Decomposition of the enveloping algebra so(5)

Čestmír Burdík and Ondřej Navrátil

Abstract The adjoint representation of so(5) on its universal enveloping algebra $U\big(\mathrm{so}(5)\big)$ is explicitly decomposed into irreducible components. It is shown that the commutant of raising operators of so(5) is generated by 7 elements. The explicit form of these elements is given.

1 Introduction

Throughout the paper we will denote by $\mathfrak{g}$ the Lie algebra and by $U(\mathfrak{g})$ its enveloping algebra [1].

We will study $U(\mathfrak{g})$ as the vector space, on which $\mathfrak{g}$ is represented via the adjoint action ρ, i.e.

$$\rho(x)a = [x,a] \quad \text{for } x \in \mathfrak{g} \quad \text{and} \quad a \in U(\mathfrak{g}) .$$

In order to decompose the representation ρ it is sufficient to determine the space $\mathscr{B}$ of the highest weight vectors. For $\mathfrak{g} = \mathrm{sl}(2)$ this program was very easily solved, see [2]. Let $\big\{\mathbf{E}_{12}, \mathbf{E}_{21}, \mathbf{H}_1\big\}$ be a basis in sl(2). Then the vector space of all highest weight vectors has basis $\mathbf{C}_2^m \mathbf{E}_{12}^n$, n, $m = 0, 1, 2, \ldots$, where $\mathbf{C}_2$ is a Casimir operator of sl(2). It is trivial to see that all such elements are commutant of $\{\mathbf{E}_{12}\}$.

The case of algebra sl(3) was solved in [3]. The commutant $\mathscr{B}$ of $\big\{\mathbf{E}_{12}, \mathbf{E}_{23}\big\}$ in $U\big(\mathrm{sl}(3)\big)$ was explicitly described. It is generated by 6 elements and each of the highest weight vectors can by written as a polynomial in these elements.

Čestmír Burdík
Department of Mathematics, Czech Technical University, Faculty of Nuclear Sciences and Physical Engineering, Trojanova 13, 120 00 Prague 2, Czech Republic, e-mail: burdik@kmalpha.fjfi.cvut.cz

Ondřej Navrátil
Department of Mathematics, Czech Technical University, Faculty of Transportation Sciences, Na Florenci 25, 110 00 Prague, Czech Republic e-mail: navratil@fd.cvut.cz

I this paper, we study the case of algebra $\mathfrak{g} = \mathrm{so}(5)$ and by construction we use the following well know fact:
Let $\mathfrak{g}$ be a Lie algebra with the basis $\mathbf{e}_k$, $k = 1, \ldots, n$, and structure constants c_{ij}^k. Then the mapping

$$\mathbf{e}_i \mapsto \sum_{j,k=1}^{n} c_{ij}^k x_k \frac{\partial}{\partial x_j} \tag{1}$$

forms a representation of algebra $\mathfrak{g}$ by differential operators of the first order.

This is the representation of $\mathrm{ad}_{\mathfrak{g}}$ on the enveloping algebra expressed in basis of symmetric polynomials in $U(\mathfrak{g})$. More precisely, if x_k is an image of $\mathbf{e}_k$ by canonical linear isomorphism $U(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{g})$, than the relation (1) gives the action of $\mathrm{ad}_{\mathbf{e}_i}$.

From definition (1) it is easy to see that for any N polynomials of the order of N form the invariant subspace of representation $\mathrm{ad}_{\mathfrak{g}}$.

2 The Lie algebra so(5)

The basis of algebra $\mathfrak{g} = \mathrm{so}(5,\mathbb{C})$ consists of the elements $\mathbf{J}_{\alpha\beta} = -\mathbf{J}_{\beta\alpha}$, $\alpha, \beta = 1, 2, \ldots, 5$, which fulfill the commutation relations

$$\left[\mathbf{J}_{\alpha\beta}, \mathbf{J}_{\mu\nu}\right] = \delta_{\beta\mu}\mathbf{J}_{\alpha\nu} + \delta_{\alpha\nu}\mathbf{J}_{\beta\mu} - \delta_{\beta\nu}\mathbf{J}_{\alpha\mu} - \delta_{\alpha\mu}\mathbf{J}_{\beta\nu} \,. \tag{2}$$

If we define

$$\begin{array}{ll}
\mathbf{H}_1 = \mathrm{i}\mathbf{J}_{12} & \mathbf{H}_2 = \mathrm{i}\mathbf{J}_{34} \\
\mathbf{E}_1 = \frac{1}{\sqrt{2}}\left(\mathbf{J}_{45} + \mathrm{i}\mathbf{J}_{35}\right), & \mathbf{E}_2 = \frac{1}{2}\left(\mathbf{J}_{23} + \mathrm{i}\mathbf{J}_{13} - \mathbf{J}_{14} + \mathrm{i}\mathbf{J}_{24}\right), \\
\mathbf{E}_3 = \frac{1}{\sqrt{2}}\left(\mathbf{J}_{15} - \mathrm{i}\mathbf{J}_{25}\right), & \mathbf{E}_4 = \frac{1}{2}\left(\mathbf{J}_{23} + \mathrm{i}\mathbf{J}_{13} + \mathbf{J}_{14} - \mathrm{i}\mathbf{J}_{24}\right), \\
\mathbf{F}_1 = \frac{1}{\sqrt{2}}\left(-\mathbf{J}_{45} + \mathrm{i}\mathbf{J}_{35}\right), & \mathbf{F}_2 = \frac{1}{2}\left(-\mathbf{J}_{23} + \mathrm{i}\mathbf{J}_{13} + \mathbf{J}_{14} + \mathrm{i}\mathbf{J}_{24}\right), \\
\mathbf{F}_3 = \frac{1}{\sqrt{2}}\left(-\mathbf{J}_{15} - \mathrm{i}\mathbf{J}_{25}\right), & \mathbf{F}_4 = \frac{1}{2}\left(-\mathbf{J}_{23} + \mathrm{i}\mathbf{J}_{13} - \mathbf{J}_{14} - \mathrm{i}\mathbf{J}_{24}\right),
\end{array}$$

we obtain from (2) the commutation relations

$$\begin{array}{llll}
\left[\mathbf{H}_1, \mathbf{E}_1\right] = 0, & \left[\mathbf{H}_1, \mathbf{E}_2\right] = \mathbf{E}_2, & \left[\mathbf{H}_1, \mathbf{E}_3\right] = \mathbf{E}_3, & \left[\mathbf{H}_1, \mathbf{E}_4\right] = \mathbf{E}_4, \\
\left[\mathbf{H}_2, \mathbf{E}_1\right] = \mathbf{E}_1, & \left[\mathbf{H}_2, \mathbf{E}_2\right] = -\mathbf{E}_2, & \left[\mathbf{H}_2, \mathbf{E}_3\right] = 0, & \left[\mathbf{H}_2, \mathbf{E}_4\right] = \mathbf{E}_4 \\
\left[\mathbf{H}_1, \mathbf{F}_1\right] = 0, & \left[\mathbf{H}_1, \mathbf{F}_2\right] = -\mathbf{F}_2, & \left[\mathbf{H}_1, \mathbf{F}_3\right] = -\mathbf{F}_3, & \left[\mathbf{H}_1, \mathbf{F}_4\right] = -\mathbf{F}_4, \\
\left[\mathbf{H}_2, \mathbf{F}_1\right] = -\mathbf{F}_1, & \left[\mathbf{H}_2, \mathbf{F}_2\right] = \mathbf{F}_2, & \left[\mathbf{H}_2, \mathbf{F}_3\right] = 0, & \left[\mathbf{H}_2, \mathbf{F}_4\right] = -\mathbf{F}_4 \\
\left[\mathbf{E}_1, \mathbf{E}_2\right] = \mathbf{E}_3, & \left[\mathbf{E}_1, \mathbf{E}_3\right] = \mathbf{E}_4, & \left[\mathbf{E}_1, \mathbf{E}_4\right] = 0, & \\
\left[\mathbf{E}_2, \mathbf{E}_3\right] = 0, & \left[\mathbf{E}_2, \mathbf{E}_4\right] = 0, & \left[\mathbf{E}_3, \mathbf{E}_4\right] = 0, & \\
\left[\mathbf{F}_1, \mathbf{F}_2\right] = -\mathbf{F}_3, & \left[\mathbf{F}_1, \mathbf{F}_3\right] = -\mathbf{F}_4, & \left[\mathbf{F}_1, \mathbf{F}_4\right] = 0, & \\
\left[\mathbf{F}_2, \mathbf{F}_3\right] = 0, & \left[\mathbf{F}_2, \mathbf{F}_4\right] = 0, & \left[\mathbf{F}_3, \mathbf{F}_4\right] = 0, & \\
\left[\mathbf{E}_1, \mathbf{F}_1\right] = \mathbf{H}_2, & \left[\mathbf{E}_1, \mathbf{F}_2\right] = 0, & \left[\mathbf{E}_1, \mathbf{F}_3\right] = -\mathbf{F}_2, & \left[\mathbf{E}_1, \mathbf{F}_4\right] = -\mathbf{F}_3
\end{array}$$

$$
\begin{array}{llll}
[\mathbf{E}_2,\mathbf{F}_1]=0, & [\mathbf{E}_2,\mathbf{F}_2]=\mathbf{H}_1-\mathbf{H}_2, & [\mathbf{E}_2,\mathbf{F}_3]=\mathbf{F}_1, & [\mathbf{E}_2,\mathbf{F}_4]=0\\
[\mathbf{E}_3,\mathbf{F}_1]=-\mathbf{E}_2, & [\mathbf{E}_3,\mathbf{F}_2]=\mathbf{E}_1, & [\mathbf{E}_3,\mathbf{F}_3]=\mathbf{H}_1, & [\mathbf{E}_3,\mathbf{F}_4]=\mathbf{F}_1,\\
[\mathbf{E}_4,\mathbf{F}_1]=-\mathbf{E}_3, & [\mathbf{E}_4,\mathbf{F}_2]=0, & [\mathbf{E}_4,\mathbf{F}_3]=\mathbf{E}_1, & [\mathbf{E}_4,\mathbf{F}_4]=\mathbf{H}_1+\mathbf{H}_2
\end{array}
$$

which we will use in further calculation.

3 The highest weight vectors

The vectors $\mathbf{v}$ with the highest weights, i.e. the commutant $\mathscr{B}$ of the elements $\mathbf{E}_1$ and $\mathbf{E}_2$ in $U(\mathfrak{g})$, are given as solutions of the equations

$$\mathrm{ad}_{\mathbf{E}_1}\mathbf{v}=\mathrm{ad}_{\mathbf{E}_2}\mathbf{v}=0\,. \tag{3}$$

If we denote x_k, h_k and y_k the images of elements $\mathbf{E}_k$, $\mathbf{H}_k$ and $\mathbf{F}_k$ by mapping $U(\mathfrak{g})\to\mathrm{Sym}(\mathfrak{g})$, we obtain from (1) and (3) the system of equations

$$
\begin{aligned}
x_3\frac{\partial f}{\partial x_2}+x_4\frac{\partial f}{\partial x_3}-x_1\frac{\partial f}{\partial h_2}+h_2\frac{\partial f}{\partial y_1}-y_2\frac{\partial f}{\partial y_3}-y_3\frac{\partial f}{\partial y_4}=0\,,\\
-x_3\frac{\partial f}{\partial x_1}-x_2\Big(\frac{\partial f}{\partial h_1}-\frac{\partial f}{\partial h_2}\Big)+\big(h_1-h_2\big)\frac{\partial f}{\partial y_2}+y_1\frac{\partial f}{\partial y_3}=0\,.
\end{aligned}
\tag{4}
$$

The general solutions of (4) are

$$f=F\big(X_1,X_2,X_3,X_4,X_5,C_2\big)\,,$$

where F is an arbitrary differentiable function in variables

$$
\begin{aligned}
X_1&=x_1\\
X_2&=2x_2x_4-x_3^2\\
X_3&=x_1x_2+x_3h_2-x_4y_1\\
X_4&=x_1^2x_2-x_1x_3\big(h_1-h_2\big)+\tfrac{1}{2}x_4\big(h_1-h_2\big)^2+\big(2x_2x_4-x_3^3\big)y_2\\
X_5&=x_1^2x_2x_3+x_1x_2x_4\big(h_1+h_2\big)-x_1x_3^2\big(h_1-h_2\big)-x_3x_4h_2\big(h_1-h_2\big)-\\
&\qquad -x_1x_3x_4y_1+\big(2x_2x_4-x_3^2\big)\big(x_3y_2+x_4y_3\big)+x_4^2y_1\big(h_1-h_2\big)\\
C_2&=2\big(x_1y_1+x_2y_2+x_3y_3+x_4y_4\big)+h_1^2+h_2^2
\end{aligned}
$$

We can say that f has the weight (N_1,N_2) when

$$
\begin{aligned}
x_2\frac{\partial f}{\partial x_2}+x_3\frac{\partial f}{\partial x_3}+x_4\frac{\partial f}{\partial x_4}-y_2\frac{\partial f}{\partial y_2}-y_3\frac{\partial f}{\partial y_3}-y_4\frac{\partial f}{\partial y_4}=N_1f\,,\\
x_1\frac{\partial f}{\partial x_1}-x_2\frac{\partial f}{\partial x_2}+x_4\frac{\partial f}{\partial x_4}-y_1\frac{\partial f}{\partial y_1}+y_2\frac{\partial f}{\partial y_2}-y_4\frac{\partial f}{\partial y_4}=N_2f\,.
\end{aligned}
$$

It is easy to see that the functions X_1 and X_4 have the weight $(1,1)$, the function X_2 has the weight $(2,0)$, X_3 has the weight $(1,0)$, X_5 has the weight $(2,1)$ and finally C_2, which is an element of the center $U(\mathfrak{g})$, has the weight $(0,0)$.

Since the determinant

$$\frac{D(X_1,X_2,X_3,X_4,X_5,C_2)}{D(x_1,x_2,x_4,y_2,y_3,y_4)} = -4x_2x_4^3(2x_2x_4 - x_3^2)^2 \neq 0,$$

the functions X_k, $k = 1, \ldots, 5$ and C_2 are functionally independent. By direct calculation we can show that the relation

$$X_5^2 + X_4^2X_2 - 2X_1X_4X_3^2 - C_2X_1X_4X_2 + C_4X_1^2X_2 = 0 \tag{5}$$

is valid, where

$$\begin{aligned} C_4 = {} & \tfrac{1}{4}(h_1^2 - h_2^2 - 2x_1y_1 + 2x_3y_3)^2 + x_2y_2(h_1+h_2)^2 + x_4y_4(h_1-h_2)^2 - \\ & -2(x_1x_2y_3 + x_3y_1y_2)(h_1+h_2) - 2(x_1x_3y_4 + x_4y_1y_3)(h_1-h_2)^2 + \\ & +2(x_1^2x_2y_4 + x_4y_1^2y_3 - x_2x_4y_3^2 - x_3^2y_2y_4 + 2x_1x_3y_1y_3 + 2x_2x_4y_2y_4) \end{aligned}$$

is an element of the center of $U(\mathfrak{g})$, and has the weight $(0,0)$.

Now we show that the vector space of all highest weight polynomials, i.e. the polynomials, which fulfill equation (4), has the basis

$$C_2^{k_1}C_4^{k_2}X_1^{n_1}X_2^{n_2}X_3^{n_3}X_4^{n_4}X_5^{n_5}, \tag{6}$$

where k_1, k_2, $n_r \geq 0$ for $r = 1, \ldots, 4$ and $n_5 = 0$, 1.

It is clear that any linear combination of the polynomials (6) are solutions of (4), thus the are highest weight polynomials.

The linear independence of the functions (6) follows from the fact, that C_4 can be expressed by using the condition (5) as a function in variables C_2 and X_k, $k = 1, \ldots, 5$, which is quadratic in variable X_5, and above proved of functional independency of these functions.

Now we will show that the linear combinations of polynomials of the form (6) generate the space of all highest weight polynomials. It is clear that the polynomial (6) has the weight (N_1,N_2), where

$$N_1 = n_1 + 2n_2 + n_3 + n_4 + 2n_5\,, \quad N_2 = n_1 + n_4 + n_5 \tag{7}$$

and is of the order

$$M = 2k_1 + 4k_2 + n_1 + 2n_2 + 2n_3 + 3n_4 + 4n_5\,. \tag{8}$$

By using the differential operators (1) corresponding to generators of the Lie algebra so(5) it is possible from any polynomial with the highest weight (N_1,N_2) of an order of M to generate

$$d_{(N_1,N_2)} = \frac{1}{6}\left(N_1+N_2+2\right)\left(N_1-N_2+1\right)\left(2N_1+3\right)\left(2N_2+1\right)$$

of the linearly independent polynomials of an order of M. By direct calculation, which is cumbersome, it is possible to show that for any $M \in \mathbb{N}$ we have

$$\sum_{\mathscr{M}_M} d_{(N_1,N_2)} = \binom{M+9}{M},$$

where the summation is with respect to any k_1, k_2, $n_r \geq 0$, $r = 1, \ldots, 4$, and $n_5 = 0, 1$, for which (8) is valid, and N_1, N_2 are equal to (7). Now, because the vector space off all polynomials of an order of M in 10 variables has dimension

$$D_M = \binom{M+9}{M},$$

the polynomials of the form (6) generate the vectors space of all highest weight polynomials.

To go back to algebra $U(\mathfrak{g})$, we use the canonical isomorphism from $\mathrm{Sym}(\mathfrak{g})$ to $U(\mathfrak{g})$. First, we define the elements

$$\begin{aligned}
\mathbf{X}_1 &= \mathbf{E}_1\\
\mathbf{X}_2 &= \mathrm{sym}\left(2\mathbf{E}_2\mathbf{E}_4-\mathbf{E}_3^2\right)\\
\mathbf{X}_3 &= \mathrm{sym}\left(\mathbf{E}_1\mathbf{E}_2+\mathbf{E}_3\mathbf{H}_2-\mathbf{E}_4\mathbf{F}_1\right)\\
\mathbf{X}_4 &= \mathrm{sym}\Big(\mathbf{E}_1^2\mathbf{E}_2-\mathbf{E}_1\mathbf{E}_3\left(\mathbf{H}_1-\mathbf{H}_2\right)+\tfrac{1}{2}\mathbf{E}_4\left(\mathbf{H}_1-\mathbf{H}_2\right)^2+\left(2\mathbf{E}_2\mathbf{E}_4-\mathbf{E}_3^3\right)\mathbf{F}_2\Big)\\
\mathbf{X}_5 &= \mathrm{sym}\Big(\mathbf{E}_1^2\mathbf{E}_2\mathbf{E}_3+\mathbf{E}_1\mathbf{E}_2\mathbf{E}_4\left(\mathbf{H}_1+\mathbf{H}_2\right)-\mathbf{E}_1\mathbf{E}_3^2\left(\mathbf{H}_1-\mathbf{H}_2\right)-\\
&\qquad -\mathbf{E}_3\mathbf{E}_4\mathbf{H}_2\left(\mathbf{H}_1-\mathbf{H}_2\right)-\mathbf{E}_1\mathbf{E}_3\mathbf{E}_4\mathbf{F}_1+\\
&\qquad +\left(2\mathbf{E}_2\mathbf{E}_4-\mathbf{E}_3^2\right)\left(\mathbf{E}_3\mathbf{F}_2+\mathbf{E}_4\mathbf{F}_3\right)+\mathbf{E}_4^2\mathbf{F}_1\left(\mathbf{H}_1-\mathbf{H}_2\right)\Big)\\
\mathbf{C}_2 &= \mathrm{sym}\Big(2\left(\mathbf{E}_1\mathbf{F}_1+\mathbf{E}_2\mathbf{F}_2+\mathbf{E}_3\mathbf{F}_3+\mathbf{E}_4\mathbf{F}_4\right)+\mathbf{H}_1^2+\mathbf{H}_2^2\Big)\\
\mathbf{C}_4 &= \mathrm{sym}\Big(\tfrac{1}{4}\left(\mathbf{H}_1^2-\mathbf{H}_2^2-2\mathbf{E}_1\mathbf{F}_1+2\mathbf{E}_3\mathbf{F}_3\right)^2+\\
&\quad +\mathbf{E}_2\mathbf{F}_2\left(\mathbf{H}_1+\mathbf{H}_2\right)^2+\mathbf{E}_4\mathbf{F}_4\left(\mathbf{H}_1-\mathbf{H}_2\right)^2-\\
&\quad -2\left(\mathbf{E}_1\mathbf{E}_2\mathbf{F}_3+\mathbf{E}_3\mathbf{F}_1\mathbf{F}_2\right)\left(\mathbf{H}_1+\mathbf{H}_2\right)-2\left(\mathbf{E}_1\mathbf{E}_3\mathbf{F}_4+\mathbf{E}_4\mathbf{F}_1\mathbf{F}_3\right)\left(\mathbf{H}_1-\mathbf{H}_2\right)^2+\\
&\quad +2\left(\mathbf{E}_1^2\mathbf{E}_2\mathbf{F}_4+\mathbf{E}_4\mathbf{F}_1^2\mathbf{F}_3-\mathbf{E}_2\mathbf{E}_4\mathbf{F}_3^2-\mathbf{E}_3^2\mathbf{F}_2\mathbf{F}_4+2\mathbf{E}_1\mathbf{E}_3\mathbf{F}_1\mathbf{F}_3+2\mathbf{E}_2\mathbf{E}_4\mathbf{F}_2\mathbf{F}_4\right)\Big).
\end{aligned}$$

Since the set of polynomials (6) is a basis in the space of the highest weight polynomials, the set

$$\mathrm{sym}\Big(\mathbf{C}_2^{k_1}\mathbf{C}_4^{k_2}\mathbf{X}_1^{n_1}\mathbf{X}_2^{n_2}\mathbf{X}_3^{n_3}\mathbf{X}_4^{n_4}\mathbf{X}_5^{n_5}\Big) \in U(\mathfrak{g}) \tag{9}$$

forms the basis in the space of the highest weight vectors.

We denote $U_n(\mathfrak{g})$ as a subspace of $U(\mathfrak{g})$ with elements of an order less or equal to n. It is a well know fact that for n_1 and n_2 one has

$$\left[U_{n_1}(\mathfrak{g}), U_{n_2}(\mathfrak{g})\right] \subset U_{n_1+n_2-1}(\mathfrak{g})\,.$$

For this reason as a basis in the vector space of the height weight vector we can use instead of (9) the set of elements

$$\mathbf{C}_2^{k_1}\mathbf{C}_4^{k_2}\mathbf{X}_1^{n_1}\mathbf{X}_2^{n_2}\mathbf{X}_3^{n_3}\mathbf{X}_4^{n_4}\mathbf{X}_5^{n_5} \in U(\mathfrak{g})\,,$$

where $k_1, k_2, n_r \geq 0$ for $r = 1, 2, 3, 4$ and $n_5 = 0, 1$.

4 Conclusion

We have studied the structure of enveloping algebra $U\big(\mathrm{so}(5)\big)$ as the vector space of adjoint representation of the Lie algebra $\mathrm{so}(5)$. We explicitly decomposed this representation into the irreducible components and found the highest weight vector in any such components.

Our result can be useful for further study of tensor products of the representations of the Lie algebras and for study of ideals in their enveloping algebras. The details of such study for $\mathrm{sl}(2)$ case see in [2].

The method described can be used for other simple Lie algebras. We really obtained some results for the Lie algebra $\mathrm{sl}(4)$, but the general solution for $\mathrm{sl}(n)$, $\mathrm{so}(n)$ etc. is still an open problem.

Acknowledgements The authors are grateful to M. Havlíček and S. Pošta for valuable and useful discussions.
Partially supported by GACR 201/05/0857.

References

1. Dixmier J.: Algebres enveloppantes. Gauthier-Vllars Editeur, (1974)
2. Kirillov A.A.: Elementy teorii predstavlenij. Nauka, Moscow (1972). in russian
3. Flath E.: J. Math. Phys. **30**, 1076–1077 (1990)
4. Konstant B.: Am. J. Math. Phys. **85**, 327–404 (1963)