Decomposition of the enveloping algebra $\text{so}(5)$

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Abstract The adjoint representation of $\text{so}(5)$ on its universal enveloping algebra $U(\text{so}(5))$ is explicitly decomposed into irreducible components. It is shown that the commutant of raising operators of $\text{so}(5)$ is generated by 7 elements. The explicit form of these elements is given.

1 Introduction

Throughout the paper we will denote by $\mathfrak{g}$ the Lie algebra and by $U(\mathfrak{g})$ its enveloping algebra [1].

We will study $U(\mathfrak{g})$ as the vector space, on which $\mathfrak{g}$ is represented via the adjoint action $\rho$, i.e.

$$\rho(x)a = [x,a] \text{ for } x \in \mathfrak{g} \text{ and } a \in U(\mathfrak{g}).$$

In order to decompose the representation $\rho$ it is sufficient to determine the space $\mathcal{B}$ of the highest weight vectors. For $\mathfrak{g} = \text{sl}(2)$ this program was very easily solved, see [2]. Let $\{E_{12}, E_{23}, H_1\}$ be a basis in $\text{sl}(2)$. Then the vector space of all highest weight vectors has basis $C_n^m E_{12}^m$, $n, m = 0, 1, 2, \ldots$, where $C_2$ is a Casimir operator of $\text{sl}(2)$. It is trivial to see that all such elements are commutant of $\{E_{12}\}$.

The case of algebra $\text{sl}(3)$ was solved in [3]. The commutant $\mathcal{B}$ of $\{E_{12}, E_{23}\}$ in $U(\text{sl}(3))$ was explicitly described. It is generated by 6 elements and each of the highest weight vectors can be written as a polynomial in these elements.

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In this paper, we study the case of algebra $\mathfrak{g} = \text{so}(5)$ and by construction we use the following well-known fact: Let $\mathfrak{g}$ be a Lie algebra with the basis $e_k$, $k = 1, \ldots, n$, and structure constants $c_{ij}^k$. Then the mapping

$$e_i \mapsto \sum_{j,k=1}^n c_{ij}^k \frac{\partial}{\partial x_j}$$

forms a representation of algebra $\mathfrak{g}$ by differential operators of the first order.

This is the representation of $\text{ad}_e$ on the enveloping algebra expressed in basis of symmetric polynomials in $U(\mathfrak{g})$. More precisely, if $x_k$ is an image of $e_k$ by canonical linear isomorphism $U(\mathfrak{g}) \to \text{Sym}(\mathfrak{g})$, then the relation (1) gives the action of $\text{ad}_{x_k}$.

From definition (1) it is easy to see that for any $N$ polynomials of the order of $N$ form the invariant subspace of representation $\text{ad}_{x_k}$.

2 The Lie algebra $\text{so}(5)$

The basis of algebra $\mathfrak{g} = \text{so}(5, \mathbb{C})$ consists of the elements $J_{\alpha\beta} = -J_{\beta\alpha}$, $\alpha, \beta = 1, 2, \ldots, 5$, which fulfill the commutation relations

$$[J_{\alpha\beta}, J_{\mu\nu}] = \delta_{\beta\mu} J_{\alpha\nu} + \delta_{\alpha\nu} J_{\beta\mu} - \delta_{\beta\nu} J_{\alpha\mu} - \delta_{\alpha\mu} J_{\beta\nu}.$$  \hspace{1cm} (2)

If we define

$$H_1 = iJ_{12}, \quad H_2 = iJ_{14}, \quad E_1 = \frac{1}{\sqrt{2}} (J_{15} + iJ_{35}), \quad E_2 = \frac{1}{\sqrt{2}} (J_{23} + iJ_{13} - J_{14} + iJ_{24}),$$

$$E_3 = \frac{1}{\sqrt{2}} (J_{15} - iJ_{35}), \quad E_4 = \frac{1}{2} (J_{23} + iJ_{13} + J_{14} - iJ_{24}),$$

$$F_1 = \frac{1}{\sqrt{2}} (J_{35} + iJ_{15}), \quad F_2 = \frac{1}{2} (J_{23} + iJ_{13} + J_{14} + iJ_{24}),$$

$$F_3 = \frac{1}{\sqrt{2}} (J_{35} - iJ_{15}), \quad F_4 = \frac{1}{2} (J_{23} + iJ_{13} - J_{14} - iJ_{24}),$$

we obtain from (2) the commutation relations

$$[H_1, E_1] = 0, \quad [H_1, E_2] = E_2, \quad [H_1, E_3] = E_3, \quad [H_1, E_4] = E_4,$$

$$[H_2, E_1] = E_1, \quad [H_2, E_2] = -E_2, \quad [H_2, E_3] = 0, \quad [H_2, E_4] = E_4,$$

$$[H_1, F_1] = 0, \quad [H_1, F_2] = -F_2, \quad [H_1, F_3] = -F_3, \quad [H_1, F_4] = -F_4,$$

$$[H_2, F_1] = -F_1, \quad [H_2, F_2] = F_2, \quad [H_2, F_3] = 0, \quad [H_2, F_4] = -F_4,$$

$$[E_1, E_2] = E_3, \quad [E_1, E_3] = E_4, \quad [E_1, E_4] = 0,$$

$$[E_2, E_1] = 0, \quad [E_2, E_4] = 0, \quad [E_3, E_1] = 0,$$

$$[F_1, F_2] = -F_3, \quad [F_1, F_3] = -F_4, \quad [F_1, F_4] = 0,$$

$$[F_2, F_1] = 0, \quad [F_2, F_4] = 0, \quad [F_3, F_4] = 0,$$

$$[E_1, F_1] = H_2, \quad [E_1, F_2] = 0, \quad [E_1, F_3] = -F_2, \quad [E_1, F_4] = -F_3.$$
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$$[E_2, F_1] = 0, \quad [E_2, F_2] = H_1 - H_2, \quad [E_2, F_3] = F_1, \quad [E_2, F_4] = 0$$
$$[E_3, F_1] = -E_2, \quad [E_3, F_2] = E_1, \quad [E_3, F_3] = H_1, \quad [E_3, F_4] = F_1,$$
$$[E_4, F_1] = -E_3, \quad [E_4, F_2] = 0, \quad [E_4, F_3] = E_1, \quad [E_4, F_4] = H_1 + H_2$$

which we will use in further calculation.

3 The highest weight vectors

The vectors $v$ with the highest weights, i.e. the commutant $\mathcal{B}$ of the elements $E_1$ and $E_2$ in $U(g)$, are given as solutions of the equations

$$\text{ad}_{E_1} v = \text{ad}_{E_2} v = 0. \quad (3)$$

If we denote $x_k, y_k$ and $z_k$ the images of elements $E_k, H_k$ and $F_k$ by mapping $U(g) \to \text{Sym}(g)$, we obtain from (1) and (3) the system of equations

$$x_3 \frac{\partial f}{\partial x_3} + x_4 \frac{\partial f}{\partial x_4} - x_1 \frac{\partial f}{\partial h_2} + h_2 \frac{\partial f}{\partial y_1} - y_2 \frac{\partial f}{\partial y_3} - y_3 \frac{\partial f}{\partial y_4} = 0,$$

$$-x_3 \frac{\partial f}{\partial x_1} - x_2 \left( \frac{\partial f}{\partial h_1} - \frac{\partial f}{\partial h_2} \right) + (h_1 - h_2) \frac{\partial f}{\partial y_2} + y_1 \frac{\partial f}{\partial y_3} = 0. \quad (4)$$

The general solutions of (4) are

$$f = F(X_1, X_2, X_3, X_4, X_5, C_2),$$

where $F$ is an arbitrary differentiable function in variables

$$X_1 = x_1$$
$$X_2 = 2x_2x_4 - x_3^2$$
$$X_3 = x_1x_2 + x_3h_2 - x_4y_1$$
$$X_4 = x_2^2 - x_1x_3(h_1 - h_2) + \frac{1}{2} x_4(h_1 - h_2)^2 + (2x_2x_4 - x_3^2)y_2$$
$$X_5 = x_1x_2x_3 + x_1x_2x_4(h_1 + h_2) - x_1x_3^2(h_1 - h_2) - x_3x_4h_2(h_1 - h_2) -$$
$$-x_1x_3x_4y_1 + \frac{1}{2} x_2x_4(x_3y_2 + x_4y_3) + x_3^2y_1(h_1 - h_2)$$

$$C_2 = 2(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4) + h_1^2 + h_2^2$$

We can say that $f$ has the weight $(N_1, N_2)$ when

$$x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + x_4 \frac{\partial f}{\partial x_4} - y_2 \frac{\partial f}{\partial y_2} - y_3 \frac{\partial f}{\partial y_3} - y_4 \frac{\partial f}{\partial y_4} = N_1 f,$$
$$x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} + x_4 \frac{\partial f}{\partial x_4} - y_1 \frac{\partial f}{\partial y_1} + y_2 \frac{\partial f}{\partial y_2} - y_4 \frac{\partial f}{\partial y_4} = N_2 f.$$
It is easy to see that the functions $X_1$ and $X_4$ have the weight $(1,1)$, the function $X_2$ has the weight $(2,0)$, $X_3$ has the weight $(1,0)$, $X_5$ has the weight $(2,1)$ and finally $C_2$, which is an element of the center $U(g)$, has the weight $(0,0)$.

Since the determinant

$$
D(X_1,X_2,X_3,X_4,X_5,C_2) = -4x_2x_4^2(2x_2x_4 - x_3^2)^2 \neq 0,
$$

the functions $X_k$, $k = 1, \ldots, 5$ and $C_2$ are functionally independent. By direct calculation we can show that the relation

$$
X_2^2 + X_3^2X_2 - 2X_1X_4X_2^2 + C_2X_1X_4X_2 + C_4X_1^2X_2 = 0
$$

is valid, where

$$
C_4 = \frac{1}{4}(h_1^2 - h_2^2 - 2x_1y_1 + 2x_3y_3)^2 + x_2y_2(h_1 + h_2)^2 + x_4y_4(h_1 - h_2)^2 - 2(x_1x_2y_3 + x_3y_1y_2)(h_1 + h_2) - 2(x_1x_3y_4 + x_4y_3y_2)(h_1 - h_2)^2 + 2(x_1x_2y_4 + x_4y_3y_2)^2 - x_2x_4y_3^2 + 2x_1x_3y_1y_3 + 2x_2x_4y_2y_4
$$

is an element of the center of $U(g)$, and has the weight $(0,0)$.

Now we show that the vector space of all highest weight polynomials, i.e. the polynomials, which fulfill equation (4), has the basis

$$
C_2^k e^{\sum_{r=1}^4 k_r X_r} C_4^{n_4} X_3^{n_3} X_4^{n_4} X_5^{n_5},
$$

where $k_1, k_2, n_r \geq 0$ for $r = 1, \ldots, 4$ and $n_5 = 0, 1$.

It is clear that any linear combination of the polynomials (6) are solutions of (4), thus the are highest weight polynomials.

The linear independence of the functions (6) follows from the fact, that $C_4$ can be expressed by using the condition (5) as a function in variables $C_2$ and $X_k$, $k = 1, \ldots, 5$, which is quadratic in variable $X_5$, and above proved of functional independence of these functions.

Now we will show that the linear combinations of polynomials of the form (6) generate the space of all highest weight polynomials. It is clear that the polynomial (6) has the weight $(N_1, N_2)$, where

$$
N_1 = n_1 + 2n_2 + n_3 + n_4 + 2n_5,
N_2 = n_1 + n_4 + n_5
$$

and is of the order

$$
M = 2k_1 + 4k_2 + n_1 + 2n_2 + 2n_3 + 3n_4 + 4n_5.
$$

By using the differential operators (1) corresponding to generators of the Lie algebra so(5) it is possible from any polynomial with the highest weight $(N_1, N_2)$ of an order of $M$ to generate
\[ d_{(N_1, N_2)} = \frac{1}{6} (N_1 + N_2 + 2)(N_1 - N_2 + 1)(2N_1 + 3)(2N_2 + 1) \]

of the linearly independent polynomials of an order of \( M \). By direct calculation, which is cumbersome, it is possible to show that for any \( M \in \mathbb{N} \) we have

\[
\sum_{d_{(N_1, N_2)}} \binom{M + 9}{M},
\]

where the summation is with respect to any \( k_1, k_2, n_r \geq 0, r = 1, \ldots, 4 \), and \( n_5 = 0, 1 \), for which (8) is valid, and \( N_1, N_2 \) are equal to (7). Now, because the vector space off all polynomials of an order of \( M \) in 10 variables has dimension

\[ D_M = \binom{M + 9}{M}, \]

the polynomials of the form (6) generate the vectors space of all highest weight polynomials.

To go back to algebra \( U(\mathfrak{g}) \), we use the canonical isomorphism from \( \text{Sym}(\mathfrak{g}) \) to \( U(\mathfrak{g}) \). First, we define the elements

\[
X_1 = E_1, \\
X_2 = \text{sym} (2E_2E_4 - E_3^2) \\
X_3 = \text{sym} (E_1E_2 + E_3H_2 - E_5F_1) \\
X_4 = \text{sym} \left( E_1^2E_2 - E_1E_3 (H_1 - H_2) + \frac{1}{2} E_4 (H_1 - H_2)^2 + (2E_2E_4 - E_3^2)F_2 \right) \\
X_5 = \text{sym} \left( E_1^2E_2E_3 + E_1E_2E_4 (H_1 + H_2) - E_1E_3^2 (H_1 - H_2) - E_1E_3H_2 (H_1 - H_2) - E_1E_3E_4F_1 + (2E_2E_4 - E_3^2) (E_3F_2 + E_6F_3) + E_5^2F_1 (H_1 - H_2) \right) \\
C_2 = \text{sym} \left( 2 (E_1F_1 + E_2F_2 + E_3F_4 + E_4F_4) + H_1^2 + H_2^2 \right) \\
C_4 = \text{sym} \left( \frac{1}{4} (H_1^2 - H_2^2 - 2E_1F_1 + 2E_3F_3)^2 + E_2F_2 (H_1 + H_2)^2 + E_4F_4 (H_1 - H_2)^2 - 2 (E_1E_2F_3 + E_4F_1F_2) (H_1 + H_2) - 2 (E_1E_3F_4 + E_6F_1F_3) (H_1 - H_2)^2 + 2 (E_2^2E_4F_4 - E_4^2F_2F_3 - E_5^2F_2F_4 + 2E_2E_3F_1F_3 + 2E_4E_3F_2F_4) \right).
\]

Since the set of polynomials (6) is a basis in the space of the highest weight polynomials, the set

\[
\text{sym} \left( C_2^2 C_4 X_1^{x_1} X_2^{x_2} X_3^{x_3} X_4^{x_4} X_5^{x_5} \right) \in U(\mathfrak{g})
\]

forms the basis in the space of the highest weight vectors.
We denote $U_n(\mathfrak{g})$ as a subspace of $U(\mathfrak{g})$ with elements of an order less or equal to $n$. It is a well know fact that for $n_1$ and $n_2$ one has
$$[U_{n_1}(\mathfrak{g}), U_{n_2}(\mathfrak{g})] \subset U_{n_1+n_2-1}(\mathfrak{g}).$$
For this reason as a basis in the vector space of the height weight vector we can use instead of (9) the set of elements
$$C_{1}^{k_1} C_{2}^{k_2} X_1^{n_1} X_2^{n_2} X_3^{n_3} X_4^{n_4} X_5^{n_5} \in U(\mathfrak{g}),$$
where $k_1, k_2, n_r \geq 0$ for $r = 1, 2, 3, 4$ and $n_5 = 0, 1$.

4 Conclusion

We have studied the structure of enveloping algebra $U(\mathfrak{so}(5))$ as the vector space of adjoint representation of the Lie algebra $\mathfrak{so}(5)$. We explicitly decomposed this representation into the irreducible components and found the highest weight vector in any such components.

Our result can be useful for further study of tensor products of the representations of the Lie algebras and for study of ideals in their enveloping algebras. The details of such study for $\mathfrak{sl}(2)$ case see in [2].

The method described can be used for other simple Lie algebras. We really obtained some results for the Lie algebra $\mathfrak{sl}(4)$, but the general solution for $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$ etc. is still an open problem.

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References