q-Analog of Gelfand–Graev basis for the noncompact quantum algebra $U_q(u(n,1))$*

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Abstract

For the quantum algebra $U_q(\mathfrak{gl}(n+1))$ in its reduction on the subalgebra $U_q(gl(n))$ an explicit description of a Mickelsson–Zhelobenko reduction algebra $Z_q(\mathfrak{gl}(n+1),\mathfrak{gl}(n))$ is given in terms of the generators and their defining relations. Using this Z-algebra we describe Hermitian irreducible representations of a discrete series for the quantum algebra $U_q(u(n,1))$ which is a real form of $U_q(\mathfrak{gl}(n))$. Namely, an orthonormal Gelfand–Tsetlin basis is constructed in explicit form.

1 Introduction

In 1950 I.M. Gelfand and M.L. Tsetlin [1] proposed a formal description of finite-dimensional irreducible representations (IR) for the compact Lie algebra $u(n)$. This description is a generalization of results for $u(2)$ and $u(3)$ on the case $u(n)$. It is the following. In the IR space of $u(n)$ there is a orthonormalized basis which is numerated by the following formal schemes

$$\begin{pmatrix} m_{1n} & & m_{2n} & & \ldots & & m_{n-1,n} & & m_{nn} \\ & m_{1,n-1} & & m_{2,n-1} & & \ldots & & m_{n-1,n-1} & \\ & & \ldots & & \ldots & & \ldots & & \\ & & & m_{12} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{pmatrix} \qquad (1.1)$$

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†Deceased.

where all numbers m_{ij} $(1 \leq i < j \leq n)$ are nonnegative integers and they satisfy the standard inequalities, "between conditions":

$$m_{ij+1} \geq m_{ij} \geq m_{i+1j+1} \quad \text{for} \;\; 1 \leq i \leq j \leq n-1. \tag{1.2}$$

The first line of this scheme is defined by the components of the highest weight of $u(n)$ IR, the second line is defined by the components of the highest weight of $u(n-1)$ IR and so on.

Later this basis was constructed in many papers (for example, see [2, 3, 4]) by using one-step lowering and raising operators.

In 1965 I.M. Gelfand and M.I. Graev [5] using analytic continuation of the results for $u(n)$ obtained some results for non-compact Lie algebra $u(n, m)$. They shown that some class of Hermitian IR of $u(n, m)$ is characterized by a "extremal weight" parametrized by a set of integers $m_N = (m_{1N}, \ldots, m_{NN})$ $(N = n + m)$ such that $m_{1N} \geq m_{1N} \geq \cdots \geq m_{NN}$, and by a representation type which is defined by a partition of n in the sum of two nonnegative integers α and β, $n = \alpha + \beta$ (also see [6]).

For simplicity we consider the case $u(2, 1)$. In this case we have three type of scheme

$$\left(\begin{array}{ccccc} & m_{13} & & m_{23} & m_{33} \\ m_{12} & & m_{22} & & \\ & m_{11} & & & \end{array}\right) \quad \text{for} \;\; (\alpha, \beta) \; = \; (2, 0) \; , \tag{1.3}$$

$$\left(\begin{array}{ccccc} & m_{13} & m_{23} & m_{33} & \\ m_{12} & & & & m_{22} \\ & & m_{11} & & \end{array}\right) \quad \text{for} \;\; (\alpha, \beta) \; = \; (1, 1) \; , \tag{1.4}$$

$$\left(\begin{array}{ccccc} m_{13} & m_{23} & & m_{33} & \\ & & m_{12} & & m_{22} \\ & & & m_{11} & \end{array}\right) \quad \text{for} \;\; (\alpha, \beta) \; = \; (0, 2) \; . \tag{1.5}$$

The numbers m_{ij} of the first scheme satisfy the following inequalities

$$m_{12} \geq m_{13} + 1 \; , \qquad m_{13} + 1 \geq m_{22} \geq m_{23} + 1 \; , \qquad m_{12} \geq m_{11} \geq m_{22} \; . \tag{1.6}$$

The numbers m_{ij} of the second scheme satisfy the following inequalities

$$m_{12} \geq m_{13} + 1 \; , \qquad m_{13} - 1 \geq m_{22} \; , \qquad m_{12} \geq m_{11} \geq m_{22} \; . \tag{1.7}$$

The numbers of the third scheme satisfy the following inequalities

$$m_{23} - 1 \geq m_{12} \geq m_{33} - 1 \; , \qquad m_{33} - 1 \geq m_{22} \; , \qquad m_{12} \geq m_{11} \geq m_{22} \; . \tag{1.8}$$

Construction of Gelfand–Graev basis for $u(n, m)$ in terms of one-step lowering and raising operators is more complicated then in the compact case $u(n + m)$.

In 1975 T.J. Enright and V.S. Varadarajan [7] obtained some classification of discrete series of non-compact Lie algebras. Later A. Molev [8] shown that for the case $u(n, m)$ Gelfand–Graev modules are some part of Enright–Varadarajan modules and Molev constructed Gelfand–Graev basis for $u(n, m)$ in terms of Mickelsson S-algebra [9].

A goal of this work to obtain analogous results for the non-compact quantum algebra $U_q(u(n, m))$. Because the general case is very complicated we at first consider the case $U_q(u(n, 1))$. It should be noted that the special case $U_q(u(2, 1))$ was considered in [10, 11].

2 Quantum algebra $U_q(gl(N))$ and its noncompact real forms $U_q(u(n,m))$ $(n+m=N)$

The quantum algebra $U_q(\mathfrak{gl}(N))$ is generated by the Chevalley elements $q^{\pm e_{ii}}$ $(i=1,\ldots,N)$, $e_{i,i+1}$, $e_{i+1,i}$ $(i=1,2,\ldots,N-1)$ with the defining relations:

$$
\begin{aligned}
q^{e_{ii}}q^{-e_{ii}} &= q^{-e_{ii}}q^{e_{ii}}=1\ , && (2.1)\\
q^{e_{ii}}q^{e_{jj}} &= q^{e_{jj}}q^{e_{ii}}\ , && (2.2)\\
q^{e_{ii}}e_{jk}q^{-e_{ii}} &= q^{\delta_{ij}-\delta_{ik}}e_{jk} \qquad (|j-k|=1)\ , && (2.3)\\
[e_{i,i+1},\, e_{j+1,j}] &= \delta_{ij}\,\frac{q^{e_{ii}-e_{i+1,i+1}}\,-\,q^{e_{i+1,i+1}-e_{ii}}}{q\,-\,q^{-1}}\ , && (2.4)\\
[e_{i,i+1},\, e_{j,j+1}] &= 0 \qquad \text{for}\ \ |i-j|\geq 2\ , && (2.5)\\
[e_{i+1,i},\, e_{j+1,j}] &= 0 \qquad \text{for}\ \ |i-j|\geq 2\ , && (2.6)\\
[[e_{i,i+1},\, e_{j,j+1}]_q,\, e_{j,j+1}]_q &= 0 \qquad \text{for}\ \ |i-j|=1\ , && (2.7)\\
[[e_{i+1,i},\, e_{j+1,j}]_q,\, e_{j+1,j}]_q &= 0 \qquad \text{for}\ \ |i-j|=1\ . && (2.8)
\end{aligned}
$$

where $[e_\beta,\, e_\gamma]_q$ denotes the q-commutator:

$$
[e_\beta,\, e_\gamma]_q\ :=\ e_\beta e_\gamma - q^{(\beta,\gamma)}e_\gamma e_\beta\ . \qquad (2.9)
$$

The definition of a quantum algebra also includes operations of a comultiplication Δ_q, an antipode S_q, and a co-unit ε_q. Because explicit formulas of these operations will not used in our later calculations, they are not given here.

For construction of the composite root vectors e_{ij} for $|i-j|\geq 2$ we fix the following normal (convex) ordering of the positive root system Δ_+ (see [12])

$$
\begin{aligned}
&\epsilon_1-\epsilon_2\prec\epsilon_1-\epsilon_3\prec\epsilon_2-\epsilon_3\prec\epsilon_1-\epsilon_4\prec\epsilon_2-\epsilon_4\prec\epsilon_3-\epsilon_4\prec\ldots\prec\\
&\epsilon_1-\epsilon_k\prec\epsilon_2-\epsilon_k\prec\ldots\prec\epsilon_{k-1}-\epsilon_k\prec\ldots\prec\epsilon_1-\epsilon_N\prec\epsilon_2-\epsilon_N\prec\ldots\prec\epsilon_{N-1}-\epsilon_N\ .
\end{aligned} \qquad (2.10)
$$

According to this ordering we set

$$
e_{ij}\ :=\ [e_{ik},\, e_{kj}]_{q^{-1}}, \qquad e_{ji}\ :=\ [e_{jk},\, e_{ki}]_q\ , \qquad (2.11)
$$

where $1\leq i<k<j\leq N$. It should be stressed that the structure of the composite root vectors does not dependent of choice of the index k in the r.h.s. of the definition (2.10). In particular, we have

$$
\begin{aligned}
e_{ij}\ &:=\ [e_{i,i+1},\, e_{i+1,j}]_{q^{-1}}\ =\ [e_{i,j-1},\, e_{j-1,j}]_{q^{-1}}\ ,\\
e_{ji}\ &:=\ [e_{j,i+1},\, e_{i+1,i}]_q\ =\ [e_{j,j-1},\, e_{j-1,i}]_q\ ,
\end{aligned} \qquad (2.12)
$$

where $2\leq i+1<j\leq N$.

Using these explicit constructions and the defining relations (2.1)–(2.8) for the Chevalley basis it is not hard to calculate the following relations between the Cartan–Weyl generators e_{ij} $(i, j = 1, 2, \ldots, N)$:

$$
\begin{aligned}
q^{e_{kk}} e_{ij} q^{-e_{kk}} &= q^{\delta_{ki}-\delta_{kj}} e_{ij} && (1 \le i, j, k \le N)\ , && (2.13)\\
[e_{ij},\, e_{ji}] &= \frac{q^{e_{ii}-e_{jj}} - q^{e_{jj}-e_{ii}}}{q - q^{-1}} && (1 \le i < j \le N)\ , && (2.14)\\
[e_{ij},\, e_{kl}]_{q^{-1}} &= \delta_{jk} e_{il} && (1 \le i < j \le k < l \le N)\ , && (2.15)\\
[e_{ik},\, e_{jl}]_{q^{-1}} &= (q - q^{-1})\, e_{jk} e_{il} && (1 \le i < j < k < l \le N)\ , && (2.16)\\
[e_{jk},\, e_{il}]_{q^{-1}} &= 0 && (1 \le i \le j < k \le l \le N)\ , && (2.17)\\
[e_{kl},\, e_{ji}] &= 0 && (1 \le i < j \le k < l \le N)\ , && (2.18)\\
[e_{il},\, e_{kj}] &= 0 && (1 \le i < j < k < l \le N)\ , && (2.19)\\
[e_{ji},\, e_{il}] &= e_{jl}\, q^{e_{ii}-e_{jj}} && (1 \le i < j < l \le N)\ , && (2.20)\\
[e_{kl},\, e_{li}] &= e_{ki}\, q^{e_{kk}-e_{ll}} && (1 \le i < k < l \le N)\ , && (2.21)\\
[e_{jl},\, e_{ki}] &= (q^{-1} - q)\, e_{kl} e_{ji}\, q^{e_{jj}-e_{kk}} && (1 \le i < j < k < l \le N)\ . && (2.22)
\end{aligned}
$$

If we apply the Cartan involution ($e_{ij}^{*} = e_{ji}$) the formulas above, we will get all relations between elements of the Cartan–Weyl basis.

The explicit formula for the extremal projector for $U_q(\mathfrak{gl}(N)$ has the form [13]

$$
\begin{aligned}
p(U_q(\mathfrak{gl}(N)) &= p(U_q(\mathfrak{gl}(N-1))(p_{1N} p_{2N} \cdots p_{N-2N} p_{N-1N})\\
&= p_{12}(p_{13}p_{23}) \cdots (p_{1i} \cdots p_{ii+1}) \cdots (p_{1N} \cdots p_{N-1N})\ ,
\end{aligned} \qquad (2.23)
$$

where the elements p_{ij} $(1 \le i < j \le N)$ are given by

$$
\begin{aligned}
p_{ij} &= \sum_{r=0}^{\infty} \frac{(-1)^s}{[r]!} \varphi_{ij,r} e_{ij}^{r} e_{ji}^{r}\ ,\\
\varphi_{ij,r} &= q^{-(j-i-1)r} \Big\{ \prod_{s=1}^{N} [e_{ii} - e_{jj} + j - i + s] \Big\}^{-1}\ .
\end{aligned} \qquad (2.24)
$$

The extremal projector $p := p(U_q(\mathfrak{gl}(N))$ satisfies the relations:

$$
e_{i,i+1} p = p e_{i+1,i} = 0 \qquad (1 \le i \le N-1)\ , \qquad\qquad p^2 = p\ . \qquad (2.25)
$$

The extremal projector p belongs to the Taylor extension $TU_q(\mathfrak{gl}(N)$ of the universal enveloping algebras $U_q(\mathfrak{gl}(N)$. The Taylor extension $TU_q(\mathfrak{gl}(N)$ is an associative algebra generated by formal Taylor series of the form

$$
\sum_{\{r'\},\{r\}} C_{\{r'\},\{r\}}(q^{e_{11}}, \ldots, q^{e_{NN}}) e_{21}^{r'_{12}} e_{31}^{r'_{13}} e_{32}^{r'_{23}} \cdots e_{N,N-1}^{r'_{N-1,N}} e_{12}^{r_{12}} e_{13}^{r_{13}} e_{23}^{r_{23}} \cdots e_{N-1,N}^{r_{N-1,N}} \qquad (2.26)
$$

provided that nonnegative integers $r'_{12}, r'_{13}, r'_{23}, \ldots, r'_{N-1,N}$ and $r_{12}, r_{13}, r_{23}, \ldots, r_{N-1,N}$ are subject to the constraints

$$\Big|\sum_{i<j} r'_{ij} - \sum_{i<j} r_{ij}\Big| \leq \text{const} \tag{2.27}$$

for each formal series and the coefficients $C_{\{r'\},\{r\}}(q^{e_{11}}, \ldots, q^{e_{NN}})$ are rational functions of the q-Cartan elements $q^{e_{ii}}$. The quantum universal enveloping algebra $U_q(\mathfrak{gl}(N))$ is a subalgebra of the Taylor extension $TU_q(\mathfrak{gl}(N))$, $U_q(\mathfrak{gl}(N)) \subset TU_q(\mathfrak{gl}(N))$.

The noncompact quantum algebra $U_q(\mathfrak{u}(n,m))$ can be considered as the quantum algebra $U_q(\mathfrak{gl}(N))$ $(N = n+m)$ endowed with the additional Cartan involution $(^*)$:

$$h_i^* \;=\; h_i \;, \qquad\qquad (i = 1, 2, \ldots n+m) \;, \tag{2.28}$$

$$e_{i,i+1}^* \;=\; e_{i+1,i} \;, \qquad e_{i+1,i}^* \;=\; e_{i+1,i} \quad \text{for} \;\; i \neq n \;, \tag{2.29}$$

$$e_{n,n+1}^* \;=\; -e_{n+1,n} \;, \quad e_{n+1,n}^* \;=\; -e_{n,n+1} \;, \tag{2.30}$$

$$q^* \;=\; q \qquad \text{or} \qquad q \;=\; q^{-1}. \tag{2.31}$$

Below we will consider the real form $U_q(\mathfrak{u}(n,1))$, i.e. the case $N = n+1$.

3 The reduction algebra $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$

In the linear space $TU_q(\mathfrak{gl}(n+1))$ we separate out a subspace of "two-sided highest vectors" with respect to the subalgebra $U_q(\mathfrak{gl}(n)) \subset U_q(\mathfrak{gl}(n+1))$, i.e.

$$\tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \;=\; \left\{x \in TU_q(\mathfrak{gl}(n+1)) \,\middle|\, e_{ij}x = xe_{ji} = 0, \;\; 1 \leq i < j \leq n\right\} . \tag{3.1}$$

It is evident that if $x \in \tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ then

$$x \;=\; pxp \;, \tag{3.2}$$

where $p := p(U_q(\mathfrak{gl}(n)))$. Again, using the annihilation properties of the projection operator p we have that any vector $x \in \tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ presents a formal Taylor series on the following monomials

$$pe_{n+1,1}^{r'_1} \cdots e_{n+1,n}^{r'_n} e_{n,n+1}^{r_n} \cdots e_{1,n+1}^{r_1} p \;. \tag{3.3}$$

It is evident that $\tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ is a subalgebra in $TU_q(\mathfrak{gl}(n+1))$. We consider a subspace $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ generated by finite series on the monomials (5.3). It is clear that $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ is a subalgebra in $\tilde{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$.

We set

$$z_i := pe_ip \;, \qquad (i = \pm 1, \pm 2, \ldots, \pm n) \;, \tag{3.4}$$

where $e_i = e_{j,n+1}$, $e_{-i} = e_{n+1,j}$ $(1 = 1, 2, \ldots, n)$.

Theorem 3.1 *The elements z_i $(i=\pm1,\pm2,\dots,\pm n)$ generates the associative algebra $Z_q(\mathfrak{gl}(n+1),\mathfrak{gl}(n))$ and satisfies the following relations*

$$z_i z_j \;=\; A_{ij} z_j z_i \quad \text{for } \; i+j\neq 0, \quad (i,j=\pm1,\pm2,\dots,\pm n)\;, \tag{3.5}$$

$$z_i z_{-i} \;=\; \sum_{j=1}^{n} B_{ij} z_{-i} z_i + \gamma_i \quad \text{for } \; i=1,2,\dots,n\;, \tag{3.6}$$

where

$$A_{ij} \;=\; 1 \qquad \text{for } \; sgn\, i \neq sgn\, j\;, \tag{3.7}$$

$$A_{ij} \;=\; A_{-j-i} \;=\; \frac{[\varphi_{ij}+1]}{[\varphi_{ij}]} \qquad \text{for } \; 1\le i<j\le n\;, \tag{3.8}$$

$$B_{ij} \;=\; -\frac{b_i^- b_j^+}{[\varphi_{ij}-1]}\;, \qquad\qquad \gamma_i \;=\; b_i^-[\varphi_{i,n+1}-1]\;, \tag{3.9}$$

$$b_i^{\pm} \;=\; \prod_{s=i+1}^{n} \frac{[\varphi_{is}\pm 1]}{[\varphi_{is}]}\;, \qquad \varphi_{ik} \;=\; e_{ii}-e_{kk}+k-i\;. \tag{3.10}$$

A proof of the theorem by direct calculations.

For construction and study of discrete series of the non-compact quantum algebra $U_q(\mathfrak{u}(n,1)$ we need another relations then (3.6). The system (3.6) expresses the elements $z_i z_{-i}$ in terms of the elements $z_{-i}z_i$ $(i=1,2,\dots,n)$ but we would like to express the elements $z_{-1}z_1,\dots,z_{-\alpha}z_\alpha$, $z_{\alpha+1}z_{-\alpha-1},\dots,z_n z_{-n}$ in terms of the elements $z_1z_{-1},\dots,z_\alpha z_{-\alpha}, z_{-\alpha-1}z_{\alpha+1},\dots,z_{-n}z_n$ for $\alpha = 0,1,\dots,n$[1]. These relations are given by the theorem.

Theorem 3.2 *The elements $z_{-1}z_1,\dots,z_{-\alpha}z_\alpha$, $z_{\alpha+1}z_{-\alpha-1},\dots,z_n z_{-n}$ are expressed in terms of the elements $z_1z_{-1},\dots,z_\alpha z_{-\alpha}$, $z_{-\alpha-1}z_{\alpha+1},\dots,z_{-n}z_n$ by the formulas*

$$z_{-i}z_i \;=\; \sum_{j=1}^{\alpha} B_{ij}^{(\alpha)} z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{il}^{(\alpha)} z_{-l} z_l + \gamma_i^{(\alpha)} \quad (i=1,2,\dots,\alpha)\;, \tag{3.11}$$

$$z_k z_{-k} \;=\; \sum_{j=1}^{\alpha} B_{kj}^{(\alpha)} z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{kl}^{(\alpha)} z_{-l} z_l + \gamma_k^{(\alpha)} \quad (k=\alpha+1,\alpha+2,\dots,n)\;, \tag{3.12}$$

where

$$B_{ij}^{(\alpha)} \;=\; \frac{b_i^{(\alpha)+} b_j^{(\alpha)-}}{[\varphi_{ij}+1]}, \qquad B_{il}^{(\alpha)} \;=\; \frac{b_i^{(\alpha)+} b_l^{(\alpha)+}}{[\varphi_{il}]}, \qquad \gamma_i^{(\alpha)} \;=\; -[\varphi_{i,n+1}-\alpha] b_i^{(\alpha)+}\;, \tag{3.13}$$

$$B_{kj}^{(\alpha)} \;=\; -\frac{b_k^{(\alpha)-} b_j^{(\alpha)-}}{[\varphi_{kj}]}, \quad B_{kl}^{(\alpha)} \;=\; -\frac{b_k^{(\alpha)-} b_l^{(\alpha)+}}{[\varphi_{kl}-1]}, \quad \gamma_k^{(\alpha)} \;=\; [\varphi_{i,n+1}-\alpha-1] b_k^{(\alpha)+}, \tag{3.14}$$

$$b_i^{(\alpha)\pm} \;=\; \prod_{s=1}^{i-1} \frac{[\varphi_{is}\pm 1]}{[\varphi_{is}]} \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is}]}{[\varphi_{is}\pm 1]}, \quad b_l^{(\alpha)\pm} \;=\; \prod_{s=1}^{\alpha} \frac{[\varphi_{ls}]}{[\varphi_{ls}\pm 1]} \prod_{s=l+1}^{n-1} \frac{[\varphi_{ls}\pm 1]}{[\varphi_{ls}]}\;. \tag{3.15}$$

[1]In the case $\alpha=0$ we have the relations (3.6) and for $\alpha=n$ we obtain the system inverse to (3.6)

Moreover the following power relations are valid

$$z_i^r z_{-j}^s = z_{-j}^s z_i^r \qquad \text{for } \forall\, i,j>0 \text{ and } r,s\in\mathbb{N}\ , \tag{3.16}$$

$$z_i^r z_j^s = z_j^s z_i^r \frac{[\varphi_{ij}+r]![\varphi_{ij}-s]!}{[\varphi_{ij}]![\varphi_{ij}+r-s]!} \quad \text{for } 1\le i<j\le n \text{ and } r,s\in\mathbb{N}\ , \tag{3.17}$$

$$z_{-i}^r z_{-j}^s = z_{-j}^s z_{-i}^r \frac{[\varphi_{ij}]![\varphi_{ij}-r+s]!}{[\varphi_{ij}-r]![\varphi_{ij}+s]!} \quad \text{for } 1\le i<j\le n \text{ and } r,s\in\mathbb{N}\ , \tag{3.18}$$

$$z_{-i} z_i^r = z_i^{r-1}\Big(\sum_{j=1}^{\alpha} B_{ij}^{(\alpha)}(r) z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{il}^{(\alpha)}(r) z_{-l} z_l + \gamma_i^{(\alpha)}(r)\Big) \quad (i=1,2,\ldots,\alpha), \tag{3.19}$$

$$z_k z_{-k}^r = z_{-k}^{r-1}\Big(\sum_{j=1}^{\alpha} B_{kj}^{(\alpha)}(r) z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{kl}^{(\alpha)}(r) z_{-l} z_l + \gamma_k^{(\alpha)}(r)\Big) \quad (k=\alpha+1,\ldots,n), \tag{3.20}$$

where

$$\begin{aligned} B_{ij}^{(\alpha)}(r) &= \frac{[r] b_i^{(\alpha)+}(r)\, b_j^{(\alpha)-}}{[\varphi_{ij}+r]}, \qquad B_{il}^{(\alpha)}(r) = \frac{[r] b_i^{(\alpha)+}(r)\, b_l^{(\alpha)+}}{[\varphi_{il}+r-1]}\ , \\ \gamma_i^{(\alpha)}(r) &= -[r][\varphi_{i,n+1}-\alpha+r-1] b_i^{(\alpha)+}(r) \quad (1\le i,j\le\alpha<l\le n)\ , \end{aligned} \tag{3.21}$$

$$\begin{aligned} B_{kj}^{(\alpha)}(r) &= -\frac{[r] b_k^{(\alpha)-}(r)\, b_j^{(\alpha)-}}{[\varphi_{kj}-r+1]}\ , \qquad B_{kl}^{(\alpha)} = -\frac{[r] b_k^{(\alpha)-}(r)\, b_l^{(\alpha)+}}{[\varphi_{kl}-r]}\ , \\ \gamma_k^{(\alpha)}(r) &= [r][\varphi_{k,n+1}-\alpha-r] b_k^{(\alpha)-}(r) \qquad (1\le j\le\alpha<k,l\le n)\ . \end{aligned} \tag{3.22}$$

Here

$$b_i^{(\alpha)+}(r) = \Big(\prod_{s=1}^{i-1}\frac{[\varphi_{is}+r]}{[\varphi_{is}+r-1]}\Big)\Big(\prod_{s=\alpha+1}^{n-1}\frac{[\varphi_{is}+r-1]}{[\varphi_{is}+r]}\Big) \quad (1\le i\le\alpha)\ , \tag{3.23}$$

$$b_i^{(\alpha)-} = \Big(\prod_{s=1}^{i-1}\frac{[\varphi_{is}-1]}{[\varphi_{ls}]}\Big)\Big(\prod_{s=\alpha+1}^{n-1}\frac{[\varphi_{is}]}{[\varphi_{is}-1]}\Big) \qquad (1\le i\le\alpha)\ , \tag{3.24}$$

$$b_l^{(\alpha)-}(r) = \Big(\prod_{s=1}^{\alpha}\frac{[\varphi_{ls}-r+1]}{[\varphi_{is}-r]}\Big)\Big(\prod_{s=l+1}^{n-1}\frac{[\varphi_{ls}-r]}{[\varphi_{ls}-r+1]}\Big) \quad (\alpha+1\le l\le n)\ , \tag{3.25}$$

$$b_l^{(\alpha)+} = \Big(\prod_{s=1}^{\alpha}\frac{[\varphi_{ls}]}{[\varphi_{ls}+1]}\Big)\Big(\prod_{s=l+1}^{n-1}\frac{[\varphi_{ls}+1]}{[\varphi_{ls}]}\Big) \qquad (\alpha+1\le l\le n)\ . \tag{3.26}$$

4 Shapovalov's forms on $Z_q^*(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$

We consider on the Z-algebra $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ two real forms: compact and noncompact.

The compact real form on $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ is defined by the involution $(^*)$ which is given as follows

$$z_{\pm i}^* \;=\; z_{\mp i}^* \qquad (i = 1, 2, \ldots, n)\;, \tag{4.1}$$

$$e_{ii}^* \;=\; e_{ii} \qquad (i = 1, 2, \ldots, n+1)\;. \tag{4.2}$$

This involution can be considered as generalization of the Cartan involution in $U_q(\mathfrak{gl}(n+1))$ to the Taylor extension, $TU_q(\mathfrak{gl}(n+1))$. The Z-algebra $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ with this involution is called the compact real form and denoted by the symbol $Z_q^{(c)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$.

The noncompact real form on $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ is defined by the involution $*$ which is given as follows

$$z_{\pm i}^* \;=\; -z_{\pm i}^* \qquad (i = 1, 2, \ldots, n)\;, \tag{4.3}$$

$$e_{ii}^* \;=\; e_{ii} \qquad (i = 1, 2, \ldots, n+1)\;. \tag{4.4}$$

This involution can be considered as generalization of the noncompact involution in $U_q(\mathfrak{gl}(n+1))$ to the Taylor extension, $TU_q(\mathfrak{gl}(n+1))$. The Z-algebra $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ with this involution is called the noncompact real form and denoted by the symbol $Z_q^{(nc)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$.

Let $p^{(\alpha)}$ be the extremal projector for $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ satisfying the relations

$$z_{-i}p^{(\alpha)} \;=\; p^{(\alpha)}z_i \qquad \text{for} \;\; i = 1, 2, \ldots, \alpha\;, \tag{4.5}$$

$$z_k p^{(\alpha)} \;=\; p^{(\alpha)}z_{-k} \qquad \text{for} \;\; k = \alpha+1, \alpha+2, \ldots, n\;, \tag{4.6}$$

$$[e_{ii},\, p^{(\alpha)}] \;=\; 0 \qquad \text{for} \;\; i = 1, 2, \ldots, n\;. \tag{4.7}$$

This extremal projector depends on the index α, which defines what elements are considered as "raising" and what elements are considered as "lowering", i.e. in our case the elements $z_{-1}, z_{-2}, \ldots, z_{-\alpha}, z_{\alpha+1}, \ldots, z_n$ are raising and the elements $z_1, z_2, \ldots, z_\alpha, z_{-\alpha-1}, \ldots, z_{-n}$ are lowering. It should be noted that the "raising" and "lowering" subsets generate disjoint subalgebras in $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$. The operator $p^{(\alpha)}$ can be constructed in explicit form.

Let us introduce on $Z_q^{(nc)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ the following sesquilinear Shapovalov form. For any elements $x, y \in Z_q^{(nc)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ we set

$$B^{(\alpha)}(x, y) \;=\; p^{(\alpha)}y^*xp^{(\alpha)}\;. \tag{4.8}$$

Therefore the Shapovalov form also depends on the index α ($\alpha = 0, 1, 2, \ldots, n$). We fix α ($\alpha = 0, 1, 2, \ldots, n$) and for each set of nonnegative integers $\{r\} = (r_1, r_2, \ldots, r_n)$ introduce a vector in the space $Z_q^{(nc)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ by the formula

$$v_{\{r\}}^{(\alpha)} \;=\; z_\alpha^{r_\alpha} \cdots z_1^{r_1} z_{-\alpha-1}^{r_\alpha+1} \cdots z_{-n}^{r_n}\;. \tag{4.9}$$

Theorem 4.1 *For each fixed α ($\alpha = 0, 1, 2, \ldots, n$) the vectors $\{v_{\{r\}}^{(\alpha)}\}$ are pairwise orthogonal with respect to the Shapovalov form (4.8)*

$$B^{(\alpha)}(v_{\{r\}}^{(\alpha)}, v_{\{r'\}}^{(\alpha)}) \;=\; \delta_{\{r\},\{r'\}}\, B^{(\alpha)}(v_{\{r\}}^{(\alpha)}, v_{\{r\}}^{(\alpha)}) \;. \tag{4.10}$$

and

$$\begin{aligned} B^{(\alpha)}(v_{\{r\}}^{(\alpha)}, v_{\{r\}}^{(\alpha)}) \;&=\; \Biggl(\prod_{i=1}^{\alpha} \frac{[r_i]![\varphi_{i,n+1}-\alpha+r_i-1]!}{[\varphi_{i,n+1}-\alpha-1]!} \prod_{l=\alpha+1}^{n} \frac{[r_l]![\varphi_{n+1,l}+\alpha+r_l]!}{[\varphi_{n+1,l}+\alpha]!} \\ &\times \prod_{1\le i<j\le\alpha} \frac{[\varphi_{ij}+r_i-r_j]![\varphi_{ij}-1]!}{[\varphi_{ij}+r_i]![\varphi_{ij}-r_j-1]!} \prod_{\alpha+1\le k<l\le n} \frac{[\varphi_{kl}-r_k+r_l]![\varphi_{kl}-1]!}{[\varphi_{kl}-r_k-1]![\varphi_{kl}+r_l]!} \\ &\times \prod_{1\le i\le\alpha<l\le n} \frac{[\varphi_{il}+r_i-1]![\varphi_{il}+r_l-1]![\varphi_{il}]}{[\varphi_{il}+r_i+r_l]![\varphi_{il}-1]!}\Biggr) p^{(\alpha)} \;. \end{aligned} \tag{4.11}$$

As consequence of this theorem we obtain that the Shapovalov form is not degenerate on $Z_q^{(nc)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$.

In the case of the compact Z-algebra $Z_q^{(c)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ the Shapovalov form $B(x, y)$ is defined by the formula (4.8) where $\alpha = 0$, $p^{(0)}$ is the standard extremal projector of the subalgebra $\mathfrak{gl}(n)$ and the involution $(^*)$ is given by the formulas (4.1). It is not difficult to see that

$$B(v_{\{r\}}, v_{\{r'\}}) \;=\; \delta_{\{r\},\{r'\}} B(v_{\{r\}}, v_{\{r\}}) \;. \tag{4.12}$$

where $v_{\{r\}} := v_{\{r\}}^{(0)}$, and

$$\begin{aligned} B(v_{\{r\}}, v_{\{r\}}) \;&=\; (-1)^{\sum_{i=1}^{n} r_i} B^{(0))}(v_{\{r\}}^{(0)}, v_{\{r\}}^{(0)}) \\ &=\; \prod_{l=1}^{n} \frac{[r_l]![\varphi_{l,n+1}-r_l]!}{[\varphi_{l,n+1}]!} \prod_{1\le k<l\le n} \frac{[\varphi_{kl}-r_k-r_l]![\varphi_{kl}-1]!}{[\varphi_{kl}-r_k-1]![\varphi_{kl}+r_l]!} \;. \end{aligned} \tag{4.13}$$

5 Discrete series of representations for $U_q(u(n,1))$

As well as in the classical case [9] each Hermitian irreducible representation of the discrete series for the noncompact quantum algebra $U_q(u(n,1))$ is defined uniquely by some extremal vector $|xw\rangle$, the vector of extremal weight[2]. This vector should be a highest vector with respect to the compact subalgebra $U_q(u(n)) \oplus U_q(u(1))$. Since the quantum algebra $U_q(u(1))$ is generated only by one Cartan element $q^{e_{n+1,n+1}}$ the vector $|xw\rangle$ should be annihilated by the raising generators e_{ij} $(1 \le i < j \le n)$ of the compact subalgebra $U_q(u(n))$. So the vector $|xw\rangle$ satisfies the relations

$$e_{ii}|xw\rangle \;=\; \mu_i|xw\rangle \qquad (i = 1, 2, \ldots n+1) \;, \tag{5.1}$$

$$e_{ij}|xw\rangle \;=\; 0 \qquad (1 \le i < j \le n) \;, \tag{5.2}$$

[2]We assume that the vector $|xw\rangle$ is orthonormalized, $\langle xw|xw\rangle = 1$

where the weight components μ_i $(i = 1, 2, \ldots, n)$ are integers subjected to the condition $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

Such weights can be compared with respect to standard lexicographic ordering. Namely, $\mu > \mu'$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\mu' = (\mu_1^{'}, \mu_2^{'}, \ldots, \mu_n^{'})$, if a first nonvanishing component of the difference $\mu - \mu'$ is positive.

The component μ_{n+1} is also an integer. In a general case for finite-dimensional irreducible representations of the compact quantum algebra $U_q(u(n)) \oplus U_q(u(1))$ the weights $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and μ_{n+1} are not ordering. If we choice some ordering for these weights, for example, as follows $(\mu_1, \ldots, \mu_\alpha, \mu_{n+1}, \mu_{\alpha+1}, \ldots, \mu_n)$, then such $n+1$-components weights can be compared.

The extremal vector $|xw\rangle$ has minimal such weight $\Lambda_{n+1}^{(\alpha)} := (\lambda_{1,n+1}, \lambda_{2,n+1}, \ldots, \lambda_{n+1,n+1})$ where $\lambda_{i,n+1} := \mu_i$ $(i = 1, 2, \ldots, \alpha)$, $\lambda_{\alpha+1,n+1} := \mu_{n+1}$, $\lambda_{l+1,n+1} := \mu_l$ $(l = \alpha + 1, \ldots, n)$.

The vector $|\Lambda_{n+1}^{(\alpha)}\rangle := |xw\rangle$ with such weight $\Lambda_{n+1}^{(\alpha)}$ satisfies the relations

$$z_{-i}|\Lambda_{n+1}^{(\alpha)}\rangle \;=\; 0, \qquad \text{for } i = 1, 2, \ldots, \alpha\ , \tag{5.3}$$

$$z_{k}|\Lambda_{n+1}^{(\alpha)}\rangle \;=\; 0, \qquad \text{for } k = \alpha+1, \alpha+2, \ldots, n\ , \tag{5.4}$$

It is evident that any highest weight vector $|\Lambda_{n+1}^{(\alpha)}; \Lambda_n)$ with respect to the compact subalgebra $U_q(u(n))$ has the form

$$|\Lambda_{n+1}^{(\alpha)}; \Lambda_n) \;=\; z_\alpha^{r_\alpha} \cdots z_1^{r_1} z_{-\alpha-1}^{r_{\alpha+1}} \cdots z_{-n}^{r_n} |\Lambda_{n+1}^{(\alpha)}\rangle\ . \tag{5.5}$$

Here the integers $\{r\}$ are defined the weights $\Lambda_{n+1}^{(\alpha)} = (\lambda_{1,n+1}, \lambda_{2,n+1}, \ldots, \lambda_{n+1,n+1})$, where $\lambda_{i,n+1} \geq \lambda_{i+1,n+1}$ $(i = 1, 2, \ldots, n)$, and $\Lambda_n = (\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{nn})$, where $\lambda_{in} \geq \lambda_{i+1,n}$ $(i = 1, 2, \ldots, n-1)$. Namely,

$$r_i \;=\; \lambda_{in} - \lambda_{i,n+1} \quad (i = 1, \ldots, \alpha)\ , \tag{5.6}$$

$$r_l \;=\; \lambda_{l+1,n+1} - \lambda_{ln} \quad (l = \alpha+1, \ldots, n)\ . \tag{5.7}$$

If we would like to calculate the scalar product two such vectors (5.5) then using results for the Shapovalov form we obtain

$$(\Lambda_n; \Lambda_{n+1}^{(\alpha)}|\Lambda_{n+1}^{(\alpha)}; \Lambda_n') \;=\; \delta_{\Lambda_n,\Lambda_n'}(\Lambda_n; \Lambda_{n+1}^{(\alpha)}|\Lambda_{n+1}^{(\alpha)}; \Lambda_n)\ , \tag{5.8}$$

$$(\Lambda_n; \Lambda_{n+1}^{(\alpha)}|\Lambda_{n+1}^{(\alpha)}; \Lambda_n) \;=\; B^{(\alpha)}(v_{\{r\}}^{(\alpha)}, v_{\{r\}}^{(\alpha)})\Big|_{\Lambda_{n+1}^{(\alpha)}} \tag{5.9}$$

where symbol $|_{\Lambda_{n+1}^{(\alpha)}}$ means that we specialize the Shapovalov form for the extremal weight Λ_{n+1}, that is we replace the Cartan elements e_{ii}, e_{jj} in the functions φ_{ij} by corresponding components $\lambda_{i,n+1}$ and $\lambda_{j,n+1}$. From the condition that

$$(\Lambda_n; \Lambda_{n+1}^{(\alpha)}|\Lambda_{n+1}^{(\alpha)}; \Lambda_n) \;>\; 0 \tag{5.10}$$

we find all admissible highest weights Λ_n of the compact subalgebra $U_q(u(n))$. We formulate this result as the theorem.

Theorem 5.1 *Every Hermitian irreducible representation of the discrete series for the noncompact quantum algebra $U_q(u(n,1))$ with the extremal weight $\Lambda^{(\alpha)}_{n+1} = (\lambda_{1,n+1}, \lambda_{2,n+1}, \ldots, \lambda_{n+1,n+1})$, where integer $\lambda_{i,n+1}$ satisfy the inequalities $\lambda_{i,n+1} \geq \lambda_{i+1,n+1}$ $(i = 1, 2, \ldots, n)$, under the restriction $U_q(u(n,1)) \downarrow U_q(u(n))$ contains all multiplicity free irreducible representations of the compact subalgebra $U_q(u(n))$ with the highest wights $\Lambda_n = (\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{nn})$ satisfying the conditions:*

$$\begin{aligned} &\lambda_{1n} \geq \lambda_{1,n+1} \geq \lambda_{2,n} \geq \lambda_{2,n+1} \geq \cdots \geq \lambda_{\alpha n} \geq \lambda_{\alpha,n+1} \ , \\ &\lambda_{\alpha+2,n+1} \geq \lambda_{\alpha+1,n} \geq \lambda_{\alpha+3,n+1} \geq \cdots \geq \lambda_{n+1,n+1} \geq \lambda_{nn} \ . \end{aligned} \tag{5.11}$$

The vectors

$$|\Lambda^{(\alpha)}_{n+1}; \Lambda_n\rangle \ = \ F^{(\alpha)}_{-}(\Lambda_n; \Lambda^{(\alpha)}_{n+1})|\Lambda^{(\alpha)}_{n+1}\rangle \ , \tag{5.12}$$

where the "lowering" operator $F^{(\alpha)}_{-}(\Lambda_n; \Lambda^{(\alpha)}_{n+1})$ is given by

$$\begin{aligned} F^{(\alpha)}_{-}(\Lambda_n; \Lambda^{(\alpha)}_{n+1}) \ &= \ N^{(\alpha)}(\Lambda_n; \Lambda^{(\alpha)}_{n+1}) \ z_{\alpha}^{\lambda_{\alpha n}-\lambda_{\alpha,n+1}} \cdots z_1^{\lambda_{1n}-\lambda_{1,n+1}} \times \\ &\times \ z_{-\alpha-1}^{\lambda_{\alpha+2,n+1}-\lambda_{\alpha+1,n}} \cdots z_{-n}^{\lambda_{n+1,n+1}-\lambda_{nn}} \ , \end{aligned} \tag{5.13}$$

for all highest wights $\Lambda_n = (\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{nn})$ constrained by the conditions (5.11) form the orthonormal basis in the space of the highest vectors with respect to the compact subalgebra $U_q(u(n))$. Here in (5.13) the normalized factor $N^{(\alpha)}(\Lambda_n; \Lambda^{(\alpha)}_{n+1})$ is given as follows

$$\begin{aligned} N^{(\alpha)}(\Lambda_n; \Lambda^{(\alpha)}_{n+1}) \ &= \ (\Lambda_n; \Lambda^{(\alpha)}_{n+1}|\Lambda^{(\alpha)}_{n+1}; \Lambda_n)^{-\frac{1}{2}} \\ &= \ \Big\{ \prod_{i=1}^{\alpha} \frac{[l_{i,n+1} - l_{\alpha+1,n+1} - 2\alpha + n - 1]!}{[l_{in} - l_{i,n+1}]![l_{i,n} - l_{\alpha+1,n+1} - 2\alpha + n - 1]!} \\ &\times \prod_{l=\alpha+1}^{n} \frac{[l_{\alpha+1,n+1} - l_{l+1,n+1} + 2\alpha - n - 1]!}{[l_{l+1,n+1} - l_{ln}]![l_{\alpha+1,n+1} - l_{ln} + 2\alpha - n]!} \\ &\times \prod_{1\leq i<j\leq\alpha} \frac{[l_{in} - l_{j,n+1}]![l_{i,n+1} - l_{jn} - 1]!}{[l_{in} - l_{jn}]![l_{i,n+1} - l_{j,n+1} - 1]!} \\ &\times \prod_{\alpha+1\leq k<l\leq n} \frac{[l_{kn} - l_{l+1,n+1} - 2]![l_{k+1,n+1} - l_{ln} - 1]!}{[l_{kn} - l_{ln}]![l_{k+1,n+1} - l_{l+1,n+1} - 1]!} \\ &\times \prod_{1\leq i\leq\alpha<l\leq n} \frac{[l_{in} - l_{ln}]![l_{i,n+1} - l_{l+1,n+1} - 2]!}{[l_{in} - l_{l+1,n+1} - 2]![l_{i,n+1} - l_{ln} - 1]![l_{i,n+1} - l_{l+1,n+1} - 1]} \Big\}^{\frac{1}{2}} \ , \end{aligned} \tag{5.14}$$

$(l_{sp} := \lambda_{sp} - s)$.

This result coincides with the classical Gelfand–Graev case [5, 6]. Using analogous construction of the Gelfand–Tsetlin basis for the compact quantum algebra $U_q(u(n))$[12] we obtain a q-analog of the Gelfand–Graev–Tsetlin basis for $U_q(u(n,1))$. Namely, in the $U_q(u(n,1))$-module with the extremal weight $\Lambda_{n+1}^{(\alpha)}$ there is the orthogonal Gelfand–Graev–Tsetlin basis consisting of all vectors of the form

$$\big|\Lambda\big\rangle := \left|\begin{array}{l}\Lambda_{n+1}^{(\alpha)}\\ \Lambda_n\\ \dots\\ \Lambda_2\\ \Lambda_1\end{array}\right\rangle = F_-(\Lambda_1;\Lambda_2)F_-(\Lambda_2;\Lambda_3)\cdots F_-(\Lambda_{n-1};\Lambda_n)|\Lambda_{n+1}^{(\alpha)};\Lambda_n\rangle, \qquad (5.15)$$

where $\Lambda_j = (\lambda_{1j}, \lambda_{2j}, \dots, \lambda_{jj})$ $(j = 1, 2, \dots, n)$ and the numbers λ_{ij} satisfy the standard "between conditions" for the quantum algebra $U_q(u(n))$, i.e.

$$\lambda_{i,j+1} \geq \lambda_{ij} \geq \lambda_{i+1,j+1} \qquad \text{for } 1 \leq i \leq j \leq n-1 \ . \qquad (5.16)$$

The lowering operators $F_-(\Lambda_k; \Lambda_{k+1})$, $(k = 1, 2 \dots, n-1)$, are given by

$$F_-(\Lambda_k;\Lambda_{k+1}) = N(\Lambda_k;\Lambda_{k+1})\, p(U_q(u(k))) \prod_{i=1}^{k} (e_{k+1i})^{\lambda_{ik+1}-\lambda_{ik}} \ , \qquad (5.17)$$

$$\begin{aligned} N(\Lambda_k;\Lambda_{k+1}) = &\left\{ \prod_{i=1}^{k} \frac{[l_{ik}-l_{k+1,k+1}-1]!}{[l_{i,k+1}-l_{ik}]![l_{i,k+1}-l_{k+1,k+1}-1]!} \times \right. \\ &\left. \times \prod_{1\leq i<j\leq k} \frac{[l_{i,k+1}-l_{jk}]![l_{ik}-l_{j,k+1}-1]!}{[l_{ik}-l_{jk}]![l_{i,k+1}-l_{j,k+1}-1]!} \right\}^{\frac{1}{2}} \end{aligned} \qquad (5.18)$$

where $l_{ij} := \lambda_{ij} - i$. This explicit construction allows to obtain formulas for the action of $U_q(u(n,1))$-generators. These results will be presented in another article.

6 Summary

Thus we obtain the explicit description of the Hermitian irreducible representations for the noncompact quantum algebra $U_q(u(n,1))$ by the reduction Z-algebras for description of which we used the standard extremal projectors.

Next step: to obtain an analogous results for the case $U_q(u(n,2))$. For this aim we need construct extremal projector $p^{(\alpha)}$ which is expressed in terms of the Z-algebra $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$.

Final aim: to consider the general case $U_q(u(n,m))$. In this case extremal projectors of new type will be used.

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