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# Generalized Nonlinear Chiral Supermultiplet

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**Abstract.** We propose the generalized nonlinear chiral supermultiplet which is defined in terms of  $2n$  bosonic  $\mathcal{N} = 2(n + 1)$  superfields subjected to the nonlinear constraints. The guiding idea of our construction is its invariance under the action of the  $SU(n + 1)$  group. With this supermultiplet we were able to find the complete  $SU(n + 1)$  invariant symplectic structure and first, linear in the fermionic variables, the terms in the supercharges which anticommute on the Hamiltonian describing the motion of the particle over the  $CP^n$  manifold.

## 1. Introduction

The irreducible supermultiplets proved to be the main ingredients for the construction of one-dimensional systems with extended  $\mathcal{N} \geq 4$  supersymmetry. Given a set of those, preferably formulated in superspace, one may immediately write the corresponding sigma-model type actions and the general potential terms. In this respect, almost complete classification of *linear* off-shell representations for one-dimensional supersymmetry (see e.g. [1] and references therein) seems to suffice for constructing any mechanics model with extended supersymmetry. However, detailed analysis shows two common restrictions. First, the classification is limited to the case of linear supermultiplets only. Second, the superspace approach is less useful in the cases with  $\mathcal{N} \geq 4$ .

The possibility of nonlinear off-shell  $\mathcal{N}=4$ ,  $d=1$  supermultiplets was firstly noted in [2]. Subsequently, in [3, 4] the first two examples of such nonlinear supermultiplets were explicitly described and analyzed. Although by now the list of nonlinear supermultiplets is a bit lengthy, their classification is still missing. Moreover, the variety of methods, by which these supermultiplets have been constructed, makes any of such a classification scheme to be rather problematic.

On the other hand, the unique example of a nontrivial mechanics with an arbitrary number of supersymmetries was constructed many years ago in [5].

In the present Letter we are going to construct the Generalized Nonlinear Chiral Supermultiplet (GNLC multiplet) which has two distinguished properties

- It can be defined for an arbitrary number  $n$  of the complex bosonic superfields living in  $\mathbf{R}^{(1|2(n+1))}$  superspace: so the number of supersymmetries is strictly correlated with the number of superfields involved;
- The defining conditions are invariant with respect to the action of  $SU(n + 1)$  group symmetry which is realized in a such way that the bosonic coordinates parametrize the

coset  $SU(n+1)/U(n)$ , while the spinor covariant derivatives, as well as the supercharges, form linear representation of the same  $SU(n+1)$  group.

Really speaking, these two basic properties are just generalization to  $n > 1$  of those already presented in the case of  $\mathcal{N}=4$  nonlinear chiral supermultiplet [3, 4]. It is quite interesting that the listed properties are sufficient to find the defining relations and, moreover, to construct (at least partially) the supercharges and the Hamiltonian (providing they are also  $SU(n+1)$  invariant).

We will start with a brief review of main peculiarities of the  $\mathcal{N} = 4$  nonlinear chiral multiplet, which is described by a single complex superfield (and, therefore, it corresponds to the  $n = 1$  case of our general construction) subjected to proper constraints. Then we consider a generalization of this multiplet to the cases with  $n > 1$ . We will demonstrate that in the case of  $\mathcal{N} = 2(n+1)$  extended supersymmetry one could uniquely obtain the irreducibility conditions selected this multiplet if our defining properties listed above are satisfied. In the rest of the Section 2 we analyze in more detail the particular system whose action is invariant with respect to  $SU(n+1)$  symmetry, i.e. the supersymmetric particle moving over the  $CP^n$  manifold. We construct the corresponding invariant symplectic structure and find first, linear in the fermionic variables, the terms in the supercharges. We finished with an explicit example of the  $CP^1$  system in which the full structure of the theory can be easily restored.

## 2. Generalized Nonlinear Chiral Multiplet

It is well known fact that the action describing the motion of the particle over the  $CP^n$  manifold admits  $\mathcal{N}=4$  supersymmetric extension [6, 7, 8]. The basic ingredients of such a construction are the  $\mathcal{N}=4$  chiral superfields. In the the same time, the simplest case with a particle moving over  $CP^1$  can be supersymmetrized in a different way, starting from the nonlinear chiral supermultiplet (NLC multiplet) [3, 4]. The new feature of the NLC multiplet is that its covariance with respect to  $SU(2)$  group is achieved by simultaneous transformations of spinor covariant derivatives (and supercharges) and the superfields. As the result, the basic bosonic  $\mathcal{N}=4$  superfields proved to obey the nonlinear chirality conditions. In this Section we, generalize the NLC multiplet to the case of  $\mathcal{N} = 2(n+1)$  supersymmetry. Our guiding principle is the invariance of this Generalized Nonlinear Chiral supermultiplet with respect to the  $SU(n+1)$  group. We supposed that by analogy with NLC multiplet, both the spinor covariant derivatives and the superfields transform under the  $SU(n+1)$  group. More precisely, the physical bosonic components of the GNLC multiplet are identified with the parameters of  $SU(n+1)/U(n)$  coset, while the  $2(n+1)$  spinor covariant derivatives transform linearly under the action of the  $SU(n+1)$  group. If we insist on the  $SU(n+1)$  symmetry of the full action, then the bosonic sector will describe the motion of the particle over the  $CP^n$  manifold. We analyze this system, which now possesses  $\mathcal{N} = 2(n+1)$  supersymmetries and find the corresponding  $SU(n+1)$  invariant symplectic structure and the linear over fermionic variables terms in the supercharges. However firstly we shortly review the basic features of the nonlinear chiral supermultiplet we are going to generalize.

### 2.1. $\mathcal{N} = 4$ Nonlinear Chiral Multiplet

Let us remind some basic properties of the  $\mathcal{N} = 4$  NLC multiplet [3, 4]. It can be defined in terms of complex scalar superfields  $Z, \bar{Z}$  satisfying the  $\mathcal{N}=4$  nonlinear chirality conditions

$$D^i Z = -Z \bar{D}^i Z, \quad \bar{D}_i \bar{Z} = \bar{Z} D_i \bar{Z}, \quad (1)$$

where the covariant spinor derivatives are defined as

$$D^i = \frac{\partial}{\partial \theta_i} + i \bar{\theta}^i \partial_t, \quad \bar{D}_j = \frac{\partial}{\partial \bar{\theta}^j} + i \theta_j \partial_t \quad \Rightarrow \quad \{D^i, \bar{D}_j\} = 2i \delta_j^i \partial_t, \quad i, j = 1, 2. \quad (2)$$

The basic feature of the defining constraints (1) is their invariance under simultaneous  $su(2)$  transformations of  $\{Z, \bar{Z}\}$

$$\delta Z = a + \bar{a}Z^2, \quad \delta \bar{Z} = \bar{a} + a\bar{Z}^2 \quad (3)$$

and the covariant derivatives

$$\delta D^i = -a\bar{D}^i, \quad \delta \bar{D}_i = \bar{a}D_i. \quad (4)$$

Here,  $a$  and  $\bar{a}$  are the parameters of the coset  $SU(2)/U(1)$  transformations.

It is clear from (3), (4), that the superfields  $\{Z, \bar{Z}\}$  parametrized the coset  $SU(2)/U(1)$ <sup>1</sup> while the spinor derivatives transform linearly under action of the whole  $S(2)$  group. Due to nonlinear transformation properties of  $\{Z, \bar{Z}\}$  (3) the constraints (1) show the manifest invariance only under  $U(1)$  transformations, while the invariance under coset  $SU(2)/U(1)$  transformations is implicit.

Note that the standard version of the  $\mathcal{N}=4$  chiral supermultiplet with the defining constraints

$$D^i Z = 0, \quad \bar{D}_i \bar{Z} = 0 \quad (5)$$

is also perfectly invariant under (3) if the spinor derivatives do not transform at all. So one may wonder why we really need such a complicated supermultiplet? As it was shown in [4], the nonlinearity of the basic constraints (1) together with the transformations of spinor derivatives under the  $SU(2)$  group (4) result in new properties of  $\mathcal{N}=4$  supersymmetric mechanics constructed with the help of NLC multiplet which include the non-standard coupling of the fermionic sector to the background, deformation of the bosonic potential and possibility to introduce in the action the interaction with the magnetic field. Usually, the appearance of a magnetic field breaks supersymmetry, but in the theory with NLC multiplet  $\mathcal{N}=4$  supersymmetry is preserved even in the presence of the magnetic field. This allow one to include in the class of  $N=4$  supersymmetric systems the Landau problem on the sphere and the Higgs oscillator on the Lobachevsky plane [4].

All these new properties are deeply related with the nonlinearity of the basic constraints. That is why in the next Subsection we will extend NLC multiplet to more general cases.

## 2.2. Generalized Nonlinear Chiral Multiplet

It should be clear from the transformation properties of our superfields  $\{Z, \bar{Z}\}$  (3) that if we insist on this invariance, then the bosonic part of the corresponding action is completely fixed to be

$$S \sim \int dt \frac{\dot{z}\dot{\bar{z}}}{(1+z\bar{z})^2} + \dots \quad (6)$$

where  $\{z, \bar{z}\}$  are just the first components of  $\{Z, \bar{Z}\}$  in  $\theta, \bar{\theta}$  expansion. This is an action of the particle moving the sphere  $SU(2)/U(1)$ .

The immediate, almost straightforward generalization of the action (6) is the action for the particle moving over the  $CP^n = SU(n+1)/U(n)$  manifold

$$S \sim \int dt g_{\alpha\beta} \dot{z}^\alpha \dot{\bar{z}}^\beta + \dots, \quad \alpha, \beta = 1, \dots, n, \quad (7)$$

where the metric  $g_{\alpha\beta}$  has the standard Fubini-Study form

$$g_{\alpha\beta} = \frac{1}{(1+z\cdot\bar{z})} \left[ \delta_{\alpha\beta} - \frac{\bar{z}_\alpha z^\beta}{(1+z\cdot\bar{z})} \right], \quad z\cdot\bar{z} \equiv z^\alpha \bar{z}_\alpha. \quad (8)$$

<sup>1</sup> Really,  $\{Z, \bar{Z}\}$  are related with the coordinates  $\{x, \bar{x}\}$  of the coset  $SU(2)/U(1)$  as  $Z = x \tan(\sqrt{x\bar{x}})/\sqrt{x\bar{x}}$ .

The action (7) is invariant with respect to the  $SU(n+1)$  group realized on the bosonic fields  $z^\alpha, \bar{z}_\alpha$ , similarly to (3), as

$$\delta z^\alpha = a^\alpha + z^\alpha (z^\beta \bar{a}_\beta), \quad \delta \bar{z}_\alpha = \bar{a}_\alpha + \bar{z}_\alpha (a^\beta \bar{z}_\beta), \quad (9)$$

where  $a^\alpha, \bar{a}_\alpha$  are now the parameters of the coset  $SU(n+1)/U(n)$  transformations.

To construct the generalization of the NLC supermultiplet (1) to the case of the  $CP^n$  manifold, let us introduce the covariant spinor derivatives  $(D^A, \bar{D}_A)$  which linearly transform under  $SU(n+1)$  and obey standard relations

$$\{D^A, \bar{D}_B\} = 2i\delta_B^A \partial_t, \quad A = 1, \dots, n+1. \quad (10)$$

The action (7) is manifestly invariant only under  $U(n)$  group, so we will split the covariant derivatives (10) as

$$D^A = (D^\alpha, D), \quad \bar{D}_A = (\bar{D}_\alpha, \bar{D}), \quad \alpha = 1, \dots, n, \quad (11)$$

The coset  $SU(n+1)/U(n)$  transformations of the covariant spinor derivatives in this basis read

$$\delta D = a^\alpha \bar{D}_\alpha, \quad \delta D^\alpha = -a^\alpha \bar{D}, \quad \delta \bar{D} = \bar{a}_\alpha D^\alpha, \quad \delta \bar{D}_\alpha = -\bar{a}_\alpha D, \quad (12)$$

while  $U(n)$  symmetry is just rotated  $D^\alpha$  and  $\bar{D}_\alpha$  derivatives. The anticommutators of these spinor derivatives have the standard form

$$\{D^\alpha, \bar{D}_\beta\} = 2i\delta_\beta^\alpha \partial_t, \quad \{D, \bar{D}\} = 2i\partial_t. \quad (13)$$

Thus, we have the superfields and spinor derivatives which transform under the  $SU(n+1)$  group in full analogy with the the NLC supermultiplet. Now we can find the  $SU(n+1)$  invariant constraints which generalized NLC conditions (1). These constraints define the generalized nonlinear chiral multiplet and they are unambiguously restored to be

$$DZ^\alpha = Z^\beta \bar{D}_\beta Z^\alpha, \quad \bar{D}\bar{Z}_\alpha = \bar{Z}_\beta D^\beta \bar{Z}_\alpha, \\ D^\alpha Z^\beta = -Z^\alpha \bar{D}Z^\beta, \quad \bar{D}_\alpha \bar{Z}_\beta = -\bar{Z}_\alpha D\bar{Z}_\beta.$$

These constraints are invariant with respect to the  $SU(n+1)$  group if the covariant spinor derivatives transform together with the superfields  $Z^\alpha, \bar{Z}_\alpha$  as (12) and (9), respectively.

### 2.3. Supercharges: beginning

One should stress that now we deal with  $\mathcal{N} = 2(n+1)$  supersymmetries. So the straightforward construction of the invariant action, in contrast with the case of  $\mathcal{N} = 4$  supersymmetry, seems to be problematic. Moreover, the superspace approach could not give so much while we deal with such high supersymmetry. Nevertheless, one could get some information about the structure of supercharges and Hamiltonian which contain only physical degrees of freedom.

Let us define the physical bosonic and fermionic components of the GNLC multiplet as follows

$$z^\alpha = Z^\alpha|, \quad \bar{z}_\alpha = \bar{Z}_\alpha|, \quad \psi_\alpha = D\bar{Z}_\alpha|, \quad \bar{\psi}^\alpha = \bar{D}Z^\alpha|, \quad \xi_\alpha^\beta = D^\beta \bar{Z}_\alpha|, \quad \bar{\xi}_\beta^\alpha = \bar{D}_\beta Z^\alpha|, \quad (14)$$

where  $|$  denotes, as usual,  $\theta = \bar{\theta} = 0$  limit. Thus we have  $2n$  bosonic and  $2n(n+1)$  fermionic variables. The rest of the components in the superfields  $\{Z^\alpha, \bar{Z}_\alpha\}$  are auxiliary ones and they will never appear in the supercharges.

Under our extended  $\mathcal{N} = 2(n+1)$  supersymmetry the superfields  $(Z^\alpha, \bar{Z}_\alpha)$  transform in the standard way as

$$\delta Z^\alpha = -\varepsilon_\beta Q^\beta Z^\alpha - \bar{\varepsilon}^\beta \bar{Q}_\beta Z^\alpha - \mu S Z^\alpha - \bar{\mu} \bar{S} Z^\alpha, \\ \delta \bar{Z}_\alpha = -\varepsilon_\beta Q^\beta \bar{Z}_\alpha - \bar{\varepsilon}^\beta \bar{Q}_\beta \bar{Z}_\alpha - \mu S \bar{Z}_\alpha - \bar{\mu} \bar{S} \bar{Z}_\alpha, \quad (15)$$

where  $\varepsilon_\alpha$  and  $\mu$  are the parameters corresponding to the generators  $Q^\alpha$  and  $S$ , respectively. Using our definition of physical variables (14), from (15) one can immediately find the transformation properties of the  $\theta = \bar{\theta} = 0$  components of the superfield, i.e. physical bosons:

$$\begin{aligned}\delta z^\alpha &= \varepsilon_\beta z^\beta \bar{\psi}^\alpha - \bar{\varepsilon}^\beta \bar{\xi}_\beta^\alpha - \mu z^\beta \bar{\xi}_\beta^\alpha - \bar{\mu} \bar{\psi}^\alpha, \\ \delta \bar{z}_\alpha &= -\varepsilon_\beta \xi_\alpha^\beta + \bar{\varepsilon}^\beta \bar{z}_\beta \psi_\alpha - \mu \psi_\alpha - \bar{\mu} \bar{z}_\beta \xi_\alpha^\beta.\end{aligned}\quad (16)$$

In the Hamiltonian formalism the variations of components are defined as

$$\delta z^\alpha = -\{\varepsilon_\beta Q^\beta, z^\alpha\} - \{\bar{\varepsilon}^\beta \bar{Q}_\beta, z^\alpha\} - \{\mu S, z^\alpha\} - \{\bar{\mu} \bar{S}, z^\alpha\}.\quad (17)$$

To calculate r.h.s of (17), one has to know the Dirac brackets, i.e. symplectic structure between all our variables. Unfortunately, for the time being, we know almost nothing about these brackets. All what we know is that the bosonic coordinates  $(z^\alpha, \bar{z}_\alpha)$  and their momenta  $(p_\alpha, \bar{p}^\alpha)$  should obey the canonical brackets <sup>2</sup>

$$\{z^\alpha, p_\beta\} = \delta_\beta^\alpha, \quad \{\bar{z}_\alpha, \bar{p}^\beta\} = \delta_\alpha^\beta.\quad (18)$$

This information is sufficient to restore the leading terms in the supercharges which are linear in the bosonic momenta and the spinor variables

$$Q^\alpha = z^\alpha \bar{\psi}^\gamma p_\gamma - \bar{p}^\gamma \xi_\gamma^\alpha + \dots, \quad \bar{Q}_\beta = \bar{z}_\beta \bar{p}^\gamma \psi_\gamma - \bar{\xi}_\beta^\gamma p_\gamma + \dots, \quad (19)$$

$$S = -z^\beta \bar{\xi}_\beta^\gamma p_\gamma - \bar{p}^\gamma \psi_\gamma + \dots, \quad \bar{S} = -\bar{p}^\gamma \xi_\gamma^\beta \bar{z}_\beta - \bar{\psi}^\gamma p_\gamma + \dots.\quad (20)$$

#### 2.4. Symplectic structure

It is funny enough that we can go much further to find all Dirac brackets between all our physical variables. The idea is to use some additional information we already know. First, as we stressed above we know the bosonic part of our theory, providing we are interested in the  $SU(n+1)$  invariant system. The bosonic part of the Hamiltonian corresponding to the Lagrangian (7) reads

$$H = p^\alpha (g^{-1})_\alpha^\beta \bar{p}_\beta, \quad (21)$$

where  $(g^{-1})_\alpha^\beta$  is the inverse metric to the Fubini-Study one given by

$$(g^{-1})_\alpha^\beta = (1 + z \cdot \bar{z}) (\delta_\alpha^\beta + \bar{z}_\alpha z^\beta). \quad (22)$$

Second, we suppose that our supercharges obey the standard  $\mathcal{N} = 2(n+1)$  Poincaré superalgebra

$$\begin{aligned}\{Q^\alpha, \bar{Q}_\beta\} &= 2i\delta_\beta^\alpha H, \quad \{Q^\alpha, Q^\beta\} = 0, \quad \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0, \\ \{S, \bar{S}\} &= 2iH, \quad \{S, S\} = 0, \quad \{\bar{S}, \bar{S}\} = 0.\end{aligned}\quad (23)$$

Now we can check relations (23) in zero-order in fermionic variables using the already known terms in the supercharges (19). Assuming the most general ansatz for the Dirac brackets between fermionic variables

$$\begin{aligned}\{\psi_\alpha, \bar{\psi}^\beta\} &= f_1 \delta_\alpha^\beta + f_2 z^\beta \bar{z}_\alpha, \\ \{\xi_\alpha^\beta, \bar{\xi}_\gamma^\rho\} &= h_1 \delta_\alpha^\beta \delta_\gamma^\rho + h_2 \delta_\alpha^\rho \delta_\gamma^\beta + h_3 \delta_\alpha^\beta z^\rho \bar{z}_\gamma + h_4 \delta_\alpha^\rho z^\beta \bar{z}_\gamma + h_5 \delta_\gamma^\beta z^\rho \bar{z}_\alpha + h_6 \delta_\gamma^\rho z^\beta \bar{z}_\alpha + h_7 z^\beta \bar{z}_\alpha z^\rho \bar{z}_\gamma\end{aligned}$$

<sup>2</sup> For the bosonic coordinates  $z^\alpha$  and conjugated momenta  $p_\alpha$  the Poisson and Dirac brackets are the same.

with some arbitrary at the moment functions  $f$  and  $h$  depending on  $z \cdot \bar{z}$  only, one may fix these functions completely by checking relations (23) with the Hamiltonian (21). The resulting Dirac brackets read

$$\{\psi_\alpha, \bar{\psi}^\beta\} = \frac{i}{1+z \cdot \bar{z}} (g^{-1})_\alpha^\beta, \quad \{\xi_\alpha^\beta, \bar{\xi}_\gamma^\rho\} = i(1+z \cdot \bar{z}) g_\gamma^\beta (g^{-1})_\alpha^\rho. \quad (24)$$

Thus, we have at hand half of all nonzero Dirac brackets for the physical variables. It is interesting that we can proceed even further. Indeed, the brackets (24) completely fixed the kinetic terms for the fermionic variables in the Lagrangian to be

$$\mathcal{L} = g_\alpha^\beta \dot{z}^\alpha \dot{\bar{z}}_\beta + \frac{i(1+z \cdot \bar{z})}{2} g_\alpha^\beta (\dot{\bar{\psi}}^\alpha \psi_\beta - \bar{\psi}^\alpha \dot{\psi}_\beta) + \frac{i}{2(1+z \cdot \bar{z})} g_\rho^\alpha (g^{-1})_\beta^\gamma (\dot{\bar{\xi}}_\gamma^\rho \xi_\alpha^\beta - \bar{\xi}_\gamma^\rho \dot{\xi}_\alpha^\beta). \quad (25)$$

The explicit form of the bosonic momenta cannot be restored from (25) because the full Lagrangian could contain additional terms which are linear in the bosonic velocities. But we know for sure the first terms in  $\{p_\alpha, \bar{p}^\alpha\}$

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{z}^\alpha} = g_\alpha^\beta \dot{\bar{z}}_\beta + \dots, \quad \bar{p}^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\bar{z}}_\alpha} = \dot{z}^\beta g_\beta^\alpha + \dots \quad (26)$$

Assuming now the standard canonical relations between momenta and the conjugated coordinate one can get the full symplectic structure of our theory

$$\begin{aligned} \{z^\alpha, p_\beta\} &= \delta_\beta^\alpha, \quad \{\psi_\alpha, \bar{\psi}^\beta\} = \frac{i}{1+z \cdot \bar{z}} (g^{-1})_\alpha^\beta, \quad \{\xi_\alpha^\beta, \bar{\xi}_\gamma^\rho\} = i(1+z \cdot \bar{z}) g_\gamma^\beta (g^{-1})_\alpha^\rho, \\ \{p_\alpha, \psi_\beta\} &= -\frac{1}{2} (1+z \cdot \bar{z}) \bar{z}_\beta g_\alpha^\sigma \psi_\sigma, \quad \{p_\alpha, \bar{\psi}^\beta\} = -\frac{1}{2(1+z \cdot \bar{z})} \delta_\alpha^\beta \bar{\psi}^\sigma \bar{z}_\sigma, \\ \{p_\alpha, \xi_\beta^\gamma\} &= \frac{1}{2} \delta_\alpha^\gamma \xi_\beta^\sigma \bar{z}_\sigma - \frac{1}{2(1+z \cdot \bar{z})^2} \bar{z}_\alpha \xi_\beta^\sigma (g^{-1})_\sigma^\gamma - \frac{1}{2} (1+z \cdot \bar{z}) \bar{z}_\beta g_\alpha^\sigma \xi_\sigma^\gamma, \\ \{p_\alpha, \bar{\xi}_\beta^\gamma\} &= \frac{1}{2(1+z \cdot \bar{z})} (\bar{z}_\beta \bar{\xi}_\alpha^\gamma - \bar{z}_\alpha \bar{\xi}_\beta^\gamma - \delta_\alpha^\gamma \bar{\xi}_\beta^\sigma \bar{z}_\sigma), \\ \{p_\alpha, p_\beta\} &= \frac{i}{4(1+z \cdot \bar{z})^2} (\bar{z}_\beta \psi_\alpha - \bar{z}_\alpha \psi_\beta) + \\ &+ \frac{i}{4(1+z \cdot \bar{z})^3} (\bar{z}_\beta (\bar{\xi} g \xi g^{-1} \bar{z})_\alpha - \bar{z}_\alpha (\bar{\xi} g \xi g^{-1} \bar{z})_\beta + \bar{z}_\beta (g \xi g^{-1} \bar{\xi} \bar{z})_\alpha - \bar{z}_\alpha (g \xi g^{-1} \bar{\xi} \bar{z})_\beta), \\ \{p_\alpha, \bar{p}^\beta\} &= \frac{i}{4(1+z \cdot \bar{z})} ((g\psi)_\alpha (z g^{-1} \bar{z}) (\bar{\psi} g)^\beta - g_\alpha^\beta (z\psi) (\bar{\psi} \bar{z})) + \\ &+ \frac{i}{4(1+z \cdot \bar{z})^3} ((z g^{-1} \bar{z}) [(\bar{\xi} g \xi)_\alpha^\beta + (g \xi g^{-1} \bar{\xi} g)_\alpha^\beta] - g_\alpha^\beta [(z \xi g^{-1} \bar{\xi} \bar{z}) + (z g^{-1} \bar{\xi} g \xi g^{-1} \bar{z})]). \end{aligned}$$

These brackets are invariant with respect to the  $SU(n+1)$  group.

The final and the most complicated step is to find the full supercharges which should contain the terms cubic in the fermionic fields. The full solution of this problem is yet unknown. In the next Section we will demonstrate how these terms could be found in the simplest case of GNLC which corresponds to the  $CP^1$  model.

### 2.5. $CP^1$ model and $SU(2)$ symmetry group

Let us remind that a symmetry group of the  $CP^n$  manifold is the  $SU(n+1)$  one. In the simplest case of the  $CP^1$  manifold this group is  $SU(2)$  one and their the generators  $(R, \bar{R}, U)$  satisfying the commutation relations

$$[U, R] = iR, \quad [U, \bar{R}] = -i\bar{R}, \quad [R, \bar{R}] = 2iU. \quad (27)$$

Under the action of the generators  $R, \bar{R}$  the bosonic and fermionic coordinates transform as in (3), while the action of the last generator  $U$  could be obtained as the corresponding commutator of  $R$  and  $\bar{R}$  ones. For the case at hand, it is convenient to pass to another basis of spinor variables

$$\tilde{\psi} = \sqrt{1+z\cdot\bar{z}}\psi, \quad \tilde{\bar{\psi}} = \sqrt{1+z\cdot\bar{z}}\bar{\psi}, \quad \tilde{\xi} = \sqrt{1+z\cdot\bar{z}}\xi, \quad \tilde{\bar{\xi}} = \sqrt{1+z\cdot\bar{z}}\bar{\xi}. \quad (28)$$

It is easy to find that these spinors transform under the action of the generators  $R$  and  $\bar{R}$  as

$$\delta\tilde{\psi} = \frac{3}{2}a\bar{z}\tilde{\psi} + \frac{1}{2}\bar{a}z\tilde{\psi}, \quad \delta\tilde{\xi} = \frac{3}{2}a\bar{z}\tilde{\xi} + \frac{1}{2}\bar{a}z\tilde{\xi}. \quad (29)$$

Then, defining the new bosonic momenta as

$$\tilde{p} = p + i\frac{\bar{z}}{(1+z\cdot\bar{z})^3}(\tilde{\psi}\tilde{\bar{\psi}} + \tilde{\xi}\tilde{\bar{\xi}}), \quad \bar{\tilde{p}} = \bar{p} - i\frac{z}{(1+z\cdot\bar{z})^3}(\tilde{\psi}\tilde{\bar{\psi}} + \tilde{\xi}\tilde{\bar{\xi}}), \quad (30)$$

one can rewrite the nonvanishing Dirac brackets (27) in terms of these new variables

$$\begin{aligned} \{z, \tilde{p}\} &= 1, \quad \{\bar{z}, \bar{\tilde{p}}\} = 1, \quad \{\tilde{p}, \bar{\tilde{p}}\} = \frac{2i}{(1+z\cdot\bar{z})^4}(\tilde{\psi}\tilde{\bar{\psi}} + \tilde{\xi}\tilde{\bar{\xi}}), \\ \{\tilde{p}, \tilde{\psi}\} &= -\frac{2\bar{z}}{1+z\cdot\bar{z}}\tilde{\psi}, \quad \{\tilde{p}, \tilde{\xi}\} = -\frac{2\bar{z}}{1+z\cdot\bar{z}}\tilde{\xi}, \\ \{\bar{\tilde{p}}, \tilde{\bar{\psi}}\} &= -\frac{2z}{1+z\cdot\bar{z}}\tilde{\bar{\psi}}, \quad \{\bar{\tilde{p}}, \tilde{\bar{\xi}}\} = -\frac{2z}{1+z\cdot\bar{z}}\tilde{\bar{\xi}}, \\ \{\tilde{\psi}, \tilde{\bar{\psi}}\} &= i(1+z\cdot\bar{z})^2, \quad \{\tilde{\xi}, \tilde{\bar{\xi}}\} = i(1+z\cdot\bar{z})^2. \end{aligned} \quad (31)$$

The generators of the  $SU(2)$  group in these variables acquire the form

$$\begin{aligned} R &= \tilde{p} + \bar{z}^2\bar{\tilde{p}} - \frac{i}{2}\frac{1-3z\cdot\bar{z}}{(1+z\cdot\bar{z})^3}\bar{z}(\tilde{\psi}\tilde{\bar{\psi}} + \tilde{\xi}\tilde{\bar{\xi}}), \\ \bar{R} &= \bar{\tilde{p}} + z^2\tilde{p} + \frac{i}{2}\frac{1-3z\cdot\bar{z}}{(1+z\cdot\bar{z})^3}z(\tilde{\psi}\tilde{\bar{\psi}} + \tilde{\xi}\tilde{\bar{\xi}}), \\ U &= iz\tilde{p} - i\bar{z}\bar{\tilde{p}} + \frac{1+5z\cdot\bar{z}}{2(1+z\cdot\bar{z})^3}(\tilde{\psi}\tilde{\bar{\psi}} + \tilde{\xi}\tilde{\bar{\xi}}). \end{aligned} \quad (32)$$

Now we are ready to write the most general ansatz for the cubic terms in the supercharges, compatible with the manifest  $U(1)$  symmetry

$$\begin{aligned} Q &= \frac{z}{\sqrt{1+z\cdot\bar{z}}}\tilde{p}\tilde{\bar{\psi}} - \frac{1}{\sqrt{1+z\cdot\bar{z}}}\bar{\tilde{p}}\tilde{\xi} - if_1\tilde{\xi}\tilde{\bar{\xi}}\tilde{\bar{\psi}} - if_2z\tilde{\xi}\tilde{\bar{\psi}}\tilde{\bar{\psi}}, \\ \bar{Q} &= \frac{\bar{z}}{\sqrt{1+z\cdot\bar{z}}}\bar{\tilde{p}}\tilde{\psi} - \frac{1}{\sqrt{1+z\cdot\bar{z}}}\tilde{p}\tilde{\bar{\xi}} + if_1\tilde{\xi}\tilde{\bar{\xi}}\tilde{\psi} + if_2\bar{z}\tilde{\xi}\tilde{\bar{\psi}}\tilde{\bar{\psi}}, \\ S &= -\frac{z}{\sqrt{1+z\cdot\bar{z}}}\tilde{p}\tilde{\bar{\xi}} - \frac{1}{\sqrt{1+z\cdot\bar{z}}}\bar{\tilde{p}}\tilde{\psi} + if_3\tilde{\xi}\tilde{\bar{\xi}}\tilde{\bar{\psi}} - if_4z\tilde{\psi}\tilde{\bar{\xi}}\tilde{\bar{\xi}}, \\ \bar{S} &= -\frac{\bar{z}}{\sqrt{1+z\cdot\bar{z}}}\bar{\tilde{p}}\tilde{\xi} - \frac{1}{\sqrt{1+z\cdot\bar{z}}}\tilde{p}\tilde{\bar{\psi}} - if_3\tilde{\xi}\tilde{\bar{\xi}}\tilde{\psi} + if_4\bar{z}\tilde{\psi}\tilde{\bar{\xi}}\tilde{\bar{\xi}}. \end{aligned} \quad (33)$$

The  $U(1)$  covariance implies that all arbitrary functions  $f_i$  in the supercharges (33) have to be neutral under the action of  $U(1)$  and therefore these functions depend on the combination  $(z\cdot\bar{z})$

only. Requiring that the supercharges form the superalgebra (23), one could find these functions explicitly

$$f_1 = f_3 = \frac{3z \cdot \bar{z}}{2\sqrt{1+z \cdot \bar{z}}(1+z \cdot \bar{z})^3}, \quad f_2 = f_4 = \frac{3}{2\sqrt{1+z \cdot \bar{z}}(1+z \cdot \bar{z})^3}. \quad (34)$$

The corresponding Hamiltonian, which appears in the right-hand side of anticommutators  $\{Q, \bar{Q}\} = 2iH$  and  $\{S, \bar{S}\} = 2iH$ , reads

$$\begin{aligned} H = & (1+z \cdot \bar{z})^2 \tilde{p}\tilde{p} + \frac{3i}{2(1+z \cdot \bar{z})} (z\tilde{p} - \bar{z}\tilde{p})(\tilde{\psi}\tilde{\psi} + \tilde{\xi}\tilde{\xi}) \\ & + \frac{i}{1+z \cdot \bar{z}} \tilde{p}\tilde{\psi}\tilde{\xi} + \frac{i}{1+z \cdot \bar{z}} \tilde{p}\tilde{\psi}\tilde{\xi} + \frac{2+9z \cdot \bar{z}}{(1+z \cdot \bar{z})^4} \tilde{\psi}\tilde{\psi}\tilde{\xi}\tilde{\xi}. \end{aligned} \quad (35)$$

One can check that the Hamiltonian (35) commutes with the  $SU(2)$  generators  $\{R, \bar{R}, U\}$ , as it should be. Finally, introducing new bosonic momenta

$$\begin{aligned} P = & \tilde{p} - \frac{3i}{(1+z \cdot \bar{z})^3} \bar{z}(\tilde{\psi}\tilde{\psi} + \tilde{\xi}\tilde{\xi}) + \frac{i}{(1+z \cdot \bar{z})^3} \tilde{\psi}\tilde{\xi}, \\ \bar{P} = & \tilde{p} + \frac{3i}{(1+z \cdot \bar{z})^3} (z\tilde{\psi}\tilde{\psi} + \tilde{\xi}\tilde{\xi}) + \frac{i}{(1+z \cdot \bar{z})^3} \tilde{\psi}\tilde{\xi}, \end{aligned} \quad (36)$$

one can rewrite the Hamiltonian as a free bosonic one

$$H = (1+z \cdot \bar{z})^2 P\bar{P}, \quad (37)$$

in full agreement with the analysis performed in [3, 4].

### 3. Conclusion

In this Letter we proposed the nonlinear constraints which pick up a Generalized Nonlinear Chiral Multiplet. The basic  $2n$  scalar superfields are defined in  $\mathcal{N} = 2(n+1)$  superspace and transform nonlinearly under the action of the  $SU(n+1)$  group. The  $2(n+1)$  spinor derivatives (as well as supercharges) form a linear representation of the same  $SU(n+1)$  group. Selecting the simplest bosonic action describing the motion of a particle over the  $CP^n$  manifold we were able to make the first steps in its  $\mathcal{N} = 2(n+1)$  supersymmetrization. Insisting on the  $SU(n+1)$  symmetry of the system we construct the corresponding invariant symplectic structure and find the first, linear in the fermionic variables, terms in the supercharges. We explicitly demonstrate that in the simplest case of the  $CP^1$  manifold and, therefore, in the case of  $\mathcal{N} = 4$  supersymmetry, the cubic terms in the supercharges can be easily restored. The resulting system coincides with the  $\mathcal{N} = 4$  supersymmetric mechanics with nonlinear chiral supermultiplet [3, 4], as it should be.

The results we reported here are, of course, just the preparatory ones for a more detailed study of this system. The main question is the construction of the full supercharges, including the terms of the third order in spinors, as well as the complete Hamiltonian. This task is under consideration now.

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