Trace–formula for the RTT–algebra of sp(4) type

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Abstract

This paper continues our recent results on the Bethe vectors for the RTT–algebra of sp(4) type. We show how it is possible to rewrite the Bethe vectors in a different which is similar to the trace–formula for the Bethe vectors of the RTT-algebra of the gl(3) type.

1 Introduction

In this paper we continue our study of the Bethe vectors for the RTT–algebra. In a recent work, [1], we have found the Bethe vectors and the Bethe conditions for the RTT–algebra of sp(4) type. We will show how one can rewrite these Bethe vectors in another form, which is similar to the trace–formula for the Bethe vectors of the RTT-algebra of the gl(3) type, [2].
We first show how the trace–formula for the RTT–algebra of gl(3) can be derived from the algebraic nested Bethe ansatz for this RTT–algebra, [1, 3]. Then we use analogous considerations to derive the trace–formula for Bethe vectors for RTT–algebra of the type sp(4).
It turns out that for the trace–formula for the RTT–algebra of sp(4) type we have to use, in contrast to the trace–formula for the RTT–algebra of gl(3) type, the R–matrix of the auxiliary RTT–algebra $\tilde{\mathcal{A}}_2$, see section 3.1.

2 Trace–formula for RTT–algebra of gl(3) type

2.1 Algebraic nested Bethe ansatz for RTT–algebra of gl(3) type

We first recall the construction of Bethe vectors and Bethe conditions for the RTT–algebra of gl(3) type. To make it eaasy to follow the construction of the Bethe vectors and trace–formula for the RTT–algebra of sp(4) type, we use the notation and formulation of the algebraic nested Bethe ansatz, which is given in [1].
Let $\mathbf{E}^i_k$ be matrices of type 3×3 with the components $\left(\mathbf{E}^i_k\right)^r_s=\delta^i_s\delta^r_k$. Then the equality $\mathbf{E}^i_k\mathbf{E}^r_s=\delta^i_s\mathbf{E}^r_k$ hold for them and $\mathbf{I}=\sum\limits_{i=1}^{3}\mathbf{E}^i_i$ is a unit matrix.

The R–matrix for the RTT–algebra of gl(3) type is

$$\mathbf{R}(x,y) = \frac{1}{f(x,y)} \Big(\mathbf{I}\otimes\mathbf{I} + g(x,y) \sum_{i,k=1}^{3} \mathbf{E}^i_k \otimes \mathbf{E}^k_i \Big),$$

where

$$g(x,y) = \frac{1}{x-y}\,, \qquad f(x,y) = \frac{x-y+1}{x-y} = 1 + g(x,y)\,.$$

The corresponding RTT–algebra $\mathcal{A}$ is an associative algebra with unity, which is generated by the elemets $T^i_k(x)$, where i, $k = 1, 2, 3$, and satisfies the RTT–equation

$$\mathbf{R}_{1,2}(x,y)\mathbf{T}_1(x)\mathbf{T}_2(y) = \mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(x,y)\,, \quad \text{whwrw} \quad \mathbf{T}(x) = \sum_{i,k=1}^{3} \mathbf{E}^k_i \otimes T^i_k(x)$$

Let ω be a vacuum vector for which

$$T^1_2(x)\omega = T^1_3(x)\omega = T^2_3(x)\omega\,, \qquad T^i_i(x)\omega = \lambda_i(x)\omega\,, \quad i = 1,\,2,\,3\,,$$

hold and $\mathcal{W} = \mathcal{A}\omega$. On the vector space $\mathcal{W}$ we will look for eigenvalues and eigenvectors of the operators

$$H(x) = \mathrm{Tr}\big(\mathbf{T}(x)\big) = T^1_1(x) + T^2_2(x) + T^3_3(x)\,.$$

In the three–dimensional vector space $\mathcal{V}$ we choose the basis $\mathbf{e}_r$, $r = 1, 2, 3$, such that $\mathbf{E}^i_k\mathbf{e}_r = \delta^i_r\mathbf{e}_k$, and the dual basis in the space $\mathcal{V}^*$ is denoted by $\mathbf{f}^r$, i.e. $\langle \mathbf{f}^r, \mathbf{e}_i \rangle = \delta^r_i$ holds. Let $\vec{u} = (u_1,\ldots,u_N)$ and $\vec{v} = (v_1,\ldots,v_M)$ are ordered N–tuple and M–tuple of mutually different complex numbers and $\overline{u} = \{u_1,\ldots,u_N\}$ and $\overline{v} = (v_1,\ldots,v_M\}$ sets of their elements.
Consider the R–matrix and the monodromy operator bounded to space with the basis $\mathbf{e}_1$ a $\mathbf{e}_2$, i.e.

$$\tilde{\mathbf{R}}(x,y) = \frac{1}{f(x,y)} \Big(\tilde{\mathbf{I}}\otimes\tilde{\mathbf{I}} + g(x,y) \sum_{i,k=1}^{2} \mathbf{E}^i_k \otimes \mathbf{E}^k_i \Big), \qquad \tilde{\mathbf{T}}(x) = \sum_{i,k=1}^{2} \mathbf{E}^k_i \otimes T^i_k(x)$$

For given $\vec{v}$, we construct on the vector space $\widehat{V} = \mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_M \otimes \mathcal{W}$ the operators $\widehat{T}^i_k(x;\vec{v})$ using the relations

$$\widehat{\mathbf{T}}_{0;1,\ldots,M}(x;\vec{v}) = \tilde{\mathbf{T}}_0(x)\tilde{\mathbf{R}}_{0,M}(x,v_M)\ldots\tilde{\mathbf{R}}_{0,1}(x,v_1) = \sum_{i,k=1}^{2} \mathbf{E}^k_i \otimes \widehat{T}^i_k(x;\vec{v})\,.$$

Let us define

$$\widehat{\mathbf{\Omega}} = \underbrace{\mathbf{e}_2 \otimes \cdots \otimes \mathbf{e}_2}_{M\times} \otimes \omega\,, \qquad \mathbf{\Phi}(\vec{u};\vec{v}) = \widehat{T}^2_1(u_1;\vec{v})\widehat{T}^2_1(u_2;\vec{v})\ldots\widehat{T}^2_1(u_N;\vec{v})\widehat{\mathbf{\Omega}} \in \widehat{V}$$

and denote $\overline{u}_k = \overline{u}\setminus\{u_k\}$, $\overline{v}_r = \overline{v}\setminus\{v_r\}$ and

$$F(x;\overline{u}) = \prod_{u_i\in\overline{u}} f(x,u_i)\,, \qquad F(\overline{u};x) = \prod_{u_i\in\overline{u}} f(u_i,x)\,.$$

The statement of the algebraic Bethe ansatz is that if the Bethe conditions

$$\begin{aligned} \lambda_1(u_k)F(u_k;\overline{u}_k) &= \lambda_2(u_k)F(\overline{u}_k;u_k)F(u_k;\overline{v}) \qquad && \forall\, u_k \in \overline{u} \\ \lambda_3(v_r)F(\overline{v}_r;v_r) &= \lambda_2(v_r)F(v_r;\overline{v}_r)F(\overline{u};v_r) \qquad && \forall\, v_r \in \overline{v} \end{aligned}$$

are satisfied, the Bethe vector

$$\mathfrak{B}(\vec{u},\vec{v}) = \sum_{k_1,\dots,k_M=1}^{2} \Big\langle \mathbf{f}^{k_1} \otimes \ldots \otimes \mathbf{f}^{k_M} \otimes T^3_{k_1}(v_1)\ldots T^3_{k_M}(v_M), \mathbf{\Phi}(\vec{u};\vec{v}) \Big\rangle \tag{1}$$

is for each x eigenvector of the operator $H(x)$ with eigenvalue

$$E_3(x;\overline{u};\overline{v}) = \lambda_3(x)F(\overline{v};x) + \lambda_2(x)F(\overline{u};x)F(x;\overline{v}) + \lambda_1(x)F(x;\overline{u})\,.$$

2.2 Trace–formula for RTT–algebra of gl(3) type

The aim of this part is to write the Bethe vector (1) using the R–matrix $\mathbf{R}(x,y)$ and monodromy operators $\mathbf{T}(x)$ of the RTT–algebra of gl(3) type. For this purpose, we introduce an ordered M–tuple $\vec{b} = (b_1,\dots,b_M)$ and vector space

$$\mathcal{V}_{\vec{b}} = \mathcal{V}_{b_1} \otimes \mathcal{V}_{b_2} \otimes \ldots \otimes \mathcal{V}_{b_M} = \mathcal{V}^{\otimes\vec{b}}.$$

From the definition of $\widehat{T}^2_1(u,\vec{v})$ it is clear that we can write

$$\widehat{T}^2_1(u_1;\vec{v})\widehat{T}^2_1(u_2;\vec{v})\ldots\widehat{T}^2_1(u_N;\vec{v}) = \sum_{\alpha_1,\dots\alpha_M,\beta_1,\dots,\beta_M=1}^{2} \mathbf{E}^{\alpha_1}_{\beta_1} \otimes \ldots \otimes \mathbf{E}^{\alpha_M}_{\beta_M} \otimes Z^{\beta_1,\dots,\beta_M}_{\alpha_1,\dots,\alpha_M}\,,$$

where $Z^{\beta_1,\dots,\beta_M}_{\alpha_1,\dots,\alpha_M}$ are the operators on the space $\mathcal{W}$. Therefore,

$$\begin{aligned} \mathfrak{B}(\vec{u},\vec{v}) = \sum_{k_1,\dots,k_M=1}^{2}\ \sum_{\beta_1,\dots,\beta_M=1}^{2} \Big\langle &\mathbf{f}^{k_1} \otimes \ldots \otimes \mathbf{f}^{k_M} \otimes T^3_{k_1}(v_1)\ldots T^3_{k_M}(v_M), \\ &\mathbf{e}_{\beta_1} \otimes \ldots \otimes \mathbf{e}_{\beta_M} \otimes Z^{\beta_1,\dots,\beta_M}_{2,\dots,2}\,\omega \Big\rangle = \\ = \sum_{k_1,\dots,k_M=1}^{2} T^3_{k_1}(v_1)&\ldots T^3_{k_M}(v_M) Z^{k_1,\dots,k_M}_{2,\dots,2}\,\omega\,. \end{aligned}$$

This expression of the Bethe vectors can be rewriten as

$$\mathfrak{B}(\vec{u},\vec{v}) = \mathrm{Tr}_{\vec{b}}\Big(\mathbf{T}_{b_1}(v_1)\ldots\mathbf{T}_{b_M}(v_M)\widehat{T}^2_1(u_1;\vec{v})\ldots\widehat{T}^2_1(u_N;\vec{v})\big(\mathbf{E}^3_2 \otimes \ldots \otimes \mathbf{E}^3_2\big)\Big)\omega$$

According to the definition we have

$$\widehat{T}^2_1(u;\vec{v}) = \mathrm{Tr}_a\big(\widehat{\mathbf{T}}_{a;1,\dots,M}(u;\vec{v})\mathbf{E}^2_1\big)\,,$$

where $\mathbf{E}^2_1$ acts on the space $\mathcal{V}_a$. If we introduce an ordered N–tuple $\vec{a} = (a_1,\dots,a_N)$ and vector space

$$\mathcal{V}_{\vec{a}} = \mathcal{V}_{a_1} \otimes \ldots \otimes \mathcal{V}_{a_N} = \mathcal{V}^{\otimes\vec{a}}\,,$$

it is possible to write

$$\widehat{T}_1^2(u_1;\vec{v})\dots\widehat{T}_1^2(u_N;\vec{v}) = \mathrm{Tr}_{\vec{a}}\Big(\widehat{\mathbf{T}}_{a_1;1,\dots,M}(u_1;\vec{v})\dots\widehat{\mathbf{T}}_{a_N;1,\dots,M}(u_M;\vec{v})\big(\mathbf{E}_1^2\otimes\dots\otimes\mathbf{E}_1^2\big)\Big)$$

If we introduce $\mathcal{V}_{\vec{a};\vec{b}} = \mathcal{V}^{\otimes\vec{a}}\otimes\mathcal{V}^{\otimes\vec{b}}$, we can use this relations and the definition $\widehat{\mathbf{T}}_{a;1,\dots,M}(u;\vec{v})$ to write the Bethe vector as

$$\mathfrak{B}(\vec{u};\vec{v}) = \mathrm{Tr}_{\vec{a},\vec{b}}\Big(\mathbf{T}_{\overrightarrow{b}}(\vec{v})\tilde{\mathbf{T}}_{\overrightarrow{a}}(\vec{u})\tilde{\mathbf{R}}_{\overrightarrow{a};\overleftarrow{b}}(\vec{u},\vec{v})\big((\mathbf{E}_1^2)^{\otimes\vec{a}}\otimes(\mathbf{E}_2^3)^{\otimes\vec{b}}\big)\Big)\omega\,,$$

where

$$\begin{aligned}
&\mathbf{T}_{\overrightarrow{b}}(\vec{v}) = \mathbf{T}_{b_1}(v_1)\dots,\mathbf{T}_{b_M}(v_m)\\
&\tilde{\mathbf{T}}_{\overrightarrow{a}}(\vec{u}) = \tilde{\mathbf{T}}_{a_1}(u_1)\dots\tilde{\mathbf{T}}_{a_N}(u_N)\\
&\tilde{\mathbf{R}}_{\overrightarrow{a};\overleftarrow{b}}(\vec{u},\vec{v}) = \tilde{\mathbf{R}}_{\overrightarrow{a},b_M}(\vec{u},v_M)\dots\tilde{\mathbf{R}}_{\overrightarrow{a},b_1}(\vec{u},v_1) = \tilde{\mathbf{R}}_{a_1,\overleftarrow{b}}(u_1,\vec{v}\dots\tilde{\mathbf{R}}_{a_N,\overleftarrow{b}}(u_N,\vec{v})\\
&\tilde{\mathbf{R}}_{\overrightarrow{a},b}(\vec{u},v) = \tilde{\mathbf{R}}_{a_1,b}(u_1,v)\dots\tilde{\mathbf{R}}_{a_N,b}(u_M,v)\\
&\tilde{\mathbf{R}}_{a,\overleftarrow{b}}(u,\vec{v}) = \tilde{\mathbf{R}}_{a,b_M}(u,v_M)\dots\tilde{\mathbf{R}}_{a,b_1}(u,v_1)
\end{aligned}$$

We show that in this expression instead of the restricted R–matrices $\tilde{\mathbf{R}}_{a,b}(u,v)$ and the monodromy operators $\tilde{\mathbf{T}}_a(u)$ of the R–matrices and we can use the R–matrices and the monodromy operators for the whole RTT–algebra of the type gl(3) and write Bethe the vectors as

$$\mathfrak{B}(\vec{u};\vec{v}) = \mathrm{Tr}_{\vec{a},\vec{b}}\Big(\mathbf{T}_{\overrightarrow{b}}(\vec{v})\mathbf{T}_{\overrightarrow{a}}(\vec{u})\mathbf{R}_{\overrightarrow{a};\overleftarrow{b}}(\vec{u},\vec{v})\big((\mathbf{E}_1^2)^{\otimes\vec{a}}\otimes(\mathbf{E}_2^3)^{\otimes\vec{b}}\big)\Big)\omega\,. \qquad (2)$$

Equation (2) is called trace–formula.
If we write

$$\mathbf{R}(x,y) = \sum_{r,s,p,q=1}^{3} R_{s,q}^{r,p}(x,y)\mathbf{E}_r^s\otimes\mathbf{E}_p^q\,,\qquad \tilde{\mathbf{R}}(x,y) = \sum_{r,s,p,q=1}^{2} R_{s,q}^{r,p}(x,y)\mathbf{E}_r^s\otimes\mathbf{E}_p^q\,,$$

where

$$R_{s,q}^{r,p} = \frac{\delta_s^r\delta_q^p + g(x,y)\delta_q^r\delta_s^p}{f(x,y)}.\,,$$

Hence we obtain the relations

$$\begin{aligned}
\mathrm{Tr}_a\Big(\mathbf{T}_a(u)\mathbf{R}_{a;\overleftarrow{b}}(u,\vec{v})\mathbf{E}_1^2\Big) = &\sum_{\alpha_1,\dots,\alpha_M=1}^{3}\ \sum_{p_1,\dots,p_M=1}^{3}\ \sum_{q_1,\dots,q_M=1}^{3}\\
&R_{\alpha_{M-1},q_M}^{\alpha_M,p_M}(u,v_M)R_{\alpha_{M-2},q_{M-1}}^{\alpha_{M-1},p_{M-1}}(u,v_{M-1})R_{\alpha_{M-3},q_{M-2}}^{\alpha_{M-2},p_{M-2}}(u,v_{M-2})\dots\\
&\dots R_{\alpha_2,q_3}^{\alpha_3,p_3}(u,v_3)R_{\alpha_1,q_2}^{\alpha_2,p_2}(u,v_2)R_{1,q_1}^{\alpha_1,p_1}(u,v_1)\\
&\mathbf{E}_{p_1}^{q_1}\otimes\mathbf{E}_{p_2}^{q_2}\otimes\mathbf{E}_{p_3}^{q_3}\otimes\dots\otimes\mathbf{E}_{p_{M-2}}^{q_{M-2}}\otimes\mathbf{E}_{p_{M-1}}^{q_{M-1}}\otimes\mathbf{E}_{p_M}^{q_M}\otimes T_{\alpha_M}^2(u)
\end{aligned}$$

$$\begin{aligned}
\tilde{T}_1^2(u;\vec{v}) = \mathrm{Tr}_a\Big(\tilde{\mathbf{T}}_a(u)\tilde{\mathbf{R}}_{a;\overleftarrow{b}}(u,\vec{v})\mathbf{E}_1^2\Big) = &\sum_{\alpha_1,\dots,\alpha_M=1}^{2}\ \sum_{p_1,\dots,p_M=1}^{2}\ \sum_{q_1,\dots,q_M=1}^{2}\\
&R_{\alpha_{M-1},q_M}^{\alpha_M,p_M}(u,v_M)R_{\alpha_{M-2},q_{M-1}}^{\alpha_{M-1},p_{M-1}}(u,v_{M-1})R_{\alpha_{M-3},q_{M-2}}^{\alpha_{M-2},p_{M-2}}(u,v_{M-2})\dots\\
&\dots R_{\alpha_2,q_3}^{\alpha_3,p_3}(u,v_3)R_{\alpha_1,q_2}^{\alpha_2,p_2}(u,v_2)R_{1,q_1}^{\alpha_1,p_1}(u,v_1)\\
&\mathbf{E}_{p_1}^{q_1}\otimes\mathbf{E}_{p_2}^{q_2}\otimes\mathbf{E}_{p_3}^{q_3}\otimes\dots\otimes\mathbf{E}_{p_{M-2}}^{q_{M-2}}\otimes\mathbf{E}_{p_{M-1}}^{q_{M-1}}\otimes\mathbf{E}_{p_M}^{q_M}\otimes T_{\alpha_M}^2(u)
\end{aligned}$$

Therefore, we have

$$\mathrm{Tr}_{\vec a}\Big(\mathbf{T}_{\vec a}(\vec u)\mathbf{R}_{\overrightarrow{a};\overleftarrow{b}}(\vec u,\vec v)\big((\mathbf{E}^2_1)^{\otimes\vec a}\otimes(\mathbf{E}^3_2)^{\otimes\vec b}\big)\Big)=\sum_{\alpha_{1,1},\dots,\alpha_{M,N}=1}^{3}\ \sum_{\beta_{1,1},\dots,\beta_{M,N}=1}^{3}$$
$$R^{\alpha_{M,1},\beta_{M,1}}_{\alpha_{M-1,1},\beta_{M,2}}(u_1,v_M)R^{\alpha_{M-1,1},\beta_{M-1,1}}_{\alpha_{M-2,1},\beta_{M-1,2}}(u_1,v_{M-1})\dots R^{\alpha_{2,1},\beta_{2,1}}_{\alpha_{1,1},\beta_{2,2}}(u_1,v_2)R^{\alpha_{1,1},\beta_{1,1}}_{1,\beta_{1,2}}(u_1,v_1)$$
$$R^{\alpha_{M,2},\beta_{M,2}}_{\alpha_{M-1,2},\beta_{M,3}}(u_2,v_M)R^{\alpha_{M-1,2},\beta_{M-1,2}}_{\alpha_{M-2,2},\beta_{M-1,3}}(u_2,v_{M-1})\dots R^{\alpha_{2,2},\beta_{2,2}}_{\alpha_{1,2},\beta_{2,3}}(u_2,v_2)R^{\alpha_{1,2},\beta_{1,2}}_{1,\beta_{1,3}}(u_2,v_1)$$
$$\dots$$
$$R^{\alpha_{M,N-1},\beta_{M,N-1}}_{\alpha_{M-1,N-1},\beta_{M,N}}(u_{N-1},v_M)R^{\alpha_{M-1,N-1},\beta_{M-1,N-1}}_{\alpha_{M-2,N-1},\beta_{M-1,N}}(u_{N-1},v_{M-1})\dots$$
$$\dots R^{\alpha_{2,N-1},\beta_{2,N-1}}_{\alpha_{1,N-1},\beta_{2,N}}(u_{N-1},v_2)R^{\alpha_{1,N-1},\beta_{1,N-1}}_{1,\beta_N}(u_{N-1},v_1)$$
$$R^{\alpha_{M,N},\beta_{M,N}}_{\alpha_{M-1,N},2}(u_N,v_M)R^{\alpha_{M-1,N},\beta_{M-1,N}}_{\alpha_{M-2,N},2}(u_N,v_{M-1})\dots$$
$$\dots R^{\alpha_{2,N-1},\beta_{2,N}}_{\alpha_{1,N},2}(u_N,v_2)R^{\alpha_{1,N},\beta_{1,N}}_{1,2}(u_N,v_1)$$
$$\mathbf{E}^3_{\beta_{1,1}}\otimes\mathbf{E}^3_{\beta_{2,1}}\otimes\dots\otimes\mathbf{E}^3_{\beta_{M-1,1}}\otimes\mathbf{E}^3_{\beta_{M,1}}\otimes$$
$$\otimes T^2_{\alpha_{M,1}}(u_1)T^2_{\alpha_{M,2}}(u_2)T^2_{\alpha_{M,3}}(u_3)\dots T^2_{\alpha_{M,N-2}}(u_{N-2})T^2_{\alpha_{M,N-1}}(u_{N-1})T^2_{\alpha_{M,N}}(u_N)$$

But for α, $\beta=1$, 2 and r or $s=3$ we have $R^{r,s}_{\alpha,\beta}(x,y)=0$. Therefore, the relation

$$\mathrm{Tr}_{\vec a}\Big(\mathbf{T}_{\vec a}(\vec u)\mathbf{R}_{\overrightarrow{a};\overleftarrow{b}}(\vec u,\vec v)\big((\mathbf{E}^2_1)^{\otimes\vec a}\otimes(\mathbf{E}^3_2)^{\otimes\vec b}\big)\Big)=\sum_{\alpha_{1,1},\dots,\alpha_{M,N}=1}^{2}\ \sum_{\beta_{1,1},\dots,\beta_{M,N}=1}^{2}$$
$$R^{\alpha_{M,1},\beta_{M,1}}_{\alpha_{M-1,1},\beta_{M,2}}(u_1,v_M)R^{\alpha_{M-1,1},\beta_{M-1,1}}_{\alpha_{M-2,1},\beta_{M-1,2}}(u_1,v_{M-1})\dots R^{\alpha_{2,1},\beta_{2,1}}_{\alpha_{1,1},\beta_{2,2}}(u_1,v_2)R^{\alpha_{1,1},\beta_{1,1}}_{1,\beta_{1,2}}(u_1,v_1)$$
$$R^{\alpha_{M,2},\beta_{M,2}}_{\alpha_{M-1,2},\beta_{M,3}}(u_2,v_M)R^{\alpha_{M-1,2},\beta_{M-1,2}}_{\alpha_{M-2,2},\beta_{M-1,3}}(u_2,v_{M-1})\dots R^{\alpha_{2,2},\beta_{2,2}}_{\alpha_{1,2},\beta_{2,3}}(u_2,v_2)R^{\alpha_{1,2},\beta_{1,2}}_{1,\beta_{1,3}}(u_2,v_1)$$
$$\dots$$
$$R^{\alpha_{M,N-1},\beta_{M,N-1}}_{\alpha_{M-1,N-1},\beta_{M,N}}(u_{N-1},v_M)R^{\alpha_{M-1,N-1},\beta_{M-1,N-1}}_{\alpha_{M-2,N-1},\beta_{M-1,N}}(u_{N-1},v_{M-1})\dots$$
$$\dots R^{\alpha_{2,N-1},\beta_{2,N-1}}_{\alpha_{1,N-1},\beta_{2,N}}(u_{N-1},v_2)R^{\alpha_{1,N-1},\beta_{1,N-1}}_{1,\beta_N}(u_{N-1},v_1)$$
$$R^{\alpha_{M,N},\beta_{M,N}}_{\alpha_{M-1,N},2}(u_N,v_M)R^{\alpha_{M-1,N},\beta_{M-1,N}}_{\alpha_{M-2,N},2}(u_N,v_{M-1})\dots$$
$$\dots R^{\alpha_{2,N-1},\beta_{2,N}}_{\alpha_{1,N},2}(u_N,v_2)R^{\alpha_{1,N},\beta_{1,N}}_{1,2}(u_N,v_1)$$
$$\mathbf{E}^3_{\beta_{1,1}}\otimes\mathbf{E}^3_{\beta_{2,1}}\otimes\dots\otimes\mathbf{E}^3_{\beta_{M-1,1}}\otimes\mathbf{E}^3_{\beta_{M,1}}\otimes$$
$$\otimes T^2_{\alpha_{M,1}}(u_1)T^2_{\alpha_{M,2}}(u_2)T^2_{\alpha_{M,3}}(u_3)\dots T^2_{\alpha_{M,N-2}}(u_{N-2})T^2_{\alpha_{M,N-1}}(u_{N-1})T^2_{\alpha_{M,N}}(u_N)=$$
$$=\mathrm{Tr}_{\vec a}\Big(\tilde{\mathbf{T}}_{\vec a}(\vec u)\tilde{\mathbf{R}}_{\overrightarrow{a};\overleftarrow{b}}(\vec u,\vec v)\big((\mathbf{E}^2_1)^{\otimes\vec a}\otimes(\mathbf{E}^3_2)^{\otimes\vec b}\big)\Big)$$

holds, which proves the trace–formula (2).

3 Algebraic nested Bethe ansatz for RTT–algebra of sp(4) type

In this part, we will briefly repeat the main results of the paper [1] on the construction of the Bethe vectors for the RTT–algebra of sp(4) type.

The R–matrix for the RTT–algebra $\mathcal{A}$ of type sp(4) is

$$\mathbf{R}(x,y) = \frac{1}{f(x,y)}\Big(\mathbf{I}\otimes\mathbf{I} + g(x,y)\sum_{i,k=-2}^{2}\mathbf{E}_k^i\otimes\mathbf{E}_i^k - h(x,y)\sum_{i,k=-2}^{2}\epsilon_i\epsilon_k\mathbf{E}_k^i\otimes\mathbf{E}_{-k}^{-i}\Big),$$

where the indices i and k take the values ±1, ±2,

$$g(x,y) = \frac{1}{x-y}\,, \qquad f(x,y) = \frac{x-y+1}{x-y} = 1+g(x,y) \qquad \epsilon_i = \mathrm{sgn}(i)$$
$$h(x,y) = \frac{1}{x-y+3}\,, \qquad k(x,y) = \frac{1}{x-y-1}\,,$$

and the monodromy operator $\mathbf{T}(x) = \sum_{i,k=-2}^{2}\mathbf{E}_i^k\otimes T_k^i(x)$.
We define the vacuum vector ω by the relations

$$T_k^i(x)\omega = 0 \quad \text{for} \quad i<k\,, \qquad T_i^i(x)\omega = \lambda_i(x)\omega\,, \quad \text{for} \quad i=\pm1,\,\pm2\,,$$

and on the vector space $\mathcal{W} = \mathcal{A}\omega$ we look for common eigenvectors of the operators

$$H(x) = \mathrm{Tr}\big(\mathbf{T}(x)\big) = \sum_{i=-2}^{2}T_i^i(x)\,.$$

3.1 Auxiliary RTT–algebra $\tilde{\mathcal{A}}_2$

In [1], we showned that the RTT–algebra generated by the element $\tilde{T}_k^i(x)$ and $\tilde{T}_{-k}^{-i}(x)$, where $i,\,k = 1,\,2$, which we denoted by $\tilde{\mathcal{A}}_2$ is essential for construction of the Bethe vectors.[1] The R–matrix $\tilde{\mathbf{R}}(x,y)$ and the monodromy operators $\tilde{\mathbf{T}}(x)$ in RTT–algebra $\tilde{\mathcal{A}}_2$ are defined by the following relations:

$$\mathbf{I}_+ = \sum_{i=1}^{2}\mathbf{E}_i^i\,, \qquad \mathbf{I}_- = \sum_{i=1}^{2}\mathbf{E}_{-i}^{-i}$$
$$\mathbf{R}^{(+,+)}(x,y) = \frac{1}{f(x,y)}\Big(\mathbf{I}_+\otimes\mathbf{I}_+ + g(x,y)\sum_{i,k=1}^{2}\mathbf{E}_k^i\otimes\mathbf{E}_i^k\Big),$$
$$\mathbf{R}^{(+,-)}(x,y) = \mathbf{I}_+\otimes\mathbf{I}_- - k(x,y)\sum_{i,k=1}^{2}\mathbf{E}_k^i\otimes\mathbf{E}_{-k}^{-i}\,,$$
$$\mathbf{R}^{(-,+)}(x,y) = \mathbf{I}_-\otimes\mathbf{I}_+ - h(x,y)\sum_{i,k=1}^{2}\mathbf{E}_{-k}^{-i}\otimes\mathbf{E}_k^i,$$
$$\mathbf{R}^{(-,-)}(x,y) = \frac{1}{f(x,y)}\Big(\mathbf{I}_-\otimes\mathbf{I}_- + g(x,y)\sum_{i,k=1}^{2}\mathbf{E}_{-k}^{-i}\otimes\mathbf{E}_{-i}^{-k}\Big),$$
$$\mathbf{T}^{(+)}(x) = \sum_{i,k=1}^{2}\mathbf{E}_i^k\otimes\tilde{T}_k^i(x)\,, \qquad \mathbf{T}^{(-)}(x) = \sum_{i,k=1}^{2}\mathbf{E}_{-i}^{-k}\otimes\tilde{T}_{-k}^{-i}(x)\,,$$
$$\tilde{\mathbf{R}}(x,y) = \mathbf{R}^{(+,+)}(x,y) + \mathbf{R}^{(+,-)}(x,y) + \mathbf{R}^{(-,+)}(x,y) + \mathbf{R}^{(-,-)}(x,y)$$
$$\tilde{\mathbf{T}}(x) = \mathbf{T}^{(+)}(x) + \mathbf{T}^{(-)}(x)$$

[1] Note that the RTT–algebra $\tilde{\mathcal{A}}_2$ is not a RTT–subalgebra of the RTT–algebra $\mathcal{A}$, but their RTT–subalgebras $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ are also RTT–subalgebras $\mathcal{A}$.

In the coordinates the R–matrix $\tilde{\mathbf{R}}(x,y)$ is

$$\mathbf{R}^{(+,+)}(x,y) = \sum_{i,k,r,s=1}^{2} \tilde{R}^{i,r}_{k,s}(x,y)\mathbf{E}^k_i \otimes \mathbf{E}^s_r, \qquad \tilde{R}^{i,r}_{k,s}(x,y) = \frac{\delta^i_k\delta^r_s + g(x,y)\delta^i_s\delta^r_k}{f(x,y)}$$

$$\mathbf{R}^{(+,-)}(x,y) = \sum_{i,k,r,s=1}^{2} \tilde{R}^{i,-r}_{k,-s}(x,y)\mathbf{E}^k_i \otimes \mathbf{E}^{-s}_{-r}, \qquad \tilde{R}^{i,-r}_{k,-s}(x,y) = \delta^i_k\delta^r_s - k(x,y)\delta^{i,r}\delta_{k,s}$$

$$\mathbf{R}^{(-,+)}(x,y) = \sum_{i,k,r,s=1}^{2} \tilde{R}^{-i,r}_{-k,s}(x,y)\mathbf{E}^{-k}_{-i} \otimes \mathbf{E}^s_r, \qquad \tilde{R}^{-i,r}_{-k,s}(x,y) = \delta^i_k\delta^r_s - h(x,y)\delta^{i,r}\delta_{k,s}$$

$$\mathbf{R}^{(-,-)}(x,y) = \sum_{i,k,r,s=1}^{2} \tilde{R}^{-i,-r}_{-k,-s}(x,y)\mathbf{E}^{-k}_{-i} \otimes \mathbf{E}^{-s}_{-r}, \qquad \tilde{R}^{-i,-r}_{-k,-s}(x,y) = \frac{\delta^i_k\delta^r_s + g(x,y)\delta^i_s\delta^r_k}{f(x,y)}$$

The R–matrix $\tilde{\mathbf{R}}(x,y)$ is invertible and its inverse matrix is

$$\big(\mathbf{R}(x,y)\big)^{-1} = \big(\mathbf{R}^{(+,+)}(x,y)\big)^{-1} + \big(\mathbf{R}^{(+,-)}(x,y)\big)^{-1} + \big(\mathbf{R}^{(-,+)}(x,y)\big)^{-1} + \big(\mathbf{R}^{(-,-)}(x,y)\big)^{-1}$$

where using the components

$$\big(\mathbf{R}^{(+,+)}(x,y)\big)^{-1} = \sum_{i,k,r,s=1}^{2} \tilde{S}^{i,r}_{k,s}(x,y)\mathbf{E}^k_i \otimes \mathbf{E}^s_r, \qquad \tilde{S}^{i,r}_{k,s}(x,y) = \tilde{R}^{i,r}_{k,s}(y,x)$$

$$\big(\mathbf{R}^{(+,-)}(x,y)\big)^{-1} = \sum_{i,k,r,s=1}^{2} \tilde{S}^{i,-r}_{k,-s}(x,y)\mathbf{E}^k_i \otimes \mathbf{E}^{-s}_{-r}, \qquad \tilde{S}^{i,-r}_{k,-s}(x,y) = \tilde{R}^{-r,i}_{-s,k}(y,x)$$

$$\big(\mathbf{R}^{(-,+)}(x,y)\big)^{-1} = \sum_{i,k,r,s=1}^{2} \tilde{S}^{-i,r}_{-k,s}(x,y)\mathbf{E}^{-k}_{-i} \otimes \mathbf{E}^s_r, \qquad \tilde{S}^{-i,r}_{-k,s}(x,y) = \tilde{R}^{r,-i}_{s,-k}(y,x)$$

$$\big(\mathbf{R}^{(-,-)}(x,y)\big)^{-1} = \sum_{i,k,r,s=1}^{2} \tilde{S}^{-i,-r}_{-k,-s}(x,y)\mathbf{E}^{-k}_{-i} \otimes \mathbf{E}^{-s}_{-r}, \qquad \tilde{S}^{-i,-r}_{-k,-s}(x,y) = \tilde{R}^{-i,-r}_{-k,-s}(y,x)$$

3.2 Constrction of the Bethe vectors for RTT–algebra of sp(4) type

We denote the bases in two dimensional spaces $\mathcal{V}_{(+)}$ and $\mathcal{V}_{(-)}$ by $\mathbf{e}_k$ and $\mathbf{e}_{-k}$, where $k = 1,\,2$, and their dual bases in the dual spaces $\mathcal{V}^*_{(+)}$ and $\mathcal{V}^*_{(-)}$ by $\mathbf{f}^k$ and $\mathbf{f}^{-k}$.
Let $\vec{u} = (u_1,\ldots,u_N)$ be an ordered N–tuple of mutually different complex numbers. We will look for Bethe vectors in the form

$$\mathfrak{B}(\vec{u}) = \Big\langle \mathbf{B}_{1,\ldots,N}(\vec{u}), \mathbf{\Phi} \Big\rangle,$$

where

$$\begin{aligned}
&\mathbf{e}_{\vec{i}} = \mathbf{e}_{i_1}\otimes\ldots\otimes\mathbf{e}_{i_N} \in \mathcal{V}_{1_+}\otimes\ldots\otimes\mathcal{V}_{N_+} = \mathcal{V}_+\,,\\
&\mathbf{f}^{-\vec{k}} = \mathbf{f}^{-k_1}\otimes\ldots\otimes\mathbf{f}^{-k_N} \in \mathcal{V}^*_{1_-}\otimes\ldots\otimes\mathcal{V}^*_{N_-} = \mathcal{V}^*_-\,,\\
&\mathbf{f}^{\vec{s}} = \mathbf{f}^{s_1}\otimes\ldots\otimes\mathbf{f}^{s_N} \in \mathcal{V}^*_{1_+}\otimes\ldots\otimes\mathcal{V}^*_{N_+} = \mathcal{V}^*_+\,,\\
&\mathbf{e}_{-\vec{r}} = \mathbf{e}_{-r_1}\otimes\ldots\otimes\mathbf{e}_{-r_N} \in \mathcal{V}_{1_-}\otimes\ldots\otimes\mathcal{V}_{N_-} = \mathcal{V}_-\,,\\
&\mathbf{B}_{1,\ldots,N}(\vec{u}) = \sum_{\vec{i},\vec{k}} \mathbf{e}_{\vec{i}}\otimes\mathbf{f}^{-\vec{k}}\otimes T^{\vec{i}}_{-\vec{k}}(\vec{u})\\
&T^{\vec{i}}_{-\vec{k}}(\vec{u}) = T^{i_1}_{-k_1}(u_1)\ldots T^{i_N}_{-k_N}(u_N) \in \mathcal{A}\,,\\
&\mathbf{\Phi} = \mathbf{f}^{\vec{s}}\otimes\mathbf{e}_{-\vec{r}}\otimes\Phi^{-\vec{r}}_{\vec{s}} \in \mathcal{V}^*_+\otimes\mathcal{V}_-\otimes\mathcal{W}_0 = \widehat{\mathcal{W}_0}\,,\\
&\Phi^{-\vec{r}}_{\vec{s}} = \Phi^{-r_1,\ldots,-r_N}_{s_1,\ldots,s_N} \in \mathcal{W}_0 \subset \mathcal{W}\,.
\end{aligned}$$

We introduce

$$\widehat{\mathbf{R}}^{(+,+)}_{0,1^*_+}(x,u) = \sum_{i,k,r,s=1}^{2} \tilde{S}^{i,r}_{k,s}(x,u)\mathbf{E}^k_i \otimes \mathbf{F}^s_r$$
$$\widehat{\mathbf{R}}^{(-,+)}_{0,1^*_+}(x,u) = \sum_{i,k,r,s=1}^{2} \tilde{S}^{-i,r}_{-k,s}(x,u)\mathbf{E}^{-k}_{-i} \otimes \mathbf{F}^s_r$$
$$\mathbf{R}^{(+,-)}_{0_+,1_-}(x,u) = \sum_{i,k,r,s=1}^{2} \tilde{R}^{i,-r}_{k,-s}(x,u)\mathbf{E}^k_i \otimes \mathbf{E}^{-s}_{-r}$$
$$\mathbf{R}^{(-,-)}_{0_-,1_-}(x,u) = \sum_{i,k,r,s=1}^{2} \tilde{R}^{-i,-r}_{-k,-s}(x,u)\mathbf{E}^{-k}_{-i} \otimes \mathbf{E}^{-s}_{-r}\,,$$

where $\mathbf{F}^r_s$ is a linear mapping on the space $\mathcal{V}^*$ defined as $\mathbf{F}^r_s\mathbf{f}^k = \delta^k_s\mathbf{f}^r$, and for $\epsilon = \pm$

$$\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\ldots,N}(x;\vec{u}) = \widehat{\mathbf{R}}^{(\epsilon,+)}_{0;1^*_+,\ldots,N^*_+}(x;\vec{u})\mathbf{T}^{(\epsilon)}_0(x)\mathbf{R}^{(\epsilon,-)}_{0;1_-,\ldots,N_-}(x;\vec{u})$$
$$\widehat{\mathbf{R}}^{(\epsilon,+)}_{0;1^*_+,\ldots,N^*_+}(x;\vec{u}) = \widehat{\mathbf{R}}^{(\epsilon,+)}_{0,1^*_+}(x,u_1)\widehat{\mathbf{R}}^{(\epsilon,+)}_{0,2^*_+}(x,u_2)\ldots\widehat{\mathbf{R}}^{(\epsilon,+)}_{0,N^*_+}(x,u_N)$$
$$\mathbf{R}^{(\epsilon,-)}_{0;1_-,\ldots,N_-}(x;\vec{u}) = \mathbf{R}^{(\epsilon,-)}_{0,N_-}(x,u_N)\ldots\mathbf{R}^{(\epsilon,-)}_{0,2_-}(x,u_2)\mathbf{R}^{(\epsilon,-)}_{0,1_-}(x,u_1)$$

It turns out that the operators $\widehat{T}^i_k(x;\vec{u})$ and $\widehat{T}^{-i}_{-k}(x;\vec{u})$, where

$$\widehat{\mathbf{T}}^{(+)}_{0;1,\ldots,N}(x;\vec{u}) = \sum_{i,k=1}^{2}\mathbf{E}^k_i \otimes \widehat{T}^i_k(x;\vec{u})\,, \qquad \widehat{\mathbf{T}}^{(-)}_{0;1,\ldots,N}(x;\vec{u}) = \sum_{i,k=1}^{2}\mathbf{E}^{-k}_{-i} \otimes \widehat{T}^{-i}_{-k}(x;\vec{u})$$

form the RTT–algebra $\tilde{\mathcal{A}}_2$.
Let $\vec{v} = (v_1,\ldots,v_P)$ and $\vec{w} = (w_1,\ldots,w_Q)$ be ordered P– and Q–tuples of mutually different complex numbers,

$$\widehat{T}^2_1(\vec{v};\vec{u}) = \widehat{T}^2_1(v_1;\vec{u})\widehat{T}^2_1(v_2;\vec{u})\ldots\widehat{T}^2_1(v_P;\vec{u})\,,$$
$$\widehat{T}^{-1}_{-2}(\vec{w};\vec{u}) = \widehat{T}^{-1}_{-2}(w_1;\vec{u})\widehat{T}^{-1}_{-2}(w_2;\vec{u})\ldots\widehat{T}^{-1}_{-2}(w_Q;\vec{u})\,.$$

and consider the vector

$$\widehat{\mathbf{\Omega}} = \underbrace{\mathbf{f}^1\otimes\ldots\otimes\mathbf{f}^1}_{N\times}\otimes\underbrace{\mathbf{e}_{-1}\otimes\ldots\otimes\mathbf{e}_{-1}}_{N\times}\otimes\,\omega \in \widehat{\mathcal{W}}_0\,.$$

The main result of [1] is the statement that if for each $u_k \in \overline{u}$, $v_r \in \overline{v}$ and $w_s \in \overline{w}$ the Bethe conditions

$$\lambda_1(u_k)F(\overline{u}_k,u_k-1)F(\overline{u}_k,u_k)F(u_k,\overline{v})F(u_k-2,\overline{w}) =$$
$$= \lambda_{-1}(u_k)F(u_k+1,\overline{u}_k)F(u_k,\overline{u}_k)F(\overline{v},u_k+2)F(\overline{w},u_k)$$
$$\lambda_1(v_r)F(\overline{u},v_r-1)F(\overline{u},v_r)F(v_r,\overline{v}_r)F(v_r-2,\overline{w}) = \lambda_2(v_r)F(\overline{v}_r,v_r)F(\overline{w},v_r-2)$$
$$\lambda_{-1}(w_s)F(w_s+1,\overline{u})F(w_s,\overline{u})F(\overline{v},w_s+2)F(\overline{w}_s,w_s) = \lambda_{-2}(w_s)F(w_s+2,\overline{v})F(w_s,\overline{w}_s)\,.$$

are satisfied, the vector

$$\mathfrak{B}(\vec{u};\vec{v};\vec{w}) = \Big\langle \mathbf{B}_{1,\ldots,N}(\vec{u}), \widehat{T}^2_1(\vec{v};\vec{u})\widehat{T}^{-1}_{-2}(\vec{w};\vec{u})\widehat{\mathbf{\Omega}}\Big\rangle \qquad (3)$$

is a common eigenvector of the operators $H(x)$ with the eigenvalue

$$E(x;\vec{u},\overline{v},\overline{w}) = \lambda_1(x)F(\overline{u},x)F(\overline{u},x-1)F(x,\overline{v})F(x-2,\overline{w}) + \lambda_2(x)F(\overline{v},x)F(\overline{w},x-2)+$$
$$+\lambda_{-1}(x)F(x,\overline{u})F(x+1,\overline{u})F(\overline{v},x+2)F(\overline{w},x) + \lambda_{-2}(x)F(x+2,\overline{v})F(x,\overline{w})$$

4 Trace–formula for RTT–algebra of sp(4) type

The aim of this part is to write the Bethe vector (3) using the trace of the R-matrices and monodromy operators. For this purpose, we will introduce, similarly as in section 2.2,

$$\begin{aligned} \vec{u} &= (u_1, \ldots, u_N)\,, & \vec{A} &= (A_1, \ldots, A_N)\,, & \mathcal{V}^{\otimes \vec{A}} &= \mathcal{V}_{A_1} \otimes \ldots \otimes \mathcal{V}_{A_N} \\ \vec{v} &= (v_1, \ldots, v_P)\,, & \vec{B} &= (B_1, \ldots, B_P)\,, & \mathcal{V}^{\otimes \vec{B}} &= \mathcal{V}_{B_1} \otimes \ldots \otimes \mathcal{V}_{B_P} \\ \vec{w} &= (w_1, \ldots, w_Q)\,, & \vec{C} &= (C_1, \ldots, C_Q)\,, & \mathcal{V}^{\otimes \vec{C}} &= \mathcal{V}_{C_1} \otimes \ldots \otimes \mathcal{V}_{C_Q}\,, \end{aligned}$$

the vector spaces

$$\mathcal{V} = \mathcal{V}^{\otimes \vec{B}} \otimes \mathcal{V}^{\otimes \vec{C}} \otimes \mathcal{V}^{\otimes \vec{A}}$$

and denote

$$\begin{aligned} &\mathbf{T}_{\vec{A}}(\vec{u}) = \mathbf{T}_{A_1}(u_1) \ldots \mathbf{T}_{A_N}(u_N) \\ &\mathbf{T}_{\vec{B}}(\vec{v}) = \mathbf{T}_{B_1}(v_1) \ldots \mathbf{T}_{B_P}(v_P) \\ &\mathbf{T}_{\vec{C}}(\vec{w}) = \mathbf{T}_{C_1}(w_1) \ldots \mathbf{T}_{C_Q}(w_Q) \\ &\tilde{\mathbf{R}}_{\vec{B},\overleftarrow{A}}(\vec{v}, \vec{u}) = \tilde{\mathbf{R}}_{\vec{B},A_N}(\vec{v}, u_N) \ldots \tilde{\mathbf{R}}_{\vec{B},A_1}(\vec{v}, u_1) = \tilde{\mathbf{R}}_{B_1,\overleftarrow{A}}(v_1, \vec{u}) \ldots \tilde{\mathbf{R}}_{B_P,\overleftarrow{A}}(v_P, \vec{u}) \\ &\tilde{\mathbf{R}}_{\vec{C},\overleftarrow{A}}(\vec{w}, \vec{u}) = \tilde{\mathbf{R}}_{\vec{C},A_N}(\vec{w}, u_N) \ldots \tilde{\mathbf{R}}_{\vec{C},A_1}(\vec{w}, u_1) = \tilde{\mathbf{R}}_{C_1,\overleftarrow{A}}(w_1, \vec{u}) \ldots \tilde{\mathbf{R}}_{C_Q,\overleftarrow{A}}(w_Q, \vec{u}) \\ &\tilde{\mathbf{R}}_{\vec{B},A}(\vec{v}, u) = \tilde{\mathbf{R}}_{B_1,A}(v_1, u) \ldots \tilde{\mathbf{R}}_{B_P,A}(v_P, u) \\ &\tilde{\mathbf{R}}_{B,\overleftarrow{A}}(v, \vec{u}) = \tilde{\mathbf{R}}_{B,A_N}(v, u_N) \ldots \tilde{\mathbf{R}}_{B,A_1}(v, u_1) \\ &\tilde{\mathbf{R}}_{\vec{C},A}(\vec{w}, u) = \tilde{\mathbf{R}}_{C_1,A}(w_1, u) \ldots \tilde{\mathbf{R}}_{C_Q,A}(w_Q, u) \\ &\tilde{\mathbf{R}}_{C,\overleftarrow{A}}(w, \vec{u}) = \tilde{\mathbf{R}}_{C,A_N}(w, u_N) \ldots \tilde{\mathbf{R}}_{C,A_1}(w, u_1)\,. \end{aligned}$$

The main result of this paper is the following trace–formula:

Theorem. The Bethe vectors for the RTT–algebra of sp(4) type are

$$\begin{aligned} \mathfrak{B}(\vec{u}; \vec{v}; \vec{w}) = \mathrm{Tr}_{\vec{A},\vec{B},\vec{C}} \Big(&\mathbf{T}_{\vec{A}}(\vec{u}) \mathbf{T}_{\vec{B}}(\vec{v}) \mathbf{T}_{\vec{C}}(\vec{w}) \tilde{\mathbf{R}}_{\vec{B},\overleftarrow{A}}(\vec{v}, \vec{u}) \tilde{\mathbf{R}}_{\vec{C},\overleftarrow{A}}(\vec{w}, \vec{u}) \\ &\big((\mathbf{E}_1^2)^{\otimes \vec{B}} \otimes (\mathbf{E}_{-2}^{-1})^{\otimes \vec{C}} \otimes \big(\mathbf{E}_{-1}^1)^{\otimes \vec{A}}\big) \big(\tilde{\mathbf{R}}_{\vec{B},\overleftarrow{A}}(\vec{v}, \vec{u}) \tilde{\mathbf{R}}_{\vec{C},\overleftarrow{A}}(\vec{w}, \vec{u}) \big)^{-1} \Big) \omega\,. \end{aligned} \tag{4}$$

Proof. We do this by comparing relations (3) and (4).
If for $\epsilon = \pm$ we introduce

$$\begin{aligned} \widehat{\mathbf{R}}^{(\epsilon,+)}_{B;\overrightarrow{A}^*_+}(x; \vec{u}) &= \widehat{\mathbf{R}}^{(\epsilon,+)}_{B,A^*_{1_+}}(x, u_1) \ldots \widehat{\mathbf{R}}^{(\epsilon,+)}_{B,A^*_{N_+}}(x, u_N) \\ \mathbf{R}^{(\epsilon,-)}_{B;\overleftarrow{A}_-}(x; \vec{u}) &= \mathbf{R}^{(\epsilon,-)}_{B,A_{N_-}}(x, u_N) \ldots \mathbf{R}^{(\epsilon,-)}_{B,A_{1_-}}(x, u_1) \end{aligned},$$

it is possible to write

$$\begin{aligned} \widehat{T}^2_1(x; \vec{u}) &= \mathrm{Tr}_B \Big(\widehat{\mathbf{R}}^{(+,+)}_{B;\overrightarrow{A}^*_+}(x; \vec{u}) \mathbf{T}^{(+)}_B(x) \mathbf{R}^{(+,-)}_{B;\overleftarrow{A}_-}(x; \vec{u}) \big(\mathbf{E}^2_1 \otimes \mathbf{I}^*_{\vec{A}_+} \otimes \mathbf{I}_{\vec{A}_-} \big) \Big) \\ \widehat{T}^{-1}_{-2}(x; \vec{u}) &= \mathrm{Tr}_C \Big(\widehat{\mathbf{R}}^{(-,+)}_{C;\overrightarrow{A}^*_+}(x; \vec{u}) \mathbf{T}^{(-)}_C(x) \mathbf{R}^{(-,-)}_{C;\overleftarrow{A}_-}(x; \vec{u}) \big(\mathbf{E}^{-1}_{-2} \otimes \mathbf{I}^*_{\vec{A}_+} \otimes \mathbf{I}_{\vec{A}_-} \big) \Big) \end{aligned}$$

Then we have

$$
\begin{aligned}
&\widehat{T}_1^2(\vec{v};\vec{u})\widehat{T}_{-2}^{-1}(\vec{w};\vec{u}) = \\
&\quad = \mathrm{Tr}_{\vec{B},\vec{C}}\Big(\widehat{\mathbf{R}}^{(+,+)}_{B_1;\overrightarrow{A}^*_+}(v_1;\vec{u})\mathbf{T}^{(+)}_{B_1}(v_1)\mathbf{R}^{(+,-)}_{B_1;\overleftarrow{A}_-}(v_1;\vec{u})\dots \\
&\qquad\qquad \dots\widehat{\mathbf{R}}^{(+,+)}_{B_P;\overrightarrow{A}^*_+}(v_P;\vec{u})\mathbf{T}^{(+)}_{B_P}(v_P)\mathbf{R}^{(+,-)}_{B_P;\overleftarrow{A}_-}(v_P;\vec{u}) \\
&\qquad \widehat{\mathbf{R}}^{(-,+)}_{C_1;\overrightarrow{A}^*_+}(w_1;\vec{u})\mathbf{T}^{(-)}_{C_1}(w_1)\mathbf{R}^{(-,-)}_{C_1;\overleftarrow{A}_-}(w_1;\vec{u})\dots \\
&\qquad\qquad \dots\widehat{\mathbf{R}}^{(-,+)}_{C_Q;\overrightarrow{A}^*_+}(w_Q;\vec{u})\mathbf{T}^{(-)}_{C_Q}(w_Q)\mathbf{R}^{(-,-)}_{C_Q;\overleftarrow{A}_-}(w_Q;\vec{u}) \\
&\qquad\qquad \big((\mathbf{E}_1^2)^{\otimes\vec{B}}\otimes(\mathbf{E}_{-2}^{-1})^{\otimes\vec{C}}\otimes\mathbf{I}^*_{\vec{A}_+}\otimes\mathbf{I}_{\vec{A}_-}\big)\Big) = \\
&\quad = \mathrm{Tr}_{\vec{B},\vec{C}}\Big(\widehat{\mathbf{R}}^{(+,+)}_{\vec{B},A^*_{1_+}}(\vec{v},u_1)\widehat{\mathbf{R}}^{(-,+)}_{\vec{C},A^*_{1_+}}(\vec{w},u_1)\dots\widehat{\mathbf{R}}^{(+,+)}_{\vec{B},A_N}(\vec{v},u_N)\widehat{\mathbf{R}}^{(-,+)}_{\vec{C},A_N}(\vec{w},u_N) \\
&\qquad\qquad\qquad \mathbf{T}^{(+)}_{\vec{B}}(\vec{v})\mathbf{T}^{(-)}_{\vec{C}}(\vec{w}) \\
&\qquad \mathbf{R}^{(+,-)}_{\vec{B},A_{N_-}}(\vec{v},u_N)\mathbf{R}^{(-,-)}_{\vec{C},A_{N_-}}(\vec{w},u_N)\dots\mathbf{R}^{(+,-)}_{\vec{B},A_{1_-}}(\vec{v},u_1)\mathbf{R}^{(-,-)}_{\vec{C},A_{1_-}}(\vec{w},u_1) \\
&\qquad\qquad \big((\mathbf{E}_1^2)^{\otimes\vec{B}}\otimes(\mathbf{E}_{-2}^{-1})^{\otimes\vec{C}}\otimes\mathbf{I}^*_{\vec{A}_+}\otimes\mathbf{I}_{\vec{A}_-}\big)\Big) \\
&\widehat{T}_1^2(\vec{v};\vec{u})\widehat{T}_{-2}^{-1}(\vec{w};\vec{u})\widehat{\mathbf{\Omega}} = \\
&\quad = \mathrm{Tr}_{\vec{B},\vec{C}}\Big(\widehat{\mathbf{R}}^{(+,+)}_{\vec{B},A^*_{1_+}}(\vec{v},u_1)\widehat{\mathbf{R}}^{(-,+)}_{\vec{C},A^*_{1_+}}(\vec{w},u_1)\mathbf{f}^1\dots\widehat{\mathbf{R}}^{(+,+)}_{\vec{B},A_N}(\vec{v},u_N)\widehat{\mathbf{R}}^{(-,+)}_{\vec{C},A_N}(\vec{w},u_N)\mathbf{f}^1 \\
&\qquad\qquad\qquad \mathbf{T}^{(+)}_{\vec{B}}(\vec{v})\mathbf{T}^{(-)}_{\vec{C}}(\vec{w})\omega \\
&\qquad \mathbf{R}^{(+,-)}_{\vec{B},A_{N_-}}(\vec{v},u_N)\mathbf{R}^{(-,-)}_{\vec{C},A_{N_-}}(\vec{w},u_N)\mathbf{e}_{-1}\dots\mathbf{R}^{(+,-)}_{\vec{B},A_{1_-}}(\vec{v},u_1)\mathbf{R}^{(-,-)}_{\vec{C},A_{1_-}}(\vec{w},u_1)\mathbf{e}_{-1} \\
&\qquad\qquad \big((\mathbf{E}_1^2)^{\otimes\vec{B}}\otimes(\mathbf{E}_{-2}^{-1})^{\otimes\vec{C}}\Big)
\end{aligned}
$$

The Bethe vector can then be written in the form

$$
\begin{aligned}
\mathfrak{B}(\vec{u};\vec{v};\vec{w}) = &\sum_{i_1,\dots,i_N,k_1,\dots,k_N=1}^{2}\mathrm{Tr}_{\vec{B},\vec{C}}\Big(\big\langle\mathbf{e}_{i_1},\widehat{\mathbf{R}}^{(+,+)}_{\vec{B},A^*_{1_+}}(\vec{v},u_1)\widehat{\mathbf{R}}^{(-,+)}_{\vec{C},A^*_{1_+}}(\vec{w},u_1)\mathbf{f}^1\big\rangle\dots \\
&\qquad \dots\big\langle\mathbf{e}_{i_N},\widehat{\mathbf{R}}^{(+,+)}_{\vec{B},A_N}(\vec{v},u_N)\widehat{\mathbf{R}}^{(-,+)}_{\vec{C},A_N}(\vec{w},u_N)\mathbf{f}^1\big\rangle \\
& T^{i_1}_{-k_1}(u_1)\dots T^{i_N}_{-k_N}(u_N)\mathbf{T}^{(+)}_{\vec{B}}(\vec{v})\mathbf{T}^{(-)}_{\vec{C}}(\vec{w})\omega \\
& \big\langle\mathbf{f}^{-k_N},\mathbf{R}^{(+,-)}_{\vec{B},A_{N_-}}(\vec{v},u_N)\mathbf{R}^{(-,-)}_{\vec{C},A_{N_-}}(\vec{w},u_N)\mathbf{e}_{-1}\big\rangle\dots \\
&\qquad \dots\big\langle\mathbf{f}^{-k_1},\mathbf{R}^{(+,-)}_{\vec{B},A_{1_-}}(\vec{v},u_1)\mathbf{R}^{(-,-)}_{\vec{C},A_{1_-}}(\vec{w},u_1)\mathbf{e}_{-1}\big\rangle\big((\mathbf{E}_1^2)^{\otimes\vec{B}}\otimes(\mathbf{E}_{-2}^{-1})^{\otimes\vec{C}}\Big).
\end{aligned}
$$

Using the relations

$$\left\langle \mathbf{e}_i, \widehat{\mathbf{R}}^{(+,+)}_{\vec{B},A^*_+}(\vec{v},u)\widehat{\mathbf{R}}^{(-,+)}_{\vec{C},A^*_+}(\vec{w},u)\mathbf{f}^1 \right\rangle =$$

$$= \sum_{r_1,\dots,r_P,s_1,\dots,s_P=1}^{2} \sum_{p_1,\dots,p_Q,q_1,\dots,q_Q=1}^{2} \sum_{\alpha_1,\dots,\alpha_P=1}^{2} \sum_{\beta_1,\dots,\beta_{Q-1}=1}^{2}$$

$$\tilde{S}^{r_1,\alpha_1}_{s_1,i}(v_1,u)\tilde{S}^{r_2,\alpha_2}_{s_2,\alpha_1}(v_2,u)\tilde{S}^{r_3,\alpha_3}_{s_3,\alpha_2}(v_3,u)\dots$$

$$\dots\tilde{S}^{r_{P-2},\alpha_{P-2}}_{s_{P-2},\alpha_{P-3}}(v_{P-2},u)\tilde{S}^{r_{P-1},\alpha_{P-1}}_{s_{P-1},\alpha_{P-2}}(v_{P-1},u)\tilde{S}^{r_P,\alpha_P}_{s_P,\alpha_{P-1}}(v_P,u)$$

$$\tilde{S}^{-p_1,\beta_1}_{-q_1,\alpha_P}(w_1,u)\tilde{S}^{-p_2,\beta_2}_{-q_2,\beta_1}(w_2,u)\tilde{S}^{-p_3,\beta_3}_{-q_3,\beta_2}(w_3,u)\dots$$

$$\dots\tilde{S}^{-p_{Q-2},\beta_{Q-2}}_{-q_{Q-2},\beta_{Q-3}}(w_{Q-2},u)\tilde{S}^{-p_{Q-1},\beta_{Q-1}}_{-q_{Q-1},\beta_{Q-2}}(w_{Q-1},u)\tilde{S}^{-p_Q,1}_{-q_Q,\beta_{Q-1}}(w_Q,u)$$

$$\mathbf{E}^{s_1}_{r_1}\otimes\mathbf{E}^{s_2}_{r_2}\otimes\mathbf{E}^{s_3}_{r_3}\otimes\dots\otimes\mathbf{E}^{s_{P-2}}_{r_{P-2}}\otimes\mathbf{E}^{s_{P-1}}_{r_{P-1}}\otimes\mathbf{E}^{s_P}_{r_P}\otimes$$

$$\otimes\mathbf{E}^{-q_1}_{-p_1}\otimes\mathbf{E}^{-q_2}_{-p_2}\otimes\mathbf{E}^{-q_3}_{-p_3}\otimes\dots\otimes\mathbf{E}^{-q_{Q-2}}_{-p_{Q-2}}\otimes\mathbf{E}^{-q_{Q-1}}_{-p_{Q-1}}\otimes\mathbf{E}^{-q_Q}_{-p_Q}$$

$$\left\langle \mathbf{f}^{-k}, \mathbf{R}^{(+,-)}_{\vec{B},A_-}(\vec{v},u)\mathbf{R}^{(-,-)}_{\vec{C},A_-}(\vec{w},u)\mathbf{e}_{-1} \right\rangle =$$

$$= \sum_{r_1,\dots,r_P,s_1,\dots,s_P=1}^{2} \sum_{a_1,\dots,a_Q,b_1,\dots,b_Q=1}^{2} \sum_{\gamma_1,\dots,\gamma_P=1}^{2} \sum_{\delta_1,\dots,\delta_{Q-1}=1}^{2}$$

$$\tilde{R}^{r_1,-k}_{s_1,-\gamma_1}(v_1,u)\tilde{R}^{r_2,-\gamma_1}_{s_2,-\gamma_2}(v_2,u)\tilde{R}^{r_3,-\gamma_2}_{s_3,-\gamma_3}(v_3,u)\dots$$

$$\dots\tilde{R}^{r_{P-2},-\gamma_{P-3}}_{s_{P-2},-\gamma_{P-2}}(v_{P-2},u)\tilde{R}^{r_{P-1},-\gamma_{P-2}}_{s_{P-1},-\gamma_{P-1}}(v_{P-1},u)\tilde{R}^{r_P,-\gamma_{P-1}}_{s_P,-\gamma_P}(v_P,u)$$

$$\tilde{R}^{-a_1,-\gamma_P}_{-b_1,-\delta_1}(w_1,u)\tilde{R}^{-a_2,-\delta_1}_{-b_2,-\delta_2}(w_2,u)\tilde{R}^{-a_3,-\delta_2}_{-b_3,-\delta_3}(w_3,u)\dots$$

$$\dots\tilde{R}^{-a_{Q-2},-\delta_{Q-3}}_{-b_{Q-2},-\delta_{Q-2}}(w_{Q-2},u)\tilde{R}^{-a_{Q-1},-\delta_{Q-2}}_{-b_{Q-1},-\delta_{Q-1}}(w_{Q-1},u)\tilde{R}^{-a_Q,-\delta_{Q-1}}_{-b_Q,-1}(w_Q,u)$$

$$\mathbf{E}^{s_1}_{r_1}\otimes\mathbf{E}^{s_2}_{r_2}\otimes\mathbf{E}^{s_3}_{r_3}\otimes\dots\otimes\mathbf{E}^{s_{P-2}}_{r_{P-2}}\otimes\mathbf{E}^{s_{P-1}}_{r_{P-1}}\otimes\mathbf{E}^{s_P}_{r_P}\otimes$$

$$\mathbf{E}^{-b_1}_{-a_1}\otimes\mathbf{E}^{-b_2}_{-a_2}\otimes\mathbf{E}^{-b_3}_{-a_3}\otimes\dots\otimes\mathbf{E}^{-b_{Q-2}}_{-a_{Q-2}}\otimes\mathbf{E}^{-b_{Q-1}}_{-a_{Q-1}}\otimes\mathbf{E}^{-b_Q}_{-a_Q}$$

it is easy to verify that

$$\sum_{i,k=1}^{2}\left\langle \mathbf{e}_i, \widehat{\mathbf{R}}^{(+,+)}_{\vec{B},A^*_+}(\vec{v},u)\widehat{\mathbf{R}}^{(-,+)}_{\vec{C},A^*_+}(\vec{w},u)\mathbf{f}^1 \right\rangle T^i_{-k}(u)\left\langle \mathbf{f}^{-k}, \mathbf{R}^{(+,-)}_{\vec{B},A_-}(\vec{v},u)\mathbf{R}^{(-,-)}_{\vec{C},A_-}(\vec{w},u)\mathbf{e}_{-1} \right\rangle =$$

$$= \mathrm{Tr}_A\Big(\left(\mathbf{R}^{(-,+)}_{C_Q,A}(w_Q,u)\right)^{-1}\dots\left(\mathbf{R}^{(-,+)}_{C_1,A}(w_1,u)\right)^{-1}\left(\mathbf{R}^{(+,+)}_{B_P,A}(v_p,u)\right)^{-1}\dots\left(\mathbf{R}^{(+,+)}_{B_1,A}(v_1,u)\right)^{-1}$$

$$\mathbf{T}_A(u)\mathbf{R}^{(+,-)}_{B_1,A}(v_1,u)\dots\mathbf{R}^{(+,-)}_{B_P,A}(v_P,u)\mathbf{R}^{(-,-)}_{C_1,A}(w_1,u)\dots\mathbf{R}^{(-,-)}_{C_Q,A}(w_Q,u)$$

$$\left(\mathbf{I}_{\vec{B}}\otimes\mathbf{I}_{\vec{C}}\otimes\mathbf{E}^1_{-1}\right)\Big)$$

hold. These considerations lead to the fact that the Bethe vector can be written as

$$
\begin{aligned}
\mathfrak{B}(\vec{u};\vec{v};\vec{w}) = \mathrm{Tr}_{\vec{A},\vec{B};\vec{C}}\Big(&\big(\mathbf{R}^{(-,+)}_{C_Q,A_1}(w_Q,u_1)\big)^{-1}\dots\big(\mathbf{R}^{(-,+)}_{C_1,A_1}(w_1,u_1)\big)^{-1}\\
&\big(\mathbf{R}^{(-,+)}_{C_Q,A_2}(w_Q,u_2)\big)^{-1}\dots\big(\mathbf{R}^{(-,+)}_{C_1,A_2}(w_1,u_2)\big)^{-1}\dots\\
&\dots\big(\mathbf{R}^{(-,+)}_{C_Q,A_N}(w_Q,u_N)\big)^{-1}\dots\big(\mathbf{R}^{(-,+)}_{C_1,A_N}(w_1,u_N)\big)^{-1}\\
&\big(\mathbf{R}^{(+,+)}_{B_P,A_1}(v_P,u_1)\big)^{-1}\dots\big(\mathbf{R}^{(+,+)}_{B_1,A_1}(v_1,u_1)\big)^{-1}\\
&\big(\mathbf{R}^{(+,+)}_{B_P,A_2}(v_P,u_2)\big)^{-1}\dots\big(\mathbf{R}^{(+,+)}_{B_1,A_2}(v_1,u_2)\big)^{-1}\dots\\
&\dots\big(\mathbf{R}^{(+,+)}_{B_P,A_N}(v_P,u_N)\big)^{-1}\dots\big(\mathbf{R}^{(+,+)}_{B_1,A_N}(v_1,u_N)\big)^{-1}\mathbf{T}_{\vec{A}}(\vec{u})\mathbf{T}^{(+)}_{\vec{B}}(\vec{v})\mathbf{T}^{(-)}_{\vec{C}}(\vec{w})\\
&\mathbf{R}^{(+,-)}_{\overrightarrow{B},\overleftarrow{A}}(\vec{v},\vec{u})\mathbf{R}^{(-,-)}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\big((\mathbf{E}^2_1)^{\otimes\vec{B}}\otimes(\mathbf{E}^{-1}_{-2})^{\otimes\vec{C}}\otimes(\mathbf{E}^1_{-1})^{\otimes\vec{A}}\big)\Big) =\\
= \mathrm{Tr}_{\vec{A},\vec{B},\vec{C}}\Big(&\mathbf{T}_{\vec{A}}(\vec{u})\mathbf{T}^{(+)}_{\vec{B}}(\vec{v})\mathbf{T}^{(-)}_{\vec{C}}(\vec{w})\mathbf{R}^{(+,-)}_{\overrightarrow{B},\overleftarrow{A}}(\vec{v},\vec{u})\mathbf{R}^{(-,-)}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\\
&\big((\mathbf{E}^2_1)^{\otimes\vec{B}}\otimes(\mathbf{E}^{-1}_{-2})^{\otimes\vec{C}}\otimes(\mathbf{E}^1_{-1})^{\otimes\vec{A}}\big)\big(\mathbf{R}^{(+,+)}_{\overrightarrow{B},\overleftarrow{A}}(\vec{v},\vec{u})\mathbf{R}^{(-,+)}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\big)^{-1}\Big)
\end{aligned}
$$

To prove Theorem, it is sufficient to show that in this expression we can use the monodromy operators $\mathbf{T}(x)$ and the R-matrices $\tilde{\mathbf{R}}(x,y)$ instead of the monodromy operators $\mathbf{T}^{(\pm)}(x)$ and the R–matrices $\mathbf{R}^{(\pm,\pm)}(x,y)$.
If the indices i and k or r and s have different signs, the compoments of the R–matrix $\tilde{R}^{i,r}_{k,s}(w,u)=\tilde{S}^{i,r}_{k,s}(w,u)=0$. Therefore, for $i,\ k,\ r,\ s=1,\ 2$ the relation

$$
\begin{aligned}
\tilde{\mathbf{R}}_{C,A}(w,u)\big(\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^{r}_{-s}\big)\big(\tilde{\mathbf{R}}_{C,A}(w,u)\big)^{-1} &= \sum_{a,b,c,d=1}^{2}\tilde{R}^{-b,-d}_{-k,-s}(w,u)\tilde{S}^{-i,r}_{-a,c}(w,u)\mathbf{E}^{-a}_{-b}\otimes\mathbf{E}^{c}_{-d} =\\
&= \mathbf{R}^{(-,-)}_{C,A}(w,u)\big(\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^{r}_{-s}\big)\big(\mathbf{R}^{(-,+)}_{C,A}(w,u)\big)^{-1}
\end{aligned}
$$

$$
\mathrm{Tr}_C\big(\mathbf{T}_C(w)\mathbf{E}^{-i}_{-k}\big)=T^{-i}_{-k}(w)=\mathrm{Tr}_C\big(\mathbf{T}^{(-)}_C(w)\mathbf{E}^{-i}_{-k}\big)
$$

holds. Hence, the equality

$$
\begin{aligned}
\mathrm{Tr}_{\vec{C}}\Big(\mathbf{T}^{(-)}_{\vec{C}}(\vec{w})\mathbf{R}^{(-,-)}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\big((\mathbf{E}^{-1}_{-2})^{\otimes\vec{C}}\otimes(\mathbf{E}^1_{-1})^{\otimes\vec{A}}\big)\big(\mathbf{R}^{(-,+)}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\big)^{-1}\Big) &=\\
= \mathrm{Tr}_{\vec{C}}\Big(\tilde{\mathbf{R}}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\big((\mathbf{E}^{-1}_{-2})^{\otimes\vec{C}}\otimes(\mathbf{E}^1_{-1})^{\otimes\vec{A}}\big)\big(\tilde{\mathbf{R}}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\big)^{-1}\Big).&
\end{aligned}
$$

holds.
Similarly, for $i,\ k,\ r,\ s=1,\ 2$

$$
\begin{aligned}
\tilde{\mathbf{R}}_{B,A}(v,u)\big(\mathbf{E}^{i}_{k}\otimes\mathbf{E}^{r}_{-s}\big)\big(\tilde{\mathbf{R}}_{B,A}(v,u)\big)^{-1} &= \sum_{a,b,c,d=1}^{2}\tilde{R}^{b,-d}_{k,-s}(v,u)\tilde{S}^{i,r}_{a,c}(v,u)\mathbf{E}^{a}_{b}\otimes\mathbf{E}^{c}_{-d} =\\
&= \mathbf{R}^{(+,-)}_{B,A}(v,u)\big(\mathbf{E}^{i}_{k}\otimes\mathbf{E}^{r}_{-s}\big)\big(\mathbf{R}^{(+,+)}_{B,A}(v,u)\big)^{-1}
\end{aligned}
$$

$$
\mathrm{Tr}_B\big(\mathbf{T}_B(v)\mathbf{E}^{i}_{k}\big)=T^{i}_{k}(v)=\mathrm{Tr}_B\big(\mathbf{T}^{(+)}_B(v)\mathbf{E}^{i}_{k}\big).
$$

hold, and therefore,

$$
\begin{aligned}
\mathfrak{B}(\vec{u};\vec{v};\vec{w}) = \mathrm{Tr}_{\vec{A},\vec{B},\vec{C}}\Big(&\mathbf{T}_{\vec{A}}(\vec{u})\mathbf{T}^{(+)}_{\vec{B}}(\vec{v})\mathbf{T}^{(-)}_{\vec{C}}(\vec{w})\mathbf{R}^{(+,-)}_{\overrightarrow{B},\overleftarrow{A}}(\vec{v},\vec{u})\mathbf{R}^{(-,-)}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\\
&\big((\mathbf{E}^2_1)^{\otimes\vec{B}}\otimes(\mathbf{E}^{-1}_{-2})^{\otimes\vec{C}}\otimes(\mathbf{E}^1_{-1})^{\otimes\vec{A}}\big)\big(\mathbf{R}^{(+,+)}_{\overrightarrow{B},\overleftarrow{A}}(\vec{v},\vec{u})\mathbf{R}^{(-,+)}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\big)^{-1}\Big) =\\
= \mathrm{Tr}_{\vec{A},\vec{B},\vec{C}}\Big(&\mathbf{T}_{\vec{A}}(\vec{u})\mathbf{T}_{\vec{B}}(\vec{v})\mathbf{T}_{\vec{C}}(\vec{w})\tilde{\mathbf{R}}_{\overrightarrow{B},\overleftarrow{A}}(\vec{v},\vec{u})\tilde{\mathbf{R}}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\\
&\big((\mathbf{E}^2_1)^{\otimes\vec{B}}\otimes(\mathbf{E}^{-1}_{-2})^{\otimes\vec{C}}\otimes(\mathbf{E}^1_{-1})^{\otimes\vec{A}}\big)\big(\tilde{\mathbf{R}}_{\overrightarrow{C},\overleftarrow{A}}(\vec{w},\vec{u})\big)^{-1}\big(\tilde{\mathbf{R}}_{\overrightarrow{B},\overleftarrow{A}}(\vec{v},\vec{u})\big)^{-1}\Big),
\end{aligned}
$$

is true, which proves the statements of the Theorem.

5 Conclusion

In this paper, we have found a trace–formula for the Bethe vectors of RTT–algebra of the sp(4) type. Unlike the trace–formula for the RTT–algebra of gl(3) type, the Bethe vectors are not expressed using the R–matrices of the RTT–algebra of sp(4) type but rather the R–matrices of the auxiliary algebra $\tilde{\mathcal{A}}_2$. Our preliminary results show that a similar trace–formula can be derived for RTT-algebras of the sp($2n$) and so($2n$) types. We plan to publish trace-formulas for these RTT–algebras in the near future.

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