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The adjoint representation of quantum algebra $U_q\big(\mathrm{sl}(2)\big)$

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Abstract

Starting from any representation of the Lie algebra $\mathfrak{g}$ on the finite dimensional vector space V we can construct the representation on the space $\mathrm{Aut}(V)$. These representations are of the type of ad. That is one of the reasons, why it is important to study the adjoint representation of the Lie algebra $\mathfrak{g}$ on the universal enveloping algebra $U(\mathfrak{g})$. A similar situation is for the quantum groups $U_q(\mathfrak{g})$. In this paper, we study the adjoint representation for the simplest quantum algebra $U_q\big(\mathrm{sl}(2)\big)$ in the case that q is not a root of unity.

1 Introduction

It is a well-known fact that the adjoint representation of the algebra sl(2) on its enveloping algebra $U\big(\mathrm{sl}(2)\big)$ is fully reducible and can be decomposed into a direct sum of finite dimensional representations, [1]. Specifically, let the generators e, f and h fulfil the commutation relations

$$\big[h,e\big]=2e\,,\qquad \big[h,f\big]=-2f\,,\qquad \big[e,f\big]=h.$$

Then the center $U\big(\mathrm{sl}(2)\big)$ is generated by an element

$$C=ef+fe+\tfrac{1}{2}\,h^2\,,$$

and for any n, $k\in\mathbb{N}_0$, $\mathbb{N}_0=0,\,1,\,2,\,\ldots$, the vector spaces

$$\mathcal{V}_{(n,k)}=\mathrm{ad}_{U(\mathrm{sl}(2))}\big(e^nC^k\big)$$

are invariant with respect to the adjoint representation and $(2n+1)$–dimensional. Moreover, we can write

$$U\big(\mathrm{sl}(2)\big) = \bigoplus_{n,k=0}^{\infty} \mathcal{V}_{(n,k)} \,. \tag{1.1}$$

A similar situation is for complex semisimple Lie algebras, see eg. [1, 2, 3, 4].

On the other hand, in the case of the quantum group $U_q\big(\mathrm{sl}(2)\big)$ there exists an element a for which the space $\mathrm{ad}_{U_q(\mathrm{sl}(2))}a$ is infinite dimensional. The aim of this paper is to find the decomposition of the quantum group $U_q\big(\mathrm{sl}(2)\big)$, where q is not a root of unity, which is similar to the decomposition (1.1).

2 The quantum group $U_q\big(\mathrm{sl}(2)\big)$

Let as have the quantum group $U_q\big(\mathrm{sl}(2)\big)$, [5], which is generated by the elements E, F, K and K^{-1} fulfilling the relations

$$\begin{aligned} &KE = q^2EK\,, \qquad KF = q^{-2}FK\,,\\ &EF - FE = \frac{K-K^{-1}}{q-q^{-1}}\,, \qquad KK^{-1} = K^{-1}K = 1\,. \end{aligned} \tag{2.1}$$

The coproduct $\triangle$, the antipod S and the counit ϵ are defined by

$$\begin{array}{lll} \triangle E = E\otimes K + 1\otimes E\,, & S(E) = -EK^{-1}\,, & \epsilon(E) = 0\,,\\ \triangle F = F\otimes 1 + K^{-1}\otimes F\,, & S(F) = -KF\,, & \epsilon(F) = 0\,,\\ \triangle K = K\otimes K\,, & S(K) = K^{-1}\,, & \epsilon(K) = 1\,,\\ \triangle K^{-1} = K^{-1}\otimes K^{-1}\,, & S\big(K^{-1}\big) = K\,, & \epsilon\big(K^{-1}\big) = 1\,. \end{array}$$

It is a well-known fact that the basis in $U_q\big(\mathrm{sl}(2)\big)$ consists of the elements $E^{n_1}F^{n_2}K^{n_3}$, where n_1, $n_2 \in \mathbb{N}_0$ and $n_3 \in \mathbb{Z}$.

The adjoint action of the Drinfeld–Jimbo algebras is given by the formula (see e.g. [6])

$$\mathrm{ad}_a b = \textstyle\sum a_{(1)}bS\big(a_{(2)}\big)\,.$$

In particular, in the case of the algebra $U_q\big(\mathrm{sl}(2)\big)$ we have

$$\begin{array}{ll} \mathrm{ad}_E a = EaK^{-1} - aEK^{-1}\,, & \mathrm{ad}_F a = Fa - K^{-1}aKF\,,\\ \mathrm{ad}_K a = KaK^{-1}\,, & \mathrm{ad}_{K^{-1}} a = K^{-1}aK\,. \end{array} \tag{2.2}$$

This shows that $\mathrm{ad}_E E = \mathrm{ad}_F\big(KF\big) = 0$ holds. We denote

$$\begin{aligned} &X = E\,, \qquad Y = KF\,,\\ &Z = \mathrm{ad}_E Y = q^{-2}EF - FE = \frac{K-K^{-1}}{q-q^{-1}} - q^{-1}\big(q-q^{-1}\big)EF\,, \end{aligned} \tag{2.3}$$

and take the element

$$C = EF + \frac{q^{-1}K + qK^{-1}}{\big(q-q^{-1}\big)^2}\,, \tag{2.4}$$

which generates the center of $U_q\big(\mathrm{sl}(2)\big)$.

From equations (2.3) and (2.4) we obtain

$$K=\frac{q-q^{-1}}{q+q^{-1}}\Big(qZ+(q-q^{-1})C\Big)\,. \tag{2.5}$$

Notice that the element K^{-1} can not be to expressed by these equations.

3 The new basis in $U_q\big(\mathrm{sl}(2)\big)$

The next step is to express $U_q\big(\mathrm{sl}(2)\big)$ using the elements X, Y, Z, C and $W=K^{-1}$. From relations (2.3), (2.4), (2.5) and the commutation relations (2.1) we obtain

$$\begin{aligned}
&CX=CX\,,\qquad CY=YC\,,\qquad CZ=ZC\,,\qquad CW=WC\,,\\
&WX=q^{-2}XW\,,\qquad WY=q^{2}YW\,,\qquad WZ=ZW\,,\\
&ZX=q^{2}XZ+(q-q^{-1})^{2}XC\,,\quad YZ=q^{2}ZY+(q-q^{-1})^{2}YC\,,\\
&YX-XY+\frac{q-q^{-1}}{q+q^{-1}}\Big(qZ+(q-q^{-1})C\Big)Z=0\,,\\
&\Big(qZ+(q-q^{-1})C\Big)W-\frac{q+q^{-1}}{q-q^{-1}}=0\,,\\
&\Big(qZ+(q-q^{-1})C\Big)\Big(q^{-1}Z-(q-q^{-1})C\Big)+q(q+q^{-1})^{2}XY+\left(\frac{q+q^{-1}}{q-q^{-1}}\right)^{2}=0\,.
\end{aligned} \tag{3.1}$$

The original elements E, F, K and K^{-1} in $U_q\big(\mathrm{sl}(2)\big)$ can be written in the following way:

$$E=X,\quad F=WY,\quad K=\frac{q-q^{-1}}{q+q^{-1}}\Big(qZ+(q-q^{-1})C\Big),\quad K^{-1}=W\,.$$

If we denote by $\mathcal{T}$ a free algebra generated by X, Y, Z, C and W, and take the two-sided ideal $\mathcal{J}$ in $\mathcal{T}$ generated by relations (3.1), the algebra $U_q\big(\mathrm{sl}(2)\big)$ is equal to $\mathcal{T}/\mathcal{J}$.

It is easy to show the following analogy of the Poincaré–Birkhoff–Witt theorem:

Lemma. *The elements $X^{n_1}Y^{n_2}Z^{n_3}C^{n_4}W^{n_5}$, where $n_k\geq 0$, $n_1n_2=0$ and $n_3n_5=0$, form the basis in $U_q\big(\mathrm{sl}(2)\big)$.*

4 The adjoint representation in the new basis

The adjoint representation (2.2) can be rewritten in the form

$$\begin{aligned}
\mathrm{ad}_E a&=XaW-aXW\,,\qquad \mathrm{ad}_F a=WYa-WaY\,,\\
\mathrm{ad}_K a&=\frac{q-q^{-1}}{q+q^{-1}}\Big(qZ+(q-q^{-1})C\Big)aW\,,\\
\mathrm{ad}_{K^{-1}} a&=\frac{q-q^{-1}}{q+q^{-1}}\,Wa\Big(qZ+(q-q^{-1})C\Big)\,.
\end{aligned} \tag{4.1}$$

4.1 The finite dimensional representations

From relations (3.1) and (4.1) it is possible by direct calculations, to show that the vector space $\mathcal{V}_N$ with the basis $X^{n_1}Y^{n_2}Z^{n_3}C^{n_4}$, where $n_1n_2 = 0$ and $n_1+n_2+n_3+n_4 \le N$, is invariant with respect to the adjoint representation for any N. The dimension of this space is

$$\dim \mathcal{V}_N = \tfrac{1}{6}\,(N+1)(N+2)(2N+3)\,.$$

In the space $\mathcal{V}_N$ there are the highest weight elements $X^{n_1}C^{n_4}$, $n_1+n_4 \le N$. It is well known, see eg. [5], p. 61, that the invariant subspace

$$\mathcal{V}_{(n_1,n_4)} = \mathrm{ad}_{U_q(\mathrm{sl}(2))}\big(X^{n_1}C^{n_4}\big)$$

has the dimension $2n_1+1$ and is irreducible. Since

$$\sum_{n_1+n_4\le N}(2n_1+1) = \tfrac{1}{6}\,(N+1)(N+2)(2N+3) = \dim \mathcal{V}_N\,,$$

the subspace $\mathcal{V}_{\mathrm{fin}}$ with the basis $X^{n_1}Y^{n_2}Z^{n_3}C^{n_4}$, $n_1n_2 = 0$, can be decomposed in to a direct sum of invariant subspaces

$$\mathcal{V}_{\mathrm{fin}} = \bigoplus_{n_1,n_4} \mathcal{V}_{(n_1,n_4)}\,.$$

The eigenvalue of ad_C on the space $\mathcal{V}_{(n_1,n_4)}$ is determined by the relation

$$\mathrm{ad}_C\big(X^{n_1}C^{n_4}\big) = \frac{q^{2n_1+1}+q^{-2n_1-1}}{\big(q-q^{-1}\big)^2}\,X^{n_1}C^{n_4}\,.$$

4.2 The infinite dimensional representations

By direct calculations we obtain the action of the adjoint representation on the elements of the basis $X^{n_1}C^{n_4}W^{n_5}$ and $Y^{n_2}C^{n_4}W^{n_5}$ for $n_5 \ge 1$ of the following form (in the formulas where X^{n_1-1} resp. Y^{n_2-1} occurs we suppose that $n_1>0$ resp. $n_2>0$):

$$\begin{aligned}
\mathrm{ad}_E\big(X^{n_1}C^{n_4}W^{n_5}\big) &= \big(1-q^{-2n_5}\big)X^{n_1+1}C^{n_4}W^{n_5+1},\\
\mathrm{ad}_F\big(X^{n_1}C^{n_4}W^{n_5}\big) &= \frac{q^{-2n_1+n_5+1}\big[n_5\big]}{q-q^{-1}}\,X^{n_1-1}C^{n_4}W^{n_5+1} +\\
&\quad +\big(1-q^{-2n_1+2n_5}\big)X^{n_1-1}C^{n_4+1}W^{n_5} -\\
&\quad -\frac{q^{n_5-1}\big[2n_1-n_5\big]}{q-q^{-1}}\,X^{n_1-1}C^{n_4}W^{n_5-1},\\
\mathrm{ad}_K\big(X^{n_1}C^{n_4}W^{n_5}\big) &= q^{2n_1}X^{n_1}C^{n_4}W^{n_5},\\
\mathrm{ad}_{K^{-1}}\big(X^{n_1}C^{n_4}W^{n_5}\big) &= q^{-2n_1}X^{n_1}C^{n_4}W^{n_5},\\
\mathrm{ad}_E\big(Y^{n_2}C^{n_4}W^{n_5}\big) &= -\frac{q^{-n_5-1}\big[n_5\big]}{q-q^{-1}}\,Y^{n_2-1}C^{n_4}W^{n_5+1} +\\
&\quad +\big(q^{-2n_2}-q^{-2n_5}\big)Y^{n_2-1}C^{n_4+1}W^{n_5} +\\
&\quad +\frac{q^{-2n_2-n_5+1}\big[2n_2-n_5\big]}{q-q^{-1}}\,Y^{n_2-1}C^{n_4}W^{n_5-1},\\
\mathrm{ad}_F\big(Y^{n_2}C^{n_4}W^{n_5}\big) &= q^{2n_2+2}\big(1-q^{2n_5}\big)Y^{n_2+1}C^{n_4}W^{n_5+1},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{ad}_K\bigl(Y^{n_2}C^{n_4}W^{n_5}\bigr) &= q^{-2n_2}Y^{n_2}C^{n_4}W^{n_5},\\
\mathrm{ad}_{K^{-1}}\bigl(Y^{n_2}C^{n_4}W^{n_5}\bigr) &= q^{2n_2}Y^{n_2}C^{n_4}W^{n_5}\,.
\end{aligned}
$$

By using these relations we obtain

$$
\begin{aligned}
\mathrm{ad}_C\bigl(X^{n_1}C^{n_4}W^{n_5}\bigr) &= \frac{q^{2n_1-2n_5+1}+q^{-2n_1+2n_5-1}}{\bigl(q-q^{-1}\bigr)^2}\,X^{n_1}C^{n_4}W^{n_5}+\\
&\quad +q^{-2n_1}\bigl[n_5\bigr]\bigl[n_5+1\bigr]X^{n_1}C^{n_4}W^{n_5+2}+\\
&\qquad +q^{-n_1}\bigl[n_5\bigr]\bigl[n_1-n_5\bigr]\bigl(q-q^{-1}\bigr)^2X^{n_1}C^{n_4+1}W^{n_5+1}\,,\\
\mathrm{ad}_C\bigl(Y^{n_2}C^{n_4}W^{n_5}\bigr) &= \frac{q^{2n_2-2n_5+1}+q^{-2n_2+2n_5-1}}{\bigl(q-q^{-1}\bigr)^2}\,Y^{n_2}C^{n_4}W^{n_5}+\\
&\quad +q^{2n_2}\bigl[n_5\bigr]\bigl[n_5+1\bigr]Y^{n_2}C^{n_4}W^{n_5+2}+\\
&\qquad +q^{n_2}\bigl[n_5\bigr]\bigl[n_2-n_5\bigr]\bigl(q-q^{-1}\bigr)^2Y^{n_2}C^{n_4+1}W^{n_5+1}\,.
\end{aligned}
$$

4.2.1 The eigenvalues and the eigenfunctions of ad_C

Since $\mathcal{V}_{\mathrm{fin}}$ is invariant subspace, we can define factor–representation of ad on the space

$$
\mathcal{W}=U_q\bigl(\mathrm{sl}(2)\bigr)/\mathcal{V}_{\mathrm{fin}}\,.
$$

We denote this representation by the same symbol ad, no confusion arises from this abuse of notation.

In the space $\mathcal{W}$ we introduce the notation $T_n=\bigl[n-1\bigr]!\,W^n$ for $n\geq 1$ and the basis consisting of elements

$$
X^{n_1}T_{n_5}C^{n_4}\,,\qquad T_{n_5}Y^{n_2}C^{n_4}=\bigl[n_5-1\bigr]!\,W^{n_5}Y^{n_2}C^{n_4}=\bigl[n_5-1\bigr]!\,q^{2n_2n_5}Y^{n_2}C^{n_4}W^{n_5}\,.
$$

In this basis the representation has the form

$$
\begin{aligned}
\mathrm{ad}_E\bigl(X^{n_1}T_{n_5}C^{n_4}\bigr) &= q^{-n_5}\bigl(q-q^{-1}\bigr)X^{n_1+1}T_{n_5+1}C^{n_4},\\
\mathrm{ad}_F\bigl(X^{n_1}T_{n_5}C^{n_4}\bigr) &= \frac{q^{-2n_1+n_5+1}}{q-q^{-1}}\,X^{n_1-1}T_{n_5+1}C^{n_4}+\\
&\quad +q^{-n_1+n_5}\bigl[n_1-n_5\bigr]\bigl(q-q^{-1}\bigr)X^{n_1-1}T_{n_5}C^{n_4+1}-\\
&\qquad -\frac{q^{n_5-1}\bigl[2n_1-n_5\bigr]\bigl[n_5-1\bigr]}{q-q^{-1}}\,X^{n_1-1}T_{n_5-1}C^{n_4},\\
\mathrm{ad}_K\bigl(X^{n_1}T_{n_5}C^{n_4}\bigr) &= q^{2n_1}X^{n_1}T_{n_5}C^{n_4},\\
\mathrm{ad}_{K^{-1}}\bigl(X^{n_1}T_{n_5}C^{n_4}\bigr) &= q^{-2n_1}X^{n_1}T_{n_5}C^{n_4},\\
\mathrm{ad}_E\bigl(T_{n_5}Y^{n_2}C^{n_4}\bigr) &= -\frac{q^{-2n_2+n_5+1}}{q-q^{-1}}\,T_{n_5+1}Y^{n_2-1}C^{n_4}-\\
&\quad -q^{-n_2+n_5}\bigl[n_2-n_5\bigr]\bigl(q-q^{-1}\bigr)T_{n_5}Y^{n_2-1}C^{n_4+1}+\\
&\qquad +\frac{q^{n_5-1}\bigl[2n_2-n_5\bigr]\bigl[n_5-1\bigr]}{q-q^{-1}}\,T_{n_5-1}Y^{n_2-1}C^{n_4},\\
\mathrm{ad}_F\bigl(T_{n_5}Y^{n_2}C^{n_4}\bigr) &= -q^{-n_5}\bigl(q-q^{-1}\bigr)T_{n_5+1}Y^{n_2+1}C^{n_4},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{ad}_K\big(T_{n_5}Y^{n_2}C^{n_4}\big) &= q^{-2n_2}T_{n_5}Y^{n_2}C^{n_4},\\
\mathrm{ad}_{K^{-1}}\big(T_{n_5}Y^{n_2}C^{n_4}\big) &= q^{2n_2}T_{n_5}Y^{n_2}C^{n_4},\\
\mathrm{ad}_C\big(X^{n_1}T_{n_5}C^{n_4}\big) &= \frac{q^{2n_1-2n_5+1}+q^{-2n_1+2n_5-1}}{\big(q-q^{-1}\big)^2}X^{n_1}T_{n_5}C^{n_4}+q^{-2n_1}X^{n_1}T_{n_5+2}C^{n_4}+\\
&\qquad +q^{-n_1}\big[n_1-n_5\big]\big(q-q^{-1}\big)^2X^{n_1}T_{n_5+1}C^{n_4+1},\\
\mathrm{ad}_C\big(T_{n_5}Y^{n_2}C^{n_4}\big) &= \frac{q^{2n_2-2n_5+1}+q^{-2n_2+2n_5-1}}{\big(q-q^{-1}\big)^2}T_{n_5}Y^{n_2}C^{n_4}+q^{-2n_2}T_{n_5+2}Y^{n_2}C^{n_4}+\\
&\qquad +q^{-n_2}\big[n_2-n_5\big]\big(q-q^{-1}\big)^2T_{n_5+1}Y^{n_2}C^{n_4+1}.
\end{aligned}
$$

To find the eigenfunctions of ad_C, we add certain vectors of the form

$$
\sum_{n_1=0}^{N_1}\sum_{n_5=1}^{\infty}\sum_{n_4=0}^{\infty}a_{n_1,n_5,n_4}X^{n_1}T_{n_5}C^{n_4}\qquad\text{and}\qquad\sum_{n_2=0}^{N_2}\sum_{n_5=1}^{\infty}\sum_{n_4=0}^{\infty}b_{n_2,n_5,n_4}T_{n_5}C^{n_4}Y^{n_2}
$$

to the space $\mathcal{W}$ and denote this new space by $\overline{\mathcal{W}}$. In the space $\overline{\mathcal{W}}$ we find the elements

$$
\begin{aligned}
|\,N_1,0,N_4,S\rangle &= \sum_{k=0}^{\infty}\sum_{r=0}^{k}X^{N_1}\Big(a_{k,r}^{(N_1,N_4,S)}T_{N_1+S+2k}C^{N_4+2r}+\\
&\qquad +b_{k,r}^{(N_1,N_4,S)}T_{N_1+S+2k+1}C^{N_4+2r+1}\Big),\\
|\,0,N_2,N_4,S\rangle &= \sum_{k=0}^{\infty}\sum_{r=0}^{k}\Big(c_{k,r}^{(N_2,N_4,S)}T_{N_2+S+2k}C^{N_4+2r}+\\
&\qquad +d_{k,r}^{(N_2,N_4,S)}T_{N_2+S+2k+1}C^{N_4+2r+1}\Big)Y^{N_2},
\end{aligned}
$$

where $N_1+S\geq 1$, $N_2+S\geq 1$ and $a_{0,0}^{(N_1,N_4,S)}=1$, for which the relations

$$
\begin{aligned}
\mathrm{ad}_C\,|\,N_1,0,N_4,S\rangle &= \frac{q^{2S-1}+q^{-2S+1}}{\big(q-q^{-1}\big)^2}\,|\,N_1,0,N_4,S\rangle\,,\\
\mathrm{ad}_C\,|\,0,N_2,N_4,S\rangle &= \frac{q^{2S-1}+q^{-2S+1}}{\big(q-q^{-1}\big)^2}\,|\,0,N_2,N_4,S\rangle
\end{aligned}
$$

are valid.

For the constants $a_{k,r}^{(N_1,N_4,S)}$ and $b_{k,r}^{(N_1,N_4,S)}$ we obtain the system of equations

$$
\begin{aligned}
&a_{k+1,0}^{(N_1,N_4,S)}\big[2k+2\big]\big[2S+2k+1\big]+q^{-2N_1}a_{k,0}^{(N_1,N_4,S)}=0,\\
&a_{k+1,k+1}^{(N_1,N_4,S)}\big[2k+2\big]\big[2S+2k+1\big]=q^{-N_1}b_{k,k}^{(N_1,N_4,S)}\big[S+2k+1\big]\big(q-q^{-1}\big)^2,\\
&a_{k+1,r}^{(N_1,N_4,S)}\big[2k+2\big]\big[2S+2k+1\big]+q^{-2N_1}a_{k,r}^{(N_1,N_4,S)}=\\
&\qquad\qquad =q^{-N_1}b_{k,r-1}^{(N_1,N_4,S)}\big[S+2k+1\big]\big(q-q^{-1}\big)^2\qquad\text{for}\quad 1\leq r\leq k,\\
&b_{k,k}^{(N_1,N_4,S)}\big[2k+1\big]\big[2S+2k\big]=q^{-N_1}a_{k,k}^{(N_1,N_4,S)}\big[S+2k\big]\big(q-q^{-1}\big)^2,\\
&b_{k+1,r}^{(N_1,N_4,S)}\big[2k+3\big]\big[2S+2k+2\big]+q^{-2N_1}b_{k,r}^{(N_1,N_4,S)}=\\
&\qquad\qquad =q^{-N_1}a_{k+1,r}^{(N_1,N_4,S)}\big[S+2k+2\big]\big(q-q^{-1}\big)^2\qquad\text{for}\quad 0\leq r\leq k\,.
\end{aligned}
$$

The constants $c_{k,r}^{(N_2,N_4,S)}$ and $d_{k,r}^{(N_2,N_4,S)}$ fulfil a similar system of equations.

The system of equations for $a_{k,r}$ and $b_{k,r}$ (for simplicity we omit the upper indices) can be solved in the following way:

Let $a_{0,0}=1$. If we know any $a_{s,t}$ for $0\le t\le s\le k$ and $b_{s,t}$ for $0\le t\le s\le k-1$, we find $b_{k,r}$ from the equations

$$\begin{aligned}
\big[2k+1\big]\big[2S+2k\big]b_{k,k} &= q^{-N_1}\big(q-q^{-1}\big)^2\big[S+2k\big]a_{k,k},\\
\big[2k+1\big]\big[2S+2k\big]b_{k,r} &= q^{-N_1}\big(q-q^{-1}\big)^2\big[S+2k\big]a_{k,r}-q^{-2N_1}b_{k-1,r}\\
&\quad \text{for}\quad 0\le r\le k-1.
\end{aligned}\tag{4.2}$$

If we know any $a_{s,t}$ and $b_{s,t}$ for $0\le t\le s\le k$, we find $a_{k+1,r}$ from the equations

$$\begin{aligned}
\big[2k+2\big]\big[2S+2k+1\big]a_{k+1,0} &= -q^{-2N_1}a_{k,0},\\
\big[2k+2\big]\big[2S+2k+1\big]a_{k+1,r} &= q^{-N_1}\big(q-q^{-1}\big)^2\big[S+2k+1\big]b_{k,r-1}-q^{-2N_1}a_{k,r}\\
&\quad \text{for}\quad 1\le r\le k,\\
\big[2k+2\big]\big[2S+2k+1\big]a_{k+1,k+1} &= q^{-N_1}\big(q-q^{-1}\big)^2\big[S+2k+1\big]b_{k,k}.
\end{aligned}\tag{4.3}$$

As far as $S>0$, $S+k\neq 0$ for any k and from equations (4.2) and (4.3) we can evaluate $a_{k,r}$ and $b_{k,r}$ with the help of $a_{0,0}=1$.

For $S=-L\le 0$ we obtain zero in (4.2) for $k=L$ and we are not able to find $b_{L,r}$. For $L=0$ and $k=0$ equation (4.2) is trivially valid and gives no other condition for $b_{0,0}$. For $L\ge 1$ the systems (4.2) and (4.3) look like

$$\begin{aligned}
\big[2k+1\big]\big[2L-2k\big]b_{k,k} &= q^{-N_1}\big(q-q^{-1}\big)^2\big[L-2k\big]a_{k,k},\\
\big[2k+1\big]\big[2L-2k\big]b_{k,r} &= q^{-N_1}\big(q-q^{-1}\big)^2\big[L-2k\big]a_{k,r}+q^{-2N_1}b_{k-1,r}\\
&\quad \text{for}\quad 0\le r\le k-1,\\
\big[2k+2\big]\big[2L-2k-1\big]a_{k+1,0} &= q^{-2N_1}a_{k,0},\\
\big[2k+2\big]\big[2L-2k-1\big]a_{k+1,r} &= q^{-N_1}\big(q-q^{-1}\big)^2\big[L-2k-1\big]b_{k,r-1}+q^{-2N_1}a_{k,r}\\
&\quad \text{for}\quad 1\le r\le k,\\
\big[2k+2\big]\big[2L-2k-1\big]a_{k+1,k+1} &= q^{-N_1}\big(q-q^{-1}\big)^2\big[L-2k-1\big]b_{k,k}.
\end{aligned}\tag{4.4}$$

For $0\le k\le L-1$ and $k=L$ the system (4.2) is reduced to

$$a_{L,L}=0\,,\qquad b_{L-1,r}=q^{N_1}\big(q-q^{-1}\big)^2\big[L\big]a_{L,r}\quad \text{for}\quad 0\le r\le L-1\,. \tag{4.5}$$

In the appendix we show that the system (4.4) has a solution for which $a_{0,0}=1$ and the relations (4.5) are a consequence of it.

We make the following choice. The constants $b_{L,r}$, $0\le r\le L$ which are not determined by the system are set to zero. The constants $a_{k,r}$ and $b_{k,r}$ for $k\ge L$ are uniquely determined by equations (4.3) for $k\ge L$ and by equations (4.2) for $k\ge L+1$. Just this element is denoted by $|\,N_1,0,N_4,S\rangle$ for $S\le 0$.

Analogously, we define the elements $|\,0,N_2,N_4,S\rangle$.

4.2.2 The decomposition of the adjoint representation in $\widehat{\mathcal{W}}$

Next we introduce the space $\widehat{\mathcal{W}} \subset \overline{\mathcal{W}}$ which is generated by the elements $|\,N_1,0,N_4,S\rangle$ and $|\,0,N_2,N_4,S\rangle$. Now we find the action of the adjoint representation on these vectors. We obtain

$$\begin{aligned}\mathrm{ad}_E\,|\,N_1,0,N_4,S\rangle &= q^{-N_1-S}\big(q-q^{-1}\big)X^{N_1+1}\times\\ &\times\sum_{k=0}^{\infty}\sum_{r=0}^{k}\Big(q^{-2k}a_{k,r}^{(N_1,N_4,S)}T_{N_1+S+2k+1}C^{N_4+2r}+\\ &+q^{-2k-1}b_{k,r}^{(N_1,N_4,S)}T_{N_5+S+2k+2}C^{N_4+2r+1}\Big)\,.\end{aligned}$$

If we denote

$$a_{k,r}=q^{-2k}a_{k,r}^{(N_1,N_4,S)}\,,\qquad b_{k,r}=q^{-2k-1}b_{k,r}^{(N_1,N_4,S)}\,,$$

we can easily show that if $a_{k,r}^{(N_1,N_4,S)}$ and $b_{k,r}^{(N_1,N_4,S)}$ fulfill equations (4.2) and (4.3) for N_1 and S, then $a_{k,r}$, $b_{k,r}$ fulfil these systems for N_1+1 and S. In addition for $S=-L\le 0$ we have $b_{L,r}=0$. Since $a_{0,0}=a_{0,0}^{(N_1,N_4,S)}=1$,

$$\mathrm{ad}_E\,|\,N_1,0,N_4,S\rangle=\big(q-q^{-1}\big)q^{-N_1-S}\,|\,N_1+1,0,N_4,S\rangle\ . \tag{4.6}$$

Similarly, we show

$$\mathrm{ad}_F\,|\,0,N_2,N_4,S\rangle=-\big(q-q^{-1}\big)q^{-N_2-S}\,|\,0,N_2+1,N_4,S\rangle\ . \tag{4.7}$$

If we apply ad_F on $|\,N_1,0,N_4,S\rangle$ where $N_1\ge 1$, similar but the more cumbersome calculations gives

$$\mathrm{ad}_F\,|\,N_1,0,N_4,S\rangle=-\frac{q^{N_1+S-1}\big[N_1-S\big]\big[S+N_1-1\big]}{q-q^{-1}}\,|\,N_1-1,0,N_4,S\rangle \tag{4.8}$$

and for action ad_E on $|\,0,N_2,N_4,S\rangle$ we can show that for $N_2\ge 1$

$$\mathrm{ad}_E\,|\,0,N_2,N_4,S\rangle=\frac{q^{N_2+S-1}\big[N_2-S\big]\big[S+N_2-1\big]}{q-q^{-1}}\,|\,0,N_2-1,0,N_4,S\rangle\,. \tag{4.9}$$

Moreover, we have

$$\begin{aligned}\mathrm{ad}_K\,|\,N_1,0,N_4,S\rangle &= q^{2N_1}\,|\,N_1,0,N_4,S\rangle\,,\\ \mathrm{ad}_K\,|\,0,N_2,N_4,S\rangle &= q^{-2N_2}\,|\,0,N_2,N_4,S\rangle\,.\end{aligned}$$

The representation preserves the value of N_4 and S but changes N_1, and N_2. Nevertheless, we see that the lowest and highest weight vectors are

$$\begin{aligned}&\mathrm{ad}_F\,|\,N_1,0,N_4,S\rangle=0 \text{ when } S=N_1\ge 1 \text{ or } S=1-N_1\le 0,\\ &\mathrm{ad}_E\,|\,0,N_2,N_4,S\rangle=0 \text{ when } S=N_2\ge 1 \text{ or } S=1-N_2\le 0.\end{aligned}$$

From the computations above it follows that for $N_4 \geq 0$ the subspaces $\widehat{\mathcal{W}}$ defined as

$$\begin{aligned}
\widehat{\mathcal{W}}_{(S,0,N_4,S)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,S,0,N_4,S\rangle = \big\{|\,N_1,0,N_4,S\rangle\,;\; N_1 \geq S \geq 1\big\} \\
&\quad \text{for } S \geq 1\,, \\
\widehat{\mathcal{W}}_{(0,S,N_4,S)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,0,S,N_4,S\rangle = \big\{|\,0,N_2,N_4,S\rangle\,;\; N_2 \geq S \geq 1\big\} \\
&\quad \text{for } S \geq 1\,, \\
\widehat{\mathcal{W}}_{(1-S,0,N_4,S)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,1-S,0,N_4,S\rangle = \big\{|\,N_1,0,N_4,S\rangle\,;\; N_1 \geq 1-S\big\} \\
&\quad \text{for } S \leq 0\,, \\
\widehat{\mathcal{W}}_{(0,1-S,N_4,S)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,0,1-S,N_4,S\rangle = \big\{|\,0,N_2,N_4,S\rangle\,;\; N_2 \geq 1-S\big\} \\
&\quad \text{for } S \leq 0
\end{aligned} \tag{4.10}$$

are invariant.

Except the elements from the subspaces (4.10) in the space $\widehat{\mathcal{W}}$ there are the elements of the form $|\,N_1,0,N_4,S\rangle$ with $0 \leq N_1 < S$ and $|\,0,N_2,N_4,S\rangle$, where $0 \leq N_2 < S$. For the vectors $|\,0,0,N_4,S\rangle$, where $N_4 \geq 0$ and $S \geq 1$ we have from relations (4.6) and (4.7)

$$\begin{aligned}
\mathrm{ad}_{E^k}\,|\,0,0,N_4,S\rangle &= \big(q-q^{-1}\big)^k q^{-S-k(k-1)/2}\,|\,k,0,N_4,S\rangle\,, \\
\mathrm{ad}_{F^k}\,|\,0,0,N_4,S\rangle &= \big(q^{-1}-q\big)^k q^{-S-k(k-1)/2}\,|\,0,k,N_4,S\rangle\,.
\end{aligned}$$

For $N_4 \geq 0$ and $S \geq 1$ we consider the invariant subspace

$$\widehat{\mathcal{W}}_{(0,0,N_4,S)} = \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,0,0,N_4,S\rangle\ .$$

This subspace contains all vectors $|\,N_1,0,N_4,S\rangle$ and $|\,0,N_2,N_4,S\rangle$ and is reducible. It is clear that its irreducible components are $\widehat{\mathcal{W}}_{(S,0,N_4,S)}$ and $\widehat{\mathcal{W}}_{(0,S,N_4,S)}$.

If we bring together the previous results, we obtain

Proposition. *The space $\widehat{\mathcal{W}}$ can be expressed as a direct sum of the invariant subspaces in the form*

$$\widehat{\mathcal{W}} = \bigoplus_{N_4=0}^{\infty}\bigg(\bigoplus_{N_1=1}^{\infty}\Big(\widehat{\mathcal{W}}_{(N_1,0,N_4,1-N_1)} \oplus \widehat{\mathcal{W}}_{(0,N_1,N_4,1-N_1)}\Big) \oplus \bigoplus_{S=1}^{\infty}\widehat{\mathcal{W}}_{(0,0,N_4,S)}\bigg)\,.$$

The subspaces $\widehat{\mathcal{W}}_{(0,0,N_4,S)}$ are reducible, their irreducible components are $\widehat{\mathcal{W}}_{(S,0,N_4,S)}$ and $\widehat{\mathcal{W}}_{(0,S,N_4,S)}$, and the dimension of the factor–space $\widehat{\mathcal{W}}_{(0,0,N_4,S)}/\big(\widehat{\mathcal{W}}_{(S,0,N_4,S)} + \widehat{\mathcal{W}}_{(0,S,N_4,S)}\big)$ is $2S-1$.

4.2.3 The infinite dimensional representation for the adjoint representation of $U_q\big(\mathrm{sl}(2)\big)$

In section 4.2.1, we introduced the factor–space $\mathcal{W}$, its modification $\widehat{\mathcal{W}}$ and factor representation on the space $\widehat{\mathcal{W}}$. In this section we return to the adjoint representation on the $U_q\big(\mathrm{sl}(2)\big)$. This representation acts on the vectors $X^{n_1}T_{n_5}C^{n_4}$ and $T_{n_5}Y^{n_2}C^{n_4}$ for $n_5 > 1$, as was described in 4.2.1, but for $n_5=1$ we obtain

$$\begin{aligned}
\mathrm{ad}_F\big(X^{n_1}T_1C^{n_4}\big) &= \frac{q^{-2n_1+2}}{q-q^{-1}}\,X^{n_1-1}T_2C^{n_4} + \\
&\quad +q^{-n_1+1}\big[n_1-1\big]\big(q-q^{-1}\big)X^{n_1-1}T_1C^{n_4+1} - \frac{\big[2n_1-1\big]}{q-q^{-1}}\,X^{n_1-1}C^{n_4}, \\
\mathrm{ad}_E\big(T_1Y^{n_2}C^{n_4}\big) &= -\frac{q^{-2n_2+2}}{q-q^{-1}}\,T_2Y^{n_2-1}C^{n_4} - \\
&\quad -q^{-n_2+1}\big[n_2-1\big]\big(q-q^{-1}\big)T_1Y^{n_2-1}C^{n_4+1} + \frac{\big[2n_2-1\big]}{q-q^{-1}}\,Y^{n_2-1}C^{n_4}.
\end{aligned}$$

To get the eigenvectors and eigenvalues of the operator of ad_C back in the game, we modify $U_q\big(\mathrm{sl}(2)\big)$ as we did in the case of the space $\mathcal{W}$. This space will be denoted by $\widehat{U}_q\big(\mathrm{sl}(2)\big)$.

It is easy to show that for $N_1+S>1$ the representation is defined by relations (4.6), (4.7), (4.8), (4.9) and (4.10), but for $N_1+S=1$ we have relations (4.7), (4.9) and

$$\begin{aligned}
\mathrm{ad}_F\,|\,N_1,0,N_4,1-N_1\rangle &= -\frac{\big[2N_1-1\big]}{q-q^{-1}}\,X^{N_1-1}C^{N_4}\,, \\
\mathrm{ad}_E\,|\,0,N_2,N_4,1-N_2\rangle &= \frac{\big[2N_2-1\big]}{q-q^{-1}}\,Y^{N_2-1}C^{N_4}\,.
\end{aligned} \tag{4.11}$$

Consequently, the spaces

$$\begin{aligned}
\mathcal{V}_{(N_1,0,N_4,1-N_1)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,N_1,0,N_4,1-N_1\rangle\,, \\
\mathcal{V}_{(0,N_2,N_4,1-N_2)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,0,N_2,N_4,1-N_2\rangle
\end{aligned}$$

are no longer irreducible but contain the finite dimensional invariant subspaces

$$\mathrm{ad}_{U_q(\mathrm{sl}(2))}\big(X^{N_1-1}C^{N_4}\big) \quad \text{and} \quad \mathrm{ad}_{U_q(\mathrm{sl}(2))}\big(Y^{N_2-1}C^{N_4}\big).$$

If we now define for any $N_1\geq 1$, $S\geq 1$ and $N_4\geq 0$ the invariant subspaces of $\widehat{U}_q\big(\mathrm{sl}(2)\big)$

$$\begin{aligned}
\mathcal{V}_{(N_1-1,N_4)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\big(X^{N_1-1}C^{N_4}\big)\,, \\
\mathcal{V}_{(N_1,0,N_4,1-N_1)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,N_1,0,N_4,1-N_1\rangle\,, \\
\mathcal{V}_{(0,N_1,N_4,1-N_1)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,0,N_1,N_4,1-N_1\rangle\,, \\
\mathcal{V}_{(S,0,N_4,S)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,S,0,N_4,S\rangle\,, \\
\mathcal{V}_{(0,S,N_4,S)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,0,S,N_4,S\rangle\,, \\
\mathcal{V}_{(0,0,N_4,S)} &= \mathrm{ad}_{U_q(\mathrm{sl}(2))}\,|\,0,0,N_4,S\rangle\,,
\end{aligned}$$

we can formulate the following theorem.

Theorem. *The space $\widehat{U}_q\big(\mathrm{sl}(2)\big)$ can be decomposed to*

$$\widehat{U}_q\big(\mathrm{sl}(2)\big) = \bigoplus_{N_4=0}^{\infty}\left(\bigoplus_{N_1=1}^{\infty}\Big(\mathcal{V}_{(N_1,0,N_4,1-N_1)}+\mathcal{V}_{(0,N_1,N_4,1-N_1)}\Big)\oplus\bigoplus_{S=1}^{\infty}\mathcal{V}_{(0,0,N_4,S)}\right).$$

Moreover, $\mathcal{V}_{(N_1-1,N_4)}=\mathcal{V}_{(N_1,0,N_4,1-N_1)}\cap\mathcal{V}_{(0,N_1,N_4,1-N_1)}$ is the irreducible subspace of dimension $2N_1-1$, and the subspace $\mathcal{V}_{(0,0,N_4,S)}$ has the invariant infinite subspaces $\mathcal{V}_{(S,0,N_4,S)}$ and $\mathcal{V}_{(0,S,N_4,S)}$, and the factor–space $\mathcal{V}_{(0,0,N_4,S)}/\big(\mathcal{V}_{(S,0,N_4,S)}+\mathcal{V}_{(0,S,N_4,S)}\big)$ has the dimension $2S-1$.

5 Conclusion

In the presented paper, we have studied the decomposition of the adjoint representation of the quantum group $U_q\big(\mathrm{sl}(2)\big)$. By construction, we modified this quantum group $U_q\big(\mathrm{sl}(2)\big)$ by adding some formal series. Let us mention that this deformation can be obtained from $U_h\big(\mathrm{sl}(2)\big)$ that contains the formal series. The quantum group $U_q\big(\mathrm{sl}(2)\big)$ is a subgroup in $U_h\big(\mathrm{sl}(2)\big)$ where we restrict ourselves to the special formal series, especially to the formal series for K, K^{-1} and $\big[E,F\big] = \dfrac{K-K^{-1}}{q-q^{-1}}$.

We showed that the decomposition of the quantum algebra $\widehat{U}_q\big(\mathrm{sl}(2)\big)$ is more complicated than the non-deformed case. The modified space $\widehat{U}_q\big(\mathrm{sl}(2)\big)$ contains finite and infinite dimensional invariant subspaces and, in addition, this space is not fully reducible, i.e., we cannot write it as a direct sum of invariant irreducible subspaces.

Of much interest is a duality between the subspaces $\widehat{U}_q\big(\mathrm{sl}(2)\big)$ with $S > 0$ and $S \le 0$. It would be useful to study whether this duality is a property of $U_q\big(\mathrm{sl}(2)\big)$ only or it is a general property for all quantum groups $U_q(\mathfrak{g})$.

Appendix

In this appendix we show that the system (4.4) has a solution for which $a_{0,0} = 1$ and that the solution fulfils relation (4.5). First, we define the new variables $\alpha_{k,r}$ and $\beta_{k,r}$ by the relations

$$a_{k,r} = q^{-2kN_1}\big(q-q^{-1}\big)^{2r}\alpha_{k,r}\,, \qquad b_{k,r} = q^{-(2k+1)N_1}\big(q-q^{-1}\big)^{2r+2}\beta_{k,r}\,.$$

With respect to these new variables the system (4.4) has the form

$$\begin{aligned} \big[2k+1\big]\big[2L-2k\big]\beta_{k,k} &= \big[L-2k\big]\alpha_{k,k}\,, \\ \big[2k+1\big]\big[2L-2k\big]\beta_{k,r} &= \big[L-2k\big]\alpha_{k,r} + \beta_{k-1,r}\,, \qquad 0 \le r \le k-1\,, \\ \big[2k+2\big]\big[2L-2k-1\big]\alpha_{k+1,0} &= \alpha_{k,0}\,, \\ \big[2k+2\big]\big[2L-2k-1\big]\alpha_{k+1,r} &= \big[L-2k-1\big]\beta_{k,r-1} + \alpha_{k,r}\,, \qquad 1 \le r \le k\,, \\ \big[2k+2\big]\big[2L-2k-1\big]\alpha_{k+1,k+1} &= \big[L-2k-1\big]\beta_{k,k}\,, \end{aligned} \tag{5.1}$$

which is valid for $0 \le k \le L-1$, and the conditions (4.5) are of the form

$$\alpha_{L,L} = 0\,, \qquad \beta_{L-1,r} = \big[L\big]\alpha_{L,r} \quad \text{for} \quad 0 \le r \le L-1\,. \tag{5.2}$$

For $k \le L-1$ we obtain from the first and last equations of (5.1)

$$\begin{aligned} \beta_{k,k} &= \frac{\big[L-2k\big]}{\big[2k+1\big]\cdot\big[2L-2k\big]}\,\alpha_{k,k}, \\ \alpha_{k+1,k+1} &= \frac{\big[L-2k-1\big]\big[L-2k\big]}{\big[2k+1\big]\big[2k+2\big]\cdot\big[2L-2k-1\big]\big[2L-2k\big]}\,\alpha_{k,k} \end{aligned}$$

which results in

$$\begin{aligned} \alpha_{k,k} &= \frac{1}{[2k]!}\cdot\frac{[2L-2k]!}{[2L]!}\cdot\frac{[L]!}{[L-2k]!}\,, \\ \beta_{k,k} &= \frac{1}{[2k+1]!}\cdot\frac{[2L-2k-1]!}{[2L]!}\cdot\frac{[L]!}{[L-2k-1]!}\,. \end{aligned} \tag{5.3}$$

In particular, from (5.3) we obtain $\alpha_{k,k}=0$ for $k>\frac{1}{2}L$ and $\beta_{k,k}=0$ for $k>\frac{1}{2}(L-1)$, and thus $a_{L,L}=0$.

Next for $k\geq r$ we put

$$\alpha_{k,r}=\frac{[2r]!!}{[2k]!!}\cdot\frac{[2L-2k-1]!!}{[2L-2r-1]!!}\,R_{k,r}\,,\qquad \beta_{k,r}=\frac{[2r+1]!!}{[2k+1]!!}\cdot\frac{[2L-2k-2]!!}{[2L-2r-2]!!}\,S_{k,r}\,,$$

(and we define $[0]!!=[-1]!!=1$), we obtain from (5.1) the system

$$\begin{aligned} R_{k+1,0}-R_{k,0} &= 0\,,\\ R_{k+1,r}-R_{k,r} &= \frac{[2k]!!}{[2k+1]!!}\cdot\frac{[2r-1]!!}{[2r]!!}\cdot\frac{[2L-2k-2]!!}{[2L-2k-1]!!}\times\\ &\quad\times\frac{[2L-2r-1]!!}{[2L-2r]!!}[L-2k-1]\,S_{k,r-1}\,, \qquad r\geq 1\,,\\ S_{k+1,r}-S_{k,r} &= \frac{[2k+1]!!}{[2k+2]!!}\cdot\frac{[2r]!!}{[2r+1]!!}\cdot\frac{[2L-2k-3]!!}{[2L-2k-2]!!}\times\\ &\quad\times\frac{[2L-2r-2]!!}{[2L-2r-1]!!}[L-2k-2]\,R_{k+1,r}\,, \qquad r\geq 0\,. \end{aligned}$$

This systems results in

$$\begin{aligned} R_{k,0} &= 1\,,\\ R_{k,r} &= \frac{[2r-1]!!}{[2r]!!}\cdot\frac{[2L-2r-1]!!}{[2L-2r]!!}\sum_{s=r-1}^{k-1}\frac{[2s]!!}{[2s+1]!!}\times\\ &\quad\times\frac{[2L-2s-2]!!}{[2L-2s-1]!!}\cdot[L-2s-1]\,S_{s,r-1}\,, \qquad r\geq 1\,,\\ S_{k,r} &= \frac{[2r]!!}{[2r+1]!!}\cdot\frac{[2L-2r-2]!!}{[2L-2r-1]!!}\times\\ &\quad\times\sum_{s=r}^{k}\frac{[2s-1]!!}{[2s]!!}\cdot\frac{[2L-2s-1]!!}{[2L-2s]!!}\cdot[L-2s]\,R_{s,r}\,, \qquad r\geq 0 \end{aligned} \tag{5.4}$$

and the following relations:

$$\begin{aligned} &R_{k_0,r}=0 \quad\text{and}\quad S_{k',r-1}=0 \quad \text{for all}\quad k'\geq k_0 \Rightarrow R_{k,r}=0 \quad\text{for all}\quad k>k_0,\\ &S_{k_0,r}=0 \quad\text{and}\quad R_{k',r}=0 \quad \text{for all}\quad k'>k_0 \Rightarrow S_{k,r}=0 \quad\text{for all}\quad k>k_0\,. \end{aligned}$$

To prove (5.2) it suffices to show the relation

$$S_{L-1,0}=\frac{[L]}{[2L]}\,R_{L,0}=\frac{[L]}{[2L]} \tag{5.5}$$

and that for any r, $1 \le r \le \frac{1}{2}\,L$ we have

$$R_{L-r+1,r} = S_{L-r,r} = 0\,. \tag{5.6}$$

First, we prove (5.5). In accordance with (5.4) we have

$$S_{L-1,0} = \frac{\big[L\big]}{\big[2L\big]} + \frac{\big[2L-2\big]!!}{\big[2L-1\big]!!}\sum_{s=1}^{L-1}\frac{\big[2s-1\big]!!}{\big[2s\big]!!}\cdot\frac{\big[2L-2s-1\big]!!}{\big[2L-2s\big]!!}\cdot\big[L-2s\big]\,.$$

If we in substitute $s \to L-s$ in the sum, we will discover that equation (5.5) holds.

The conditions $R_{L-r+1,r} = S_{L-1,r} = 0$ for $1 \le r \le \frac{1}{2}\,L$ are equivalent to the equations

$$\begin{aligned}&\sum_{s=r-1}^{L-r}\frac{\big[2s\big]!!}{\big[2s+1\big]!!}\cdot\frac{\big[2L-2s-2\big]!!}{\big[2L-2s-1\big]!!}\cdot\big[L-2s-1\big]S_{s,r-1} = 0,\\&\sum_{s=r}^{L-r}\frac{\big[2s-1\big]!!}{\big[2s\big]!!}\cdot\frac{\big[2L-2s-1\big]!!}{\big[2L-2s\big]!!}\cdot\big[L-2s\big]R_{s,r} = 0\,.\end{aligned} \tag{5.7}$$

To prove these equations, we use (5.4). After an appropriate arrangement of the terms in the sums we can deduce that (5.7) is valid if the following statement holds:
Let $1 \le r \le \frac{1}{2}\,L$. Then for $L = 2J$ we have

$$\begin{aligned}&S_{J+s,r-1} - S_{J-s-1,r-1} = 0 \quad \text{for} \quad 0 \le s \le J-r\,,\\&R_{J+s,r} - R_{J-s,r} = 0 \qquad\quad\;\; \text{for} \quad 1 \le s \le J-r\,,\end{aligned} \tag{5.8}$$

and for $L = 2J+1$ is

$$\begin{aligned}&S_{J+s,r-1} - S_{J-s,r-1} = 0 \quad \text{for} \quad 1 \le s \le J-r+1\,,\\&R_{J+s+1,r} - R_{J-s,r} = 0 \quad\;\; \text{for} \quad 0 \le s \le J-r\,.\end{aligned} \tag{5.9}$$

We will study the first case (5.8). Let $1 \le r \le J$. The relation $R_{J+s,r} - R_{J-s,r} = 0$ for $1 \le s \le J-r$ with the use of (5.4) is equivalent to the equation

$$\sum_{t=J-s}^{J+s-1}\frac{\big[2t\big]!!}{\big[2t+1\big]!!}\cdot\frac{\big[4J-2t-2\big]!!}{\big[4J-2t-1\big]!!}\cdot\big[2J-2t-1\big]\Big(S_{t,r-1} - S_{2J-t-1,r-1}\Big) = 0\,,$$

which is the consequence of the relation $S_{J+s,r-1} - S_{J-s-1,r-1} = 0$ for $0 \le s \le J-r$. On the other hand, for $2 \le r$ the equality $S_{J+s,r-1} - S_{J-s-1,r-1} = 0$ for $0 \le s \le J-r$ is equivalent to the equation

$$\sum_{t=J-s}^{J+s}\frac{\big[2t-1\big]!!}{\big[2t\big]!!}\cdot\frac{\big[4J-2t-1\big]!!}{\big[4J-2t\big]!!}\cdot\big[2J-2t\big]\Big(R_{t,r-1} - R_{2J-t,r-1}\Big) = 0$$

and this is the consequence of the relation $R_{J+s,r-1} - R_{J-s,r-1} = 0$ for $1 \le s \le J-r+1$.

It follows from this consideration that for $L = 2J$ all relations (5.8) are result of the relations $S_{J+s,0} - S_{J-s-1,0} = 0$ for $0 \le s \le J-1$.

Similarly, we can see, that for $L = 2J+1$ the validity (5.8) follows from $S_{J+s,0} - S_{J-s,0} = 0$ for $1 \le s \le J$.

For $L = 2J$ with the use of (5.4) the condition $S_{J+s,0} - S_{J-s-1,0} = 0$ is equivalent to the equation

$$\sum_{r=J-s}^{J+s} \frac{\big[2r-1\big]!!}{\big[2r\big]!!} \cdot \frac{\big[4J-2r-1\big]!!}{\big[4J-2r\big]!!} \cdot \big[2J-2r\big] = 0\,,$$

which we can easily prove by substitution $r \to 2J - r$, and for $L = 2J + 1$ the condition $S_{J+s,0} - S_{J-s,0} = 0$ is equivalent to the equation

$$\sum_{r=J-s+1}^{J+s} \frac{\big[2r-1\big]!!}{\big[2r\big]!!} \cdot \frac{\big[4J-2r+1\big]!!}{\big[4J-2r+2\big]!!} \cdot \big[2J-2r+1\big] = 0\,,$$

which is easily seen to hold after substitution $r \to 2J - r + 1$.

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