$q$-Analog of Gelfand–Graev basis for the noncompact quantum algebra $U_q(u(n, 1))$

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Abstract

For the quantum algebra $U_q(gl(n+1))$ in its reduction on the subalgebra $U_q(gl(n))$ an explicit description of a Mickelsson–Zhelobenko reduction algebra $Z_q(gl(n+1), gl(n))$ is given in terms of the generators and their defining relations. Using this $Z$-algebra we describe Hermitian irreducible representations of a discrete series for the quantum algebra $U_q(u(n, 1))$ which is a real form of $U_q(gl(n))$. Namely, an orthonormal Gelfand–Tsetlin basis is constructed in explicit form.

1 Introduction

In 1950 I.M. Gelfand and M.L. Tsetlin [1] proposed a formal description of finite-dimensional irreducible representations (IR) for the compact Lie algebra $u(n)$. This description is a generalization of results for $u(2)$ and $u(3)$ on the case $u(n)$. It is the following. In the IR space of $u(n)$ there is a orthonormalized basis which is numerated by the following formal schemes

$$
\begin{pmatrix}
m_{1n} & m_{2n} & \cdots & m_{n-1,n} & m_{nn} \\
m_{1,n-1} & m_{2,n-1} & \cdots & m_{n-1,n-1} & \\
\vdots & \vdots & \ddots & \vdots & \\
m_{12} & m_{22} & \cdots & m_{n-1,n-1} & \\
m_{11} & & & & \\
\end{pmatrix}
$$

(1.1)
where all numbers $m_{ij}$ ($1 \leq i < j \leq n$) are nonnegative integers and they satisfy the standard inequalities, "between conditions":

$$m_{ij+1} \geq m_{ij} \geq m_{i+1j+1} \quad \text{for} \quad 1 \leq i \leq j \leq n-1.$$

The first line of this scheme is defined by the components of the highest weight of $u(n)$ IR, the second line is defined by the components of the highest weight of $u(n-1)$ IR and so on.

Later this basis was constructed in many papers (for example, see [2, 3, 4]) by using one-step lowering and raising operators.

In 1965 I.M. Gelfand and M.I. Graev [5] using analytic continuation of the results for $u(n)$ obtained some results for non-compact Lie algebra $u(n, m)$. They shown that some class of Hermitian IR of $u(n, m)$ is characterized by a "extremal weight" parametrized by a set of integers $m_N = (m_{1N}, \ldots, m_{NN})$ ($N = n + m$) such that $m_{1N} \geq m_{1N} \geq \cdots \geq m_{NN}$, and by a representation type which is defined by a partition of $n$ in the sum of two nonnegative integers $\alpha$ and $\beta$, $n = \alpha + \beta$ (also see [6]).

For simplicity we consider the case $u(2, 1)$. In this case we have three type of scheme

$$
\begin{pmatrix}
  m_{12} & m_{13} & m_{23} & m_{33} \\
  m_{12} & m_{22} & m_{23} & m_{33} \\
  m_{11} & m_{12} & m_{22} & m_{23} \\
  m_{11} & m_{12} & m_{13} & m_{23}
\end{pmatrix}
\quad \text{for} \quad (\alpha, \beta) = (2, 0),
$$

$$
\begin{pmatrix}
  m_{12} & m_{13} & m_{23} & m_{33} \\
  m_{12} & m_{22} & m_{23} & m_{33} \\
  m_{11} & m_{12} & m_{22} & m_{23} \\
  m_{11} & m_{12} & m_{13} & m_{23}
\end{pmatrix}
\quad \text{for} \quad (\alpha, \beta) = (1, 1),
$$

$$
\begin{pmatrix}
  m_{13} & m_{23} & m_{33} \\
  m_{12} & m_{33} & m_{23} \\
  m_{11} & m_{12} & m_{22}
\end{pmatrix}
\quad \text{for} \quad (\alpha, \beta) = (0, 2).
$$

The numbers $m_{ij}$ of the first scheme satisfy the following inequalities

$$m_{12} \geq m_{13} + 1, \quad m_{13} + 1 \geq m_{22} \geq m_{23} + 1, \quad m_{12} \geq m_{11} \geq m_{22} .$$

The numbers $m_{ij}$ of the second scheme satisfy the following inequalities

$$m_{12} \geq m_{13} + 1, \quad m_{13} - 1 \geq m_{22}, \quad m_{12} \geq m_{11} \geq m_{22} .$$

The numbers of the third scheme satisfy the following inequalities

$$m_{23} - 1 \geq m_{12} \geq m_{33} - 1, \quad m_{33} - 1 \geq m_{22}, \quad m_{12} \geq m_{11} \geq m_{22} .$$

Construction of Gelfand–Graev basis for $u(n, m)$ in terms of one-step lowering and raising operators is more complicated then in the compact case $u(n + m)$.

In 1975 T.J. Enright and V.S. Varadarajan [7] obtained some classification of discrete series of non-compact Lie algebras. Later A. Molev [8] shown that for the case $u(n, m)$ Gelfand–Graev modules are some part of Enright–Varadarajan modules and Molev constructed Gelfand–Graev basis for $u(n, m)$ in terms of Mickelsson S-algebra [9].

A goal of this work to obtain analogous results for the non-compact quantum algebra $U_q(u(n, m))$. Because the general case is very complicated we at first consider the case $U_q(u(2, 1))$. It should be noted that the special case $U_q(u(2, 1))$ was considered in [10, 11].
2 Quantum algebra \( U_q(gl(N)) \) and its noncompact real forms \( U_q(u(n, m)) \) \((n + m = N)\)

The quantum algebra \( U_q(gl(N)) \) is generated by the Chevalley elements \( q^{\pm e_{ii}} \) \((i = 1, \ldots, N)\), \( e_{i,i+1}, e_{i+1,i} \) \((i = 1, 2, \ldots, N - 1)\) with the defining relations:

\[
q^{e_{ii}} q^{-e_{ii}} = q^{-e_{ii}} q^{e_{ii}} = 1 , \\
q^{e_{ii}} q^{e_{jj}} = q^{e_{jj}} q^{e_{ii}} , \\
q^{e_{ii}} e_{jk} q^{-e_{ii}} = q^{\delta_{ij} - \delta_{ik}} e_{jk} \quad (|j - k| = 1) , \\
[e_{i,i+1}, e_{j+1,j}] = \delta_{ij} \frac{q^{e_{ii}} - q^{e_{i+1,i+1}} - q^{e_{i+1,i+1} - e_{ii}}}{q - q^{-1}} , \\
[e_{i,i+1}, e_{jj+1}] = 0 \quad \text{for} \quad |i - j| \geq 2 , \\
[e_{i+1,i}, e_{j+1,j}] = 0 \quad \text{for} \quad |i - j| \geq 2 , \\
[[e_{i,i+1}, e_{jj+1}]_q, e_{jj+1}]_q = 0 \quad \text{for} \quad |i - j| = 1 , \\
[[e_{i+1,i}, e_{j+1,j}]_q, e_{j+1,j}]_q = 0 \quad \text{for} \quad |i - j| = 1 .
\]

where \([e_\beta, e_\gamma]_q\) denotes the \(q\)-commutator:

\[
[e_\beta, e_\gamma]_q := e_\beta e_\gamma - q^{(\beta \gamma)} e_\gamma e_\beta . \quad (9.2)
\]

The definition of a quantum algebra also includes operations of a comultiplication \(\Delta_q\), an antipode \(S_q\), and a co-unit \(\varepsilon_q\). Because explicit formulas of these operations will not used in our later calculations, they are not given here.

For construction of the composite root vectors \(e_{ij}\) for \(|i - j| \geq 2\) we fix the following normal (convex) ordering of the positive root system \(\Delta_+\) (see [12])

\[
\epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_2 - \epsilon_3 < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4 < \ldots < \\
\epsilon_1 - \epsilon_k < \epsilon_2 - \epsilon_k < \ldots < \epsilon_{k-1} - \epsilon_k < \ldots < \epsilon_1 - \epsilon_N < \epsilon_2 - \epsilon_N < \ldots < \epsilon_{N-1} - \epsilon_N . \quad (2.10)
\]

According to this ordering we set

\[
e_{ij} := [e_{ik}, e_{kj}]_{q^{-1}} , \quad e_{ji} := [e_{jk}, e_{ki}]_q , \quad (11.2)
\]

where \(1 \leq i < k < j \leq N\). It should be stressed that the structure of the composite root vectors does not dependent of choice of the index \(k\) in the r.h.s. of the definition \((2.10)\). In particular, we have

\[
e_{ij} := [e_{i,i+1}, e_{i+1,j}]_{q^{-1}} = [e_{i,j-1}, e_{j-1,j}]_{q^{-1}} , \\
e_{ji} := [e_{j,i+1}, e_{i+1,j}]_q = [e_{j,j-1}, e_{j-1,j}]_q , \quad (12.2)
\]

where \(2 \leq i + 1 < j \leq N\).
Using these explicit constructions and the defining relations (2.1)–(2.8) for the Chevalley basis it is not hard to calculate the following relations between the Cartan–Weyl generators $e_{ij}$ ($i, j = 1, 2, \ldots, N$):

\[
q^{e_{kk}}e_{ij}q^{-e_{kk}} = q^{\delta_{ii}-\delta_{jj}}e_{ij} \quad (1 \leq i, j, k \leq N) ,
\tag{2.13}
\]

\[
[e_{ij}, e_{kl}] = \frac{q^{e_{ii}}-q^{e_{jj}} - q^{e_{jj}}e_{ii}}{q-q^{-1}} \quad (1 \leq i < j \leq N) ,
\tag{2.14}
\]

\[
[e_{ij}, e_{kl}]q^{-1} = \delta_{jk}e_{il} \quad (1 \leq i < j \leq k < l \leq N) ,
\tag{2.15}
\]

\[
[e_{ik}, e_{jl}]q^{-1} = (q-q^{-1})e_{ij}e_{kl} \quad (1 \leq i < j < k < l \leq N) ,
\tag{2.16}
\]

\[
[e_{ij}, e_{kl}] = 0 \quad (1 \leq i < j \leq k < l \leq N) ,
\tag{2.17}
\]

\[
[e_{kl}, e_{nj}] = 0 \quad (1 \leq i < j < k < l \leq N) ,
\tag{2.18}
\]

\[
[e_{il}, e_{kj}] = 0 \quad (1 \leq i < j < k < l \leq N) ,
\tag{2.19}
\]

\[
[e_{ii}, e_{jj}] = e_{jj}q^{e_{ii}}-e_{jj} \quad (1 \leq i < j \leq N) ,
\tag{2.20}
\]

\[
[e_{kl}, e_{ii}] = e_{kl}q^{e_{kk}}-e_{ii} \quad (1 \leq i < k < l \leq N) ,
\tag{2.21}
\]

\[
[e_{ij}, e_{kl}] = (q^{-1}-q)e_{kl}e_{ji}q^{e_{jj}-e_{kk}} \quad (1 \leq i < j < k < l \leq N) .
\tag{2.22}
\]

If we apply the Cartan involution ($e^*_{ij} = e_{ji}$) the formulas above, we will get all relations between elements of the Cartan–Weyl basis.

The explicit formula for the extremal projector for $U_q(\mathfrak{gl}(N))$ has the form [13]

\[
p(U_q(\mathfrak{gl}(N))) = p(U_q(\mathfrak{gl}(N-1))(p_{11}p_{22} \cdots p_{N-2Np_{N-1N}})
\tag{2.23}
\]

\[
= p_{12}(p_{13}p_{23}) \cdots (p_{i1} \cdots p_{ii+1}) \cdots (p_{1N} \cdots p_{N-1N}) ,
\]

where the elements $p_{ij}$ ($1 \leq i < j \leq N$) are given by

\[
p_{ij} = \sum_{r=0}^{\infty} (-1)^r \frac{(1)^{s}}{[r]^!} \varphi_{ij,r} e_{ij}^r ,
\tag{2.24}
\]

\[
\varphi_{ij,r} = q^{-(j-i-1)^r} \left\{ \prod_{s=1}^{N} [e_{ii} - e_{jj} + j - i + s] \right\}^{-1} .
\]

The extremal projector $p := p(U_q(\mathfrak{gl}(N))$ satisfies the relations:

\[
e_{i,i+1}p = p e_{i+1,i} = 0 \quad (1 \leq i \leq N-1) ,
\tag{2.25}
\]

\[
p^2 = p .
\]

The extremal projector $p$ belongs to the Taylor extension $TU_q(\mathfrak{gl}(N)$ of the universal enveloping algebras $U_q(\mathfrak{gl}(N)$. The Taylor extension $TU_q(\mathfrak{gl}(N)$ is an associative algebra generated by formal Taylor series of the form

\[
\sum_{\{r'\}, \{r\}} C_{\{r'\}, \{r\}} (q^{e_{11}}, \ldots, q^{e_{NN}}) e_{21}^{r'_{12}} e_{31}^{r'_{13}} e_{32}^{r'_{32}} \cdots e_{NN-1,N}^{r'_{N-1,N}} e_{N,1}^{r_{12}} e_{13}^{r_{13}} e_{23}^{r_{23}} \cdots e_{N-1,N}^{r_{N-1,N}}
\tag{2.26}
\]
provided that nonnegative integers \( r'_{12}, r'_{13}, r'_{23}, \ldots, r'_{N-1,N} \) and \( r_{12}, r_{13}, r_{23}, \ldots, r_{N-1,N} \) are subject to the constraints
\[
\left| \sum_{i<j} r'_{ij} - \sum_{i<j} r_{ij} \right| \leq \text{const} \tag{2.27}
\]
for each formal series and the coefficients \( C_{\{r',\},\{r\}}(q^{e^{u_1}}, \ldots, q^{e^{u_N}}) \) are rational functions of the \( q \)-Cartan elements \( q^{e^{u_i}} \). The quantum universal enveloping algebra \( U_q(\mathfrak{gl}(N)) \) is a subalgebra of the Taylor extension \( TU_q(\mathfrak{gl}(N)) \), \( U_q(\mathfrak{gl}(N)) \subset TU_q(\mathfrak{gl}(N)) \).

The noncompact quantum algebra \( U_q(\mathfrak{u}(n,m)) \) can be considered as the quantum algebra \( U_q(\mathfrak{gl}(N)) \) \( (N = n + m) \) endowed with the additional Cartan involution (*)
\[
h^*_i = h_i, \quad (i = 1, 2, \ldots n + m) \tag{2.28}
\]
\[
e^*_{i,i+1} = e_{i+1,i}, \quad e^*_{i+1,i} = e_{i+1,i} \quad \text{for } i \neq n, \tag{2.29}
\]
\[
e^*_n,_{n+1} = -e_{n+1,n}, \quad e^*_{n+1,n} = -e_{n,n+1}, \tag{2.30}
\]
\[
q^* = q \quad \text{or} \quad q = q^{-1}. \tag{2.31}
\]

Below we will consider the real form \( U_q(\mathfrak{u}(n,1)) \), i.e. the case \( N = n + 1 \).

3 The reduction algebra \( Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \)

In the linear space \( TU_q(\mathfrak{gl}(n+1)) \) we separate out a subspace of "two-sided highest vectors" with respect to the subalgebra \( U_q(\mathfrak{gl}(n)) \subset U_q(\mathfrak{gl}(n+1)) \), i.e.
\[
\tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) = \left\{ x \in TU_q(\mathfrak{gl}(n+1)) \mid e_{ij}x = xe_{ji} = 0, \ 1 \leq i < j \leq n \right\}. \tag{3.1}
\]

It is evident that if \( x \in \tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \) then
\[
x = pxp, \tag{3.2}
\]
where \( p := p(U_q(\mathfrak{gl}(n)) \). Again, using the annihilation properties of the projection operator \( p \) we have that any vector \( x \in \tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \) presents a formal Taylor series on the following monomials
\[
p e^r_{n+1,1} \cdots e^r_{n,n+1} e^e_{n,n+1} \cdots e^e_{1,n+1} p . \tag{3.3}
\]
It is evident that \( \tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \) is a subalgebra in \( TU_q(\mathfrak{gl}(n+1)) \). We consider a subspace \( Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \) generated by finite series on the monomials (5.3). It is clear that \( Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \) is a subalgebra in \( \tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \).

We set
\[
\zeta_i := pe_i p, \quad (i = \pm 1, \pm 2, \ldots, \pm n) \tag{3.4}
\]
where \( e_i = e_{j,n+1}, e_{-i} = e_{n+1,j} \ (1 = 1, 2, \ldots, n) \).
Theorem 3.1  The elements $z_i (i = \pm 1, \pm 2, \ldots, \pm n)$ generates the associative algebra $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ and satisfies the following relations

$$z_i z_j = A_{ij} z_j z_i \quad \text{for} \quad i + j \neq 0, \quad (i, j = \pm 1, \pm 2, \ldots, \pm n) ,$$

(3.5)

$$z_i z_{-i} = \sum_{j=1}^{n} B_{ij} z_{-i} z_i + \gamma_i \quad \text{for} \quad i = 1, 2, \ldots, n ,$$

(3.6)

where

$$A_{ij} = 1 \quad \text{for} \quad \text{sgn} \, i \neq \text{sgn} \, j ,$$

(3.7)

$$A_{ij} = A_{-j-i} = \frac{[\varphi_{ij} + 1]}{[\varphi_{ij}]} \quad \text{for} \quad 1 \leq i < j \leq n ,$$

(3.8)

$$B_{ij} = - \frac{b_i^+ b_j^+}{[\varphi_{ij} - 1]} , \quad \gamma_i = b_i^- [\varphi_{i,n+1} - 1] ,$$

(3.9)

$$b_i^\pm = \prod_{s=i+1}^{n} \frac{[\varphi_{is} \pm 1]}{[\varphi_{is}]} , \quad \varphi_{ik} = e_{ii} - e_{kk} + k - i .$$

(3.10)

A proof of the theorem by direct calculations.

For construction and study of discrete series of the non-compact quantum algebra $U_q(u(n, 1)$ we need another relations then (3.6). The system (3.6) expresses the elements $z_i z_{-i}$ in terms of the elements $z_{-i} z_i$ ($i = 1, 2, \ldots, n$) but we would like to express the elements $z_{-1} z_1, \ldots, z_{-\alpha} z_{\alpha}, z_{\alpha+1} z_{-\alpha-1}, \ldots, z_n z_{-n}$ in terms of the elements $z_1 z_{-1}, \ldots, z_\alpha z_{-\alpha}, z_{\alpha-1} z_{\alpha+1}, \ldots, z_n z_{-n}$ for $\alpha = 0, 1, \ldots, n$\(^1\). These relations are given by the theorem.

Theorem 3.2 The elements $z_{-1} z_1, \ldots, z_{-\alpha} z_{\alpha}, z_{\alpha+1} z_{-\alpha-1}, \ldots, z_n z_{-n}$ are expressed in terms of the elements $z_1 z_{-1}, \ldots, z_\alpha z_{-\alpha}, z_{\alpha-1} z_{\alpha+1}, \ldots, z_n z_{-n}$ by the formulas

$$z_{-i} z_i = \sum_{j=1}^{\alpha} B_{ij}^{(\alpha)} z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{il}^{(\alpha)} z_{-l} z_i + \gamma_i^{(\alpha)} \quad (i = 1, 2, \ldots, \alpha) ,$$

(3.11)

$$z_{-k} z_k = \sum_{j=1}^{\alpha} B_{kj}^{(\alpha)} z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{kl}^{(\alpha)} z_{-l} z_k + \gamma_k^{(\alpha)} \quad (k = \alpha + 1, \alpha + 2, \ldots, n) ,$$

(3.12)

where

$$B_{ij}^{(\alpha)} = \frac{b_i^{(\alpha)+} b_j^{(\alpha)-}}{[\varphi_{ij} + 1]} , \quad B_{il}^{(\alpha)} = \frac{b_i^{(\alpha)+} b_l^{(\alpha)+}}{[\varphi_{il}]} , \quad \gamma_i^{(\alpha)} = -[\varphi_{i,n+1} - \alpha] b_i^{(\alpha)+} ,$$

(3.13)

$$B_{kj}^{(\alpha)} = - \frac{b_k^{(\alpha)-} b_j^{(\alpha)-}}{[\varphi_{kj}]} , \quad B_{kl}^{(\alpha)} = - \frac{b_k^{(\alpha)-} b_l^{(\alpha)+}}{[\varphi_{kl} - 1]} , \quad \gamma_k^{(\alpha)} = [\varphi_{i,n+1} - \alpha - 1] b_k^{(\alpha)+} ,$$

(3.14)

$$b_i^{(\alpha)\pm} = \prod_{s=1}^{i-1} \frac{[\varphi_{is} \pm 1]}{[\varphi_{is}]} \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is}]}{[\varphi_{is} \pm 1]} , \quad b_i^{(\alpha)\pm} = \prod_{s=1}^{\alpha} \frac{[\varphi_{ls}]}{[\varphi_{ls} + 1]} \prod_{s=l+1}^{n-1} \frac{[\varphi_{ls} \pm 1]}{[\varphi_{ls}]} .$$

(3.15)

\(^1\)In the case $\alpha = 0$ we have the relations (3.6) and for $\alpha = n$ we obtain the system inverse to (3.6)
Moreover the following power relations are valid

\[ z_i^rz_j^s = z_j^rz_i^r \quad \text{for } \forall i, j > 0 \text{ and } r, s \in \mathbb{N} , \]  

(3.16)

\[ z_i^rz_j^s = z_j^rz_i^r [\varphi_{ij} + r][\varphi_{ij} - s]! \quad \text{for } 1 \leq i < j \leq n \text{ and } r, s \in \mathbb{N} , \]  

(3.17)

\[ z_i^rz_j^r = z_j^rz_i^r [\varphi_{ij}][\varphi_{ij} + r + s]! \quad \text{for } 1 \leq i < j \leq n \text{ and } r, s \in \mathbb{N} , \]  

(3.18)

\[ z_i^rz_j^r = z_j^rz_i^r \sum_{l=\alpha+1}^n B_{ij}^{(\alpha)}(r)z_jz_{-l}^rz_l + \sum_{l=\alpha+1}^n B_{il}^{(\alpha)}(r)z_{-i}z_l + \gamma_i^{(\alpha)}(r) \quad (i = 1, 2, \ldots, \alpha), \]  

(3.19)

\[ z_k^rz_k^r = z_{-k}^rz_{-k}^r \sum_{j=1}^\alpha B_{kj}^{(\alpha)}(r)z_jz_{-l}^rz_l + \sum_{l=\alpha+1}^n B_{kl}^{(\alpha)}(r)z_{-i}z_l + \gamma_k^{(\alpha)}(r) \quad (k = \alpha+1, \ldots, n), \]  

(3.20)

where

\[ B_{ij}^{(\alpha)}(r) = \frac{[r][\varphi_{ij} + r] b_{ij}^{(\alpha)-}}{[\varphi_{ij} + r]}, \quad B_{il}^{(\alpha)}(r) = \frac{[r][\varphi_{il} + r] b_{il}^{(\alpha)-}}{[\varphi_{il} + r - 1]}, \]  

\[ \gamma_i^{(\alpha)}(r) = -[r][\varphi_{i,n+1} - \alpha + r - 1] b_{i}^{(\alpha)+}(r) \quad (1 \leq i, j \leq \alpha < l \leq n) , \]  

(3.21)

\[ B_{kj}^{(\alpha)}(r) = -\frac{[r][\varphi_{kj} - r] b_{kj}^{(\alpha)-}}{[\varphi_{kj} - r + 1]}, \quad B_{kl}^{(\alpha)}(r) = -\frac{[r][\varphi_{kl} - r] b_{k}^{(\alpha)-}}{[\varphi_{kl} - r]}, \]  

\[ \gamma_k^{(\alpha)}(r) = [r][\varphi_{k,n+1} - \alpha - r] b_{k}^{(\alpha)-}(r) \quad (1 \leq j \leq \alpha < k, l \leq n) . \]  

Here

\[ b_{ij}^{(\alpha)+}(r) = \left( \prod_{s=1}^{i-1} \frac{[\varphi_{is} + r]}{[\varphi_{is} + r - 1]} \right) \left( \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is} + r - 1]}{[\varphi_{is} + r]} \right) \quad (1 \leq i \leq \alpha) , \]  

(3.23)

\[ b_{ij}^{(\alpha)-} = \left( \prod_{s=1}^{i-1} \frac{[\varphi_{is} - 1]}{[\varphi_{is}]} \right) \left( \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is}]}{[\varphi_{is} - 1]} \right) \quad (1 \leq i \leq \alpha) , \]  

(3.24)

\[ b_{il}^{(\alpha)-}(r) = \left( \prod_{s=1}^{\alpha} \frac{[\varphi_{is} - r + 1]}{[\varphi_{is} - r]} \right) \left( \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is} - r]}{[\varphi_{is} - r + 1]} \right) \quad (\alpha + 1 \leq l \leq n) , \]  

(3.25)

\[ b_{il}^{(\alpha)+} = \left( \prod_{s=1}^{\alpha} \frac{[\varphi_{is}]}{[\varphi_{is} + 1]} \right) \left( \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is} + 1]}{[\varphi_{is}]} \right) \quad (\alpha + 1 \leq l \leq n) . \]  

(3.26)
4 Shapovalov’s forms on \( Z_q^*(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \)

We consider on the \( Z \)-algebra \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) two real forms: compact and noncompact.

The compact real form on \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) is defined by the involution \((^*)\) which is given as follows

\[
Z_{\pm i}^* = Z_{\mp i} \quad (i = 1, 2, \ldots, n),
\]
\[
e_{ii}^* = e_{ii} \quad (i = 1, 2, \ldots, n + 1).
\]

This involution can be considered as generalization of the Cartan involution in \( U_q(\mathfrak{gl}(n + 1)) \) to the Taylor extension, \( TU_q(\mathfrak{gl}(n + 1)) \). The \( Z \)-algebra \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) with this involution is called the compact real form and denoted by the symbol \( Z_q^{(c)}(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \).

The noncompact real form on \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) is defined by the involution \(*\) which is given as follows

\[
Z_{\pm i}^* = -Z_{\mp i} \quad (i = 1, 2, \ldots, n),
\]
\[
e_{ii}^* = e_{ii} \quad (i = 1, 2, \ldots, n + 1).
\]

This involution can be considered as generalization of the noncompact involution in \( U_q(\mathfrak{gl}(n + 1)) \) to the Taylor extension, \( TU_q(\mathfrak{gl}(n + 1)) \). The \( Z \)-algebra \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) with this involution is called the noncompact real form and denoted by the symbol \( Z_q^{(nc)}(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \).

Let \( p^{(\alpha)} \) be the extremal projector for \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) satisfying the relations

\[
z_{-i}p^{(\alpha)} = p^{(\alpha)}z_i \quad \text{for} \quad i = 1, 2, \ldots, \alpha,
\]
\[
z_kp^{(\alpha)} = p^{(\alpha)}z_{-k} \quad \text{for} \quad k = \alpha + 1, \alpha + 2, \ldots, n,
\]
\[
[e_{ii}, p^{(\alpha)}] = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

This extremal projector depends on the index \( \alpha \), which defines what elements are considered as ”raising” and what elements are considered as ”lowering”, i.e. in our case the elements \( z_{-1}, z_{-2}, \ldots, z_{-\alpha}, z_{\alpha+1}, \ldots, z_n \) are raising and the elements \( z_1, z_2, \ldots, z_{\alpha}, z_{-\alpha-1}, \ldots, z_{-n} \) are lowering. It should be noted that the ”raising” and ”lowering” subsets generate disjoint subalgebras in \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \). The operator \( p^{(\alpha)} \) can be constructed in explicit form.

Let us introduce on \( Z_q^{(nc)}(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) the following sesquilinear Shapovalov form. For any elements \( x, y \in Z_q^{(nc)}(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) we set

\[
B^{(\alpha)}(x, y) = p^{(\alpha)}y^*xp^{(\alpha)}.
\]

Therefore the Shapovalov form also depends on the index \( \alpha \) \( (\alpha = 0, 1, 2, \ldots, n) \). We fix \( \alpha \) \( (\alpha = 0, 1, 2, \ldots, n) \) and for each set of nonnegative integers \( \{r\} = (r_1, r_2, \ldots, r_n) \) introduce a vector in the space \( Z_q^{(nc)}(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) by the formula

\[
q_{\{r\}}^{(\alpha)} = z_{\alpha}^{r_\alpha}z_1^{r_1}z_{-\alpha-1}^{r_{-\alpha-1}}\cdots z_{-n}^{r_{-n}}.
\]
As well as in the classical case \[ \text{Theorem 4.1} \]
\[ B^{(\alpha)}(v^{(\alpha)}_{\{ r \}}, v^{(\alpha)}_{\{ r' \}}) = \delta_{\{ r \}, \{ r' \}} B^{(\alpha)}(v^{(\alpha)}_{\{ r \}}, v^{(\alpha)}_{\{ r \}}) . \] (4.10)

and
\[ B^{(\alpha)}(v^{(\alpha)}_{\{ r \}}, v^{(\alpha)}_{\{ r \}}) = \left( \prod_{i=1}^{\alpha} [r_i]! \frac{[\varphi_{i,n+1} - \alpha + r_i - 1]!}{[\varphi_{i,n+1} - \alpha - 1]!} \right) \frac{\prod_{l=\alpha+1}^{n} [r_l]! \frac{[\varphi_{n+1,l} + \alpha + r_l]!}{[\varphi_{n+1,l} + \alpha]!}}{\prod_{1 \leq i < j \leq n} \frac{[\varphi_{ij} + r_i - r_j]!}{[\varphi_{ij} - r_j - 1]!} \prod_{\alpha+1 \leq k \leq l \leq n} \frac{[\varphi_{kl} - r_k + r_l]!}{[\varphi_{kl} - r_k - 1]!} \frac{[\varphi_{kl} + r_l]!}{[\varphi_{kl} + r_l - 1]!}} \prod_{1 \leq i \leq \alpha \leq \leq n} \frac{[\varphi_{il} + r_i - 1]!}{[\varphi_{il} + r_i + r_l]!} [\varphi_{il} - 1]! p^{(\alpha)}. \] (4.11)

As consequence of this theorem we obtain that the Shapovalov form is not degenerate on \( Z_q^{(n)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \).

In the case of the compact \( Z \)-algebra \( Z_q^{(c)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \) the Shapovalov form \( B(x, y) \) is defined by the formula (4.8) where \( \alpha = 0 \), \( p^{(0)} \) is the standard extremal projector of the subalgebra \( \mathfrak{gl}(n) \) and the involution (*) is given by the formulas (4.1). It is not difficult to see that
\[ B(v_{\{ r \}}, v_{\{ r' \}}) = \delta_{\{ r \}, \{ r' \}} B(v_{\{ r \}}, v_{\{ r \}}) . \] (4.12)

where \( v_{\{ r \}} := v^{(0)}_{\{ r \}} \), and
\[ B(v_{\{ r \}}, v_{\{ r \}}) = (-1)^{\sum_{i=1}^{n} r_i} B^{(0)}(v^{(0)}_{\{ r \}}, v^{(0)}_{\{ r \}}) \]
\[ = \prod_{l=1}^{n} [r_l]! \frac{[\varphi_{l,n+1} - r_l]!}{[\varphi_{l,n+1}]!} \prod_{1 \leq k \leq l \leq n} \frac{[\varphi_{kl} - r_k + r_l]!}{[\varphi_{kl} - r_k - 1]!} \frac{[\varphi_{kl} + r_l]!}{[\varphi_{kl} + r_l - 1]!} . \] (4.13)

5 Discrete series of representations for \( U_q(\mathfrak{u}(n, 1)) \)

As well as in the classical case \([9]\) each Hermitian irreducible representation of the discrete series for the noncompact quantum algebra \( U_q(\mathfrak{u}(n, 1)) \) is defined uniquely by some extremal vector \( |xw\rangle \), the vector of extremal weight\(^{2}\). This vector should be a highest vector with respect to the compact subalgebra \( U_q(\mathfrak{u}(n)) \oplus U_q(\mathfrak{u}(1)) \). Since the quantum algebra \( U_q(\mathfrak{u}(1)) \) is generated only by one Cartan element \( q^{\mathfrak{g}^{\mathfrak{u}(n+1)}} \) the vector \( |xw\rangle \) should be annihilated by the raising generators \( e_{ij} \) (\( 1 \leq i < j \leq n \)) of the compact subalgebra \( U_q(\mathfrak{u}(n)) \). So the vector \( |xw\rangle \) satisfies the relations
\[ e_{ii} |xw\rangle = \mu_i |xw\rangle \quad (i = 1, 2, \ldots n + 1), \] (5.1)
\[ e_{ij} |xw\rangle = 0 \quad (1 \leq i < j \leq n), \] (5.2)
\(^{2}\)We assume that the vector \( |xw\rangle \) is orthonormalized, \( \langle xw| xw \rangle = 1 \).
where the weight components $\mu_i$ ($i = 1, 2, \ldots, n$) are integers subjected to the condition $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

Such weights can be compared with respect to standard lexicographic ordering. Namely, $\mu > \mu'$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\mu' = (\mu_1', \mu_2', \ldots, \mu'_n)$, if a first nonvanishing component of the difference $\mu - \mu'$ is positive.

The component $\mu_{n+1}$ is also an integer. In a general case for finite-dimensional irreducible representations of the compact quantum algebra $U_q(u(n)) \oplus U_q(u(1))$ the weights $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\mu_{n+1}$ are not ordering. If we choice some ordering for these weights, for example, as follows $(\mu_1, \ldots, \mu_\alpha, \mu_{n+1}, \mu_{\alpha+1}, \ldots, \mu_n)$, then such $n + 1$-components weights can be compared.

The extremal vector $|xw\rangle$ has minimal such weight $\Lambda^{(\alpha)}_{n+1} := (\lambda_{1,n+1}, \lambda_{2,n+1}, \ldots, \lambda_{n+1,n+1})$ where $\lambda_{i,n+1} := \mu_i$ ($i = 1, 2, \ldots, \alpha$), $\lambda_{\alpha+1,n+1} := \mu_{n+1}$, $\lambda_{\alpha+1,n+1} := \mu_{l}$ ($l = \alpha + 1, \ldots, n$).

The vector $|\Lambda^{(\alpha)}_{n+1}\rangle := |xw\rangle$ with such weight $\Lambda^{(\alpha)}_{n+1}$ satisfies the relations

$$ z_{-i}|\Lambda^{(\alpha)}_{n+1}\rangle = 0, \quad \text{for } i = 1, 2, \ldots, \alpha, \quad (5.3) $$

$$ z_{k}|\Lambda^{(\alpha)}_{n+1}\rangle = 0, \quad \text{for } k = \alpha + 1, \alpha + 2, \ldots, n, \quad (5.4) $$


It is evident that any highest weight vector $|\Lambda^{(\alpha)}_{n+1}; \Lambda_n\rangle$ with respect to the compact subalgebra $U_q(u(n))$ has the form

$$ |\Lambda^{(\alpha)}_{n+1}; \Lambda_n\rangle = z_{\alpha}^{r_{\alpha}} \cdots z_1^{r_1} z_{-\alpha-1}^{r_{-\alpha-1}} \cdots z_{-n}^{r_n}|\Lambda^{(\alpha)}_{n+1}\rangle. \quad (5.5) $$

Here the integers $\{r\}$ are defined the weights $\Lambda^{(\alpha)}_{n+1} = (\lambda_{1,n+1}, \lambda_{2,n+1}, \ldots, \lambda_{n+1,n+1})$, where $\lambda_{i,n+1} \geq \lambda_{i+1,n+1}$ ($i = 1, 2, \ldots, n$), and $\Lambda_n = (\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{nn})$, where $\lambda_m \geq \lambda_{i,n}$ ($i = 1, 2, \ldots, n - 1$). Namely,

$$ r_{i} = \lambda_{i,n} - \lambda_{i,n+1} \quad (i = 1, \ldots, \alpha), \quad (5.6) $$

$$ r_{l} = \lambda_{l+1,n+1} - \lambda_{lm} \quad (l = \alpha + 1, \ldots, n). \quad (5.7) $$

If we would like to calculate the scalar product two such vectors (5.5) then using results for the Shapovalov form we obtain

$$ (\Lambda_n; \Lambda^{(\alpha)}_{n+1}|\Lambda^{(\alpha)}_{n+1}; \Lambda'_n) = \delta_{\lambda_n\lambda'_n}(\Lambda_n; \Lambda^{(\alpha)}_{n+1}|\Lambda^{(\alpha)}_{n+1}; \Lambda_n), \quad (5.8) $$

$$ (\Lambda_n; \Lambda^{(\alpha)}_{n+1}|\Lambda^{(\alpha)}_{n+1}; \Lambda_n) = B^{(\alpha)}(v^{(\alpha)}_{\{r\}}, v^{(\alpha)}_{\{r\}}) |_{\Lambda^{(\alpha)}_{n+1}} \quad (5.9) $$

where symbol $|\Lambda^{(\alpha)}_{n+1}\rangle$ means that we specialize the Shapovalov form for the extremal weight $\Lambda_{n+1}$, that is we replace the Cartan elements $e_{ii}$, $e_{jj}$ in the functions $\varphi_{ij}$ by corresponding components $\lambda_{i,n+1}$ and $\lambda_{j,n+1}$. From the condition that

$$ (\Lambda_n; \Lambda^{(\alpha)}_{n+1}|\Lambda^{(\alpha)}_{n+1}; \Lambda_n) > 0 \quad (5.10) $$

we find all admissible highest weights $\Lambda_n$ of the compact subalgebra $U_q(u(n))$. We formulate this result as the theorem.
Theorem 5.1  Every Hermitian irreducible representation of the discrete series for the noncompact quantum algebra $U_q(u(n, 1))$ with the extremal weight $\Lambda_{n+1}^{(a)} = (\lambda_{1,n+1}, \lambda_{2,n+1}, \ldots, \lambda_{n+1,n+1})$, where integer $\lambda_{i,n+1}$ satisfy the inequalities $\lambda_{i,n+1} \geq \lambda_{i+1,n+1}$ ($i = 1, 2, \ldots, n$), under the restriction $U_q(u(n, 1)) \downarrow U_q(u(n))$ contains all multiplicity free irreducible representations of the compact subalgebra $U_q(u(n))$ with the highest weights $\Lambda_n = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying the conditions:

$$\lambda_1 \geq \lambda_{1,n+1} \geq \lambda_{2,n+1} \geq \cdots \geq \lambda_{\alpha n} \geq \lambda_{\alpha,n+1},$$

$$\lambda_{\alpha+2,n+1} \geq \lambda_{\alpha+1,n+1} \geq \lambda_{\alpha+3,n+1} \geq \cdots \geq \lambda_{n+1,n+1} \geq \lambda_{n, n}.$$  

(5.11)

The vectors

$$|\Lambda_n^{(a)}; \Lambda_n\rangle = F^{(a)}_{-}(\Lambda_n^{(a)}; \Lambda_{n+1}^{(a)}|\Lambda_{n+1}^{(a)}),$$

(5.12)

where the "lowering" operator $F^{(a)}_{-}(\Lambda_n^{(a)}; \Lambda_{n+1}^{(a)})$ is given by

$$F^{(a)}_{-}(\Lambda_n^{(a)}; \Lambda_{n+1}^{(a)}) = N^{(a)}(\Lambda_n^{(a)}; \Lambda_{n+1}^{(a)}) z^{\lambda_{n}-\lambda_{\alpha,n+1}} \cdots z^{\lambda_{1}-\lambda_{1,n+1}} \times$$

$$\times z^{\lambda_{\alpha+2,n+1}-\lambda_{\alpha+1,n+1}} \cdots z^{\lambda_{n+1,n+1}-\lambda_{n,n}},$$

(5.13)

for all highest weights $\Lambda_n = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ constrained by the conditions (5.11) form the orthonormal basis in the space of the highest vectors with respect to the compact subalgebra $U_q(u(n))$. Here in (5.13) the normalized factor $N^{(a)}(\Lambda_n^{(a)}; \Lambda_{n+1}^{(a)})$ is given as follows

$$N^{(a)}(\Lambda_n^{(a)}; \Lambda_{n+1}^{(a)}) = (\Lambda_n^{(a)}; \Lambda_{n+1}^{(a)}|\Lambda_{n+1}^{(a)}; \Lambda_{n+1}^{(a)})^{-\frac{1}{2}}$$

$$= \left\{ \prod_{i=1}^{\alpha} \frac{[l_{i,n+1} - l_{\alpha+1,n+1} - 2\alpha + n - 1]}{[l_{i,n} - l_{\alpha+1,n+1} - 2\alpha + n - 1]} \right\} \times$$

$$\times \prod_{l=\alpha+1}^{n} \frac{[l_{\alpha+1,n+1} - l_{l+1,n+1} + 2\alpha - n - 1]}{[l_{l,n+1} - l_{l+1,n+1} + 2\alpha - n - 1]} \times$$

$$\times \prod_{1 \leq i < j \leq \alpha} \frac{[l_{i,n} - l_{j,n+1}] [l_{i,n+1} - l_{j,n} - 1]}{[l_{i,n} - l_{j,n}] [l_{i,n+1} - l_{j,n} - 1]} \times$$

$$\times \prod_{\alpha+1 \leq k < l \leq n} \frac{[l_{kn} - l_{l+1,n+1} - 2]}{[l_{kn} - l_{l,n}]} \frac{[l_{kn+1} - l_{l,n+1} - 1]}{[l_{kn+1} - l_{l+1,n+1} - 1]} \times$$

$$\times \prod_{1 \leq i < k \leq l \leq n} \frac{[l_{in} - l_{n+1} - 2]}{[l_{in} - l_{n+1} - 2]} \frac{[l_{in+1} - l_{n+1} - 2]}{[l_{in+1} - l_{n+1} - 2]} \right)^{\frac{1}{2}},$$

$$l_{sp} := \lambda_{sp} - s.$$

This result coincides with the classical Gelfand–Graev case [5, 6]. Using analogous construction of the Gelfand–Tsetlin basis for the compact quantum algebra $U_q(u(n))[12]$ we obtain a q-analog of the Gelfand–Graev–Tsetlin basis for $U_q(u(n, 1))$. Namely, in the $U_q(u(n, 1))$-module
with the extremal weight $\Lambda_{n+1}^{(\alpha)}$ there is the orthogonal Gelfand–Graev–Tsetlin basis consisting of all vectors of the form

$$
|\Lambda\rangle := \begin{pmatrix}
\Lambda_{n+1}^{(\alpha)} \\
\Lambda_n \\
\vdots \\
\Lambda_2 \\
\Lambda_1
\end{pmatrix} = F_-(\Lambda_1; \Lambda_2)F_-(\Lambda_2; \Lambda_3) \cdots F_-(\Lambda_{n-1}; \Lambda_n)|\Lambda_{n+1}^{(\alpha)}; \Lambda_n\rangle,
$$

(5.15)

where $\Lambda_j = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{jj})$ $(j = 1, 2, \ldots, n)$ and the numbers $\lambda_{ij}$ satisfy the standard "between conditions" for the quantum algebra $U_q(u(n))$, i.e.

$$
\lambda_{i,j+1} \geq \lambda_{ij} \geq \lambda_{i+1,j+1} \quad \text{for} \quad 1 \leq i \leq j \leq n-1.
$$

(5.16)

The lowering operators $F_-(\Lambda_k; \Lambda_{k+1})$, $(k = 1, 2, \ldots, n-1)$, are given by

$$
F_-(\Lambda_k; \Lambda_{k+1}) = N(\Lambda_k; \Lambda_{k+1}) p(U_q(u(k))) \prod_{i=1}^k (e_{k+1,i})^ {\lambda_{k+1,i} - \lambda_{ik}},
$$

(5.17)

where $l_{ij} := \lambda_{ij} - i$. This explicit construction allows to obtain formulas for the action of $U_q(u(n,1))$-generators. These results will be presented in another article.

### 6 Summary

Thus we obtain the explicit description of the Hermitian irreducible representations for the noncompact quantum algebra $U_q(u(n,1))$ by the reduction $Z$-algebras for description of which we used the standard extremal projectors.

Next step: to obtain an analogous results for the case $U_q(u(n,2))$. For this aim we need construct extremal projector $p^{(\alpha)}$ which is expressed in terms of the $Z$-algebra $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$.

Final aim: to consider the general case $U_q(u(n,m))$. In this case extremal projectors of new type will be used.

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