Highest Weight Representation for Sklyanin Algebra sl(3)(u) with Application to the Gaudin Model.

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Abstrakt

We study the infinite dimensional Sklyanin algebra sl(3)(u). Specificaly we construct the highest weight representation for this algebra in an explicit form. Its application to the Gaudin model is mentioned.

1 Introduction

In 1973, M. Gaudin [1, 2] proposed a new class of integrable quantum models. This sl(2) Gaudin model was studied by many authors [1–9] from different points of view. Let us remember the well-known Bethe ansatz method for finding the eigenvectors and the eigenvalues [1, 10–12] in this case.
Let the generators e, f and h form a standard basis in sl(2) which fulfils the commutation relations

$$\big[h,e\big]=2e\,,\qquad \big[h,f\big]=-2f\,,\qquad \big[e,f\big]=h. \tag{1}$$

In the algebraic approach to the Bethe ansatz construction a fundamental role is played by the infinite dimensional Sklyanin algebra

$$\begin{aligned}
\big[E(u),E(w)\big] &= \big[F(u),F(w)\big]=\big[H(u),H(w)\big]=0\,,\\
\big[E(u),F(w)\big] &= -\frac{H(u)-H(w)}{u-w}\,,\\
\big[H(u),E(w)\big] &= -2\,\frac{E(u)-E(w)}{u-w}\,,\\
\big[H(u),F(w)\big] &= 2\,\frac{F(u)-F(w)}{u-w}\,.
\end{aligned}$$

We fix the notation

$$F(\mathbf{w}) = F(w_1)F(w_2)\dots F(w_n)\,,$$
$$F(\mathbf{w}-w_r) = F(w_1)\dots F(w_{r-1})F(w_{r+1})\dots F(w_n)\,,$$
$$F(\mathbf{w}+u) = F(u)F(w_1)\dots F(w_n)$$

and following [7] we can define on the space $|\,\mathbf{w}\rangle = F(\mathbf{w})\,|\,0\rangle$ the highest weight representation:

$$E(u)\,|\,0\rangle = 0\,, \qquad H(u)\,|\,0\rangle = \lambda(u)\,|\,0\rangle\,,$$
$$F(u)\,|\,\mathbf{w}\rangle = |\,\mathbf{w}+u\rangle\,,$$
$$H(u)\,|\,\mathbf{w}\rangle = \Big(\lambda(u) - \sum_r \frac{2}{u-w_r}\Big)\,|\,\mathbf{w}\rangle + 2\sum_r \frac{|\,\mathbf{w}+u-w_r\rangle}{u-w_r}\,,$$
$$E(u)\,|\,\mathbf{w}\rangle = -\sum_r \frac{|\,\mathbf{w}-w_r\rangle}{u-w_r}\Big(\lambda(u)-\lambda(w_r)\Big) + $$
$$+\sum_{r\neq\hat{r}} \frac{2\,|\,\mathbf{w}-w_r\rangle}{(u-w_{\hat{r}})(w_{\hat{r}}-w_r)} - \sum_{r\neq\hat{r}} \frac{|\,\mathbf{w}+u-w_r-w_{\hat{r}}\rangle}{(u-w_r)(u-w_{\hat{r}})}\,.$$

The aim of this paper is to obtain this highest weight representation in an explicit form for sl(3)(u). The formulas are given in section 3 theorem 1. In section 4, we remember our recent result [13] for the Gaudin model in the sl(3) case where this kind of representation was used implicitly.

2 The infinite-dimensional Sklyanin algebra sl(3)(u)

We will start with a basis in gl(3), e_{ij}, $i, j = 1, 2, 3$, where

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}\,.$$

The standard basis in sl(3) is then given by

$$e_1 = e_{12}\,,\quad e_2 = e_{23}\,,\quad e_3 = e_{13}\,,\quad f_1 = e_{21}\,,\quad f_2 = e_{32}\,,\quad f_3 = e_{31}\,,$$
$$h_1 = [e_1, f_1] = e_{11} - e_{22} \quad \text{and} \quad h_2 = [e_2, f_2] = e_{22} - e_{33}\,.$$

In Appendix A you can find the explicit commutation relations.
Again, the conversion to the Sklyanin algebra is given by

$$\big[X(u), Y(w)\big] = -\frac{Z(u)-Z(w)}{u-w} \quad \text{iff} \quad [x, y] = z\,.$$

We will not write here explicitly these formulas for sl(3)(u) but you can see some in the following lemma in section 3 and appendix B.

3 The highest weight representation for the Sklyanin algebra sl(3)(u)

To fix the notation, we write

$$F_j(\mathbf{w}_j) = F_j(w_{j,1})F_j(w_{j,2})\dots F_j(w_{j,k_j}), \qquad F(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3) = F_1(\mathbf{w}_1)F_2(\mathbf{w}_2)F_3\mathbf{w}_3)$$

and

$$F_k(\mathbf{w_k}) = F_k(w_{k,1})F_k(w_{k,2})\dots F_k(w_{k,n})\,,$$
$$F_k(\mathbf{w_k}-w_{k,r}) = F(w_{k,1})\dots F_k(w_{k,r-1})F_k(w_{k,r+1})\dots F_k(w_{k,n})\,,$$
$$F_k(\mathbf{w_k}+u) = F_k(u)F_k(w_{k,1})\dots F_k(w_{k,n})\,.$$

so we will work with the space

$$|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle = F(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3)\,|\,\mathbf{0},\mathbf{0},\mathbf{0}\rangle\,.$$

Again, we start with

$$E_i(u)\,|\,\mathbf{0},\mathbf{0},\mathbf{0}\rangle = 0\,,\quad H_j(u)\,|\,\mathbf{0},\mathbf{0},\mathbf{0}\rangle = \lambda_j(u)\,|\,\mathbf{0},\mathbf{0},\mathbf{0}\rangle\,.$$

The idea of construction is to use the PBW theorem just as in construction of Verma modules; so by calculation of the action $X(u)$ on $F(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3)\,|\,\mathbf{0},\mathbf{0},\mathbf{0}\rangle$ we must commute this element with $F_k(\mathbf{w_k})$. In the following lemma we show the case $F_1(\mathbf{w_1})$:

Lemma:

$$\begin{aligned}
F_1(u)F_1(\mathbf{w}_1) &= F_1(\mathbf{w}_1+u)\,,\\
F_2(u)F_1(\mathbf{w}_1) &= F_1(\mathbf{w}_1)F_2(u) - \sum_r \frac{F_1(\mathbf{w}_1-w_r)}{u-w_r}\big(F_3(u)-F_3(w)\big)\,,\\
F_3(u)F_1(\mathbf{w}_1) &= F_1(\mathbf{w}_1)F_3(u)\,,\\
H_1(u)F_1(\mathbf{w}_1) &= F_1(\mathbf{w}_1)\Big(H_1(u) - \sum_r \frac{2}{u-w_r}\Big) + 2\sum_r \frac{F_1(\mathbf{w}_1+u-w_r)}{u-w_r}\,,\\
H_2(u)F_1(\mathbf{w}_1) &= F_1(\mathbf{w}_1)\Big(H_2(u) + \sum_r \frac{1}{u-w_r}\Big) - \sum_r \frac{F_1(\mathbf{w}_1+u-w_r)}{u-w_r}\,,\\
E_1(u)F_1(\mathbf{w}_1) &= F_1(\mathbf{w}_1)E_1(u) - \sum_r \frac{F_1(\mathbf{w}_1-w_r)}{u-w_r}\Big(H_1(u)-H_1(w_r)\Big) +\\
&\quad + \sum_{r\neq\hat{r}} \frac{2F_1(\mathbf{w}_1-w_r)}{(u-w_{\hat{r}})(w_{\hat{r}}-w_r)} - \sum_{r\neq\hat{r}} \frac{F_1(\mathbf{w}_1+u-w_r-w_{\hat{r}})}{(u-w_r)(u-w_{\hat{r}})}\,,\\
E_2(u)F_1(\mathbf{w}_1) &= F_1(\mathbf{w}_1)E_2(u)\,,\\
E_3(u)F_1(\mathbf{w}_1) &= F_1(\mathbf{w}_1)E_3(u) + \sum_r \frac{F_1(\mathbf{w}_1-w_r)}{u-w_r}\Big(E_2(u)-E_2(w_r)\Big)
\end{aligned} \tag{2}$$

Proof: Direct calculations by using the mathematical induction with respect to k_1. For $k_1 = 1$ it is a definition of the commutation relation in sl(3)(u).

The other cases for $F_2(\mathbf{w_2})$ and $F_3(\mathbf{w_3})$ are written explicitly in Appendix B.

It is easy to see that

$$\begin{aligned}
&F_1(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle = |\,\mathbf{w}_1+u,\mathbf{w}_2,\mathbf{w}_3\rangle\\
&F_2(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle =\\
&\qquad = |\,\mathbf{w}_1,\mathbf{w}_2+u,\mathbf{w}_3\rangle - \sum_r \frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3+u\rangle}{u-w_{1,r}} + \sum_r \frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3+w_{1,r}\rangle}{u-w_{1,r}}\\
&F_3(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle = |\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3+u\rangle
\end{aligned}$$

By calculation of the action $H_1(u)$ we have

$$H_1(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle =$$
$$= F_1(\mathbf{w}_1)H_1(u)\,|\,\mathbf{0},\mathbf{w}_2,\mathbf{w}_3\rangle - \sum_r \frac{2\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}} + \sum_r \frac{2\,|\,\mathbf{w}_1+u-w_{1,r}\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}} =$$
$$= F_1(\mathbf{w}_1)F_2(\mathbf{w}_2)H_1(u)\,|\,\mathbf{0},\mathbf{0},\mathbf{w}_3\rangle + \sum_s \frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{2,s}} - \sum_s \frac{|\,\mathbf{w}_1,\mathbf{w}_2+u-w_{2,s},\mathbf{w}_3\rangle}{u-w_{2,s}} -$$
$$-\sum_r \frac{2\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}} + \sum_r \frac{2\,|\,\mathbf{w}_1+u-w_{1,r}\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}} =$$
$$= F_1(\mathbf{w}_1)F_2(\mathbf{w}_2)F_3(\mathbf{w}_3)H_1(u)\,|\,\mathbf{0},\mathbf{0},\mathbf{0}\rangle - \sum_t \frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{3,t}} +$$
$$+\sum_t \frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3+u-w_{3,t}\rangle}{u-w_{3,t}} + \sum_s \frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{2,s}} - \sum_s \frac{|\,\mathbf{w}_1,\mathbf{w}_2+u-w_{2,s},\mathbf{w}_3\rangle}{u-w_{2,s}} -$$
$$-\sum_r \frac{2\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}} + \sum_r \frac{2\,|\,\mathbf{w}_1+u-w_{1,r}\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}}$$

If we order it, we obtain

$$H_1(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle = \Big(\lambda_1 - \sum_r \frac{2}{u-w_{1,r}} + \sum_s \frac{1}{u-w_{2,s}} - \sum_t \frac{1}{u-w_{3,t}}\Big)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle +$$
$$+2\sum_r \frac{|\,\mathbf{w}_1+u-w_{1,r}\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}} - \sum_s \frac{|\,\mathbf{w}_1,\mathbf{w}_2+u-w_{2,s},\mathbf{w}_3\rangle}{u-w_{2,s}} + \sum_t \frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3+u-w_{3,t}\rangle}{u-w_{3,t}}$$

Similar calculations give us the result for $H_2(u)$. The explicit form is written in theorem 1.

It is little more complicated to express the actions for $E_j(u)$

$$E_1(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle = F_1(\mathbf{w}_1)E_1(u)\,|\,\mathbf{0},\mathbf{w}_2,\mathbf{w}_3\rangle -$$
$$-\sum_r \frac{F_1(\mathbf{w}_1-w_{1,r})}{u-w_{1,r}}\Big(H_1(u)-H_1(w_{1,r})\Big)\,|\,\mathbf{0},\mathbf{w}_2,\mathbf{w}_3\rangle +$$
$$+\sum_{r\neq\hat{r}} \frac{2\,|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{1,\hat{r}})(w_{1,\hat{r}}-w_{1,r})} - \sum_{r\neq\hat{r}} \frac{|\,\mathbf{w}_1+u-w_{1,r}-w_{1,\hat{r}},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{1,r})(u-w_{1,\hat{r}})} =$$
$$= F_1(\mathbf{w}_1)F_2(\mathbf{w}_2)E_1(u)\,|\,\mathbf{0},\mathbf{0},\mathbf{w}_3\rangle -$$
$$-\sum_r \frac{F_1(\mathbf{w}_1-w_{1,r})F_2(\mathbf{w}_2)}{u-w_{1,r}}\Big(H_1(u)-H_1(w_{1,r})\Big)\,|\,\mathbf{0},\mathbf{0},\mathbf{w}_3\rangle -$$
$$-\sum_{r,s} \frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{2,s})(w_{2,s}-w_{1,r})} + \sum_{r,s} \frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2+u-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{1,r})(u-w_{2,s})} -$$
$$-\sum_{r,s} \frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2+w_{1,r}-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{1,r})(w_{1,r}-w_{2,s})} +$$
$$+\sum_{r\neq\hat{r}} \frac{2\,|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{1,\hat{r}})(w_{1,\hat{r}}-w_{1,r})} - \sum_{r\neq\hat{r}} \frac{2\,|\,\mathbf{w}_1+u-w_{1,r}-w_{1,\hat{r}},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{1,\hat{r}})(w_{1,\hat{r}}-w_{1,r})} =$$
$$= \sum_t \frac{|\,\mathbf{w}_1,\mathbf{w}_2+u,\mathbf{w}_3-w_{3,t}\rangle}{u-w_{3,t}} - \sum_t \frac{|\,\mathbf{w}_1,\mathbf{w}_2+w_{3,t},\mathbf{w}_3-w_{3,t}\rangle}{u-w_{3,t}} -$$
$$-\sum_r \Big(\lambda_1(u)-\lambda_1(w_{1,r})\Big)\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}} + \sum_{r,t} \frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{3,t})(w_{3,t}-w_{1,r})} -$$
$$-\sum_{r,t} \frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3+u-w_{3,t}\rangle}{(u-w_{1,r})(u-w_{3,t})} + \sum_{r,t} \frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3+w_{1,r}-w_{3,t}\rangle}{(u-w_{1,r})(w_{1,r}-w_{3,t})} -$$

$$
\begin{aligned}
&-\sum_{r,s}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{2,s})(w_{2,s}-w_{1,r})}+\sum_{r,s}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2+u-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{1,r})(u-w_{2,s})}-\\
&-\sum_{r,s}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2+w_{1,r}-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{1,r})(w_{1,r}-w_{2,s})}+\\
&+\sum_{r\neq\hat{r}}\frac{2\,|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{1,\hat{r}})(w_{1,\hat{r}}-w_{1,r})}-\sum_{r\neq\hat{r}}\frac{2\,|\,\mathbf{w}_1+u-w_{1,r}-w_{1,\hat{r}},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{1,\hat{r}})(w_{1,\hat{r}}-w_{1,r})}
\end{aligned}
$$

By calculation of the action $E_2(u)$ we use the fact that $\big[F_2(w_2),F_3(w_3)\big]=0$, so we can write

$$
\begin{aligned}
E_2(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle &= F_1(\mathbf{w}_1)E_2(u)F_3(\mathbf{w}_3)F_2(\mathbf{w}_2)\,|\,\mathbf{0},\mathbf{0},\mathbf{0}\rangle=\\
&=-\sum_{s}\Big(\lambda_2(u)-\lambda_2(w_{2,s})\Big)\frac{|\,\mathbf{w}_1,\mathbf{w}_2-w_{2,s},\mathbf{w}_3\rangle}{u-w_{3,s}}-\\
&-\sum_{t}\frac{|\,\mathbf{w}_1+u,\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{u-w_{3,t}}+\sum_{t}\frac{|\,\mathbf{w}_1+w_{3,t},\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{u-w_{3,t}}+\\
&+\sum_{s\neq\hat{s}}\frac{2\,|\,\mathbf{w}_1,\mathbf{w}_2-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{2,\hat{s}})(w_{2,\hat{s}}-w_{2,s})}-\sum_{s\neq\hat{s}}\frac{2\,|\,\mathbf{w}_1,\mathbf{w}_2+u-w_{2,s}-w_{2,\hat{s}},\mathbf{w}_3\rangle}{(u-w_{2,\hat{s}})(w_{2,\hat{s}}-w_{2,s})}
\end{aligned}
$$

The calculation of $E_3(u)$ is lengthy and troublesome, so we give only the final answer in the following theorem which summarizes our calculation.

Theorem 1.
The following operators defined on the space $|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle=F(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3)\,|\,\mathbf{0},\mathbf{0},\mathbf{0}\rangle$

$$
\begin{aligned}
&F_1(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle=|\,\mathbf{w}_1+u,\mathbf{w}_2,\mathbf{w}_3\rangle\\
&F_2(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle=|\,\mathbf{w}_1,\mathbf{w}_2+u,\mathbf{w}_3\rangle-\sum_{r}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3+u\rangle}{u-w_{1,r}}+\\
&\quad+\sum_{r}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3+w_{1,r}\rangle}{u-w_{1,r}}\\
&F_3(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle=|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3+u\rangle\\
&H_1(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle=\Big(\lambda_1(u)-\sum_{r}\frac{2}{u-w_{1,r}}+\sum_{s}\frac{1}{u-w_{2,s}}-\sum_{t}\frac{1}{u-w_{3,t}}\Big)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle+\\
&\quad+2\sum_{r}\frac{|\,\mathbf{w}_1+u-w_{1,r}\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}}-\sum_{s}\frac{|\,\mathbf{w}_1,\mathbf{w}_2+u-w_{2,s},\mathbf{w}_3\rangle}{u-w_{2,s}}+\sum_{t}\frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3+u-w_{3,t}\rangle}{u-w_{3,t}}\\
&H_2(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle=\Big(\lambda_2(u)+\sum_{r}\frac{1}{u-w_{1,r}}-\sum_{s}\frac{2}{u-w_{2,s}}-\sum_{t}\frac{1}{u-w_{3,t}}\Big)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle-\\
&\quad-\sum_{r}\frac{|\,\mathbf{w}_1+u-w_{1,r}\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}}+2\sum_{s}\frac{|\,\mathbf{w}_1,\mathbf{w}_2+u-w_{2,s},\mathbf{w}_3\rangle}{u-w_{2,s}}+\sum_{t}\frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3+u-w_{3,t}\rangle}{u-w_{3,t}}\\
&E_1(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle=-\sum_{r}\Big(\lambda_1(u)-\lambda_1(w_{1,r})\Big)\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{u-w_{1,r}}+\\
&\quad+\sum_{t}\frac{|\,\mathbf{w}_1,\mathbf{w}_2+u,\mathbf{w}_3-w_{3,t}\rangle}{u-w_{3,t}}-\sum_{t}\frac{|\,\mathbf{w}_1,\mathbf{w}_2+w_{3,t},\mathbf{w}_3-w_{3,t}\rangle}{u-w_{3,t}}+\\
&\quad+\sum_{r\neq\hat{r}}\frac{2\,|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{1,\hat{r}})(w_{1,\hat{r}}-w_{1,r})}-\sum_{r\neq\hat{r}}\frac{2\,|\,\mathbf{w}_1+u-w_{1,r}-w_{1,\hat{r}},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{1,\hat{r}})(w_{1,\hat{r}}-w_{1,r})}-\\
&\quad-\sum_{r,s}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{2,s})(w_{2,s}-w_{1,r})}+\sum_{r,s}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2+u-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{1,r})(u-w_{2,s})}-
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{r,s}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2+w_{1,r}-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{1,r})(w_{1,r}-w_{2,s})}+\sum_{r,t}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3\rangle}{(u-w_{3,t})(w_{3,t}-w_{1,r})}-\\
&-\sum_{r,t}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3+u-w_{3,t}\rangle}{(u-w_{1,r})(u-w_{3,t})}+\sum_{r,t}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2,\mathbf{w}_3+w_{1,r}-w_{3,t}\rangle}{(u-w_{1,r})(w_{1,r}-w_{3,t})}
\end{aligned}
$$

$$
\begin{aligned}
E_2(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle = &-\sum_{s}\Big(\lambda_2(u)-\lambda_2(w_{2,s})\Big)\frac{|\,\mathbf{w}_1,\mathbf{w}_2-w_{2,s},\mathbf{w}_3\rangle}{u-w_{2,s}}-\\
&-\sum_{t}\frac{|\,\mathbf{w}_1+u,\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{u-w_{3,t}}+\sum_{t}\frac{|\,\mathbf{w}_1+w_{3,t},\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{u-w_{3,t}}+\\
&+\sum_{s\neq\widehat{s}}\frac{2\,|\,\mathbf{w}_1,\mathbf{w}_2-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{2,\widehat{s}})(w_{2,\widehat{s}}-w_{2,s})}-\sum_{s\neq\widehat{s}}\frac{2\,|\,\mathbf{w}_1,\mathbf{w}_2+u-w_{2,s}-w_{2,\widehat{s}},\mathbf{w}_3\rangle}{(u-w_{2,\widehat{s}})(w_{2,\widehat{s}}-w_{2,s})}
\end{aligned}
$$

$$
\begin{aligned}
E_3(u)\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle = &-\sum_{t}\Big(\lambda_3(u)-\lambda_3(w_{3,t})\Big)\frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{u-w_{3,t}}-\\
&-\sum_{r,s}\Big(\lambda_2(u)-\lambda_2(w_{1,r})\Big)\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{1,r})(w_{1,r}-w_{2,s})}-\\
&-\sum_{r,s}\Big(\lambda_2(u)-\lambda_2(w_{2,s})\Big)\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{2,s})(w_{2,s}-w_{1,r})}-\\
&-\sum_{r,t}\frac{|\,\mathbf{w}_1+u-w_{1,r},\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{(u-w_{1,r})(u-w_{3,t})}+\sum_{r,t}\frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{(u-w_{1,r})(w_{1,r}-w_{3,t})}+\\
&+\sum_{r,t}\frac{|\,\mathbf{w}_1+w_{3,t}-w_{1,r},\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{(u-w_{3,t})(w_{3,t}-w_{1,r})}-\\
&-\sum_{s,t}\frac{|\,\mathbf{w}_1,\mathbf{w}_2+u-w_{2,s},\mathbf{w}_3-w_{3,t}\rangle}{(u-w_{2,s})(u-w_{3,t})}+\sum_{s,t}\frac{|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{(u-w_{2,s})(w_{2,s}-w_{3,t})}+\\
&+\sum_{s,t}\frac{|\,\mathbf{w}_1,\mathbf{w}_2+w_{3,t}-w_{2,s},\mathbf{w}_3-w_{3,t}\rangle}{(u-w_{3,t})(w_{3,t}-w_{2,s})}+\\
&+\sum_{t\neq\widehat{t}}\frac{2\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3-w_{3,t}\rangle}{(u-w_{3,\widehat{t}})(w_{3,\widehat{t}}-w_{3,t})}-\sum_{t\neq\widehat{t}}\frac{2\,\Big|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3+u-w_{3,t}-w_{3,\widehat{t}}\Big\rangle}{(u-w_{3,\widehat{t}})(w_{3,\widehat{t}}-w_{3,t})}+\\
&+\sum_{r}\sum_{s\neq\widehat{s}}\frac{2\,|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2-w_{2,s},\mathbf{w}_3\rangle}{(u-w_{2,\widehat{s}})(w_{2,\widehat{s}}-w_{1,r})(w_{2,\widehat{s}}-w_{2,s})}-\\
&-\sum_{r}\sum_{s\neq\widehat{s}}\frac{2\,|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2+u-w_{2,s}-w_{2,\widehat{s}},\mathbf{w}_3\rangle}{(u-w_{1,r})(u-w_{2,\widehat{s}})(w_{2,\widehat{s}}-w_{2,s})}+\\
&+\sum_{r}\sum_{s\neq\widehat{s}}\frac{2\,|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2+w_{1,r}-w_{2,s}-w_{2,\widehat{s}},\mathbf{w}_3\rangle}{(u-w_{1,r})(w_{1,r}-w_{2,\widehat{s}})(w_{2,\widehat{s}}-w_{2,s})}
\end{aligned}
$$

give the representation of the Sklyanin algebra sl(3)(u).

4 The Gaudin model for sl(3)

.
The central role for the Gaudin model in the sl(3) case is played by the operator

$$
\begin{aligned}
T(u) = &\tfrac{1}{2}\left(F_1(u)E_1(u)+F_2(u)E_2(u)+F_3(u)E_3(u)+\right.\\
&\left.+E_1(u)F_1(u)+E_2(u)F_2(u)+E_3(u)F_3(u)\right)+\\
&+\tfrac{1}{3}\left(H_1^2(u)+H_1(u)H_2(u)+H_2^2(u)\right).
\end{aligned}
$$

By easy calculation we can rewrite this in the form

$$T(u)=F_1(u)E_1(u)+F_2(u)E_2(u)+F_3(u)E_3(u)-H_3'(u)+\tfrac{1}{3}\left(H_3^2(u)-H_1(u)H_2(u)\right)$$

In our paper [13], we were able to construct the eigenvectors and calculate the eigenvalues. We defined a linear operator $\mathbf{P}$ by

$$\begin{aligned}\mathbf{P}\,|\,0,\mathbf{w}_2,\mathbf{w}_3\rangle &= \mathbf{P}\,|\,\mathbf{w}_1,0,\mathbf{w}_3\rangle = 0\,,\\ \mathbf{P}\,|\,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\rangle &= \sum_{r,s}\frac{|\,\mathbf{w}_1-w_{1,r},\mathbf{w}_2-w_{2,s},\mathbf{w}_3+w_{1,r}\rangle}{w_{2,s}-w_{1,r}}\end{aligned}$$

and a vector

$$|\,\mathbf{w}_1,\mathbf{w}_2)=\sum_{n=0}^{\infty}\frac{\mathbf{P}^n}{n!}\,|\,\mathbf{w}_1,\mathbf{w}_2,0\rangle$$

for any $\mathbf{w}_1$ and $\mathbf{w}_2$. We showed by the fulfilment of the Bethe ansatz equations for $\lambda_1(w_{1,r})$ and $\lambda_2(w_{2,s})$ that $|\,\mathbf{w}_1,\mathbf{w}_2)$ is the eigenvector of $T(u)$ for any u. Here is our main theorem from paper [13].

Theorem 2.
If the Bethe ansatz conditions: for any r

$$\lambda_1(w_{1,r})-\sum_{\hat r\neq r}\frac{2}{w_{1,r}-w_{1,\hat r}}+\sum_{s}\frac{1}{w_{1,r}-w_{2,s}}=0 \tag{3}$$

and for any s

$$\lambda_2(w_{2,s})+\sum_{r}\frac{1}{w_{2,s}-w_{1,r}}-\sum_{\hat s\neq s}\frac{2}{w_{2,s}-w_{2,\hat s}}=0 \tag{4}$$

are fulfilled, then the vector

$$|\,\mathbf{w}_1,\mathbf{w}_2)=\sum_{n=0}^{\infty}\frac{\mathbf{P}^n}{n!}\,|\,\mathbf{w}_1,\mathbf{w}_2,0\rangle$$

is the eigenvector of the sl(3) *Gaudin model and*

$$T(u)\,|\,\mathbf{w}_1,\mathbf{w}_2)=\tau(u;\mathbf{w}_1,\mathbf{w}_2)\,|\,\mathbf{w}_1,\mathbf{w}_2)\,,$$

where

$$\begin{aligned}\tau(u;\mathbf{w}_1,\mathbf{w}_2) = \tau(u) &-\sum_{r}\Big(\lambda_1(u)-\sum_{\hat r\neq r}\frac{2}{w_{1,r}-w_{1,\hat r}}+\sum_{s}\frac{1}{w_{1,r}-w_{2,s}}\Big)\frac{1}{u-w_{1,r}}-\\ &-\sum_{s}\Big(\lambda_2(u)+\sum_{r}\frac{1}{w_{2,s}-w_{1,r}}-\sum_{\hat s\neq s}\frac{2}{w_{2,s}-w_{2,\hat s}}\Big)\frac{1}{u-w_{2,s}}\,.\end{aligned} \tag{5}$$

5 Concluding remarks and open problems

In the present paper, we have studied the infinite dimensional Sklyanin algebra sl(3)(u) which is connected with the Gaudin model for sl(3). Specifically, we constructed the highest weight representation for this algebra in an explicit form. Evidently this representation was implicitly used in our recent paper [13].

We hope that the result of this paper can be useful in further study of this model in sl(3) case, e. g., correlators etc. For details of sl(2)(u), see [7, 12].
It will be very interesting to generalize this construction to other semisimple Lie algebras. Some calculation for so(5) was really made and the result is being prepared for publication [14].

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Appendix A

We will work in the basis $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{e}_k$ and $\mathbf{f}_k$, $k = 1,\ 2,\ 3$, and use the explicit commutation relations

$$
\begin{array}{llll}
\big[\mathbf{h}_1,\mathbf{e}_1\big] = 2\mathbf{e}_1\,, & \big[\mathbf{h}_1,\mathbf{e}_2\big] = -\mathbf{e}_2\,, & \big[\mathbf{h}_1,\mathbf{e}_3\big] = \mathbf{e}_3\,, & \big[\mathbf{h}_1,\mathbf{f}_1\big] = -2\mathbf{f}_1\,, \\
\big[\mathbf{h}_1,\mathbf{f}_2\big] = \mathbf{f}_2\,, & \big[\mathbf{h}_1,\mathbf{f}_3\big] = -\mathbf{f}_3\,, & \big[\mathbf{h}_2,\mathbf{e}_1\big] = -\mathbf{e}_1\,, & \big[\mathbf{h}_2,\mathbf{e}_2\big] = 2\mathbf{e}_2 \\
\big[\mathbf{h}_2,\mathbf{e}_3\big] = \mathbf{e}_3\,, & \big[\mathbf{h}_2,\mathbf{f}_1\big] = \mathbf{f}_1\,, & \big[\mathbf{h}_2,\mathbf{f}_2\big] = -2\mathbf{f}_2\,, & \big[\mathbf{h}_2,\mathbf{f}_3\big] = -\mathbf{f}_3\,, \\
\big[\mathbf{e}_1,\mathbf{e}_2\big] = \mathbf{e}_3\,, & \big[\mathbf{e}_1,\mathbf{e}_3\big] = 0\,, & \big[\mathbf{e}_2,\mathbf{e}_3\big] = 0\,, & \\
\big[\mathbf{e}_1,\mathbf{f}_1\big] = \mathbf{h}_1\,, & \big[\mathbf{e}_1,\mathbf{f}_2\big] = 0\,, & \big[\mathbf{e}_1,\mathbf{f}_3\big] = \mathbf{f}_2\,, & \\
\big[\mathbf{e}_2,\mathbf{f}_1\big] = 0\,, & \big[\mathbf{e}_2,\mathbf{f}_2\big] = \mathbf{h}_2\,, & \big[\mathbf{e}_2,\mathbf{f}_3\big] = \mathbf{f}_1\,, & \\
\big[\mathbf{f}_2,\mathbf{f}_3\big] = 0\,, & \big[\mathbf{e}_3,\mathbf{f}_1\big] = -\mathbf{e}_2\,, & \big[\mathbf{e}_3,\mathbf{f}_2\big] = \mathbf{e}_1\,, & \\
\big[\mathbf{e}_3,\mathbf{f}_3\big] = \mathbf{h}_1+\mathbf{h}_2\,, & \big[\mathbf{f}_1,\mathbf{f}_2\big] = \mathbf{f}_3\,, & \big[\mathbf{e}_1,\mathbf{f}_3\big] = 0\,, & \big[\mathbf{e}_2,\mathbf{f}_3\big] = 0
\end{array}
$$

Appendix B

In this Appendix we collect the actions of $F_1(u), F_2(u), F_3(u), E_1(u), E_2(u),\ E_3(u), H_1(u)$ and $H_2(u)$ on $F_2(\mathbf{w}_2)$

$$
\begin{aligned}
F_2(\mathbf{w}_2)F_1(u) &= F_1(u)F_2(\mathbf{w}_2) - \sum_s \frac{F_2(\mathbf{w}_2 - w_s)}{u - w_s}\Big(F_3(u) - F_3(w_s)\Big)\,, \\
F_2(u)F_2(\mathbf{w}_2) &= F_2(\mathbf{w}_2 + u)\,, \\
F_3(u)F_2(\mathbf{w}_2) &= F_2(\mathbf{w}_2)F_3(u)\,, \\
H_1(u)F_2(\mathbf{w}_2) &= F_2(\mathbf{w}_2)\Big(H_1(u) + \sum_s \frac{1}{u - w_s}\Big) - \sum_s \frac{F_2(\mathbf{w}_2 + u - w_s)}{u - w_s}\,, \\
H_2(u)F_2(\mathbf{w}_2) &= F_2(\mathbf{w}_2)\Big(H_2(u) - \sum_s \frac{2}{u - w_s}\Big) + 2\sum_s \frac{F_2(\mathbf{w}_2 + u - w_s)}{u - w_s}\,, \\
E_1(u)F_2(\mathbf{w}_2) &= F_2(\mathbf{w}_2)E_1(u)\,,
\end{aligned}
$$

$$
\begin{aligned}
E_2(u)F_2(\mathbf{w}_2) &= F_2(\mathbf{w}_2)E_2(u) - \sum_s \frac{F_2(\mathbf{w}_2 - w_s)}{u - w_s}\Big(H_2(u) - H_2(w_s)\Big) + \\
&\quad + \sum_{s\neq\widehat{s}} \frac{2F_2(\mathbf{w}_2 - w_s)}{(u - w_{\widehat{s}})(w_{\widehat{s}} - w_s)} - \sum_{s\neq\widehat{s}} \frac{F_2(\mathbf{w}_2 + u - w_s - w_{\widehat{s}})}{(u - w_s)(u - w_{\widehat{s}})}\,, \\
E_3(u)F_2(\mathbf{w}_2) &= F_2(\mathbf{w}_2)E_3(u) - \sum_s \frac{F_2(\mathbf{w}_2 - w_s)}{u - w_s}\Big(E_1(u) - E_1(w_s)\Big)\,,
\end{aligned}
$$

and $F_3(\mathbf{w}_3)$

$$
\begin{aligned}
F_3(\mathbf{w}_3)F_1(u) &= F_1(u)F_3(\mathbf{w}_3)\,, \\
F_3(\mathbf{w}_3)F_2(u) &= F_2(u)(\mathbf{w}_3)\,, \\
F_3(u)F_3(\mathbf{w}_3) &= F_3(\mathbf{w}_3 + u)\,, \\
H_1(u)F_3(\mathbf{w}_3) &= F_3(\mathbf{w}_3)\Big(H_1(u) - \sum_t \frac{1}{u - w_t}\Big) + \sum_t \frac{F_3(\mathbf{w}_3 + u - w_t)}{u - w_t}\,, \\
H_2(u)F_3(\mathbf{w}_3) &= F_3(\mathbf{w}_3)\Big(H_2(u) - \sum_t \frac{1}{u - w_t}\Big) + \sum_t \frac{F_3(\mathbf{w}_3 + u - w_t)}{u - w_t}\,, \\
E_1(u)F_3(\mathbf{w}_3) &= F_3(\mathbf{w}_3)E_1(u) + \sum_t \Big(F_2(u) - F_2(w_t)\Big)\frac{F_3(\mathbf{w}_3 - w_t)}{u - w_t}\,, \\
E_2(u)F_3(\mathbf{w}_3) &= F_3(\mathbf{w}_3)E_2(u) - \sum_t \Big(F_1(u) - F_1(w_t)\Big)\frac{F_3(\mathbf{w}_3 - w_t)}{u - w_t}\,, \\
E_3(u)F_3(\mathbf{w}_3) &= F_3(\mathbf{w}_3)E_3(u) - \sum_t \frac{|\,\mathbf{w}_3 - w_t\rangle}{u - w_t}\Big(H_3(u) - H_3(w_t)\Big) + \\
&\quad + \sum_{t\neq\widehat{t}} \frac{2F_3(\mathbf{w}_3 - w_t)}{(u - w_{\widehat{t}})(w_{\widehat{t}} - w_t)} - \sum_{t\neq\widehat{t}} \frac{(\mathbf{w}_3 + u - w_t - w_{\widehat{t}})}{(u - w_t)(u - w_{\widehat{t}})}\,.
\end{aligned}
$$

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