^{i+}$. For the aim of LF construction it is enough to have a simpler, without central charge $\tilde{m}^2_b$ (so called *modified* HS symmetry algebra $\mathcal{A}_m(Y(k), AdS_d)$), with operator l_0 (8) instead of $\tilde{l}_0$, so that AdS-mass term, $\tilde{m}^2_b + r((g^1_0 - 2\beta - 2)g^1_0 - \sum_{i=2}^k g^i_0)$, will be restored as usual later within conversion procedure and properly construction of LF.

Algebra $\mathcal{A}_m(Y(k), AdS_d)$) of the operators o_I from the Hamiltonian analysis of the dynamical systems viewpont contains 1 first-class constraint l_0, $2k$ differential l_i, l^+_i and $2k^2$ algebraic $t^{i_1 j_1}, t^+_{i_1 j_1}, l^{ij}, l^+_{ij}$ second-class constraints o_a and operators g^i_0, composing an invertible matrix $\Delta_{ab}(g^i_0)$ for topological gauge system because of

$$[o_a,\ o_b] = f^c_{ab}o_c + f^{cd}_{ab}o_c o_d + \Delta_{ab}(g^i_0),\ [o_a,\ l_o] = f^c_{a[l_0]}o_c + f^{cd}_{a[l_0]}o_c o_d, . \tag{15}$$

Here $f^c_{ab}, f^{cd}_{ab}, f^c_{a[l_0]}, f^{cd}_{a[l_0]}, \Delta_{ab}$ are antisymmetric with respect to permutations of lower indices constant quantities and the operators $\Delta_{ab}(g^i_0)$ form the non-vanishing $2k(k+1) \times 2k(k+1)$ matrix $\|\Delta_{ab}\|$ in the Fock space $\mathcal{H}$ on the surface $\Sigma \subset \mathcal{H}$: $\|\Delta_{ab}\|_{|\Sigma} \neq 0$, which is determined by the equations, $o_\alpha|\Phi\rangle = 0$. The set of o_I satisfies the non-linear relations (additional to ones for $sp(2k)$) given by the multiplication table 1.

First note that, in the table 1 we did not include the columns with $[\ ,\ \}$-products of all o_I with g^j_0 which may be obtained from the rows with g^j_0 as follows: $[b, o_I\}^+ = -[o^+_I, b\}$, with account of closedness of HS algebra with respect to Hermitian conjugation. Second the operators $t^{i_2 j_2}, t^+_{i_2 j_2}$ satisfy by the definition the properties

$$(t^{i_2 j_2}, t^+_{i_2 j_2}) \equiv (t^{i_2 j_2}, t^+_{i_2 j_2})\theta^{j_2 i_2},\ \theta^{j_2 i_2} = (1, 0)\ \texttt{for}\ (j_2 > i_2, j_2 \leq i_2) \tag{16}$$

with Heaviside θ-symbol[4] θ^{ji}. Third, the products $B^{i_2 j_2}_{i_1 j_1}, A^{i_2 j_2, i_1 j_1}, F^{i_1 j_1, i}, L^{i_2 j_2, i_1 j_1}$ are

[3] one should not confuse the term "*higher-spin symmetry algebra*" using here for free HS formulation with the algebraic structure known as "*higher-spin algebra*" (see, e.g. Ref.[45]) arising to describe the HS interactions

[4] there are no summation with respect to the indices i_2, j_2 in the Eqs.(16), the figure brackets for the indices i_1, i_2 in the quantity $A^{\{i_1}B^{i_2\}i_3}\theta^{i_3 i_2\}}$ mean the symmetrization $A^{\{i_1}B^{i_2\}i_3}\theta^{i_3 i_2\}} = A^{i_1}B^{i_2 i_3}\theta^{i_3 i_2} + A^{i_2}B^{i_1 i_3}\theta^{i_3 i_1}$ as well as these indices are raising and lowering by means of Euclidian metric tensors δ^{ij}, δ_{ij}, δ^i_j

determined by the explicit relations,

$$\begin{aligned}
B^{i_2j_2}{}_{i_1j_1} &= (g_0^{i_2}-g_0^{j_2})\delta^{i_2}_{i_1}\delta^{j_2}_{j_1} + (t_{j_1}{}^{j_2}\theta^{j_2}{}_{j_1} + t^{j_2+}_{j_1}\theta_{j_1}{}^{j_2})\delta^{i_2}_{i_1} - (t^{+}_{i_1}{}^{i_2}\theta^{i_2}{}_{i_1} + t^{i_2}{}_{i_1}\theta_{i_1}{}^{i_2})\delta^{j_2}_{j_1}\,, \quad (17)\\
A^{i_2j_2,i_1j_1} &= t^{i_1j_2}\delta^{i_2j_1} - t^{i_2j_1}\delta^{i_1j_2}, \qquad F^{i_2j_2,i} = t^{i_2j_2}(\delta^{j_2i}-\delta^{i_2i}), \quad (18)\\
L^{i_2j_2,i_1j_1} &= \tfrac{1}{4}\Big\{\delta^{i_2i_1}\delta^{j_2j_1}\Big[2g_0^{i_2}\delta^{i_2j_2} + g_0^{i_2} + g_0^{j_2}\Big] - \delta^{j_2\{i_1}\Big[t^{j_1\}i_2}\theta^{i_2j_1\}} + t^{i_2j_1\}+}\theta^{j_1\}i_2}\Big]\\
&\quad -\delta^{i_2\{i_1}\Big[t^{j_1\}j_2}\theta^{j_2j_1\}} + t^{j_2j_1\}+}\theta^{j_1\}j_2}\Big]\Big\}. \quad (19)
\end{aligned}$$

They obeys the obvious additional properties of antisymmetry and Hermitian conjugation,

$$\begin{aligned}
&A^{i_2j_2,i_1j_1} = -A^{i_1j_1,i_2j_2}, \qquad A^{+}_{i_1j_1,i_2j_2} = (A_{i_1j_1,i_2j_2})^+ = t^{+}_{i_2j_1}\delta_{j_2i_1} - t^{+}_{i_1j_2}\delta_{i_2j_1}, \quad (20)\\
&(L^{i_2j_2,i_1j_1})^+ = L^{i_1j_1,i_2j_2}, \qquad F^{i_2j_2,i+} = (F^{i_2j_2,i})^+ = t^{i_2j_2+}(\delta^{j_2i}-\delta^{i_2i}), \quad (21)\\
&B^{i_2j_2}{}_{i_1j_1}{}^{+} = (g_0^{i_2}-g_0^{j_2})\delta^{i_2}_{i_1}\delta^{j_2}_{j_1} + (t^{+}_{j_1}{}^{j_2}\theta^{j_2}{}_{j_1} + t^{j_2}{}_{j_1}\theta_{j_1}{}^{j_2})\delta^{i_2}_{i_1} - (t_{i_1}{}^{i_2}\theta^{i_2}{}_{i_1} + t^{i_2+}{}_{i_1}\theta_{i_1}{}^{i_2})\delta^{j_2}_{j_1}. \quad (22)
\end{aligned}$$

Fourth, the independent quantities $\mathcal{K}_1^k, W_b^{ki}, X_b^{ki}$ in the table 1 are quadratic in o_I,

$$\begin{aligned}
W_b^{ij} &= [l^i, l^j] = 2r\Big[(g_0^j - g_0^i)l^{ij} - \sum_m (t^{m[j}\theta^{[jm} + t^{[jm+}\theta^{m[j})l^{i]m}\Big], \quad (23)\\
\mathcal{K}_1^k &= r^{-1}[l_0, l^{k+}] = \Big(4\sum_i l^{ki+}l^i + l^{k+}(2g_0^k - 1) - 2(\sum_i l^{i+}t^{+ik}\theta^{ki} + l^{i+}t^{ki}\theta^{ik})\Big), \quad (24)\\
X_b^{ij} &= \{l_0 + r(K_0^{0i} + \sum_{l=i+1}^{k}\mathcal{K}_0^{il} + \sum_{l=1}^{i-1}\mathcal{K}_0^{li})\}\delta^{ij} \quad (25)\\
&\quad -r[4\sum\nolimits_l l^{jl+}l^{li} - \sum_{l=1}^{j-1}t^{lj+}t^{li} - \sum_{l=i+1}^{k}t^{il+}t^{jl} - \sum_{l=j+1}^{i-1}t^{li}t^{jl} + (g_0^j + g_0^i - j - 1)t_{ji}]\theta^{ij}\\
&\quad -r[4\sum\nolimits_l l^{jl+}l^{li} - \sum_{l=1}^{i-1}t^{lj+}t^{li} - \sum_{l=j+1}^{k}t^{il+}t^{jl} - \sum_{l=i+1}^{j-1}t^{il+}t^{lj+} + t^{+}_{ij}(g_0^j + g_0^i - i - 1)]\theta^{ji}.
\end{aligned}$$

In writing (25) we have used the quantities $K_0^{0i}, i,j = 1,...,k$, $\mathcal{K}_0^{ij}$ composing a Casimir operator $\mathcal{K}_0(k)$ for $sp(2k)$ algebra

$$\mathcal{K}_0(k) = \sum_i K_0^{0i} + 2\sum_{i,j}\mathcal{K}_0^{ij}\theta^{ji} = \sum_i((g_0^i)^2 - 2g_0^i - 4l^{ii+}l^{ii}) + 2(\sum_{i=1}^{k}\sum_{j=i+1}^{k}(t^{+}_{ij}t^{ij} - 4l^{+}_{ij}l^{ij} - g_0^j)). \quad (26)$$

Algebra $\mathcal{A}_m(Y(k), AdS_d)$ maybe considered as non-linear deformation in power of r of the integer HS symmetry algebra in Minkowski space $\mathcal{A}(Y(k), R^{1,d-1})$ [29] as follows,

$$\mathcal{A}_m(Y(k), AdS_d) = \mathcal{A}(Y(k), R^{1,d-1})(r) = \Big(T^k \oplus T^{k*} \oplus l_0\Big)(r) \Supset sp(2k), \quad (27)$$

for k-dimensional commutative (on $R^{1,d-1}$) algebra $T^k = \{l_i\}$, its dual $T^{k*} = \{l^{i+}\}$ and represents the semidirect sum of the symplectic algebra $sp(2k)$ [as an algebra of internal derivations of $(T^k \oplus T^{k*})$]. For HS fields with single spin s_1 (for $k=1$) and with two-component spin (s_1, s_2) (for $k=2$) the algebra $\mathcal{A}_m(Y(k), AdS_d)$ coincides respectively with known HS symmetry algebras on AdS spaces given in [12] and in [31].

Now, we should to use the results of an special proposition in order to proceed to the conversion procedure of the algebra of o_I to get the algebra of O_I with only first-class constraints.

2.2. On additive conversion for polynomial algebras

In this subsection, to solve the problem of additive conversion of non-linear algebras with a subset of 2nd class constraints, we need to use some important statement which based on the following (see Ref.[14] for detailed description)

Definition: *A non-linear commutator algebra $\mathcal{A}$ of basis elements o_I, $I \in \Delta$ (with Δ to be finite or infinite set of indices) is called a* ***polynomial algebra of order*** *n, $n \in \mathbf{N}$, if a set of $\{o_I\}$ is subject to n-th order polynomial commutator relations:*

$$[o_I, o_J] = F^K_{IJ}(o)o_K,\ F^K_{IJ} = f^{(1)K}_{IJ} + \sum_{n=2}^{\infty} f^{(n)K_1\ldots K_{n-1}K}_{IJ} \prod_{i=1}^{n-1} o_{K_i}\,,$$
$$f^{(n)K_1\ldots K_n}_{IJ} \neq 0,\ \texttt{and}\ f^{(k)K_1\ldots K_nK_{n+1}\ldots K_k}_{IJ} = 0,\ k > n, \qquad (28)$$

with structural coefficients $f^{(n)K_1\ldots K_{n-1}K}_{IJ}$, to be antisymmetric with respect to permutations of lower indices, $f^{(n)K_1\cdots K_n}_{IJ} = -f^{(n)K_1\cdots K_n}_{JI}$.

Now, we may to formulate the basic statement in the subsection 2.2 in the form of

Proposition: Let $\mathcal{A}$ is the polynomial algebra of order n of basis elements o_I determined in Hilbert space $\mathcal{H}$. Then, for a set $\mathcal{A}'$ of elements o'_I given in a new Hilbert space $\mathcal{H}'$ ($\mathcal{H}\bigcap\mathcal{H}' = \emptyset$) and commuting with o_I, and for a direct sum of sets $\mathcal{A}_c = \mathcal{A} + \mathcal{A}'$ of the operators, O_I, $O_I = o_I + o'_I$ given in the tensor product $\mathcal{H}\otimes\mathcal{H}'$ from the requirement to be in involution relations,

$$[O_I, O_J] = F^K_{IJ}(o', O)O_K\,, \qquad (29)$$

it follows the sets of $\{o'_I\}$, $\{O_I\}$ form respectively the polynomial commutator algebra $\mathcal{A}'$ of order n and the non-linear commutator algebra $\mathcal{A}_c$ with composition laws:

$$[\,o'_I, o'_J\} = f^{(1)K_1}_{IJ} o'_{K_1} + \sum_{m=2}^{n} (-1)^{m-1} f^{(m)K_l\cdots K_1}_{IJ} \prod_{s=1}^{m} o'_{K_s}, \qquad (30)$$

$$[\,O_I, O_J\} = \Big(f^{(1)K}_{IJ} + \sum_{m=2}^{n} F^{(m)K}_{IJ}(o', O)\Big)O_K. \qquad (31)$$

The structural functions $F^{(m)k}_{IJ}(o', O)$ in (30) are constructed with respect to known from the Eqs. (28) coefficients $f^{(m)K_1\ldots K_m}_{IJ} \equiv f^{K_1\ldots K_m}_{IJ}$ as follows

$$F^{(m)K}_{IJ} = f^{K_1\cdots K_m}_{IJ} \prod_{p=1}^{m-1} O_{K_p} + \sum_{s=1}^{m-1} (-1)^s f^{\widehat{K_s\cdots K_1 K_{s+1}\cdots} K_m}_{ij} \prod_{p=1}^{s} o'_{K_p} \prod_{l=s+1}^{m-1} O_{K_l},\ \texttt{where}$$
$$f^{\widehat{K_s\cdots K_1 K_{s+1}\cdots} K_m}_{ij} = f^{K_s\cdots K_1K_{s+1}\cdots K_m}_{ij} + f^{K_s\cdots K_{s+1}K_1K_{s+2}\cdots K_m}_{ij} + \cdots +$$
$$f^{K_{s+1}K_s\cdots K_1K_{s+2}\cdots K_m}_{ij} + \Big(f^{K_{s+1}K_s\cdots K_{s+2}K_1K_{s+3}\cdots K_m}_{ij} +$$
$$\cdots + f^{K_{s+1}K_{s+2}K_s\cdots K_1K_{s+3}\cdots K_l}_{ij}\Big) + \cdots + f^{K_{s+1}\cdots K_mK_s\cdots K_1}_{ij}, \qquad (32)$$

where the sum in the Eq.(32) contains $\frac{m!}{s!(m-s)!}$ terms with all the possible ways of the arrangement the indices $(K_{s+1}, ..., K_m)$ among the indices $(K_s, ..., K_1)$ in $f^{K_s\cdots K_1K_{s+1}\cdots K_m}_{ij}$ without changing the separate orderings of the indices $K_{s+1}, ..., K_m$ and $K_s, ..., K_1$[5].

[5] We do not consider here the case of polynomial superalgebra for which the proposition may be easily generalized with introducing corresponding sign factors in the Eqs. (30)–(32) with the same number of summands to use it for fermionic HS fields on AdS space

The validity of the proposition is verified in the Appendix A. Turning to the structure of algebra $\mathcal{A}_c$ note that in contrary to $\mathcal{A}$ and $\mathcal{A}'$ we call it as *non-homogeneous polynomial algebra of order* n due to form of relations (32) (see footnote 13 in the Appendix A for the comments). For Lie algebra case ($n=1$) the structures of the algebras $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}_c$ coincides as it then used, e.g. for the integer spin HS symmetry algebra $\mathcal{A}(Y(k), R^{1,d-1})$ [29]. For quadratic algebras (n=2) the algebraic relations for $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}_c$ do not coincide with each other due to structural functions $f^{(2)K_1K_2}_{IJ}$ presence that was firstly shown for the algebra $\mathcal{A}(Y(1), AdS_d)$ for totally-symmetric HS tensors on AdS space in [12] and having the form,

$$[\,o_I, o_J] = f^{(1)K_1}_{IJ} o_{K_1} + f^{(2)K_1K_2}_{IJ} o_{K_1}o_{K_2}, \quad [\,o'_I, o'_J] = f^{(1)K_1}_{IJ} o'_{K_1} - f^{(2)K_2K_1}_{IJ} o'_{K_1}o'_{K_2}, \tag{33}$$

$$[\,O_I, O_J] = \Big(f^{(1)K}_{IJ} + F^{(2)K}_{IJ}(o', O)\Big)O_K, \quad F^{(2)K}_{IJ} = f^{(2)K_1K}_{IJ}O_{K_1} - (f^{(2)K_1K}_{IJ} + f^{(2)KK_1}_{IJ}){o'}_{K_1}. \tag{34}$$

Relations (33), (34) are sufficient to determine the form of the multiplication laws for the additional parts o'_I algebra $\mathcal{A}'(Y(k), AdS_d)$ and for converted operators O_I algebra $\mathcal{A}_c(Y(k), AdS_d)$.

As the new result we write down the explicit form for the cubic commutator algebras $\mathcal{A}'$, $\mathcal{A}_c$

$$\begin{aligned}
[\,o'_I, o'_J\} &= f^{(1)K_1}_{IJ} o'_{K_1} - f^{(2)K_2K_1}_{IJ} o'_{K_1}o'_{K_2} + f^{(3)K_3K_2K_1}_{IJ} o'_{K_1}o'_{K_2}o'_{K_3}, \\
[\,O_I, O_J\} &= \Big(f^{(1)K}_{IJ} + \sum_{l=2}^{n} F^{(l)K}_{IJ}(o', O)\Big)O_K, \qquad (35)\\
F^{(3)K}_{IJ} &= f^{(3)K_1K_2K}_{IJ}O_{K_1}O_{K_2} - (f^{(3)K_1K_2K}_{IJ} + f^{(3)K_2K_1K}_{IJ}){o'}_{K_1}O_{K_2} \\
&\quad + (f^{(3)K_2K_1K}_{IJ} + f^{(3)K_2KK_1}_{IJ} + f^{(3)KK_2K_1}_{IJ}){o'}_{K_1}{o'}_{K_2}, \qquad (36)
\end{aligned}$$

if the commutator relations for the initial algebra $\mathcal{A}$ given by the Eqs.(28) for $n=3$.

3. Auxiliary HS symmetry algebra $\mathcal{A}'(Y(2), AdS_d)$

The procedure of additive conversion for non-linear HS symmetry algebra $\mathcal{A}(Y(k), AdS_d)$ of the operators o_I implies finding, first, the explicit form of the algebra $\mathcal{A}'(Y(k), AdS_d)$ of the additional parts o'_I, second, the representation of $\mathcal{A}'(Y(k), AdS_d)$ in terms of some appropriate Heisenberg algebra elements acting in a new Fock space $\mathcal{H}'$. Structure of the non-linear commutators of the initial algebra leads to necessity to convert all the operators o_I to construct unconstrained LF for given HS field $\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}}$.

Considering here the case of $k=2$ family of indices only in the initial HS field $\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2}}$ (the general case of algebra $\mathcal{A}'(Y(k), AdS_d)$ is discussed in [46]) we see the former step is based on a determination of a multiplication table $\mathcal{A}'(Y(2), AdS_d)$ of operators o'_I following to the form of the algebra $\mathcal{A}(Y(k), AdS_d)$ for $k=2$ given by the table 1 and Eqs.(33).

As the result, the searched composition law for $\mathcal{A}'(Y(2), AdS_d)$ is the same as for the algebra $\mathcal{A}(Y(2), AdS_d)$ in its linear Lie part, i.e. for $sp(4)$ subalgebra of elements $(l'^{ij}, l'^{ij+}, t'^{12}, t'^{+}_{12}, g'^{i}_0)$[6], and is differed in the non-linear part of the Table 1, determined by the isometry group elements l'_i, l'^{+}_j, l'_0. The corresponding non-linear submatrix of the multiplication matrix for $\mathcal{A}'(Y(2), AdS_d)$ has the form given by the Table 2, Here the functions $\mathcal{K}'^{i+}_1$, $\mathcal{K}'^{i}_1$, W'^{ji}_b, W'^{ji+}_b $(X'^{ij}_b - l'_0)$ have the same definition as the ones (23)–(25) for initial operators o_I but with opposite sign for $(X'^{ij}_b - l'_0)$ and for $k=2$:

$$W'^{ij}_b = 2r\epsilon^{ij}\left[(g'^2_0 - g'^1_0)l'^{12} - t'^{12}l'^{11} + t'^{12+}l'^{22}\right], \tag{37}$$

[6] To turn from general algebra $\mathcal{A}(Y(k), AdS_d)$ to $\mathcal{A}(Y(2), AdS_d)$, we put $\theta^{ij} = \delta^{i2}\delta^{j1}$ and therefore only surviving operators among mixed-symmetry ones are $t^{ij} = t^{12}$, $t^{+}_{ij} = t^{+}_{12}$.

Table 2. The non-linear part of algebra $\mathcal{A}'(Y(2), AdS_d)$.

$[\downarrow, \rightarrow\}$	l'_0	l'^i	l'^{i+}
l'_0	0	$r\mathcal{K}_1'^{bi+}$	$-r\mathcal{K}_1'^i$
l'^j	$-r\mathcal{K}_1'^{j+}$	$-W_b'^{ji}$	$X_b'^{ji}$
l'^{j+}	$r\mathcal{K}_1'^j$	$-X_b'^{ij}$	$W_b'^{ji+}$

$$\mathcal{K}_1'^j = \Big(4\sum_i l'^{ji+}l'^i + l'^{j+}(2g_0'^j - 1) - 2l'^{2+}t'^{+12}\delta^{j1} - 2l'^{1+}t'^{12}\delta^{j2}\Big), \tag{38}$$

$$\begin{aligned} X_b'^{ij} &= \{l'_0 - r(K_0'^{0i} + \mathcal{K}_0'^{12})\}\delta^{ij} + r\{[4\textstyle\sum_l l'^{1l+}l'^{l2} + (g_0'^2 + g_0'^1 - 2)t'_{12}]\delta^{i2}\delta^{j1} \\ &\quad + r\{[4\textstyle\sum_l l'^{l2+}l'^{1l} + t_{12}'^{+}(g_0'^2 + g_0'^1 - 2)]\delta^{i1}\delta^{j2}, \end{aligned} \tag{39}$$

with totally antisymmetric $sp(2)$-invariant tensor ϵ^{ij}, $\epsilon^{12} = 1$ and operator $\mathcal{K}_0^{12} = (t_{12}^+ t^{12} - 4l_{12}^+ l^{12} - g_0^2)$ derived from Casimir operator for $sp(4)$ algebra in the Eqs.(26) for $k = 2$. In turn, the Lie part of the Tables 1 for $k = 2$ is the same as one for bosonic Lie subalgebra in [28] for the following expressions of only non-vanishing operators $B'^{12}{}_{12}$, A'^{1212}, $F'^{12,j}$, $F'^{12,j+}$, $L'^{i_1j_1,i_2j_2}$

$$B'^{12}{}_{12} = (g_0^1 - g_0^2), \quad A'^{12,12} = 0, \quad (F'^{12,j}, \equiv F'^j = t'^{12}(\delta^{j2} - \delta^{j1}), \tag{40}$$

$$\begin{aligned} L'^{i_1j_1,i_2j_2} &= \tfrac{1}{4}\Big\{\delta^{i_2i_1}\delta^{j_2j_1}\Big[2g_0^{i_2}\delta^{i_2j_2} + g_0^{i_2} + g_0^{j_2}\Big] - \delta^{j_2\{i_1}\Big[t^{12}\delta^{j_1\}1}\delta^{i_22} + t^{12+}\delta^{j_1\}2}\delta^{i_21}\Big] \\ &\quad - \delta^{i_2\{i_1}\Big[t^{12}\delta^{j_22}\delta^{j_1\}1} + t^{12+}\delta^{j_1\}2}\delta^{j_21}\Big]\Big\}, \end{aligned} \tag{41}$$

being identical to the same operators o_I from the initial algebra $\mathcal{A}(Y(2), AdS_d)$.

Now, we may to sketch the points to find oscillator representation for the elements o'_I of auxiliary HS symmetry algebra.

3.1. Verma module for the quadratic algebra $\mathcal{A}'(Y(2k), AdS_d)$

Here, we following to assumption that generalization of Poincare–Birkhoff–Witt theorem for the second order algebra $\mathcal{A}'(Y(2), AdS_d)$ is true (see on PBW theorem generalization for quadratic algebra [47]), we start to construct Verma module, based on Cartan-like decomposition enlarged from one for $sp(4)$ $(i \leq j)$

$$\mathcal{A}'(Y(2), AdS_d) = \{l_{ij}'^+, t_{12}'^+, l_i'^+\} \oplus \{g_0'^i, l'_0\} \oplus \{l'_{ij}, t'_{12}, l'_i\} \equiv \mathcal{E}_2^- \oplus H_2 \oplus \mathcal{E}_2^+.^7 \tag{42}$$

Note, that in contrast to the case of Lie algebra the element l'_0 does not diagonalize the elements of upper $\mathcal{E}_2^-$ (lower $\mathcal{E}_2^+$) triangular subalgebra due to quadratic relations (38) (as well as it was for totally-symmetric HS fields on AdS space [10, 12]) and in addition the negative root vectors $l_i'^+, l_j'^+$ do not commute.

Because the Verma module over a semi-simple finite-dimensional Lie algebra g (an induced module $\mathcal{U}(g)\bigotimes_{\mathcal{U}(b)}|0\rangle_V$ with a vacuum vector $|0\rangle_V$[8]) is isomorphic due to PBW theorem as a vector space to a polynomial algebra $\mathcal{U}(g^-)\bigotimes_C|0\rangle_V$, it is clear that g can be realized by first-order inhomogeneous differential operators acting on these polynomials.

[7] we may consider $sp(4)$ and generally $sp(2k)$ in Cartan-Weyl basis for unified description, however without loss of generality the basis elements and structure constants of the algebra under consideration will be chosen as in the table 1

[8] here the signs $\mathcal{U}(g)$, $\mathcal{U}(b)$, $\mathcal{U}(g^-)$ denote the universal enveloping algebras respectively for g, for its Borel subalgebra and for lower triangular subalgebra g^- like $\mathcal{E}_2^-$ in (42)

We consider generalization of *Verma module notion for the case of quadratic algebras* $g(r)$ which present the $g(r)$-deformation of Lie algebra g in such form that $g(r=0)=g$. Thus, we consider Verma module for such kinds of non-linear algebras, supposing that PBW theorem is valid for $g(r)$ as well that will be proved later by the explicit construction of the Verma module.

Doing so, we consider the quadratic algebra $\mathcal{A}'(Y(2), AdS_d)$, as r-deformation of Lie algebra $\mathcal{A}'(Y(2), R^{1,d-1})$ with use of (27) for $k=2$ (see for details [29]),

$$\mathcal{A}'(Y(2), AdS_d) = \left(T'^2 \oplus T'^{2*} \oplus l'_0\right)(r) \Supset sp(4), \quad T'^2 = \{l'_i\},\ T'^{2*} = \{l'^+_i\}. \tag{43}$$

Corresponding basis vector of Verma module $|\vec{N}(2)\rangle_V$ has, therefore the form according to (42)

$$|\vec{N}(2)\rangle_V = |\vec{n}_{ij}, n_1, p_{12}, n_2\rangle_V \equiv \prod_{i\le j}^{2}(l'^+_{ij})^{n_{ij}} \left(\frac{l'^+_1}{m_1}\right)^{n_1}(t'^+_{12})^{p_{12}}\left(\frac{l'^+_2}{m_2}\right)^{n_2}|0\rangle_V,\ \mathcal{E}^+_2|0\rangle_V = 0, \tag{44}$$

with (vacuum) highest weight vector $|0\rangle_V$, non-negative integers n_{ij}, n_l, p_{lm}, and arbitrary constants m_l with dimension of mass.

Here, in contrast to the case of Lie algebra [28] and totally-symmetric HS fields on AdS space [10], [13], the negative root vectors $l'^+_1, t'^+_{12}, l'^+_2$ do not commute, making the vector $|\vec{N}(2)\rangle_V$ by not proper one for the operators t'^+_{12}, l'^+_2 presenting therefore the essential peculiarities to construct Verma module for $\mathcal{A}'(Y(2), AdS_d)$.

First of all, by the definition the highest weight vector $|0\rangle_V$ is proper for the vectors from the Cartan-like subalgebra H_2,

$$(g'^i_0, l'_0)|0\rangle_V = (h^i, m_0^2)|0\rangle_V, \tag{45}$$

with some real numbers h^1, h^2, m_0 whose values will be determined later in the end of LF construction in. order to Lagrangian equations of motion reproduce the initial AdS group irreps conditions (2)–(5).

Then, we determine the action of the negative root vectors from the subspace $\mathcal{E}^-_2$ on the basis vector $|\vec{N}(2)\rangle_V$ which now do not present explicitly the action of raising operators and reads,

$$l'^+_{lm}\left|\vec{N}(2)\rangle_V\right. = |\vec{n}_{ij} + \delta_{ij,lm}, \vec{n}_s\rangle_V\ , \tag{46}$$

$$l'^+_l\left|\vec{N}(2)\rangle_V\right. = \delta^{l2}\prod_{i\le j}^{2}(l'^+_{ij})^{n_{ij}} l'^+_2\left|\vec{0}_{ij}, \vec{n}_s\rangle_V\right. + \delta^{l1}m_1\,|\vec{n}_{ij}, n_1+1, p_{12}, n_2\rangle_V\ ,\ l=1,2, \tag{47}$$

$$\begin{aligned} t'^+_{12}\left|\vec{N}(2)\rangle_V\right. = {}& \prod_{i\le j}^{2}(l'^+_{ij})^{n_{ij}} t'^+_{12}|\vec{0}_{ij}, \vec{n}_s\rangle_V - 2n_{11}\,|n_{11}-1, n_{12}+1, n_{22}, \vec{n}_s\rangle_V \\ & - n_{12}\,|n_{11}, n_{12}-1, n_{22}+1, \vec{n}_s\rangle_V\ . \end{aligned} \tag{48}$$

Here we have used the notations, first, $\vec{n}_s \equiv (n_1, p_{12}, n_2)$, $\vec{0}_{ij} \equiv (0,0,0)$ in accordance with definition of $|\vec{N}(2)\rangle_V$, second, $\delta_{ij,lm} = \delta_{il}\delta_{jm}$, for $i \le j, l \le m$, so that the vector $\left|\vec{N} + \delta_{ij,lm}\rangle_V\right.$ in the Eq.(46) means subject to definition (44) increasing of only the coordinate n_{ij} in the vector $|\vec{N}(2)\rangle_V$, for $i=l, j=m$, on unit with unchanged values of the rest ones.

In turn, the action of Cartan-like generators on the vector $|\vec{N}(2)\rangle_V$ are given by the relations,

$$g'^{l}_0\left|\vec{N}(2)\rangle_V\right. = \left(2n_{ll} + n_{12} + n_l + (-1)^l p_{12} + h^l\right)\left|\vec{N}(2)\rangle_V\right.\ ,\ l=1,2, \tag{49}$$

$$l'_0\left|\vec{N}(2)\rangle_V\right. = \textstyle\prod_{i\le j}^{2}(l'^+_{ij})^{n_{ij}} l'_0\left|\vec{0}_{ij}, \vec{n}_s\rangle_V\right.\ . \tag{50}$$

To derive the Eqs. (46)–(50) we have used the formula for the product of operators A, B,

$$AB^n = \sum_{k=0}^{n} C^n_k B^{n-k}\mathrm{ad}^k_B A\,,\ \mathrm{ad}^k_B A = [[...[A,\overbrace{B\},...\},B\}}^{k\,\text{times}},\ \texttt{for}\ n\geq 0, C^n_k = \frac{n!}{k!(n-k)!}. \tag{51}$$

At last, the action of the positive root vectors from the subspace $\mathcal{E}^+_2$ on the vector $|\vec{N}(2)\rangle_V$ being based on the rule (51), reads as follows,

$$\begin{aligned}
t'_{12}\left|\vec{N}(2)\right\rangle_V &= \prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}\Big(\frac{l'^+_1}{m_1}\Big)^{n_1}(t'^+_{12})^{p_{12}}t'_{12}\left|\vec{0}_{ij},0,0,n_2\right\rangle_V - \sum_l l n_{l2}|\vec{n}_{ij}-\delta_{ij,l2}+\delta_{ij,1l},\vec{n}_s\rangle_V \\
&\quad +p_{12}(h^1-h^2-n_2-p_{12}+1)\,|\vec{n}_{ij},n_1,p_{12}-1,n_2\rangle_V\ , \qquad (52)\\
l'^1\left|\vec{N}(2)\right\rangle_V &= -m_1 n_{11}|\vec{n}_{ij}-\delta_{ij,11},n_1+1,p_{12},n_2\rangle_V \\
&\quad +\Big\{\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}l'^1 - \frac{n_{12}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,12}}l'^+_2\Big\}\left|\vec{0}_{ij},\vec{n}_s\right\rangle_V\ , \qquad (53)\\
l'^2\left|\vec{N}(2)\right\rangle_V &= -m_1\frac{n_{12}}{2}|\vec{n}_{ij}-\delta_{ij,12},n_1+1,p_{12},n_2\rangle_V \\
&\quad +\Big\{\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}l'^2 - n_{22}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,22}}l'^+_2\Big\}\left|\vec{0}_{ij},\vec{n}_s\right\rangle_V\ , \\
l'^{11}\left|\vec{N}(2)\right\rangle_V &= n_{11}(n_{11}+n_{12}+n_1-p_{12}-1+h^1)\,|\vec{n}_{ij}-\delta_{ij,11},\vec{n}_s\rangle_V \\
&\quad +\frac{n_{12}(n_{12}-1)}{4}\,|\vec{n}_{ij}-2\delta_{ij,12}+\delta_{ij,22},\vec{n}_s\rangle_V \\
&\quad +\Big\{\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}l'^{11} - \frac{n_{12}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,12}}t'^+_{12}\Big\}\left|\vec{0}_{ij},\vec{n}_s\right\rangle_V\ , \qquad (54)\\
l'^{12}\left|\vec{N}(2)\right\rangle_V &= \frac{n_{12}}{4}\Big(n_{12}+\sum_l(2n_{ll}+n_l+h^l)-1\Big)\,|\vec{n}_{ij}-\delta_{ij,12},\vec{n}_s\rangle_V \\
&\quad +\frac{1}{2}p_{12}n_{11}(h^2-h^1+n_2+p_{12}-1)\,|\vec{n}_{ij}-\delta_{ij,11},n_1,p_{12}-1,n_2\rangle_V \\
&\quad +n_{11}n_{22}\,|\vec{n}_{ij}-\delta_{ij,11}-\delta_{ij,22}+\delta_{ij,12},\vec{n}_s\rangle_V \\
&\quad +\Big\{\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}l'^{12} - \frac{n_{22}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,22}}t'^+_{12}\Big\}\left|\vec{0}_{ij},\vec{n}_s\right\rangle_V \\
&\quad -\frac{n_{11}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,11}}\Big(\frac{l'^+_1}{m_1}\Big)^{n_1}(t'^+_{12})^{p_{12}}t'_{12}\left|\vec{0}_{ij},0,0,n_2\right\rangle_V \qquad (55)\\
l'^{22}\left|\vec{N}(2)\right\rangle_V &= n_{22}(n_{12}+n_{22}+p_{12}+n_2-1+h^2)\,|\vec{n}_{ij}-\delta_{ij,22},\vec{n}_s\rangle_V \\
&\quad +\frac{n_{12}p_{12}}{2}(p_{12}-1+h^2-h^1+n_2)\,|\vec{n}_{ij}-\delta_{ij,12},n_1,p_{12}-1,n_2\rangle_V \\
&\quad +\frac{n_{12}(n_{12}-1)}{4}\,|\vec{n}_{ij}+\delta_{ij,11}-2\delta_{ij,12},\vec{n}_s\rangle_V \\
&\quad +\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}\Big(\frac{l'^+_1}{m_1}\Big)^{n_1}l'^{22}\left|\vec{0}_{ij},0,p_{12},n_2\right\rangle_V \\
&\quad -\frac{n_{12}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,12}}\Big(\frac{l'^+_1}{m_1}\Big)^{n_1}(t'^+_{12})^{p_{12}}t'_{12}\left|\vec{0}_{ij},0,0,n_2\right\rangle_V\ . \qquad (56)
\end{aligned}$$

It is easy to see that to complete the calculations in Eqs. (47), (48), (53)–(56) we need to find the action of positive root vectors l'^m, l'^{1m}, $m = 1, 2$, Cartan-like vector l'_0, negative root vectors l'^+_2, t'^+_{12} on the vector $|\vec{0}_{ij}, \vec{n}_s\rangle_V$. and the rest operators t'_{12}, l'^{22} on an arbitrary vector $|\vec{0}_{lm}, 0, 0, n_2\rangle_V$ in terms of linear combinations of definite vectors. We solve this rather non-trivial technical problem explicitly with introducing auxiliary quantities which we call *primary block-operator* $\widehat{t}'_{12}$ given in (68) and *derived block-operators* $\widehat{t}'^+_{12}, \widehat{l}'^+_2, \widehat{l}'_0, \widehat{l}'_m, \widehat{l}'_{m2}, m = 1, 2$ below whose concrete expressions will be shown with details of Verma module construction for the algebra under consideration in [46]. Thus, we may to formulate the result in the form of

Theorem 1. The Verma module for the non-linear second order algebra $\mathcal{A}'(Y(2), AdS_d)$ exists, is determined by the relations (46), (49), (57)–(66), expressed with help of primary $\widehat{t}'_{12}$ and derived block-operators $\widehat{t}'^+_{12}, \widehat{l}'^+_2, \widehat{l}'_0, \widehat{l}'_m, \widehat{l}'_{m2}, m = 1, 2$ and has the final form,

$$\begin{aligned}
t'_{12}\left|\vec{N}(2)\rangle_V\right. &= p_{12}(h^1 - h^2 - n_2 - p_{12} + 1)\,|\vec{n}_{ij}, n_1, p_{12} - 1, n_2\rangle_V \\
&\quad - \sum_l l n_{l2}|\vec{n}_{ij} - \delta_{ij,l2} + \delta_{ij,1l}, \vec{n}_s\rangle_V + \hat{t}'_{12}\left|\vec{N}(2)\rangle_V\right. , && (57)\\
t'^+_{12}\left|\vec{N}(2)\rangle_V\right. &= -\sum_l (3 - l)n_{1l}\,|\vec{n}_{ij} - \delta_{ij,1l} + \delta_{ij,l2}, \vec{n}_s\rangle_V + \hat{t}'^+_{12}\left|\vec{N}(2)\rangle_V\right. , && (58)\\
l'^+_l\left|\vec{N}(2)\rangle_V\right. &= \delta^{l1} m_1\left|\vec{N}(2) + \delta_{s,1}\rangle_V\right. + \delta^{l2}\hat{l}'^+_2\left|\vec{N}(2)\rangle_V\right. , && (59)\\
l'_0\left|\vec{N}(2)\rangle_V\right. &= l'_0\left|\vec{0}_{ij}, \vec{n}_s\rangle_V\right.|_{[\vec{0}_{ij}\to\vec{n}_{ij}]}, && (60)\\
l'_1\left|\vec{N}(2)\rangle_V\right. &= l'_1\left|\vec{0}_{ij}, \vec{n}_s\rangle_V\right.|_{[\vec{0}_{ij}\to\vec{n}_{ij}]} - m_1 n_{11}\,|\vec{n}_{ij} - \delta_{ij,11}, \vec{n}_s + \delta_{s,1}\rangle_V \\
&\quad - \frac{n_{12}}{2}\widehat{l}'^+_2\,|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s\rangle_V , && (61)\\
l'_2\left|\vec{N}(2)\rangle_V\right. &= l'_2\left|\vec{0}_{ij}, \vec{n}_s\rangle_V\right.|_{[\vec{0}_{ij}\to\vec{n}_{ij}]} - m_1\frac{n_{12}}{2}\,|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s + \delta_{s,1}\rangle_V && (62)\\
&\quad - n_{22}\widehat{l}'^+_2\,|\vec{n}_{ij} - \delta_{ij,22}, \vec{n}_s\rangle_V , && (63)\\
l'_{11}\left|\vec{N}(2)\rangle_V\right. &= n_{11}(n_{11} + n_{12} + n_1 - p_{12} - 1 + h^1)\,|\vec{n}_{ij} - \delta_{ij,11}, \vec{n}_s\rangle_V \\
&\quad + \frac{n_{12}(n_{12} - 1)}{4}\,|\vec{n}_{ij} - 2\delta_{ij,12} + \delta_{ij,22}, \vec{n}_s\rangle_V + l'_{11}\left|\vec{0}_{ij}, \vec{n}_s\rangle_V\right.|_{[\vec{0}_{ij}\to\vec{n}_{ij}]} \\
&\quad - \frac{n_{12}}{2}\widehat{t}'^+_{12}\,|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s\rangle_V , && (64)\\
l'_{12}\left|\vec{N}(2)\rangle_V\right. &= \frac{n_{12}}{4}\Big(n_{12} + \sum_l (2n_{ll} + n_l + h^l) - 1\Big)\,|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s\rangle_V \\
&\quad + \frac{1}{2}p_{12}n_{11}(h^2 - h^1 + n_2 + p_{12} - 1)\,|\vec{n}_{ij} - \delta_{ij,11}, n_1, p_{12} - 1, n_2\rangle_V \\
&\quad + n_{11}n_{22}\,|\vec{n}_{ij} - \delta_{ij,11} - \delta_{ij,22} + \delta_{ij,12}, \vec{n}_s\rangle_V + l'_{12}\left|\vec{0}_{ij}, \vec{n}_s\rangle_V\right.|_{[\vec{0}_{ij}\to\vec{n}_{ij}]} \\
&\quad - \frac{n_{22}}{2}\widehat{t}'^+_{12}\,|\vec{n}_{ij} - \delta_{ij,22}, \vec{n}_s\rangle_V - \frac{n_{11}}{2}\widehat{t}'_{12}\,|\vec{n}_{ij} - \delta_{ij,11}, \vec{n}_s\rangle_V , && (65)\\
l'_{22}\left|\vec{N}(2)\rangle_V\right. &= n_{22}(n_{12} + p_{12} + n_2 + n_{22} - 1 + h^2)\,|\vec{n}_{ij} - \delta_{ij,22}, \vec{n}_s\rangle_V \\
&\quad + \frac{n_{12}p_{12}}{2}(p_{12} - 1 + h^2 - h^1 + n_2)\,|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s - \delta_{s,12}\rangle_V \\
&\quad + \frac{n_{12}(n_{12} - 1)}{4}\,|\vec{n}_{ij} + \delta_{ij,11} - 2\delta_{ij,12}, \vec{n}_s\rangle_V + \widehat{l}'_{22}\left|\vec{N}(2)\rangle_V\right. \\
&\quad - \frac{n_{12}}{2}\widehat{t}'_{12}\,|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s\rangle_V . && (66)
\end{aligned}$$

In deriving these relations the rule was used

$$\prod_{i\leq j}^{2}(l'^{+}_{ij})^{n_{ij}}(l'_0,l'_n,l'_{lm})\left|\vec{0}_{ij},\vec{n}_s\rangle_V\right. \equiv (l'_0,l'_n,l'_{lm})\left|\vec{0}_{ij},\vec{n}_s\rangle_V\right. |_{[\vec{0}_{ij}\to\vec{n}_{ij}]},\quad l,m,n=1,2,\ l\leq m, \quad (67)$$

when the multipliers $(l'^{+}_{ij})^{n_{ij}}$ act as only raising operators on vectors $(l'_0,l'_k,l'_{km})\left|\vec{0}_{ij},\vec{n}_s\rangle_V\right.$. The *primary block operator* $\hat{t}'_{12}$ (corresponding to non-Lie part of t'_{12}) is determined as follows

$$\begin{aligned}
\hat{t}'_{12}\left|\vec{N}(2)\rangle_V\right. &= \sum_{k=0}^{[(n_2-1)/2]}\left\{\sum_{^1m=0}^{[n_2/2]}\sum_{^1l=0}^{[n_2/2-(^1m+1)]}\cdots\right.\\
&\cdots\sum_{^km=0}^{\left[n_2/2-\sum_{i=1}^{k-1}(^im+^il)-(k-1)\right]}\ \sum_{^kl=0}^{\left[n_2/2-\sum_{i=1}^{k-1}(^im+^il)-^km-k\right]}(-1)^k\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^{k}(^im+^il)+k}\\
&\times C^{n_2}_{2^1m+1}C^{n_2-2^1m-1}_{2^1l+1}\ldots C^{n_2-2(\sum_{i=1}^{k-1}(^im+^il)+k-1)}_{2^km+1}C^{n_2-2(\sum_{i=1}^{k-1}(^im+^il)+k-1)-2^km-1}_{2^kl+1}\\
&\times\left.\left|\hat{A}_{\vec{n}_{ij}+[\sum_{i=1}^{k}(^im+^il)+k]\delta_{i2}\delta_{j2},n_1,n,n_2-2[\sum_{i=1}^{k}(^im+^il)+k]}\right\rangle_V\right\}, \qquad (68)
\end{aligned}$$

with the vector $\left|\hat{A}_{\vec{N}(2)}\rangle_V\right.$ given as,

$$\begin{aligned}
\left|\hat{A}_{\vec{N}(2)}\rangle_V\right. &= n_2p_{12}\left|\vec{N}(2)-\delta_{s,12}\rangle_V\right.\\
&-\frac{m_1}{m_2}\sum_{^1m=0}^{[n_2/2]}\left(\frac{-2r}{m_2^2}\right)^{^1m}C^{n_2}_{2^1m+1}\left|\vec{N}(2)+{}^1m\delta_{ij,22}+\delta_{s,1}-(2^1m+1)\delta_{s,2}\rangle_V\right.\\
&-\sum_{^1m=1}^{[n_2/2]}\left(\frac{-2r}{m_2^2}\right)^{^1m}\left\{C^{n_2}_{2^1m}\left|\vec{N}(2)+\delta_{ij,11}+(^1m-1)\delta_{ij,22}+\delta_{s,12}-2^1m\delta_{s,2}\rangle_V\right.\right.\\
&-\left[C^{n_2}_{2^1m}(h^2-h^1+2p_{12})+C^{n_2}_{2^1m+1}\right]\left|\vec{N}(2)+\delta_{ij,12}+(^1m-1)\delta_{ij,22}-2^1m\delta_{s,2}\rangle_V\right.\\
&\left.+p_{12}C^{n_2}_{2^1m}(h^2-h^1+p_{12}-1)\left|\vec{N}(2)+{}^1m\delta_{ij,22}-\delta_{s,12}-2^1m\delta_{s,2}\rangle_V\right.\right\}\\
&-\sum_{^1m=0}^{[n_2/2]}\sum_{^1l=0}^{[n_2/2-(^1m+1)]}\left(\frac{-2r}{m_2^2}\right)^{^1m+^1l+1}C^{n_2}_{2^1m+1}\left\{\left[C^{n_2-2^1m-1}_{2^1l+1}(h^2-h^1+2p_{12}+n_2\right.\right.\\
&\left.-2(^1m+^1l+1))-C^{n_2-2^1m-1}_{2^1l+2}\right]\left|\vec{N}(2)+\delta_{ij,12}+(^1m+^1l)\delta_{ij,22}-2(^1m\right.\\
&+^1l+1)\delta_{s,2}\rangle_V-p_{12}C^{n_2-2^1m-1}_{2^1l+1}(h^2-h^1+n_2-2(^1m+^1l+1)+p_{12}-1)\\
&\times\left|\vec{N}(2)+(^1m+^1l+1)\delta_{ij,22}-\delta_{s,12}-2(^1m+^1l+1)\delta_{s,2}\rangle_V\right.\\
&-C^{n_2-2^1m-1}_{2^1l+1}\left|\vec{N}(2)+\delta_{ij,11}+(^1m+^1l)\delta_{ij,22}+\delta_{s,12}-2(^1m+^1l+1)\delta_{s,2}\rangle_V\right.\\
&\left.+\frac{m_1}{m_2}C^{n_2-2^1m-1}_{2^1l+2}\left|\vec{N}(2)+(^1m+^1l+1)\delta_{ij,22}+\delta_{s,1}-2(^1m+^1l+\tfrac{3}{2})\delta_{s,2}\rangle_V\right.\right\}. \qquad (69)
\end{aligned}$$

The obtained result has obvious consequences, first, in case of reducing of $\mathcal{A}'(Y(2),AdS_d)$ to the quadratic algebra $\mathcal{A}'(Y(1),AdS_d)$, given in Ref. [10, 12] for the vanishing components

$n_{l2} = n_2 = p_{12} = 0, l = 1,2$ of an arbitrary VM vector $\left|\vec{N}(1)\right\rangle_V \equiv |n_{11},0,0,n_1,0,0\rangle_V$. Second, for reducing the AdS space to Minkowski $R^{1,d-1}$ space when the non-linear algebra $\mathcal{A}'(Y(2), AdS_d)$ for $r=0$ turns to the Lie algebra $\left(T'^2 \oplus T'^{2*} \oplus l'_0\right)(0)\,\supset\!\!\!\!+\, sp(4)$ for $k=2$ in the Eqs.(43). For former case we derive known Verma module [12] from results of Theorem 1 whereas for latter one we will get new Verma module realization (see [46]) for above Lie algebra being different from one for $sp(4)$ in [24] and in [29] for $k=2$.

3.2. Fock space realization of $\mathcal{A}'(Y(2), AdS_d)$

In this section, we will find on a base of constructed Verma module the realization of $\mathcal{A}'(Y(2), AdS_d)$ in formal power series in degrees of creation and annihilation operators $(B_a, B_a^+) = ((b_i, b_{ij}, d_{12}), (b_i^+, b_{ij}^+, d_{12}^+))$ in $\mathcal{H}'$ whose number coincides to ones of second-class constraints among o'_I, i.e. with $\dim(\mathcal{E}_2^- \oplus \mathcal{E}_2^+)$. It is solved following the results of [48] and algorithms suggested in [49] initially elaborated for a simple Lie algebra, then enlarged to a non-linear quadratic algebra $\mathcal{A}(Y(1), AdS_d)$ Ref. [10]. To this end, we make use of the mapping for an arbitrary basis vector of the Verma module and one $|\vec{n}_{ij}, \vec{n}_s\rangle$ of new Fock space $\mathcal{H}'$,

$$|\vec{n}_{ij}, \vec{n}_s\rangle_V \longleftrightarrow |\vec{n}_{ij}, \vec{n}_s\rangle = \prod_{i\le j}^2 (b_{ij}^+)^{n_{ij}} \prod_{l=1}^2 (b_l^+)^{n_l} (d_{12}^+)^{p_{12}}|0\rangle\,, \qquad \text{for } b_{ij}|0\rangle = b_l|0\rangle = d_{12}|0\rangle = 0, \quad \left(\left|\vec{N}(2)\right\rangle_{[\vec{N}=\vec{0}]} \equiv |0\rangle\right). \tag{70}$$

Here, vector $|\vec{n}_{ij}, \vec{n}_s\rangle$, for non-negative integers n_{ij}, n_l, p_{12} are the basis vectors of a Fock space $\mathcal{H}'$ generated by 6 pair of bosonic (B, B^+) operators, being the basis elements of the Heisenberg algebra A_6, with the standard (only non-vanishing) commutation relations

$$[B_a, B_b^+] = \delta_{ab} \Longleftrightarrow [b_{ij}, b_{lm}^+] = \delta_{il,jm},\ [b_i, b_m^+] = \delta_{im},\ [d_{12}, d_{12}^+] = 1. \tag{71}$$

Omitting the peculiarities of the correspondence among the Verma module vectors (especially one of $\left|\hat{A}_{\vec{N}(2)}\right\rangle_V$ for $\vec{N}(2) = (0, ..., 0, n_2)$ (69) and Fock space $\mathcal{H}'$ vector $\left|\vec{0}_{ij}, 0, 0, n_2\right\rangle$ (see [46] for details) we formulate our basic result as the

Theorem 2. The oscillator realization of the non-linear algebra $\mathcal{A}'(Y(2), AdS_d)$ over Heisenberg algebra A_6 exists in terms of formal power series in degrees of creation and annihilation operators, is given by the relations (72)–(75), (77), (78)–(83) and expressed with help of *primary block-operator* $\widehat{t}'_{12}$ (74), and *derived block-operators*, $\widehat{t}'^+_{12}, \widehat{l}'_0, \widehat{l}'_m, \widehat{l}'_{m2}, m=1,2$ (76), (84)–(88) as follows.

First, for the trivial negative root vectors, we have

$$l'^+_1 = m_1 b_1^+\,, \quad l'^+_{ij} = b_{ij}^+\,, \quad g'^i_0 = 2b_{ii}^+ b_{ii} + b_{12}^+ b_{12} + (-1)^i d_{12}^+ d_{12} + b_i^+ b_i + h^i. \tag{72}$$

Second, for the operator $t'_{12}(B, B^+)$ and *primary block-operator* $\widehat{t}'_{12}(B, B^+)$ we obtain,

$$t'_{12} = \left(h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}\right) d_{12} - b_{11}^+ b_{12} - 2b_{12}^+ b_{22} + \widehat{t}'_{12}\,, \tag{73}$$

$$\begin{aligned}
\widehat{t}'_{12} = {} & \sum_{k=0}\Bigg[\sum_{{}^1m=0}\sum_{{}^1l=0}\cdots\sum_{{}^km=0}\sum_{{}^kl=0}(-1)^k\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^k({}^im+{}^il)+k}\prod_{i=1}^k\frac{1}{(2^im+1)!}\frac{1}{(2^il+1)!} \\
& \times (b_{22}^+)^{\sum_{i=1}^k({}^im+{}^il)+k}\Bigg\{b_2^+ d_{12} b_2 - \frac{m_1}{m_2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^m\frac{(b_{22}^+)^m}{(2m+1)!}b_1^+ b_2^{2m+1} \\
& + \sum_{m=1}\left(\frac{-2r}{m_2^2}\right)^m (b_{22}^+)^{m-1}\Big[b_{12}^+\Big\{\frac{h^2-h^1+2d_{12}^+d_{12}}{(2m)!} + \frac{b_2^+b_2}{(2m+1)!}\Big\} \\
& - \frac{b_{11}^+ d_{12}^+}{(2m)!} - \frac{b_{22}^+}{(2m)!}(h^2-h^1+d_{12}^+d_{12})d_{12}\Big]b_2^{2m}
\end{aligned}$$

$$-\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}\frac{1}{(2m+1)!}(b_{22}^+)^{m+l}\Big[b_{12}^+\Big\{\frac{h^2-h^1+2d_{12}^+d_{12}+b_2^+b_2}{(2l+1)!}$$
$$-\frac{b_2^+b_2}{(2l+2)!}\Big\}-\frac{b_{11}^+d_{12}^+}{(2l+1)!}-\frac{b_{22}^+}{(2l+1)!}(h^2-h^1+b_2^+b_2+d_{12}^+d_{12})d_{12}$$
$$+\frac{m_1}{m_2}\frac{b_{22}^+}{(2l+2)!}b_1^+b_2\Big]b_2^{2(m+l+1)}\Bigg\}(b_2)^{2(\sum_{i=1}^k({}^im+{}^il)+k)}\Bigg]^9. \quad (74)$$

Third, for the operators $t_{12}^{\prime +}, l_2^{\prime +}$ and derived block-operator $\widehat{t}_{12}^{\prime +}$ we have,

$$t_{12}^{\prime +} = -2b_{12}^+b_{11}-b_{22}^+b_{12}+\widehat{t}_{12}^{\prime +}, \quad (75)$$

$$\widehat{t}_{12}^{\prime +} = \sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^m(b_{11}^+)^m\Big\{\frac{d_{12}^+}{(2m)!}-\frac{m_2}{m_1}\frac{b_2^+b_1}{(2m+1)!}\Big\}b_1^{2m}$$
$$-\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^m(b_{11}^+)^{m-1}\Bigg\{b_{12}^+\Bigg[\frac{(h^2-h^1+2d_{12}^+d_{12}+b_2^+b_2)}{(2m)!}-\frac{b_1^+b_1}{(2m+1)!}\Bigg]$$
$$+b_{22}^+\frac{(h^1-h^2-d_{12}^+d_{12}-b_2^+b_2)}{(2m)!}d_{12}+\frac{b_{22}^+}{(2m)!}\widehat{t}_{12}^{\prime}\Bigg\}b_1^{2m}, \quad (76)$$

$$l_2^{\prime +} = m_1\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}(b_{11}^+)^m\Bigg\{b_{12}^+\Bigg[\frac{(h^2-h^1+2d_{12}^+d_{12}+b_2^+b_2)}{(2m+1)!}-\frac{b_1^+b_1}{(2m+2)!}\Bigg]$$
$$+b_{22}^+\frac{(h^1-h^2-d_{12}^+d_{12}-b_2^+b_2)}{(2m+1)!}d_{12}-b_{11}^+\frac{d_{12}^+}{(2m+1)!}\Bigg\}b_1^{2m+1}$$
$$+m_2\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^m\frac{(b_{11}^+)^mb_2^+}{(2m)!}b_1^{2m}+m_1\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}\frac{(b_{11}^+)^mb_{22}^+}{(2m+1)!}\widehat{t}_{12}^{\prime}\, b_1^{2m+1}. \quad (77)$$

Fourth, for the Cartan-like vector $l_0^{\prime}$ the representation holds,

$$l_0^{\prime} = \widehat{l}_0^{\prime}+\frac{m_1}{2}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^{m+1}\frac{(b_{11}^+)^m}{(2m+1)!}\Bigg\{b_{12}^+\,\widehat{l}_2^{\prime}\;+\;b_{11}^+\,\widehat{l}_1^{\prime}$$
$$-\frac{m_2}{2}[\widehat{t}_{12}^{\prime}+(h^1-h^2-b^{2+}b_2-d_{12}^+d_{12})d_{12}]\;b_2^+\Bigg\}b_1^{2m+1}$$
$$-r\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^m(b_{11}^+)^mb_1^+\left\{\frac{2(h^1-d_{12}^+d_{12})-3}{(2m+1)!}+\frac{2b_1^+b_1}{(2m+2)!}\right\}b_1^{2m+1}$$
$$+\frac{1}{2}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^{m+1}\frac{(b_{11}^+)^m}{(2m+2)!}\Bigg\{b_{11}^+\,\widehat{l}_0^{\prime}\;-rb_{11}^+\Big([h^1-d_{12}^+d_{12}][h^1-d_{12}^+d_{12}-2]$$
$$-h^2-d_{12}^+d_{12}-b_2^+b_2+2d_{12}^+(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}\Big)$$
$$-2rb_{11}^+\widehat{t}_{12}^{\prime}\,d_{12}^++8rb_{11}^+b_{12}^+\widehat{l}_{12}^{\prime}\;+4r(b_{12}^+)^2\,\widehat{l}_{22}^{\prime}+2rb_{12}^+\Big[\,(h^1-h^2$$
$$-b_2^+b_2-d_{12}^+d_{12})d_{12}+\widehat{t}_{12}^{\prime}\Big]\;\Big[h^2+b_2^+b_2+d_{12}^+d_{12}-2\Big]\Bigg\}b_1^{2m+2}$$
$$+\frac{r}{2}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^{m+1}\frac{(b_{11}^+)^m}{(2m+2)!}b_{22}^+\;\Bigg\{\Big[(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}+\widehat{t}^{\prime}{}_{12}\Big]$$

[9] one should be noted that in (74) for $k=0$ there are no doubled sums and the products $\prod_{i=1}^0\ldots$ is equal to 1 and the only term $b_2^+d_{12}b_2$ survive

$$\times(h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12})d_{12} + t'_{12}\widehat{t}'_{12}\Big\}\, b_1^{2m+2}. \tag{78}$$

Fifth, the operators l'_m, l'_{lm}, $l, m = 1, 2, l \le m$, read as follows,

$$\begin{aligned}
l'_1 \;=\;& -m_1 b_1^+ b_{11} - \frac{1}{2} l'^{2+} b_{12} + \sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^m \frac{(b_{11}^+)^m}{(2m)!} \widehat{l}'^1 b_1^{2m} \\
& + \frac{1}{m_1}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^m \frac{(b_{11}^+)^m}{(2m+1)!}\Big\{\widehat{l}'_0 - r\Big[(h^1 - d_{12}^+ d_{12})(h^1 - d_{12}^+ d_{12} - 2) - h^2 \\
& - d_{12}^+ d_{12} - b_2^+ b_2\Big] - r d_{12}^+\Big[h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}\Big] d_{12} - r\ \widehat{t}'_{12}\ d_{12}^+\Big\} b_1^{2m+1} \\
& - 4m_1 \sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1} \frac{(b_{11}^+)^m b_{12}^+}{(2m+1)!}(4^m - \tfrac{1}{2})\ \widehat{l}'_{12}\ b_1^{2m+1} \\
& + \frac{m_1}{2}\sum_{m=1}\left(\frac{-8r}{m_1^2}\right)^m (b_{11}^+)^{m-1} b_1^+ \left\{\frac{1}{2}\frac{(2(h^1 - d_{12}^+ d_{12}) - 1)}{(2m)!} + \frac{b_1^+ b_1}{(2m+1)!}\right\} b_1^{2m} \\
& + \frac{1}{2}\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^m (4^m - 1)\frac{(b_{11}^+)^{m-1}}{(2m)!}\Big\{2 b_{12}^+ \widehat{l}'^2 - m_1 b_1^+ \\
& - m_2(\widehat{t}'_{12} + h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} b_2^+\Big\} b_1^{2m} \\
& + \frac{m_1}{2}\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^{m+1} (4^m - 1)\frac{(b_{11}^+)^{m-1}}{(2m+1)!}\Big\{b_{11}^+\ \widehat{t}'_{12}\ d_{12}^+ - 4(b_{12}^+)^2 \widehat{l}'_{22} \\
& \qquad - 2b_{12}^+\Big[(h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} + \widehat{t}'_{12}\Big](h^2 + b_2^+ b_2 + d_{12}^+ d_{12} - 2) \\
& \qquad + b_{11}^+ d_{12}^+ (h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12}\Big\} b_1^{2m+1} \\
& - \frac{m_1}{2}\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^{m+1} (4^m - 1)\frac{(b_{11}^+)^{m-1}}{(2m+1)!} b_{22}^+ \Big\{\Big[(h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} \\
& \qquad + \widehat{t}'_{12}\Big](h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} + t'_{12}\widehat{t}'_{12}\Big\} b_1^{2m+1},
\end{aligned} \tag{79}$$

$$\begin{aligned}
l'_2 \;=\;& \sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^m \frac{(b_{11}^+)^m}{(2m+2)!}\widehat{l}'_2\ (b_1^+)^2 b_1^{2m+2} - 2m_1 \sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1} \frac{(b_{11}^+)^m}{(2m+1)!}\sum_k b_{1k}^+ \widehat{l}'_{k2} b_1^{2m+1} \\
& - 2m_1 \sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1} \frac{(b_{11}^+)^m}{4}\widehat{t}'_{12}\left\{\frac{(h^1 + h^2 + b_2^+ b_2 - 2)}{(2m+1)!} + \frac{b_1^+ b_1}{(2m+2)!}\right\} b_1^{2m+1} \\
& - \frac{m_1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1} (b_{11}^+)^m \left\{\frac{(h^1 + h^2 + b_2^+ b_2 - 2)}{(2m+1)!} + \frac{b_1^+ b_1}{(2m+2)!}\right\} \times \\
& \quad \times (h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} b_1^{2m+1} - \frac{m_1}{2} b_1^+ b_{12} - l_2'^+\ b_{22},
\end{aligned} \tag{80}$$

$$\begin{aligned}
l'_{11} \;=\;& (b_{11}^+ b_{11} + b_{12}^+ b_{12} + b_1^+ b_1 - d_{12}^+ d_{12} + h^1) b_{11} + \frac{b_{22}^+}{4} b_{12}^2 - \frac{1}{2}\widehat{t}'^+_{12} b_{12} \\
& + \frac{1}{2}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^{m+1} \frac{(b_{11}^+)^m}{(2m+2)!}\Big\{\frac{1}{4r}\widehat{l}'_0\ + b_{12}^+\ \widehat{l}'_{12} - \frac{1}{4}\Big[d_{12}^+(2 - h^1 - h^2 - b_2^+ b_2) d_{12}
\end{aligned}$$

$$
\begin{aligned}
& +h^1(h^1-2)-h^2-b_2^+b_2+\widehat{t}'_{12}d_{12}^+\Big]\Big\}b_1^{2m+2} \\
& -\frac{1}{2}\sum_{m=1}\left(\frac{-8r}{m_1^2}\right)^m (b_{11}^+)^{m-1}b_1^+\left\{\frac{h^1-d_{12}^+d_{12}-\frac{1}{2}}{(2m+1)!}+\frac{b_1^+b_1}{(2m+2)!}\right\}b_1^{2m+1} \\
& -\frac{1}{m_1}\sum_{m=0}\left\{\left(\frac{-8r}{m_1^2}\right)^m\frac{(b_{11}^+)^m}{(2m+1)!}\,\widehat{l}'_1-\left(\frac{-2r}{m_1^2}\right)^m(4^m-1)\frac{(b_{11}^+)^{m-1}b_{12}^+}{(2m+1)!}\,\widehat{l}'_2\right\}b_1^{2m+1} \\
& +\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^m(4^m-1)(b_{11}^+)^{m-1}\frac{b_1^+b_1}{(2m+1)!}b_1^{2m} \\
& +\frac{m_2}{2m_1}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^m(4^m-1)\frac{(b_{11}^+)^{m-1}}{(2m+1)!}\,\Big\{t'_{12}+\sum_l l b_{1l}^+b_{l2}\Big\}b_2^+b_1^{2m+1} \\
& -\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)\frac{(b_{11}^+)^{m-1}}{(2m+2)!}\Big[b_{11}^+d_{12}^+-2b_{12}^+(h^2+b_2^+b_2+d_{12}^+d_{12}-1)\Big]\times \\
& \quad\times\Big(h^1-h^2-d_{12}^+d_{12}-b_2^+b_2\Big)d_{12}b_1^{2m+2} \\
& -\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)\frac{(b_{11}^+)^{m-1}}{(2m+2)!}\Big\{\,b_{11}^+\,\widehat{t}'_{12}\,d_{12}^+-4b_{11}^+b_{12}^+\,\widehat{l}'_{12} \\
& \quad-4(b_{12}^+)^2\,\widehat{l}'_{22}\,-2b_{12}^+\,\widehat{t}'_{12}\,(h^2+b_2^+b_2+d_{12}^+d_{12}-2)\Big\}b_1^{2m+2} \\
& +\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)\frac{(b_{11}^+)^{m-1}}{(2m+2)!}b_{22}^+\,\Big\{\Big[(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}+\widehat{t}'_{12}\Big]\times \\
& \quad\times(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}+t'_{12}\widehat{t}'_{12}\Big\}\,b_1^{2m+2}, \qquad (81)
\end{aligned}
$$

$$
\begin{aligned}
l'_{12} &= \frac{1}{4}\Big(b_{12}^+b_{12}+\sum_m(2b_{mm}^+b_{mm}+b_m^+b_m+h^m)\Big)b_{12}+b_{12}^+b_{11}b_{22}-\frac{1}{2}\left(\widehat{t}'^+_{12}\,b_{22}+\widehat{t}'_{12}\,b_{11}\right) \\
&\quad+\frac{1}{2}(h^2-h^1+b_2^+b_2+d_{12}^+d_{12})b_{11}d_{12}-\frac{1}{2m_1}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^m\frac{(b_{11}^+)^m}{(2m+1)!}\widehat{l}'_2b_1^{2m+1} \\
&\quad+\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^m\frac{(b_{11}^+)^m}{(2m)!}\widehat{l}'_{12}b_1^{2m}+\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^m\frac{(b_{11}^+)^{m-1}b_{12}^+}{(2m)!}\,\widehat{l}'_{22}b_1^{2m} \\
&\quad+\frac{1}{4}\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^m(b_{11}^+)^{m-1}\Big\{\widehat{t}'_{12}\,\left[\frac{(h^1+h^2+b_2^+b_2-2)}{(2m)!}+\frac{b_1^+b_1}{(2m+1)!}\right] \\
&\quad+\left[\frac{h^1+h^2+b_2^+b_2-2}{(2m)!}+\frac{b_1^+b_1}{(2m+1)!}\right](h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}\Big\}b_1^{2m}, \qquad (82)
\end{aligned}
$$

$$
\begin{aligned}
l'_{22} &= \widehat{l}'_{22}\,+(b_{12}^+b_{12}+d_{12}^+d_{12}+b_2^+b_2+b_{22}^+b_{22}+h^2)b_{22}+\frac{1}{4}b_{11}^+b_{12}^2-\frac{1}{2}\,\widehat{t}'_{12}\,b_{12} \\
&\quad+\frac{1}{2}(h^2-h^1+d_{12}^+d_{12}+b_2^+b_2)d_{12}b_{12}. \qquad (83)
\end{aligned}
$$

In the Eqs.(78)–(83), the *derived block-operators* $\widehat{l}'_0$, $\widehat{l}'_m$, $\widehat{l}'_{m2}$, for $m=1,2$ are written as follows,[10]

$$
\widehat{l}'_0 \;=\; m_0^2-r\sum_{m=0}\left(\frac{-8r}{m_2^2}\right)^m(b_{22}^+)^m\Big\{b_2^+\left[\frac{1}{(2m+1)!}(2h^2+2d_{12}^+d_{12}-1)\right.
$$

[10] they are directly determined through theirs action on vector $\left|\vec{0}_{ij},0,p_{12},n_2\right\rangle_V$, see [46] for details

$$
\begin{aligned}
&+\frac{2b_2^+b_2}{(2m+2)!}\Bigg]\, b_2 - 2\frac{m_1}{m_2}\frac{b_1^+d_{12}^+}{(2m+1)!}b_2\Bigg\}b_2^{2m}\\
&+\frac{1}{2}\sum_{m=0}\left(\frac{-8r}{m_2^2}\right)^{m+1}\frac{(b_{22}^+)^m}{(2m+2)!}\Big\{b_{22}^+\Big(m_0^2 - r[h^2(h^2-4)+h^1-(d_{12}^+)^2d_{12}^2\\
&\quad +2d_{12}^+d_{12}(h^1-2)]\Big) + 2rb_{12}^+d_{12}^+(h^1-d_{12}^+d_{12}-2) + rb_{11}^+(d_{12}^+)^2\Big\}b_2^{2m+2}\\
&+2r\sum_{m=0}\sum_{l=0}\left(\frac{-8r}{m_2^2}\right)^{m}\left(\frac{-2r}{m_2^2}\right)^{l+1}\frac{(b_{22}^+)^m}{(2m+1)!}\frac{(b_{22}^+)^{l+1}}{(2l+1)!}\,\widehat{t}'_{12}\, d_{12}^+b_2^{2(m+l+1)}\\
&+2r\sum_{m=0}\sum_{l=0}\left(\frac{-8r}{m_2^2}\right)^{m}\left(\frac{-2r}{m_2^2}\right)^{l+1}\frac{(b_{22}^+)^{m+l}}{(2m+1)!}\Bigg\{b_{12}^+d_{12}^+\Bigg[\frac{[h^2-h^1+2d_{12}^+d_{12}+b_2^+b_2+2]}{(2l+1)!}\\
&-\frac{b_2^+b_2}{(2l+2)!}\Bigg] - \frac{b_{11}^+(d_{12}^+)^2}{(2l+1)!} + \frac{b_{22}^+(d_{12}^+d_{12}+1)}{(2l+1)!}[h^1-h^2-b_2^+b_2-d_{12}^+d_{12}]\\
&+\frac{m_1}{m_2}\frac{b_{22}^+b_1^+d_{12}^+}{(2l+2)!}b_2\Bigg\}b_2^{2(m+l+1)},
\end{aligned}
\tag{84}
$$

$$
\widehat{l}'_1 = -\frac{m_2}{2}\sum_{m=1}\left(\frac{-2r}{m_2^2}\right)^{m+1}(b_{22}^+)^m d_{12}^+\left\{\frac{(h^1+h^2-2)}{(2m+1)!}+\frac{b_2^+b_2}{(2m+2)!}\right\}b_2^{2m+1}, \tag{85}
$$

$$
\begin{aligned}
\widehat{l}'_2 =& -\widehat{l}'_1 d_{12} - \sum_{m=0}\left(\frac{-8r}{m_2^2}\right)^m\frac{(b_{22}^+)^m}{(2m+1)!}\frac{[m_0^2-rh^2(h^2-3)]}{m_2}b_2^{2m+1}\\
&+\frac{m_2}{4}\sum_{m=1}\left(\frac{-8r}{m_2^2}\right)^m(b_{22}^+)^{m-1}b_2^+\left\{\frac{(2h^2-1)}{(2m)!}+2\frac{b_2^+b_2}{(2m+1)!}\right\}b_2^{2m}\\
&+\frac{m_2}{2}\sum_{m=1}\left(\frac{-2r}{m_2^2}\right)^{m+1}(b_{22}^+)^m d_{12}^+\left\{\frac{(h^1+h^2-2)}{(2m+1)!}+\frac{b_2^+b_2}{(2m+2)!}\right\}d_{12}b_2^{2m+1}\\
&-\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^m(4^m-1)(b_{22}^+)^{m-1}d_{12}^+\left\{\frac{m_1b_1^+}{(2m)!}-\frac{m_2b_2^+d_{12}}{(2m)!}\right\}b_2^{2m}\\
&+\frac{m_2}{2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^{m+1}(4^m-1)\frac{(b_{22}^+)^{m-1}}{(2m+1)!}\Big\{b_{22}^+\Big[2(h^1-2)d_{12}^+d_{12}\\
&\quad +h^1-h^2-(d_{12}^+)^2d_{12}^2\Big] - 2b_{12}^+d_{12}^+(h^1-d_{12}^+d_{12}-2) - b_{11}^+(d_{12}^+)^2\Big\}b_2^{2m+1}\\
&-\frac{m_2}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^{m+l-1}}{(2m)!}\Bigg\{\frac{m_1}{m_2}\frac{b_{22}^+b_1^+d_{12}^+}{(2l+2)!}b_2\\
&\quad +b_{12}^+d_{12}^+\left[\frac{(h^2-h^1+2d_{12}^+d_{12}+b_2^+b_2+2}{(2l+1)!}-\frac{b_2^+b_2}{(2l+2)!}\right]\\
&\quad -\frac{\Big(b_{11}^+(d_{12}^+)^2+b_{22}^+[h^2-h^1+b_2^+b_2]\Big)}{(2l+1)!}\Bigg\}b_2^{2(m+l)+1}\\
&+\frac{m_2}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^m}{(2m)!}\frac{(b_{22}^+)^l}{(2l+1)!}\times\\
&\quad \times d_{12}^+\left\{h^2-h^1+b_2^+b_2+d_{12}^+d_{12}+2\right\}d_{12}b_2^{2(m+l)+1}\\
&-\frac{m_2}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^m}{(2m)!}\frac{(b_{22}^+)^l}{(2l+1)!}\,\widehat{t}'_{12}\, d_{12}^+b_2^{2(m+l)+1},
\end{aligned}
\tag{86}
$$

$$\begin{aligned}
\widehat{l}'_{12} &= \frac{1}{4}\sum_{m=1}\left(\frac{-2r}{m_2^2}\right)^m (b_{22}^+)^{m-1}d_{12}^+\left\{\frac{(h^1+h^2-2)}{(2m)!}+\frac{b_2^+b_2}{(2m+1)!}\right\}b_2^{2m}, && (87)\\
\widehat{l}'_{22} &= -2\widehat{l}'_{12}d_{12}-\sum_{m=0}\left(\frac{-8r}{m_2^2}\right)^m\frac{(b_{22}^+)^m}{m_2^2(2m+2)!}\{m_0^2-rh^2(h^2-3)\}b_2^{2(m+1)}\\
&-\frac{1}{2}\sum_{m=1}\left(\frac{-8r}{m_2^2}\right)^m(b_{22}^+)^{m-1}b_2^+\left\{\frac{1}{2}\frac{(2h^2-1)}{(2m+1)!}+\frac{b_2^+b_2}{(2m+2)!}\right\}b_2^{2m+1}\\
&+\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^{m+1}(4^m-1)(b_{22}^+)^{m-1}\left\{\frac{b_{12}^+d_{12}^+}{(2m+2)!}[2(h^1-2)-2d_{12}^+d_{12}]b_2\right.\\
&+\frac{b_{22}^+}{(2m+2)!}\left[h^2-h^1-2d_{12}^+d_{12}(h^1-2)+(d_{12}^+)^2d_{12}^2\right]b_2+\frac{b_{11}^+(d_{12}^+)^2b_2}{(2m+2)!}\\
&\left.-\frac{b_2^+d_{12}^+d_{12}}{(2m+1)!}\right\}b_2^{2m+1}+\frac{m_1}{2m_2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^{m+1}\frac{(b_{22}^+)^{m-1}b_1^+d_{12}^+}{(2m+1)!}(4^m-1)b_2^{2m+1}\\
&+\frac{1}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^{m+l-1}}{(2m+1)!}\left\{\frac{m_1}{m_2}\frac{b_{22}^+b_1^+d_{12}^+}{(2l+2)!}b_2-\frac{b_{11}^+(d_{12}^+)^2}{(2l+1)!}\right.\\
&+\frac{b_{22}^+}{(2l+1)!}(d_{12}^+d_{12}+1)(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})\\
&\left.+b_{12}^+d_{12}^+\left[\frac{(h^2-h^1+2d_{12}^+d_{12}+b_2^+b_2+2)}{(2l+1)!}-\frac{b_2^+b_2}{(2l+2)!}\right]\right\}b_2^{2(m+l+1)}\\
&+\frac{1}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^{m+l}}{(2m+1)!(2l+1)!}\;\widehat{t}'_{12}\;d_{12}^+b_2^{2(m+l+1)}. && (88)
\end{aligned}$$

Note, the additional parts $o'_I(B,B^+)$ as the formal power series in the oscillators (B,B^+) do not obey the usual properties,

$$(l'_{lm})^+\neq l'^+_{lm}\,,\qquad (t'_{12})^+\neq t'^+_{12},\qquad (l'_0)^+\neq l'_0\,,\qquad (l'_m)^+\neq l'^+_m\,,\ l\leq m, \tag{89}$$

if one should use the standard rules of Hermitian conjugation for the new creation and annihilation operators, $(B_a)^+ = B_a$. Restoration the proper Hermitian conjugation properties for o'_I, is achieved by changing the scalar product in $\mathcal{H}'$ as follows,

$$\langle\Phi_1|\Phi_2\rangle_{\mathrm{new}} = \langle\Phi_1|K'|\Phi_2\rangle\,, \tag{90}$$

for any vectors $|\Phi_1\rangle, |\Phi_2\rangle$ with some non-degenerate operator K'. This operator is determined by the condition that all the operators of the algebra must have the proper Hermitian properties with respect to the new scalar product,

$$\langle\Phi_1|K'E^{-\prime\alpha}|\Phi_2\rangle = \langle\Phi_2|K'E^{\prime\alpha}|\Phi_1\rangle^*,\ \langle\Phi_1|K'G'|\Phi_2\rangle = \langle\Phi_2|K'G'|\Phi_1\rangle^*, \tag{91}$$

for $(E'^{\alpha}; E^{-\prime\alpha}) = (l'_{lm}, t'_{12}, l'_m; l'^+_{lm}, t'^+_{12}, l'^+_m)$, $G' = (g'^i_0, l'_0)$. The relations (91) lead to definition the operator K', Hermitian with respect to the standard scalar product $\langle\,|\,\rangle$, in the form

$$K' = Z^+Z,\qquad Z = \sum_{(\vec{n}_{lm},\vec{n}_s)=(\vec{0},\vec{0})}^{\infty}\Big|\vec{N}(2)\rangle_V\,\frac{1}{(\vec{n}_{lm})!(\vec{n}_s)!}\langle 0|\prod_{r=1}^{2}b_r^{n_r}d_{12}^{p_{12}}\prod_{l,m\geq l}b_{lm}^{n_{lm}}, \tag{92}$$

where $(\vec{n}_{lm})! = n_{11}!n_{12}!n_{22}!$, $(\vec{n}_s)! = n_1!n_2!p_{12}!$ and the normalization ${}_V\langle 0|0\rangle_V = 1$ is supposed.

Table 3. The non-linear part of the converted algebra $\mathcal{A}_c(Y(2), AdS_d)$.

$[\downarrow,\rightarrow]$	L_0	L^i	L^{i+}
L_0	0	$-r\hat{\mathcal{K}}_1^{i+}$	$r\hat{\mathcal{K}}_1^{i}$
L^j	$r\hat{\mathcal{K}}_1^{j+}$	$\hat{W}_b^{ji}$	$\hat{X}_b^{ji}$
L^{j+}	$-r\hat{\mathcal{K}}_1^{j}$	$-\hat{X}_b^{ij}$	$-\hat{W}_b^{ji+}$

The theorem 2 has the same consequences as ones from theorem 1 which concern, first, the flat limit of the algebra $\mathcal{A}'(Y(2), AdS_d)$ and therefore a new representation for Lie algebra $(T'^2 \oplus T'^{2*} \oplus l'_0)\ (0)\oplus\!\!\!\!\supset\ sp(4)$. Second, modulo the oscillator pairs $b_{m2}, b^+_{m2}, b_2, b^+_2, d_{12}, d^+_{12}, m = 1, 2$, the obtained representation coincides with one for totally symmetric HS fields quadratic algebra $\mathcal{A}'(Y(1), AdS_d)$ in [10].

The set of the equations which compose the results of the Theorems 1, 2 represents the general solution of mentioned in the Introduction the second problem on the LF construction for mixed-symmetry HS tensors on AdS spaces with given mass and spin $\mathbf{s} = (s_1, s_2)$.

4. Construction of Lagrangian Actions

To construct Lagrangian formulation for the HS tensor field of fixed generalized spin $\mathbf{s} = (s_1, s_2)$ we should, initially, determine explicitly the composition law for the deformed algebra $\mathcal{A}_c(Y(2), AdS_d)$ (for arbitrary $k \geq 2$ see [46]), find BRST operator for the non-linear algebra $\mathcal{A}_c(Y(2), AdS_d)$ and, finally, reproduce properly gauge-invariant Lagrangian formulation for the basic bosonic field $\Phi_{(\mu)_{s_1},(\nu)_{s_2}}$.

4.1. Explicit form for the algebra $\mathcal{A}_c(Y(2), AdS_d)$

To this end as in the case of the algebra $\mathcal{A}'(Y(2), AdS_d)$ of o'_I the only multiplication law for quadratic part of the initial algebra $\mathcal{A}(Y(k), AdS_d)$ for k=2 is changed, while its linear part is given by the same $sp(2k)$ algebra as for the maximal Lie subalgebra for $\mathcal{A}(Y(2), AdS_d)$ and $\mathcal{A}'(Y(2), AdS_d)$ with the same form of the commutators $[O_a, O_I]$ for $O_a \in sp(4)$. From the Eqs. (33), (34) and Table 1 the non-linear part of the algebra $\mathcal{A}_c(Y(2), AdS_d)$ can be restored in the form of the Table 3, Here the functions $\hat{\mathcal{K}}_1^i$, $\hat{W}_b^{ji}$, $\hat{X}_b^{ij}$ (therefore its Hermitian conjugated quantities $\hat{\mathcal{K}}_1^{i+}$, $\hat{W}_b^{ji+}$) are given as follows:

$$
\begin{aligned}
\hat{W}_b^{ij} &= 2r\epsilon^{ij}\Big\{\sum_l(-1)^l[G_0^l - g_0'^l]L^{12} - l'^{12}\sum_l(-1)^l G_0^l - [(T^{12} - t'^{12})L^{11} - l'^{11}T^{12}] \\
&\qquad +(T^+_{12} - t'^+_{12})L^{22} - l'^{22}T^+_{12}\Big\}, \qquad (93)\\
\hat{\mathcal{K}}_1^j &= 4\sum\nolimits_{i=1}^{2}\Big\{(L^{ji+} - l'^{ji+})L^i - l'^iL^{ji+}\Big\} + 2(L^{j+} - l'^{j+})G_0^j - 2(g_0'^j + \tfrac{1}{2})L^{j+} \\
&\qquad -2\Big\{[(L^{1+} - l'^{1+})T^+_{12} - t'^+_{12}L^{1+}]\delta^{j2}\delta^{i1} + [(L^{2+} - l'^{2+})T^{12} - t'^{12}L^{2+}]\delta^{j1}\delta^{i2}\Big\}, \qquad (94)\\
\hat{X}_b^{ij} &= \Big\{L_0 + r\Big[\hat{K}_0^{0i} - 2g_0'^iG_0^i + 4l'^{ii+}L^{ii} + 4l'^{ii}L^{ii+} + \Big(\hat{\mathcal{K}}_0^{12} - t'^+_{12}T^{12} - t'^{12}T^+_{12} \qquad (95)\\
&\qquad +4l'^{12+}L^{12} + 4l'^{12}L^{12+}\Big)\Big]\Big\}\delta^{ij} \\
&\quad -r\Big\{4\sum\nolimits_l\Big[(L^{jl+} - l'^{jl+})L^{li} - l'^{li}L^{jl+}\Big] + \Big(\sum\nolimits_l(G_0^{lj} - g_0'^l) - 2\Big)T^{12} - t'^{12}\sum\nolimits_l G_0^l\Big\}\delta^{i2}\delta^{j1} \\
&\quad -r\Big\{4\sum\nolimits_l\Big[(L^{jl+} - l'^{jl+})L^{li} - l'^{li}L^{jl+}\Big] + \Big(T^+_{ij} - t'^+_{ij}\Big)\sum_l G_0^l - \Big(\sum_l g_0'^l + 2\Big)T^+_{12}\Big\}\delta^{i1}\delta^{j2}.
\end{aligned}
$$

The quantities $\hat{K}_0^{0i}$, $\hat{\mathcal{K}}_0^{12}$ above are the same as ones in (26) for $k = 2$, but expressed in terms of O_I.

Following to our experience from study of the (super)algebra $\mathcal{A}_c(Y(1), AdS_d)$ in [10, 12, 13] when in order to find exact BRST operator, we choose the Weyl (symmetric) ordering for quadratic combinations of O_I in the r.h.s. of the Eqs. (93)–(95) as follows, $O_I O_J = \frac{1}{2}(O_I O_J + O_J O_I) + \frac{1}{2}[O_I\,, O_J]$. As the result, the Table 3 for such kind of ordering must contain the quantities $\hat{W}^{ij}_{bW}$, $\hat{\mathcal{K}}^{j}_{1W}$, $\hat{X}^{ij}_{bW}$ (and $\hat{W}^{ij+}_{bW}$, $\hat{\mathcal{K}}^{j+}_{1W}$) which read as

$$\hat{W}^{ij}_{bW} = r\epsilon^{ij}\left\{\sum_l (-1)^l \mathcal{G}_0^l L^{12} + \mathcal{L}^{12}\sum_l(-1)^l G_0^l - \mathcal{T}^{12}L^{11} - \mathcal{L}^{11}T^{12} + \mathcal{T}^{12+}L^{22} + \mathcal{L}^{22}T^{12+}\right\} \quad (96)$$

$$\hat{\mathcal{K}}^{j}_{1W} = \Big\{2\sum_l[\mathcal{L}^{jl+}L^l + \mathcal{L}^l L^{jl+}] + \mathcal{L}^{j+}G_0^j + \mathcal{G}_0^j L^{j+} - \{\mathcal{L}^{1+}T^+_{12} + \mathcal{T}^+_{12}L^{1+}\}\delta^{j2}$$
$$-[\mathcal{L}^{2+}T^{12} + \mathcal{T}^{12}L^{2+}]\delta^{j1}\Big\}, \quad (97)$$

$$\hat{X}^{ij}_{bW} = \Big\{L_0 + r\Big[\mathcal{G}_0^i G_0^i - 2\mathcal{L}^+_{ii}L^{ii} - 2\mathcal{L}^{ii}L^+_{ii} + \frac{1}{2}(\mathcal{T}^{12}T^+_{12} + \mathcal{T}^+_{12}T^{12} - 4\mathcal{L}^+_{12}L^{12} - 4\mathcal{L}^{12}L^+_{12})\Big]\Big\}\delta^{ij}$$
$$-r\Big\{2\textstyle\sum_l^2(\mathcal{L}^{1l+}L^{l2} + \mathcal{L}^{l2}L^{1l+}) + \displaystyle\frac{1}{2}\sum_l^2 \mathcal{G}_0^l T^{12} + \frac{1}{2}\mathcal{T}^{12}\sum_l^2 G_0^l\Big\}\delta^{i2}\delta^{j1}$$
$$-r\Big\{2\textstyle\sum_l^2(\mathcal{L}^{l2+}L^{1l} + \mathcal{L}^{1l}L^{l2+}) + \displaystyle\frac{1}{2}\mathcal{T}^+_{12}\sum_l^2 G_0^l + \frac{1}{2}\sum_l^2 \mathcal{G}_0^l T^+_{12}\Big\}\delta^{i1}\delta^{j2}, \quad (98)$$

with the notation $\mathcal{O}_I$ for the quantity $\mathcal{O}_I = (O_I - 2o'_I)$. Note, the ordering quantities (96)–(98) do not contain linear terms (except for L_0 in r.h.s of the last relation) as compared to (93)–(95).

Thus, we derive the algebra of converted operators O_I underlying HS field subject to an arbitrary unitary irreducible AdS group representation in AdS space with spin $\mathbf{s} = (s_1, s_2)$ so that the problem now to find BRST operator for $\mathcal{A}_c(Y(2), AdS_d)$.

4.2. BRST-operator for converted algebra $\mathcal{A}_c(Y(2), AdS_d)$

The non-linear algebra now has not the form of closed algebra because of the operatorial functions $F^{(2)K}_{IJ}(o', O)$ in the Eq. (34) and as it was shown in [31] it leads to appearance of higher order structural functions due to the quadratic algebraic relations (96)–(98) and their Hermitian conjugates corresponding to those quantities $F^{(2)K}_{IJ}(o', O)$ for $i, j = 1, 2$. In ref.[31] it was found new structure functions $F^{RS}_{IJK}(O)$ of the 3rd order in terminology of Ref.[17], implied by a resolution of the Jacobi identities $[[O_I, O_J], O_K] + cycl.perm.(I, J, K) = 0$, as follows (32),

$$\Big\{F^M_{IJ}F^P_{MK} + [F^{(2)P}_{IJ}, O_K] + cycl.perm.(I, J, K)\Big\} = F^{RS}_{IJK}\Big(O_R\delta^P_S - \tfrac{1}{2}F^P_{RS})\Big), \quad (99)$$

for $F^M_{IJ} \equiv (f^M_{IJ} + F^{(2)M}_{IJ})$. The structure functions $F^{RS}_{IJK}(o', O)$ are antisymmetric with respect to a permutation of any two of lower indices (I, J, K) and upper ones R, S and exist because of nontrivial Jacobi identities for the $k(2k - 1) = 6$ triples (L_i, L_j, L_0), (L^+_i, L^+_j, L_0), (L_i, L^+_j, L_0).

The construction of a BFV-BRST operator Q' for $\mathcal{A}_c(Y(2), AdS_d)$ are considered in [31] and has the general form

$$Q' = \mathcal{C}^I[O_I + \tfrac{1}{2}\mathcal{C}^J(f^P_{JI} + F^{(2)P}_{JI})\mathcal{P}_P + \tfrac{1}{12}\mathcal{C}^J\mathcal{C}^K F^{RP}_{KJI}\mathcal{P}_R\mathcal{P}_P]. \quad (100)$$

for the $(\mathcal{CP})$-ordering for the ghost coordinates $\mathcal{C}^I = \{\eta_0,\, \eta^i_G\, \eta^+_i,\, \eta_i,\, \eta^+_{ij},\, \eta_{ij},\, \vartheta_{12},\, \vartheta^+_{12}\}$, and their conjugated momenta $\mathcal{P}_I = \{\mathcal{P}_0, \mathcal{P}^i_G, \mathcal{P}_i, \mathcal{P}^+_i, \mathcal{P}_{ij}, \mathcal{P}^+_{ij}, \lambda^+_{12}, \lambda_{12}\}$[11]. Explicitly, Q' given as,

$$\begin{aligned} Q' = {} & Q'_1 + Q'_2 + r^2 \Big\{ \eta_0 \sum_{i,j} \eta_i\eta_j\varepsilon^{ij}\Big[\tfrac{1}{2}\sum_m \Big(G^m_0[\lambda_{12}\mathcal{P}^+_{22} - \lambda^+_{12}\mathcal{P}^+_{11} + i\mathcal{P}^+_{12}\sum_l(-1)^l\mathcal{P}^l_G] \\ & -i(L^+_{11}\lambda^+_{12} - L^+_{22}\lambda_{12})\mathcal{P}^m_G + 4L^{mm}\mathcal{P}^+_{m2}\mathcal{P}^+_{1m}\Big) - L^+_{12}\mathcal{P}^1_G\mathcal{P}^2_G + 2L^{12}\mathcal{P}^+_{22}\mathcal{P}^+_{11}\Big] \\ & +\eta_0\sum_{i,j}\eta^+_i\eta_j\Big[\sum_m\Big((-1)^m\tfrac{i}{2}G^m_0\sum_l\mathcal{P}^l_G + 2(L^+_{22}\mathcal{P}^{22} - L^{11}\mathcal{P}^+_{11})\Big)\lambda_{12}\delta^{1j}\delta^{2i} \\ & +\varepsilon^{\{1j}\delta^{2\}i}\Big(\imath\sum_m\Big[\tfrac{1}{2}T^{12}\lambda^+_{12} - 2L^{12}\mathcal{P}^+_{12}(-1)^m\Big]\mathcal{P}^m_G + 2\Big[L^{12}\lambda_{12} - T^{12}\mathcal{P}^{12}\Big]\mathcal{P}^+_{22} \\ & +2\Big[T^+_{12}\mathcal{P}^{12} - L^{12}\lambda^+_{12}\Big]\mathcal{P}^+_{11}\Big) - T^{12}\Big[\mathcal{P}^1_G\mathcal{P}^2_G\delta^{2i}\delta^{1j} + 2\mathcal{P}^{11}\mathcal{P}^+_{22}\delta^{1i}\delta^{2j}\Big] \\ & -2\sum_m(-1)^m\Big[(G^m_0\mathcal{P}^{11} + \imath L^{11}\mathcal{P}^m_G)\delta^{1i}\delta^{2j} - (G^m_0\mathcal{P}^{22} + \imath L^{22}\mathcal{P}^m_G)\delta^{2i}\delta^{1j}\Big]\mathcal{P}^+_{12}\Big] + h.c.\Big\}, \end{aligned} \quad (101)$$

with the standard form for linear Q'_1 and quadratic Q'_2 terms in ghosts $\mathcal{C}^I$ (see [31] for details). The Hermiticity of the nilpotent operator Q' in total Hilbert space $\mathcal{H}_{tot} = \mathcal{H}\bigotimes\mathcal{H}'\bigotimes\mathcal{H}_{gh}$ is defined by the rule,

$$Q'^+K = KQ', \quad \texttt{for}\ K = \hat{1}\otimes K'\otimes\hat{1}_{gh}, \quad (102)$$

with the operator K' given in (92).

4.3. Lagrangian formulation

Properly the construction of Lagrangians for bosonic HS fields in AdS$_d$ space, can be developed by partially following the algorithm of [11], [12], (see, as well [29]), which is a particular case of our construction, corresponding to $s_2 = 0$. As a first step, we extract the dependence of the BRST operator Q' (101) on the ghosts $\eta^i_G, \mathcal{P}^i_G$,

$$Q' = Q + \eta^i_G(\sigma^i + h^i) + \mathcal{B}^i\mathcal{P}^i_G \quad (103)$$

with some inessential in later operators $\mathcal{B}^i$ and the BRST operator Q which corresponds only to the converted first-class constraints $\{O_I\}\setminus\{G^i_0\}$,

$$\begin{aligned} Q = {} & \tfrac{1}{2}\eta_0L_0 + \sum_i\eta^+_iL^i + \sum_{l\le m}\eta^+_{lm}L^{lm} + \vartheta^+_{12}T^{12} + \tfrac{\imath}{2}\sum_l\eta^+_l\eta^l\mathcal{P}_0 \\ & -\vartheta^+_{12}\sum_n(1+\delta_{1n})\eta^{1n+}\mathcal{P}^{n2} + \vartheta^+_{12}\sum_n(1+\delta_{n2})\eta^{n2}\mathcal{P}^{+1n} + \tfrac{1}{2}\sum_n\eta^+_{n2}\eta^{1n}\lambda^{12} \\ & -\tfrac{1}{2}\sum_{l\le m}(1+\delta_{lm})\eta^m\eta^+_{lm}\mathcal{P}^l - [\vartheta_{12}\eta^{+2} + \vartheta^+_{12}\eta^{+1}]\mathcal{P}^2 \\ & +r\Big\{\eta_0\sum_i\eta^+_i\Big[2\mathcal{L}^{ii}\mathcal{P}^+_i + 2\mathcal{L}^{i+}\mathcal{P}_{ii} + \mathcal{G}^i_0\mathcal{P}_i + 2(\mathcal{L}^{12}\mathcal{P}^{\{1+} + \mathcal{L}^{\{1+}\mathcal{P}_{12})\delta^{2\}i} \\ & \quad -\delta^{1i}(\mathcal{L}^2\lambda^+_{12} + \mathcal{T}^+_{12}\mathcal{P}_2) - \delta^{2i}(\mathcal{L}^1\lambda_{12} + \mathcal{T}_{12}\mathcal{P}_1)\Big] \\ & -\tfrac{1}{2}\sum_{i,j}\eta^+_i\eta^+_j\varepsilon^{ij}\Big[\sum_m(-1)^m\mathcal{G}^m_0\mathcal{P}_{12} - (\mathcal{T}_{12}\mathcal{P}_{11} + \mathcal{L}^{11}\lambda_{12}) + \mathcal{T}^+_{12}\mathcal{P}_{22} + \mathcal{L}^{22}\lambda^+_{12}\Big] \\ & +2\eta^+_i\eta_j\Big[\sum_m\mathcal{L}^{jm+}\mathcal{P}^{im} - \tfrac{1}{8}(\mathcal{T}^+_{12}\lambda_{12} + \mathcal{T}_{12}\lambda^+_{12})\delta^{ij} + \tfrac{1}{4}\sum_m\mathcal{G}^m_0\lambda_{12}\delta^{j1}\delta^{i2}\Big]\Big\} \\ & +r^2\Big\{\eta_0\sum_{i,j}\eta_i\eta_j\varepsilon^{ij}\Big[\tfrac{1}{2}\sum_m\Big(G^m_0[\lambda_{12}\mathcal{P}^+_{22} - \lambda^+_{12}\mathcal{P}^+_{11}] + 4L^{mm}\mathcal{P}^+_{m2}\mathcal{P}^+_{1m}\Big) + 2L^{12}\mathcal{P}^+_{22}\mathcal{P}^+_{11}\Big] \\ & +\eta_0\sum_{i,j}\eta^+_i\eta_j\Big[2\sum_m(L^+_{22}\mathcal{P}^{22} - L^{11}\mathcal{P}^+_{11}))\lambda_{12}\delta^{1j}\delta^{2i} - 2T^{12}\mathcal{P}^{11}\mathcal{P}^+_{22}\delta^{1i}\delta^{2j} \end{aligned}$$

[11] they obey to independent nonvanishing anticommutation relations $\{\vartheta_{12}, \lambda^+_{12}\} = 1, \{\eta_i, \mathcal{P}^+_j\} = \delta_{ij}$, $\{\eta_{lm}, \mathcal{P}^+_{ij}\} = \delta_{li}\delta_{jm}$, $\{\eta_0, \mathcal{P}_0\} = \imath, \{\eta^i_G, \mathcal{P}^j_G\} = \imath\delta^{ij}$, possess the standard ghost number distribution, $gh(\mathcal{C}^I) = -gh(\mathcal{P}_I) = 1$, providing the property $gh(Q') = 1$, and have the Hermitian conjugation properties of zero-mode pairs, $\big(\eta_0, \eta^i_G, \mathcal{P}_0, \mathcal{P}^i_G\big)^+ = \big(\eta_0, \eta^i_G, -\mathcal{P}_0, -\mathcal{P}^i_G\big)$

$$+2\varepsilon^{\{1j}\delta^{2\}i}\Big((L^{12}\lambda_{12} - T^{12}\mathcal{P}^{12})\mathcal{P}^+_{22} + (T^+_{12}\mathcal{P}^{12} - L^{12}\lambda^+_{12})\mathcal{P}^+_{11}\Big)$$
$$-2\textstyle\sum_m(-1)^m\Big(G_0^m\mathcal{P}^{11}\delta^{1i}\delta^{2j} - G_0^m\mathcal{P}^{22}\delta^{2i}\delta^{1j}\Big)\mathcal{P}^+_{12}\Big]\Big\} + h.c.^{12}\,. \qquad (104)$$

The generalized spin operator $\vec{\sigma} = (\sigma^1, \sigma^2)$, extended by the ghost Wick-pair variables,

$$\sigma^i = G_0^i - h^i - \eta_i\mathcal{P}^+_i + \eta^+_i\mathcal{P}_i + \sum_m(1+\delta_{im})(\eta^+_{im}\mathcal{P}^{im} - \eta_{im}\mathcal{P}^+_{im}) + [\vartheta^+_{12}\lambda^{12} - \vartheta^{12}\lambda^+_{12}](-1)^i\,, \qquad (105)$$

commutes with Q, $[Q, \sigma^i] = 0$. We choose a representation for Hilbert space $\mathcal{H}_{tot}$ coordinated with decomposition (103) such that the operators $(\eta_i, \eta_{ij}, \vartheta_{12}, \mathcal{P}_0, \mathcal{P}_i, \mathcal{P}_{ij}, \lambda_{12}, \mathcal{P}^i_G)$ annihilate vacuum vector $|0\rangle = 0$, and suppose that the field vectors $|\chi\rangle$ as well as the gauge parameters $|\Lambda\rangle$ do not depend on ghosts η^i_G,

$$|\chi\rangle = \sum_n \textstyle\prod_l^2 (b^+_l)^{n_l} \prod_{i\le j}^2 (b^+_{ij})^{n_{ij}} (d^+_{12})^{p_{12}} (\eta^+_0)^{n_{f0}} \displaystyle\prod_{i,j,l\le m,n\le o}^2 (\eta^+_i)^{n_{fi}} (\mathcal{P}^+_j)^{n_{pj}} (\eta^+_{lm})^{n_{flm}} (\mathcal{P}^+_{no})^{n_{pno}}$$
$$\times\, (\vartheta^+_{12})^{n_{f12}} (\lambda^+_{12})^{n_{\lambda 12}} |\Phi(a^+_i)^{n_{f0}(n)_{fi}(n)_{pj}(n)_{flm}(n)_{pno}(n)_{f12}(n)_{\lambda 12}}_{(n)_l(n)_{ij}p_{12}}\rangle\,. \qquad (106)$$

The brackets $(n)_{fi}, (n)_{pj}, (n)_{pno}$ in the Eq.(106) means, for instance, for $(n)_{pno}$ the set of indices $(n_{p11}, n_{p12}, n_{p22})$. The sum above is taken over n_l, n_{ij}, p_{12} and running from 0 to infinity, and over the rest n's from 0 to 1. The Hilbert space $\mathcal{H}_{tot}$ is decomposed into a direct sum of Hilbert subspaces with definite ghost number: $\mathcal{H}_{tot} = \bigoplus_{k=-6}^{6}\mathcal{H}_k$. Denote by $|\chi^k\rangle \in \mathcal{H}_{-k}$, the state (106) with the ghost number $-k$, i.e. $gh(|\chi^k\rangle) = -k$. Thus, the physical state having the ghost number zero is $|\chi^0\rangle$, the gauge parameters $|\Lambda\rangle$ having the ghost number -1 is $|\chi^1\rangle$ and so on. For vanishing of all auxiliary creation operators B^+ and ghost variables $\eta_0, \eta^+_i, \mathcal{P}^+_i, \ldots$ the vector $|\chi^0\rangle$ must contain only physical string-like vector $|\Phi\rangle = |\Phi(a^+_i)^{(0)_{fo}(0)_{fi}(0)_{pj}(0)_{flm}(0)_{pno}(0)_{f12}(0)_{\lambda 12}}_{(0)_l(0)_{ij}0_{12}}\rangle$,

$$|\chi^0\rangle = |\Phi\rangle + |\Phi_A\rangle, \quad |\Phi_A\rangle\Big|_{[B^+=\eta_0=\eta^+_i=\mathcal{P}^+_i=\eta^+_{lm}=\mathcal{P}^+_{no}=\vartheta^+_{12}=\lambda^+_{12}=0]} = 0 \qquad (107)$$

One can show, using the part of equations of motion and gauge transformations, that the vector $|\Phi_A\rangle$ can be completely removed (see Ref.[46]).

The equation for the physical state $Q'|\chi^0\rangle = 0$ and the tower of the reducible gauge transformations, $\delta|\chi\rangle = Q'|\chi^1\rangle$, $\delta|\chi^1\rangle = Q'|\chi^2\rangle$, …, $\delta|\chi^{(s-1)}\rangle = Q'|\chi^{(s)}\rangle$, lead to relations:

$$Q|\chi\rangle = 0, \qquad (\sigma^i + h^i)|\chi\rangle = 0, \qquad (\varepsilon, gh)(|\chi\rangle) = (0,0), \qquad (108)$$
$$\delta|\chi\rangle = Q|\chi^1\rangle, \qquad (\sigma^i + h^i)|\chi^1\rangle = 0, \qquad (\varepsilon, gh)(|\chi^1\rangle) = (1,-1), \qquad (109)$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$\delta|\chi^{s-1}\rangle = Q|\chi^s\rangle, \qquad (\sigma^i + h^i)|\chi^s\rangle = 0, \qquad (\varepsilon, gh)(|\chi^s\rangle) = (s\,mod\,2, -s). \qquad (110)$$

Here ε means for Grassmann parity and $s = 6$ is the maximal stage of reducibility for the massive bosonic HS field, because of subspaces $H_k = \emptyset$, for all integer $k \le -7$. The middle set of equations in (108)–(110) determines the possible values of the parameters h^i and the eigenvectors of the operators σ^i. Solving spectral problem, we obtain a set of eigenvectors, $|\chi^0\rangle_{(n)_2}$, $|\chi^1\rangle_{(n)_2}$, …, $|\chi^s\rangle_{(n)_2}$, $n_1 \ge n_2 \ge 0$, and a set of eigenvalues,

$$\sigma_i|\chi\rangle_{(n)_k} = \Big(n^i + \tfrac{d-1-4i}{2}\Big)|\chi\rangle_{(n)_k}, \quad -h^i = n_i + \tfrac{d-1-4i}{2}\,,\; i = 1,2\,,\; n_1 \in \mathbf{Z}, n_2 \in \mathbf{N}_0\,. \qquad (111)$$

[12] here, in writing the coefficients depending on o'_I, O_I we have used the convention from Eqs.(96)–(98), for $\mathcal{O}_I$

It is easy to see that in order to construct Lagrangian for the field corresponding to a definite Young tableau (1) the numbers n_i must be equal to the numbers of the boxes in the i-th row of the corresponding Young tableau, i.e. $n_i = s_i$. Thus, the state $|\chi\rangle_{(s)_2}$ contains the physical field (7) and all its auxiliary fields. We fix some values of $n_i = s_i$. After substitution: $h^i \to h^i(s_i)$ operator $Q_{(s_1,s_2)} \equiv Q_{|h^i \to h^i(s_i)}$, is nilpotent on each subspace $H_{tot(s_1,s_2)}$ whose vectors satisfy to the Eqs.(108) for (111). Hence, the Lagrangian equations of motion (one to one correspond to Eqs.(2)–(5) for $k = 2$), a sequence of reducible gauge transformations have the form

$$Q_{(s_1,s_2)}|\chi^0\rangle_{(s_1,s_2)} = 0, \quad \delta|\chi^s\rangle_{(s_1,s_2)} = Q_{(s_1,s_2)}|\chi^{s+1}\rangle_{(s_1,s_2)}, \; s = 0, ..., 5. \tag{112}$$

Analogously to totally symmetric bosonic HS fields [11],[12] one can show that Lagrangian action for fixed spin $(n)_2 = (s)_2$ is defined up to an overall factor as follows

$$\mathcal{S}_{(s_1,s_2)} = \int d\eta_0 \; {}_{(s_1,s_2)}\langle\chi^0|K_{(s_1,s_2)}Q_{(s_1,s_2)}|\chi^0\rangle_{(s_1,s_2)}, \; \texttt{for} \; |\chi^0\rangle \equiv |\chi\rangle. \tag{113}$$

where the standard scalar product for the creation and annihilation operators is assumed with measure $d^dx\sqrt{|g|}$ over AdS space. The vector $|\chi^0\rangle_{(s)_2}$ and the operator $K_{(s)_2}$ in (113) are respectively the vector $|\chi\rangle$ (106) subject to spin distribution relations (111) for HS tensor field $\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2}}(x)$ and operator K (102) where the substitution $h_i \to -(n_i + \frac{d-1-4i}{2})$ is done. The corresponding LF for bosonic field with spin $\mathbf{s}$ subject to $Y(s_1, s_2)$ is a reducible gauge theory of maximally $L = 6$-th stage of reducibility.

One can prove that the equations of motion (112) indeed reproduces only the basic conditions (2)–(5) for HS fields with given spin (s_1, s_2) and mass. Therefore, the resulting equations of motion because of the representation (107) have the form,

$$L_0|\Phi\rangle_{(s)_2} = (l_0 + m_0^2)|\Phi\rangle_{(s)_2}, \; (l_i, l_{ij}, t_{12})|\Phi\rangle_{(s)_2} = (0, 0, 0), \; i \le j. \tag{114}$$

The above relations permit one to determine the parameter m_0 in a unique way in terms of $h^i(s_i)$,

$$m_0^2 = m^2 + r\Big\{\beta(\beta+1) + \tfrac{d(d-6)}{4} + \Big(h^1 - \tfrac{1}{2} + 2\beta\Big)\Big((h^1 - \tfrac{5}{2}\Big) + \Big(h^2 - \tfrac{9}{2}\Big)\Big\}, \tag{115}$$

whereas the values of parameter m_1, m_2 remain by completely arbitrary and may be used to reach special properties of the Lagrangian for given HS field.

The general action (113) gives, in principle, a direct recept to obtain the Lagrangian for any component field $\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2}}(x)$ from general vector $|\chi^0\rangle_{(s)_2}$ since the only what we should do it is a computation of vacuum expectation values of products of some number of creation and annihilation operators.

5. Conclusion

In the paper we have derived the quadratic non-linear HS symmetry algebra for description of arbitrary integer HS fields on AdS-spaces with any dimensions and subject to k row Young tableaux $Y(s_1, \ldots, s_k)$. It is shown the difference of the obtained algebras $\mathcal{A}(Y(k), AdS_d)$ for $k = 2$, $\mathcal{A}'(Y(2), AdS_d)$, $\mathcal{A}_c(Y(2), AdS_d)$ corresponding respectively to initial set of operators, their additional parts and converted set of operators within additive conversion procedure, is due to their pure non-linear parts, which are, in turn, connected to the AdS$_d$-radius $(\sqrt{r})^{-1}$ presence through the set of isometry AdS-space operators.

To obtain the algebras we start from an embedding of bosonic HS fields into vectors of an auxiliary Fock space, treat the fields as coordinates of Fock-space vectors and reformulate the theory in such terms. We realize the conditions that determine an irreducible AdS-group representation with a given mass and generalized spin in terms of differential operator constraints imposed on the Fock space vectors. These constraints generate a closed non-linear algebra of HS symmetry, which contains, with the exception of k basis generators of its Cartan subalgebra, a system of first- and second-class constraints. Above algebra coincides modulo isometry group generators with its Howe dual $sp(2k)$ symplectic algebra. The construction of a correct Lagrangian description requires a deformation of the initial symmetry algebra, into algebra $\mathcal{A}_c(Y(2), AdS_d)$ introducing the algebra $\mathcal{A}'(Y(2), AdS_d)$.

We have generalized the method of construction of Verma module [8] from the case of Lie (super)algebras [48], [49], [29] and for quadratic algebra $\mathcal{A}'(Y(1), AdS_d)$ for totally-symmetric HS fields [10], [12] on to case of non-linear algebra underlying mixed-symmetric HS bosonic fields on AdS-space with two-row Young tableaux. The Theorem 1 presents our basic result in this relation. We show that as the byproduct of Verma module derivation the Poincare–Birkhoff–Witt theorem is valid in case of the algebra under consideration, therefore providing the lifting of the Verma module for Lie algebra $\mathcal{A}(Y(2), R^{1,d-1})$ [being isomorphic to $(T^2 \oplus T^{2*}) \niplus sp(4)$] to one for quadratic algebra in a deformation parameter r. Of course, the same it is expected to be true for general algebra $\mathcal{A}'(Y(k), AdS_d)$, for which we suppose to obtain the explicit form of Verma module in the recursive procedure manner by means of new *primary and derived block-operators*, like $\hat{t}'_{12}$, $\hat{t}'^{+}_{12}$.

We have obtained the representation for the 15 generators of the algebra $\mathcal{A}'(Y(2), AdS_d)$ over Heisenberg-Weyl algebra A_6 as the formal power series in creation and annihilation operators, which in case of flat space limit ($r = 0$) takes the polynomial form, coinciding with earlier known results, at least for $m = 0$ [24] and appearing new one for massive case [50] and for $k = 2$ in [29]. The Theorem 2 finalizes our second basic result on solution of this Fock space realization problem for $\mathcal{A}'(Y(2), AdS_d)$ through Verma module construction approach.

On a base of BFV-BRST operator Q' which was found in Ref. [31] exactly up to third degree in ghost coordinates, for the nonlinear algebra $\mathcal{A}_c(Y(2), AdS_d)$ of 15 converted constraints O_I by analyzing the structure of Jacobi identities for them we present a proper construction of gauge-invariant Lagrangian formulations for the bosonic HS fields of given spin $\mathbf{s} = (s_1, s_2)$ and mass on AdS$_d$ space. The corresponding Lagrangian formulation is at most 6-th stage reducible Abelian gauge theory and is given by the Eqs.(112),(113). The last relations may be considered as the final result in solution of the general problem to construct Lagrangian formulation for non-Lagrangian initial AdS-group irreducible representations relations which describe the bosonic HS field with two rows in Young tableaux. One should be noted the unconstrained Lagrangians for the free mixed-symmetry HS fields with two rows in Young tableaux on a AdS background have not been derived until now in both “metric-like” and “frame-like” formulations. These results to be seen as the first step to interacting theory, following in part to the research [51], [52].

From a mathematical point of view the construction of the Verma module for the algebra $\mathcal{A}'(Y(2), AdS_d)$ open the possibility to study both its structure and search singular, subsingular vectors in it, so that it, in principle, will then permit to construct new (non-scalar) infinite-dimensional representations for given algebra. Besides, the above results permit to definitely understand the problems of (generalized) Verma module construction for HS symmetry algebras and superalgebras underlying HS bosonic and respectively fermionic fields on AdS-spaces subject to multi-row Young tableaux.

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Appendix

Appendix A. Proof of the Proposition

In this appendix we check the validity of the Proposition in the subsection 2.2. The proof is based on the explicit derivation of the multiplication laws (30) and (31), (32) for the sets $\mathcal{A}'$ of the operators o'_I and $\mathcal{A}_c$ of O_I. Namely, from the right-hand-side of the relations (29) we have (with account for commutativity of o_I with o'_J) the equations to determine the unknown structural functions $F^K_{IJ}(o', O)$,

$$[\,O_I, O_J\} = [\,o_I, o_J\} + [\,o'_I, o'_J\} = \sum_{m=1}^{n} f_{ij}^{K_1\cdots K_m} \prod_{l=1}^{m} o_{K_l} + [\,o'_I, o'_J\}. \tag{A.1}$$

Expressing in (A.1) the initial elements $o_{K_1}, \ldots, o_{K_n}$ through enlarged O_I and additional o'_I operators with use of $o'O$-ordering we obtain the sequence of relations for each power of o_K,

$$f^{K_1}_{IJ} o_{K_1} = f^{K_1}_{IJ} O_{K_1} - f^{K_1}_{IJ} o'_{K_1}\,, \tag{A.2}$$

$$f^{K_1K_2}_{IJ} o_{K_1} o_{K_2} = f^{K_1K_2}_{IJ} O_{K_1} O_{K_2} - (f^{K_1K_2}_{IJ} + f^{K_2K_1}_{IJ}) o'_{K_1} O_{K_2} + f^{K_2K_1}_{IJ} o'_{K_1} o'_{K_2} \tag{A.3}$$

$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ ,$$

$$\begin{aligned} f^{K_1\cdots K_n}_{IJ} \prod_{l=1}^{n} o_{K_l} &= f^{K_1\cdots K_n}_{IJ} \prod\nolimits_{m=1}^{n} O_{K_m} + \sum\nolimits_{s=1}^{n-1} (-1)^s f_{ij}^{\widehat{K_s\cdots K_1}\widehat{K_{s+1}\cdots K_n}} \\ &\times \prod\nolimits_{p=1}^{s} o'_{K_p} \prod\nolimits_{m=s+1}^{n} O_{K_m} - (-1)^n f^{(n)K_n\cdots K_1}_{IJ} \prod\nolimits_{s=1}^{n} o'_{K_s}, \end{aligned} \tag{A.4}$$

where the hats in the notation $f_{ij}^{\widehat{K_s\cdots K_1}\widehat{K_{s+1}\cdots K_n}}$ for the quantities $f_{ij}^{K_s\cdots K_1 K_{s+1}\cdots K_n}$ means the set of $C^n_s = \frac{n!}{s!(n-s)!}$ terms obtained through $f_{ij}^{\widehat{K_s\cdots K_1}\widehat{K_{s+1}\cdots K_n}}$ by the symmetrization as it is explicitly shown in the Eqs. (32). Above system (A.2)–(A.4) permits one to immediately establish, first, from the rightmost terms above in (A.1)–(A.4) that the set of o'_I form the polynomial algebra $\mathcal{A}'$ of order n subject to the algebraic relations (30). Second, the rest terms in (A.1)–(A.4) completely determine the structural functions $F^{(m)K}_{IJ}(o', O)$, $m = 1, \ldots, n$ in the form (32) and show that the set of O_I indeed determine the non-linear algebra $\mathcal{A}_c$[13].

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