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Generalized Four-Dimensional Effective Hadronic Supersymmetry based on QCD: New Results

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Abstract. New results are shown on the generalization of phenomenological hadronic supersymmetry based on QCD as an extension of spin-flavor $SU(6)$ symmetry. A more general supersymmetric extension of the group $SU(6/21)$ will be given with applications to pentaquark and tetraquark states for which there is a great deal of ongoing research.

1. Introduction

It was previously shown [1, 2] that the algebra of octonions neatly represented hadronic supersymmetry based on the extension of the spin-flavor group $SU(6)$ to the supergroup $SU(6/21)$. In considering a rotationally excited baryon (qqq), the most energetically stable configuration occurs when two of the quarks come together at a point to form a diquark $(D = qq)$ on one end of a spherical bag structure leaving the remaining quark (q) on the opposite end. Therefore, the baryonic system (qqq) can be effectively likened to a mesonic one $(q\bar{q})$ in which we identify the diquark (D) of the baryon with the antiquark $(\bar{q})$ of the meson. This symmetry between baryons and mesons served as a stepping stone into closely examining the Regge trajectories of various hadronic states. Upon doing so, one readily discovered a parallelism among all the Regge trajectories thereby extracting a universal slope parameter. The parallelism of Regge trajectories established a deep connection between fermionic and bosonic hadronic states. In the context of the work done by Catto and Gürsey [3, 4], baryonic states are fermionic and map into mesonic ones which are bosonic under the transformation of $D = qq \to \bar{q}$, or alternatively, $\bar{D} = \bar{q}\bar{q} \to q$. These symmetries were further bolstered by the use of the split octonionic algebra [5, 6]. In light of this symmetry formalism, mass calculations were carried out [7, 8] concerning low-lying baryonic states. The predicted masses obtained fell to within 5% accuracy of experimental observations. Furthermore, the predicted relationship between the mass differences among mesons and the mass differences among baryons was found to be in excellent agreement with experimental results. For example, the difference in π and ρ masses as it relates to the difference in the masses of N and Δ were within less than 1% accuracy.

The algebra of octonions is the only Heisenberg-like algebra in modern mathematics to incorporate color degrees of freedom that underlie hadronic phenomena. The use of octonions in hadronic physics gave rise to dynamical supersymmetry and provided the first description of existing quark models through the edifice of exceptional Lie algebras. Supergroups and infinite groups, like those generated by the Virasoro algebra, emerged as useful descriptions of certain properties regarding hadronic spectra. The exceptional Lie groups are G_2, F_4, E_6, E_7 and E_8.

Are supersymmetry and its accompanying algebraic structures effective properties of more complex structures? Are they to be found in an exact form only through an artificial extension of physical Hilbert spaces, or, are they really fundamental in terms of a new extension of quantum field theory? We may not be able to provide answers to such questions but our hope is to be able to shed light on these problems and define them more precisely.

The outstanding difficulty in applying supersymmetry to hadrons was that supersymmetry is badly broken, otherwise the proton and the pion would have the same mass. The arrival of broken supersymmetry stems from the underlying difference between quark and antidiquark masses, differences in spin, and also, variations in size.

It should be noted that the spin-dependence of forces in QCD plays an important role. This effect was minimized by performing an appropriate averaging over spin. The main assumption here was that the spin-dependent interaction energy between two quarks in a diquark is independent of the nature of the hadron in which the diquark is embedded. The spin-dependent contribution to the interaction energy was approximately extracted from the known values of baryon masses.

Presently in literature there exists no conclusive papers on the sizes of quarks or diquarks. Recently [9, 10] we have constructed octonionic geometries that tie together, in an algebraic way, many well-known and apparently disconnected projective geometries [11]. Through such octonionic geometries, there appears to be novel ways of relating diquark and antiquark sizes within a bag structure of a $q-D$ system. We present relevant theoretical work and await experimental results from LHC, SLAC and other labs for potential confirmations. In addition, herein we develop a broad scheme for calculating hadronic masses based on an extension of $SU(6/21)$.

2. Bilocal approximation to hadronic structure and inclusion of color.

It is known that the low-lying baryons occur in the symmetric 56 representation [12] of $SU(6)$, whereas the Pauli principle would have led to the antisymmetrical 20 representation. This fact was a crucial fact for the introduction of color degree of freedom [13] based on $SU(3)^c$. Hence, to obtain the correct representation of the $q-D$ system we must also include color degrees of freedom for the constituents. Since the quark field transforms like a color triplet and the diquark like a color antitriplet under $SU(3)^c$, the color degrees of freedom of the constituents must be included correctly in order to obtain a correct representation of the q-D system. Hadronic states must be color singlets. These are represented by bilocal operators $O(\mathbf{r}_1, \mathbf{r}_2)$ in the bilocal approximation (for (qD) system see [14]) that gives $\bar{q}(1)q(2)$ for mesons and $D(1)q(2)$ for baryons. Here $\bar{q}(1)$ represents the antiquark situated at $\mathbf{r}_1$, $q(2)$ the quark situated at $\mathbf{r}_2$, and $D(1) = q(1)q(1)$ the diquark situated at $\mathbf{r}_1$. If we denote the c.m. and the relative coordinates of the constituents by $\mathbf{R}$ and $\mathbf{r}$, where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and

$$\mathbf{R} = \frac{(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)}{(m_1 + m_2)} \tag{1}$$

with m_1 and m_2 being their masses, we can then write $O(\mathbf{R}, \mathbf{r})$ for the operator that creates hadrons out of the vacuum. The matrix element of this operator between the vacuum and the hadronic state h will be of the form

$$< h|O(\mathbf{R}, \mathbf{r})|0 >= \chi(\mathbf{R})\psi(\mathbf{r}) \tag{2}$$

where $\chi(\mathbf{R})$ is the free wave function of the hadron as a function of the c.m. coordinate and $\psi(\mathbf{r})$ is the bound-state solution of the $U(6/21)$ invariant Hamiltonian describing the $q-\bar{q}$ mesons, $q-D$ baryons, $\bar{q}-\bar{D}$ antibaryons and $D-\bar{D}$ exotic mesons, given by

$$\begin{aligned} i\partial_t\psi_{\alpha\beta} &= [\sqrt{(m_\alpha+\frac{1}{2}V_s)^2+\mathbf{p}^2}+\sqrt{(m_\beta+\frac{1}{2}V_s)^2+\mathbf{p}^2} \\ &\quad -\frac{4}{3}\frac{\alpha_s}{r}+k\frac{\mathbf{s}_\alpha\cdot\mathbf{s}_\beta}{m_\alpha m_\beta}]\ \psi_{\alpha\beta} \end{aligned} \tag{3}$$

Here $\mathbf{p} = -i\nabla$ in the c.m. system and m and $\mathbf{s}$ denote the masses and spins of the constituents, α_s the strong-coupling constant, $V_s = br$ is the scalar potential with r being the distance between the constituents in the bilocal object, and $k = |\psi(0)|^2$.

The operator product expansion [15] will give a singular part depending only on $\mathbf{r}$ and proportional to the propagator of the field binding the two constituents. There will be a finite number of singular coefficients $c_n(\mathbf{r})$ depending on the dimensionality of the constituent fields. For example, for a meson, the singular term is proportional to the propagator of the gluon field binding the two constituents. Once we subtract the singular part, the remaining part $\tilde{O}(\mathbf{R}, \mathbf{r})$ is analytic in r and thus we can write

$$\tilde{O}(\mathbf{R}, \mathbf{r}) = O_0(\mathbf{R}) + \mathbf{r}\cdot\mathbf{O}_1(\mathbf{R}) + o(r^2), \tag{4}$$

where the remainder term is of the order r^2.

Now $O_0(\mathbf{R})$ creates a hadron at its c.m. point $\mathbf{R}$ equivalent to a $\ell = 0$, s-state of the two constituents. For a baryon this is a state associated with q and $D \sim qq$ at the same point $\mathbf{R}$, hence it is essentially a 3-quark state when the three quarks are at a common location. The $\mathbf{O}_1(\mathbf{R})$ can create three $\ell = 1$ states with opposite parity to the state created by $O_0(\mathbf{R})$. Hence, if we denote the nonsingular parts of $\bar{q}(1)q(2)$ and $D(1)q(2)$ by $[\bar{q}(1)q(2)]$ and $[D(1)q(2)]$, respectively, we have

$$[\bar{q}(1)q(2)]|0 >= |M(\mathbf{R}) > +\mathbf{r}\cdot|\mathbf{M}^{'}(\mathbf{R}) > +o(r^2), \tag{5}$$

$$[D(1)q(2)]|0 >= |B(\mathbf{R}) > +\mathbf{r}\cdot|\mathbf{B}^{'}(\mathbf{R}) > +o(r^2), \tag{6}$$

and similarly for the exotic meson states $D(1)\bar{D}(2)$.

The meson states $|M(\mathbf{R})$ being $\ell = 0$ bound state of a quark and antiquark will correspond to the singlet and 35-dimensional representations of $SU(6)$. The $M^{'}(\mathbf{R})$ is an orbital excitation ($\ell = 1$) of opposite parity, which are in the $(35+1, 3)$ representation of the group $SU(6)\times O(3)$, $O(3)$ being associated with the relative angular momentum of the constituents. The $M^{'}$ states contain mesons like B, $\mathbf{A}_1$, $\mathbf{A}_2$, and scalar particles. On the whole, the $\ell = 0$ and $\ell = 1$ part $\bar{q}(1)q(2)$ contain $4\times(35+1) = 144$ meson states.

Switching to the baryon states, the requirement of antisymmetry in color, and symmetry in spin-flavor indices gives the $(56)^+$ representation for $B(\mathbf{R})$. The $\ell = 1$ multiplets have negative parity and have mixed spin-flavor symmetry. They belong to the representation $(70^-, 3)$ of $SU(6)\times O(3)$ and are represented by the states $|\mathbf{B}^{'}(\mathbf{R}) >$ which are 210 in number. On the whole, these 266 states account for all the observed low-lying baryon states obtained form $56 + 3\times 70 = 266$. A similar analysis can be carried out for the exotic meson states $D(1)\bar{D}(2)$, where the diquark and the antidiquark can be bound in a $\ell = 0$ or $\ell = 1$ state with opposite parities.

3. Algebraic treatment of color through Split Octonions.

The octonions have seven imaginary units e_α ($\alpha = 1, \ldots, 7$) which obey the algebra

$$e_\alpha e_\beta = -\delta_{\alpha\beta} + \epsilon_{\alpha\beta\gamma} e_\gamma \tag{7}$$

Here the structure constants $\epsilon_{\alpha\beta\gamma}$ are antisymmetrical in all three indices. They are equal to one for $(\alpha\beta\gamma)$ being of the seven triplets (123), (246), (435), (367), (651), (572) and (714). They vanish otherwise.

The behavior of various states under the color group are best seen if we use split octonion units defined by

$$u_0 = \frac{1}{2}(1 + ie_7), \qquad u_0^* = \frac{1}{2}(1 - ie_7), \tag{8}$$

$$u_j = \frac{1}{2}(e_j + ie_{j+3}), \quad u_j^* = \frac{1}{2}(e_j - ie_{j+3}), \quad j = 1, 2, 3. \tag{9}$$

The automorphism group of the octonion algebra is the 14-parameter exceptional group G_2. The imaginary octonion units $e_\alpha(\alpha = 1, ..., 7)$ fall into its 7-dimensional representation.

Under the $SU(3)^c$ subgroup of G_2 that leaves e_7 invariant, u_0 and u_0^* are singlets, while u_j and u_j^* correspond, respectively, to the representations $\mathbf{3}$ and $\bar{\mathbf{3}}$. The multiplication table can now be written in a manifestly $SU(3)^c$ invariant manner (together with the complex conjugate equations):

$$u_0^2 = u_0, \qquad u_0 u_0^* = 0 \tag{10}$$

$$u_0 u_j = u_j u_0^* = u_j, \qquad u_0^* u_j = u_j u_0 = 0 \tag{11}$$

$$u_i u_j = -u_j u_i = \epsilon_{ijk} u_k^*, \tag{12}$$

$$u_i u_j^* = -\delta_{ij} u_0 \tag{13}$$

where ϵ_{ijk} is completely antisymmetric with $\epsilon_{ijk} = 1$ for ijk = 123, 246, 435, 651, 572, 714, 367. Here, one sees the virtue of octonion multiplication. If we consider the direct products

$$C: \qquad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8} \tag{14}$$

$$G: \qquad \mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6} \tag{15}$$

for $SU(3)^c$, then these equations show that octonion multiplication gets rid of $\mathbf{8}$ in $\mathbf{3} \otimes \bar{\mathbf{3}}$, while it gets rid of $\mathbf{6}$ in $\mathbf{3} \otimes \mathbf{3}$. Combining Eq.(12) and Eq.(13) we find

$$(u_i u_j) u_k = -\epsilon_{ijk} u_0^* \tag{16}$$

Thus the octonion product leaves only the color part in $\mathbf{3}\otimes\bar{\mathbf{3}}$ and $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$, so that it is a natural algebra for colored quarks. We note that under the $SU(3)$ subgroup of G_2 (the 14 parameter automorphism group of octonions), the split units u_i transform like a triplet, u_j^* like an antitriplet, u_0 and u_0^* like singlets. From the octonion algebra we can extract a subalgebra satisfied by u_i and u_j^* only, i.e.

$$\{u_i, u_j\} = 0, \quad \{u_i^\dagger, u_j^\dagger\} = 0, \quad \{u_i, u_j^\dagger\} = \delta_{ij}, \tag{17}$$

where

$$u_i^\dagger = -u_i^* = \bar{u_i}^*, \tag{18}$$

the bar denoting octonionic conjugation. This tells us that the three split units u_i are Grassmann numbers. u_j^* form a conjugate set of Grassmann numbers and together they form a fermionic Heisenberg algebra. Note that the split units represent an exceptional Grassmann algebra because they are not associative. In fact their associator is

$$[u_i, u_j, u_k] = (u_i, u_j)u_k - u_i(u_j u_k) = \epsilon_{ijk}(u_0 - u_0^*) = i\epsilon_{ijk} e_7. \tag{19}$$

We also have

$$[u_i, u_j, u_k^*] = \delta_{ki}u_j - \delta_{kj}u_i. \tag{20}$$

As a result the split units, like the imaginary units e_α cannot be represented by matrices. Note that the split octonion algebra we gave above is such that the product of two triplets gives an antitriplet, while the product of a triplet and an antitriplet gives a singlet. Hence the dynamical suppression of octet and sexted states mentioned above is automatically achieved.

The quarks, being in the triplet representation of the color group $SU(3)^c$, they are represented by the local fields $q^i_\alpha(x)$, where $i = 1, 2, 3$ is the color index and α the combined spin-flavor index. Antiquarks at point y are color antitriplets $\bar{q}^i_\beta(y)$. Consider the two-body systems

$$C^{\beta i}_{\alpha j} = q^i_\alpha(x_1)\bar{q}^j_\beta(x_2) \tag{21}$$

$$G^{ij}_{\alpha\beta} = q^i_\alpha(x_1)q^j_\beta(x_2) \tag{22}$$

so that C is either a color singlet or color octet, while G is a color antitriplet or a color sextet. Now C contains meson states that are color singlets and hence observable. The octet $q - \bar{q}$ state is confined and not observed as a scattering state. In the case of two-body G states, the antitriplets are diquarks which, inside a hadron can be combined with another triplet quark to give observable, color singlet, three-quark baryon states. The color sextet part of G can only combine with a third quark to give unobservable color octet and color decuplet three-quark states. Hence the hadron dynamics is such that the $\mathbf{8}$ part of C and the $\mathbf{6}$ part of G are suppressed. This can best be achieved by the use of above split octonion algebra. The dynamical suppression of the octet and sextet states in Eq.(21) and Eq.(22) is therefore automatically achieved. The split octonion units can be contracted with color indices of triplet or antitriplet fields. For quarks and antiquarks we can define the "transverse" octonions (calling u_0 and u_0^* longitudinal units)

$$q_\alpha = u_i q^i_\alpha = \mathbf{u}\cdot\mathbf{q}_\alpha, \quad \bar{q}_\beta = u_i^\dagger \bar{q}^j_\beta = -\mathbf{u}^*\cdot\bar{\mathbf{q}}_\beta \tag{23}$$

We find

$$q_\alpha(1)\bar{q}_\beta(2) = u_0\mathbf{q}_\alpha(1)\cdot\bar{\mathbf{q}}_\beta(2) \tag{24}$$

$$\bar{q}_\alpha(1)q_\beta(2) = u_0^*\bar{\mathbf{q}}_\alpha(1)\cdot\mathbf{q}_\beta(2) \tag{25}$$

$$G_{\alpha\beta}(12) = q_\alpha(1)q_\beta(2) = \mathbf{u}^*\cdot\mathbf{q}_\alpha(1)\times\mathbf{q}_\beta(2) \tag{26}$$

$$G_{\beta\alpha}(21) = q_\beta(2)q_\alpha(1) = \mathbf{u}^*\cdot\mathbf{q}_\beta(2)\times\mathbf{q}_\alpha(1) \tag{27}$$

Because of the anticommutativity of the quark fields, we have

$$G_{\alpha\beta}(12) = G_{\beta\alpha}(21) = \frac{1}{2}\{q_\alpha(1), q_\beta(2)\} \tag{28}$$

If the diquark forms a bound state represented by a field $D_{\alpha\beta}(x)$ at the center-of-mass location x

$$x = \frac{1}{2}(x_1 + x_2), \tag{29}$$

when x_2 tends to x_1 we can replace the argument by x, and we obtain

$$D_{\alpha\beta}(x) = D_{\beta\alpha}(x) \tag{30}$$

so that the local diquark field must be in a symmetric representation of the spin-flavor group. If the latter is taken to be $SU(6)$, then $D_{\alpha\beta}(x)$ is in the 21-dimensional symmetric representation, given by

$$(\mathbf{6}\otimes\mathbf{6})_s = \mathbf{21} \tag{31}$$

If we denote the antisymmetric 15 representation by $\Delta_{\alpha\beta}$, we see that the octonionic fields single out the 21 diquark representation at the expense of $\Delta_{\alpha\beta}$. We note that without this color algebra supersymmetry would give antisymmetric configurations as noted by Salam and Strathdee [16] in their possible supersymmetric generalization of hadronic supersymmetry. Using the nonsingular part of the operator product expansion we can write

$$\tilde{G}_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2) = D_{\alpha\beta}(x) + \mathbf{r} \cdot \boldsymbol{\Delta}_{\alpha\beta}(x) \tag{32}$$

The fields $\Delta_{\alpha\beta}$ have opposite parity to $D_{\alpha\beta}$; $\mathbf{r}$ is the relative coordinate at time t if we take $t = t_1 = t_2$. They play no role in the excited baryon which becomes a bilocal system with the 21- dimensional diquark as one of its constituents.

Now consider a three-quark system at time t. The c.m. and relative coordinates are

$$\mathbf{R} = \frac{1}{\sqrt{3}}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) \tag{33}$$

$$\boldsymbol{\rho} = \frac{1}{\sqrt{6}}(2\mathbf{r}_3 - \mathbf{r}_1 - \mathbf{r}_2) \tag{34}$$

$$\mathbf{r} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2), \tag{35}$$

giving

$$\mathbf{r}_1 = \frac{1}{\sqrt{3}}\mathbf{R} - \frac{1}{\sqrt{6}}\boldsymbol{\rho} + \frac{1}{\sqrt{2}}\mathbf{r} \tag{36}$$

$$\mathbf{r}_2 = \frac{1}{\sqrt{3}}\mathbf{R} - \frac{1}{\sqrt{6}}\boldsymbol{\rho} - \frac{1}{\sqrt{2}}\mathbf{r} \tag{37}$$

$$\mathbf{r}_3 = \frac{1}{\sqrt{3}}\mathbf{R} + \frac{2}{\sqrt{6}}\boldsymbol{\rho} \tag{38}$$

The baryon state must be a color singlet, symmetric in the three pairs (α, x_1), (β, x_2), (γ, x_3). We find

$$(q_\alpha(1)q_\beta(2))q_\gamma(3) = -u_0^* F_{\alpha\beta\gamma}(123) \tag{39}$$

$$q_\gamma(3)(q_\alpha(1)q_\beta(2)) = -u_0 F_{\alpha\beta\gamma}(123) \tag{40}$$

so that

$$-\frac{1}{2}\{\{q_\alpha(1), q_\beta(2)\}, q_\gamma(3)\} = F_{\alpha\beta\gamma}(123) \tag{41}$$

The operator $F_{\alpha\beta\gamma}(123)$ is a color singlet and is symmetrical in the three pairs of coordinates. We have

$$F_{\alpha\beta\gamma}(123) = B_{\alpha\beta\gamma}(\mathbf{R}) + \boldsymbol{\rho} \cdot \mathbf{B}'(\mathbf{R}) + \mathbf{r} \cdot \mathbf{B}''(\mathbf{R}) + C \tag{42}$$

where C is of order two and higher in $\boldsymbol{\rho}$ and $\mathbf{r}$. Because $\mathbf{R}$ is symmetric in $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, the operator $B_{\alpha\beta\gamma}$ that creates a baryon at $\mathbf{R}$ is totally symmetrical in its flavor-spin indices. In the $SU(6)$ scheme it belongs to the (56) representation. In the bilocal $q - D$ approximation we have $\mathbf{r} = 0$ so that $F_{\alpha\beta\gamma}$ is a function only of $\mathbf{R}$ and $\boldsymbol{\rho}$ which are both symmetrical in $\mathbf{r}_1$ and $\mathbf{r}_2$. As before, $\mathbf{B}'$ belongs to the orbitally excited 70^- representation of $SU(6)$. The totally antisymmetrical representation (20) is absent in the bilocal approximation. It would only appear in the trilocal treatment that would involve the 15-dimensional diquarks. Hence, if we use local fields, any product of two octonionic quark fields gives a (21) diquark

$$q_\alpha(\mathbf{R})q_\beta(\mathbf{R}) = D_{\alpha\beta}(\mathbf{R}), \tag{43}$$

and any nonassociative combination of three quarks, or a diquark and a quark at the same point give a baryon in the 56^+ representation:

$$(q_\alpha(\mathbf{R})q_\beta(\mathbf{R}))q_\gamma(\mathbf{R}) = -u_0^* B_{\alpha\beta\gamma}(\mathbf{R}) \tag{44}$$

$$q_\alpha(\mathbf{R})(q_\beta(\mathbf{R})q_\gamma(\mathbf{R})) = -u_0 B_{\alpha\beta\gamma}(\mathbf{R}) \tag{45}$$

$$q_\gamma(\mathbf{R})(q_\alpha(\mathbf{R})q_\beta(\mathbf{R})) = -u_0 B_{\alpha\beta\gamma}(\mathbf{R}) \tag{46}$$

$$(q_\gamma(\mathbf{R})q_\alpha(\mathbf{R}))q_\beta(\mathbf{R}) = -u_0^* B_{\alpha\beta\gamma}(\mathbf{R}) \tag{47}$$

The bilocal approximation gives the $(35+1)$ mesons and the 70^- baryons with $\ell = 1$ orbital excitation.

4. Algebraic Realization

Octonionic Supersymmetry of Catto and Gürsey (C-G) was based on the supergroup $SU(6/21)$ acts on a quark and antidiquark situated at the same point x_1. At the point x_2 we can consider the action of the supergroup with the same parameters, or one with different parameters. In the first case we have a global symmetry. In the second case, if we only deal with bilocal fields the symmetry will be represented by $SU(6/21) \times SU(6/21)$, doubling the C-G supergroup. On the other hand, if any number of points are considered, with different parameters attached to each point, we are led to introduce a local supersymmetry $SU(6/21)$ to which we should add the local color group $SU(3)^c$. Since it is not a fundamental symmetry, we shall not deal with the local $SU(6/21)$ group here. However, the double $SU(6/21)$ supergroup is useful for bilocal fields since the decomposition of the adjoint representation of the 728-dimensional group with respect to $SU(6) \times SU(21)$ gives

$$728 = (35,1) + (1,440) + (6,21) + (\bar{6},\bar{21}) + (1,1) \tag{48}$$

A further decomposition of the double $SU(6/21)$ supergroup into its field with respect to its c.m. coordinates, as seen above (see Eq.(42)), leads to the decomposition of the 126-dimensional cosets $(6,21)$ and $(21,6)$ into $56^+ + 70^-$ of the diagonal $SU(6)$.

We would have a much tighter and more elegant scheme if we could perform such a decomposition from the start and be able to identify $(1,21)$ part of the fundamental representation of $SU(6/21)$ with the 21-dimensional representation of the $SU(6)$ subgroup, which means going beyond the $SU(6/21)$ supersymmetry to a smaller supergroup having $SU(6)$ as a subgroup.

5. Beyond $SU(6/21)$– A minimal scheme

In order to go beyond the $SU(6/21)$ supersymmetry we propose a nonsimple super-Lie algebra that has parabolic structure with the odd part being represented by a symmetric third rank tensor $F_{\alpha\beta\gamma}$ with 56 components and the even part by the 36 $U(6)$ generators L^ρ_σ. The bracket relations are

$$[L^\rho_\sigma, L^\lambda_\mu] = \delta^\rho_\mu L^\lambda_\sigma - \delta^\lambda_\sigma L^\rho_\mu \tag{49}$$

$$\{F_{\alpha\beta\gamma}, F_{\rho\sigma\tau}\} = 0 \tag{50}$$

$$[L^\rho_\sigma, F_{\alpha\beta\gamma}] = \delta^\rho_\alpha F_{\beta\gamma\sigma} + \delta^\rho_\beta F_{\gamma\alpha\sigma} + \delta^\rho_\gamma F_{\alpha\beta\sigma} \tag{51}$$

There is a conjugate superalgebra generated by L^ρ_σ and $F^{\alpha\beta\gamma}$. Note that

$$N = \sum_\rho L^\rho_\rho \tag{52}$$

acts like a number operator and traceless part of L^ρ_σ coincides with the elements of the $SU(6)$ Lie algebra. Hence this minimal superalgebra S has $56 + 35 = 91$ generators. With the addition of color, the algebra becomes $SU(3)^c \times S$.

We can immediately construct a matrix representation of this algebra by means of fermionic operators f^i_α that transform like the $(3, 6)$ representation of $SU(3)^c \times SU(6)$ and the conjugate operators $\bar{f}^\beta_j$ associated with the $(\bar{3}, \bar{6})$ representation. We have

$$\{f^i_\alpha, f^j_\beta\} = 0, \qquad \{\bar{f}^\alpha_i, \bar{f}^\beta_j\} = 0, \tag{53}$$

$$\{f^i_\alpha, \bar{f}^\beta_j\} = \delta^i_j \delta^\beta_\alpha. \tag{54}$$

Since f^i_α are Grassmann numbers they can be represented by finite matrices belonging to a Clifford algebra. The conjugate operators $\bar{f}^i_\alpha$ are represented by the corresponding Hermitian conjugate matrices. Out of this fermionic Heisenberg basis we construct the following matrices:

$$\Phi_{\alpha\beta\gamma} = \epsilon_{ijk} f^i_\alpha f^j_\beta f^k_\gamma \tag{55}$$

$$M^\rho_\sigma = \bar{f}^\rho_i f^i_\sigma - \frac{1}{6}\delta^\rho_\sigma N \tag{56}$$

where

$$N = \bar{f}^\tau_j f^j_\tau \tag{57}$$

Then $\Phi_{\alpha\beta\gamma}$ and M^ρ_α are the representations of $F_{\alpha\beta\gamma}$ and L^ρ_σ since they satisfy the algebra of Eqs.(49-51). If f^i_α are identified with the colored-quark annihilation operators, the odd elements of Φ of the algebra correspond to baryons, while the even elements M correspond to mesons. N represents the number operator.

At this point let us introduce a module that consists of the representations $(\bar{3}, \bar{6})$ and $(\bar{3}, 21)$ of $SU(3)^c \times SU(6)$. They correspond to the operators associated with the antiquarks and diquarks:

$$\bar{Q}^\alpha_j = \bar{f}^\alpha_j, \qquad D_{\alpha\beta k} = \epsilon_{ijk} f^i_\alpha f^j_\beta \tag{58}$$

This multiplet is closed under the group $SU(3)^c \times S$. Indeed we have

$$\{\Phi_{\alpha\beta\gamma}, \bar{Q}^\rho_k\} = \delta^\rho_\alpha D_{\beta\gamma k} + \delta^\rho_\beta D_{\gamma\alpha k} + \delta^\rho_\gamma D_{\alpha\beta k} \tag{59}$$

$$[\Phi_{\rho\sigma\tau}, D_{\alpha\beta k}] = 0 \tag{60}$$

$$[M^\gamma_\rho, \bar{Q}^\sigma_k] = \delta^\sigma_\rho \bar{Q}^\gamma_k \tag{61}$$

$$[M^\gamma_\rho, D_{\alpha\beta k}] = \delta^\gamma_\alpha D_{\beta\rho k} + \delta^\gamma_\beta D_{\alpha\rho k} \tag{62}$$

Eq.(61) and Eq.(62) show that $\bar{Q}$ and D transform like $(\bar{6})$ and (21) under $SU(6)$. Eq.(59) and Eq.(60) tell us that the odd part of the supergroup maps $\bar{Q}$ into D and D into zero. Letting the corresponding odd parameters be $\eta^{\alpha\beta\gamma}$, we now consider the superalgebra element

$$\Phi = \frac{1}{6}\eta^{\alpha\beta\gamma}\Phi_{\alpha\beta\gamma} \tag{63}$$

We find

$$\delta\bar{Q}^\rho_k = [\Phi, \bar{Q}^\rho_k] = \frac{1}{2}\eta^{\rho\alpha\beta} D_{\alpha\beta k} \tag{64}$$

$$\delta D_{\rho\sigma} = [\Phi, D_{\rho\sigma}] = 0 \tag{65}$$

Neglecting mass and spin differences, the Hamiltonian we wrote is invariant under the infinitesimal transformations given above.

The baryon-meson system transforms like the adjoint representation of the supergroup S. We find

$$\delta B_{\alpha\beta\gamma} = [\Phi, B_{\alpha\beta\gamma}] = 0 \tag{66}$$

$$\delta M^{\rho}_{\sigma} = [\Phi, M^{\rho}_{\sigma}] = -\frac{1}{2}\eta^{\rho\alpha\beta}B_{\alpha\beta\sigma} \tag{67}$$

Hence the odd part of S maps mesons into baryons and baryons into zero. If the conjugate supergroup is used, baryons and diquarks are replaced by antibaryons and antidiquarks, while the antiquarks are replaced by quarks.

Note that in this minimal scheme the exotic mesons do not appear. Nevertheless the transformations Eq.(66) and Eq.(67) are sufficient for establishing the equality of the Regge slopes for baryons and mesons.

Because of the form of the cubic function and the quadratic functions of the generators of S in terms of the fermionic annihilation and creation operators, it is natural to introduce the transverse octonion operators as in equation Eq.(23)

$$f_{\alpha} = u_i f^i_{\alpha}, \qquad \bar{f}^{\beta} = -u^{*j}\bar{f}^{\beta}_j \tag{68}$$

Then we obtain in a general fashion

$$\Phi_{\alpha\beta\gamma} = \frac{1}{2}\{\{f_{\alpha}, f_{\beta}\}, f_{\gamma}\} \tag{69}$$

$$\{\bar{f}^{\rho}, f_{\rho}\} = \frac{3}{2}\delta^{\rho}_{\sigma} + ie_7 M^{\rho}_{\sigma} \tag{70}$$

We also have

$$[f_{\alpha}, f_{\beta}] = 0, \qquad [\bar{f}^{\alpha}, \bar{f}^{\beta}] = 0 \tag{71}$$

$$[f_{\alpha}, \bar{f}^{\beta}] = M^{\beta}_{\alpha} + \frac{3}{2}ie_7\delta^{\beta}_{\alpha} \tag{72}$$

Eq.(71) and Eq.(72) together with the $SU(6)$ Lie algebra of the form Eq.(49) satisfied by the operators M^{β}_{α} suggest that M^{β}_{α}, f_{α}, $\bar{f}^{\beta}$, and $ie_7 = (u_0 - u^*_0)$ form an octonionic extension of $SU(6)$ that has $SU(6) \times SU(3)^c$ as a subgroup. It is related to the simple supergroup $SU(6/3)$, but not equivalent to it because of the suppression of the color octet and sextet states in the octonionic products $f_{\alpha}\bar{f}^{\beta}$ and $f_{\alpha}f_{\beta}$.

The bilocal treatment outlined earlier gets carried over unchanged to the minimal scheme. The $(\bar{q}, D)$ system forms one multiplet $R(1)$ at point x_1 under the supergroup S. The conjugate system $(q, \bar{D})$ forms a multiplet $R(2)$ at point x_2 under the conjugate group $\bar{S}$. Then the bilocal system $R(1) \times R(2)$ is expanded as in Eq.(32) and it transforms under the group $S \times \bar{S}$. Its local part with respect to c.m. coordinate x gives the meson-baryon and the meson-antibaryon multiplets (35, 56^+) and (35, 56^+), while the coefficients of the relative coordinate $\mathbf{r}$ give the $\ell = 1$ excitations of mesons in the $(35 + 1)$ representations and the $\ell = 1$ excited baryons in the 70^- representation of $SU(6)$. In the symmetric approximation, the masses of various states are proportional to the sum of the quark number plus the antiquark number.

6. Symmetries of the Three Quark System

For the three quark system we can write the generalized form of the two body Hamiltonian as

$$\begin{aligned}
\tilde{H}_{123} \quad = \quad & \tilde{V}_{12} + \tilde{V}_{23} + \tilde{V}_{31} + \gamma^{(1)}_4\sqrt{[m_1 + \frac{1}{2}S_{12}]^2 + \boldsymbol{p}^2_{12}} \\
& + \gamma^{(2)}_4\sqrt{[m_2 + \frac{1}{2}S_{12}]^2 + \boldsymbol{p}^2_{12}} + \gamma^{(2)}_4\sqrt{[m_2 + \frac{1}{2}S_{23}]^2 + \boldsymbol{p}^2_{23}} \\
& + \gamma^{(3)}_4\sqrt{[m_3 + \frac{1}{2}S_{23}]^2 + \boldsymbol{p}^2_{23}} + \gamma^{(3)}_4\sqrt{[m_3 + \frac{1}{2}S_{31}]^2 + \boldsymbol{p}^2_{31}} \\
& + \gamma^{(1)}_4\sqrt{[m_1 + \frac{1}{2}S_{31}]^2 + \boldsymbol{p}^2_{31}}
\end{aligned} \tag{73}$$

where

$$\tilde{V}_{ij} = V_{ij} + \text{spin dependent terms} \tag{74}$$

with $i, j = 1, 2, 3$.

To study the symmetry of this system we introduce fermion annihilation and creation operators for colored quarks f^i_α with $i = 1, 2, 3$ the color index, and $\alpha = 1, \ldots, 6$ the flavor index, so that

$$\{f^i_\alpha, f^j_\beta\} = 0, \quad \{\bar{f}^\alpha_i, \bar{f}^\beta_j\} = 0, \quad \{f^i_\alpha, \bar{f}^\beta_j\} = \delta^i_j \delta^\beta_\alpha \tag{75}$$

The diquark transforms like

$$D_{\alpha\beta k} = \epsilon_{ijk} f^i_\alpha f^k_\beta \tag{76}$$

It is in $(\bar{3})$ representation for color, and in 21 representation for $SU(6)$. The baryons transform like

$$\Phi_{\alpha\beta\gamma} = \epsilon_{ijk} f^i_\alpha f^j_\beta f^k_\gamma \tag{77}$$

It is a color singlet belonging to the 56 representation of $SU(6)$. Color singlet mesons that are in the 35 representation of $SU(6)$ transform like

$$M^\rho_\sigma = \bar{f}^\rho_i f^i_\sigma - \frac{1}{6}\delta^\rho_\sigma N \tag{78}$$

with N being a color singlet and an $SU(6)$ singlet, and is given by

$$N = \bar{f}^\tau_j f^j_\tau \tag{79}$$

We have the relations

$$[M^\rho_\sigma, M^\lambda_\mu] = \delta^\rho_\mu M^\lambda_\sigma - \delta^\lambda_\sigma M^\rho_\mu \tag{80}$$

$$\{\Phi_{\alpha\beta\gamma}, \Phi_{\rho\sigma\tau}\} = 0 \tag{81}$$

$$[M^\rho_\sigma, \Phi_{\rho\sigma\tau}] = \delta^\rho_\alpha \Phi_{\beta\gamma\sigma} + \delta^\rho_\beta \Phi_{\gamma\alpha\sigma} + \delta^\rho_\gamma \Phi_{\alpha\beta\sigma} \tag{82}$$

Now $\bar{f}^\rho_j$ and $D_{\alpha\beta,k}$ form a multiplet that transforms under Eqs.(80 - 82) as follows:

$$\{\Phi_{\alpha\beta\gamma}, \bar{f}^\alpha_k\} = \delta^\rho_\alpha D_{\beta\gamma,k} + \delta^\rho_\beta D_{\gamma\alpha,k} + \delta^\rho_\gamma D_{\alpha\beta,k} \tag{83}$$

$$[\Phi_{\rho\sigma\tau}, D_{\alpha\beta,k}] = 0 \tag{84}$$

$$[M^\gamma_\rho, \bar{f}^\sigma_k] = \delta^\sigma_\rho \bar{f}^\gamma_k \tag{85}$$

$$[M^\gamma_\rho, D_{\alpha\beta,k}] = \delta^\gamma_\alpha D_{\beta\rho,k} + \delta^\gamma_\beta D_{\alpha\rho,k} \tag{86}$$

When the three quark system gets excited, it becomes a $q - D$ system, but retains its symmetry

$$B_{123} = \frac{1}{\sqrt{3}}(q_1 D_{23} + q_2 D_{31} + q_3 D_{12}) \tag{87}$$

and remains in the 56 representation. The 70 representation has opposite parity (for example in $\boldsymbol{r}_1 - \boldsymbol{r}_{23}$, etc.).

When in the excited baryon quarks 2 and 3 form a diquark, and we can put

$$\boldsymbol{p}_{12} = \boldsymbol{p}_{13} = \boldsymbol{p} \quad \boldsymbol{p}_{23} = 0 \tag{88}$$

which leads to

$$\tilde{H}_{123} \longrightarrow \tilde{H}_{q_1 D_{23}} \cong 2\tilde{V}_{12} + \gamma_4^{(1)} \sqrt{[m_1 + \frac{1}{2}S]^2 + \boldsymbol{p}^2} + (\gamma_4^{(2)} + \gamma_4^{(3)}) \sqrt{[(m_2 + m_3 + \bar{S}_{23}) + \frac{1}{2}S]^2 + \boldsymbol{p}^2} \tag{89}$$

and similarly for the other two configurations obtained by cyclic permutations. Note here that $m_D = m_2 + m_3 + \bar{S}_{23}$ is the mass of the diquark.

Then the superalgebra given by Eqs.(80-82) which is symmetric in (123) becomes, for a given $q - D$ configuration the $SU(6/21)$ supersymmetry $1 - (23)$.

When $\tilde{H}_{123}$ is symmetrised it becomes invariant under the discrete group S_3. Hence its eigenstates are also eigenstates of S_3, i.e. correspond to representations of S_3. The splitting of 3 quarks into q and D breaks the S_3 symmetry down to $Z_2 \sim S_2$ which has two eigenstates (associated with ± 1). As mentioned in our earlier papers, Iachello [17, 18] has shown that the 70 representation of $SU(6)$ associated with $\ell = 1$ is parity doubled while the ground state 56 ($\ell = 0$) is not. This is supported by experiment.

We can now define eight auxiliary octonions of quadratic norm $\frac{1}{2}$ constructed out of the seven imaginary units f_α ($\alpha = 1, \cdots 7$).

$$f_1 = \frac{1}{2}(e_1 + e_4), \qquad f_2 = \frac{1}{2}(e_2 + e_5), \tag{90}$$

$$f_3 = \frac{1}{2}(e_3 + e_6), \qquad f_4 = \frac{1}{2}(e_1 - e_4), \tag{91}$$

$$f_5 = \frac{1}{2}(e_2 - e_5), \qquad f_6 = \frac{1}{2}(e_3 - e_6), \tag{92}$$

$$f_7 = \frac{1}{2}(1 - e_7), \quad f_8 = \frac{1}{2}(1 + e_7) \tag{93}$$

These are intimately related to the eight split octonionic units and to their conjugates we discussed above. If we keep one of the f's fixed, say f_j, then the difference combination $(f_i f_j f_k - f_k f_j f_i)$ is always either zero or equal to another f. We have worked out isomorphisms for such combination rules and they seem to play a fundamental role in dealing with symmetries of three quark systems we discussed, with further applications in multiquark systems. These will be discussed in another publication.

Combinations of f's also appear naturally within the root systems of groups associated with a magic square and play a profound role in superstring theories and their compactifications.

7. Particle multiplets including a giant supermultiplet

The multiplet X that sits in the adjoint representation of $SU(6/21)$ is

$$X = \begin{pmatrix} M & B \\ \bar{B} & N \end{pmatrix} \tag{94}$$

where M and N are mesons and exotics, and B and $\bar{B}$ are fermions. The M and N are square matrices, and B is a rectangular matrix. Specifically, $M = 6 \times \bar{6}$, $B = 6 \times 21$, $\bar{B} = \bar{21} \times \bar{6}$, and $N = \bar{21} \times 21$. M and N are taken to be hermitian. If we have three flavors the $SU(6)$ content of these matrices are

$M = 1 + 35$ (negative parity), $N = 1 + 35 + 405$ (positive parity), and $B = 56 + 70$ (positive parity). The fundamental representation $\mathbf{F}$ is the color triplet

$$\mathbf{F} = \begin{pmatrix} \mathbf{Q} \\ \bar{\mathbf{D}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \bar{\mathbf{q}} \times \bar{\mathbf{q}} \end{pmatrix} \tag{95}$$

with $\mathbf{q} = 6 \times 1$ and $\bar{\mathbf{q}} \times \bar{\mathbf{q}} = \bar{2}1 \times 1$. Now let Ξ be the superalgebra element of $SU(6/21)$ ($SU(3)$ singlet). Ξ is a color singlet given by

$$\Xi = \begin{pmatrix} m & b \\ \bar{b} & n \end{pmatrix} \tag{96}$$

and the transformation laws for the fundamental representation $[3, (6 + \overline{21})]$ are

$$\begin{aligned} \delta\mathbf{F} &= \Xi\mathbf{F} = \begin{pmatrix} m & b \\ \bar{b} & n \end{pmatrix} \begin{pmatrix} \mathbf{Q} \\ \bar{\mathbf{D}} \end{pmatrix} \\ &= \begin{pmatrix} m\mathbf{Q} + b\bar{\mathbf{D}} \\ \bar{b}\mathbf{Q} + n\bar{\mathbf{D}} \end{pmatrix} = \begin{pmatrix} \delta\mathbf{Q} \\ \delta\bar{\mathbf{D}} \end{pmatrix} \end{aligned} \tag{97}$$

and

$$\overline{\delta\mathbf{F}} = \overline{\mathbf{F}}\Xi = (\bar{\mathbf{Q}}m + \mathbf{D}\bar{b} \quad \bar{\mathbf{Q}}b + \mathbf{D}n) = (\delta\bar{\mathbf{Q}} \quad \delta\mathbf{D}) \tag{98}$$

or in the index notation

$$\delta q^i_\alpha = m^\beta_\alpha q^i_\beta + b_{\alpha\beta\gamma}(\bar{D}^i)^{\beta\gamma}, \tag{99}$$

$$(\delta\bar{D}^i)^{\beta\gamma} = \bar{b}^{\alpha\beta\gamma} q^i_\alpha + n^{\beta\gamma}_{\rho\sigma}(\bar{D}^i)^{\rho\sigma} \tag{100}$$

On the other hand, the transformation law for the adjoint representation is

$$\delta\Xi = i[\Xi, X], \tag{101}$$

where

$$\begin{aligned} \Xi X &= \begin{pmatrix} m & b \\ \bar{b} & n \end{pmatrix} \begin{pmatrix} M & B \\ \bar{B} & N \end{pmatrix} \\ &= \begin{pmatrix} mM + b\bar{B} & mB + bN \\ \bar{b}M + n\bar{B} & \bar{b}B + nN \end{pmatrix} \end{aligned} \tag{102}$$

$$X\Xi = \begin{pmatrix} Mm + B\bar{b} & Mb + Bn \\ \bar{B}m + N\bar{b} & \bar{B}b + Nn \end{pmatrix} \tag{103}$$

so that

$$\begin{aligned} &\begin{pmatrix} \delta M & \delta B \\ \delta\bar{B} & \delta N \end{pmatrix} = \\ &i \begin{pmatrix} [M, m] + b\bar{B} - \bar{B}b & Mb - Bn + bN - Mb \\ -\bar{B}m + n\bar{B} - N\bar{b} + \bar{b}M & [n, N] + \bar{b}B - \bar{B}b \end{pmatrix} \end{aligned} \tag{104}$$

Next we build a giant supermultiplet containing M, N, L, B, $\bar{B}$, Q, $\bar{Q}$, D, and $\bar{D}$. The fundamental representation of $SU(6/21) \times [SU(3)^c]_{triplet}$ and the adjoint representation of $U(6/21) \times [SU(3)^c]_{singlet}$

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fits in the adjoint representation of an octonionic version of $SU(6/22)$ denoted by Z, given by

$$\begin{aligned} Z &= u_0 \begin{pmatrix} M & B & 0 \\ B^\dagger & N & 0 \\ 0 & 0 & 0 \end{pmatrix} + u_0^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L \end{pmatrix} \\ &+ \mathbf{u} \cdot \begin{pmatrix} 0 & 0 & \mathbf{Q} \\ 0 & 0 & \mathbf{D}^* \\ 0 & 0 & 0 \end{pmatrix} + \mathbf{u}^* \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \epsilon\mathbf{Q}^\dagger & \epsilon\mathbf{D}^T & 0 \end{pmatrix} \\ &= \begin{pmatrix} u_0 M & u_0 B & \mathbf{u}\cdot\mathbf{Q} \\ u_0 B^\dagger & u_0 N & \mathbf{u}\cdot\mathbf{D}^* \\ \epsilon\mathbf{u}^*\cdot\mathbf{Q}^\dagger & \epsilon\mathbf{u}^*\cdot\mathbf{D}^T & u_0^* L \end{pmatrix} \end{aligned} \tag{105}$$

where mesons M $(6 \times \bar{6})$ and exotics N $(\bar{21} \times 21)$ are Hermitian; B (6×21), $\bar{B}$ $(\bar{21} \times \bar{6})$, Q (6×1), $\bar{Q}$ $(1 \times \bar{6})$, D (1×21), $\bar{D}$ $(\bar{21} \times 1)$, and L (1×1); ϵ can be taken as 1 if $u^\dagger = \bar{u}^* = -u^*$, -1 if $u^\dagger = u^*$, and zero. Closure properties of Z matrices are such that

$$[Z, Z'] = iZ'', \qquad \{Z, Z'\} = Z''' \tag{106}$$

and in general they are nonassociative (Jacobian $J = f(Q, D) \neq 0$), except in the case when $\epsilon = 0$ we have

$$[[Z, Z'], Z''] + [[Z', Z''], Z] + [[Z'', Z], Z'] = 0. \tag{107}$$

Then we have a true superalgebra (non-semisimple) which is a contraction of a simple algebra that closes but does not satisfy the Jacobi identity. In both cases we get an extension of $SU(6/21)$.

We now consider the element of the algebra

$$\Omega = \begin{pmatrix} u_0 m & u_0 b & \mathbf{u}\cdot\boldsymbol{\xi} \\ u_0 b^\dagger & u_0 n & \mathbf{u}\cdot\mathbf{d}^* \\ \epsilon\mathbf{u}^*\cdot\boldsymbol{\xi}^\dagger & \epsilon\mathbf{u}^*\cdot\mathbf{d}^T & u_0^*\ell \end{pmatrix} \tag{108}$$

where

$$\begin{pmatrix} u_0 m & u_0 b \\ u_0 b^\dagger & u_0 n \end{pmatrix} \tag{109}$$

are the $[SU(6/21)]$ color singlet parameters,

$$\begin{pmatrix} \mathbf{u}\cdot\boldsymbol{\xi} \\ \mathbf{u}\cdot\mathbf{d}^* \end{pmatrix} \tag{110}$$

and

$$\begin{pmatrix} \epsilon\mathbf{u}^*\cdot\boldsymbol{\xi}^\dagger & \epsilon\mathbf{u}^*\cdot\mathbf{d}^T \end{pmatrix} \tag{111}$$

are the colored parameters,

$$\begin{pmatrix} u_0 b & \mathbf{u}\cdot\boldsymbol{\xi} \end{pmatrix} \tag{112}$$

and

$$\begin{pmatrix} u_0 b^\dagger \\ \epsilon\mathbf{u}^*\cdot\boldsymbol{\xi}^\dagger \end{pmatrix} \tag{113}$$

are the fermionic parameters, and $m \in SU(6)$.

The change in Z is given by

$$\delta Z = [\Omega, Z] \tag{114}$$

which leads to

$$\delta \mathbf{Q} = m\mathbf{Q} - M\boldsymbol{\xi} + b\mathbf{D}^* - B\mathbf{d}^* + \boldsymbol{\xi}L - \mathbf{Q}\ell, \tag{115}$$

and

$$\delta \mathbf{D}^* = b^\dagger \mathbf{Q} - B^\dagger \boldsymbol{\xi} + n\mathbf{D}^* - N\mathbf{d}^* + \mathbf{d}^*L - \mathbf{D}^*\ell. \tag{116}$$

The $SU(6/21)$ subgroup is obtained by taking

$$\boldsymbol{\xi} = 0, \quad \mathbf{d} = 0 \quad \ell = 0 \tag{117}$$

so that

$$\delta \mathbf{Q} = m\mathbf{Q} + b\mathbf{D}^* \tag{118}$$

$$\delta \mathbf{D}^* = b^\dagger \mathbf{Q} + n\mathbf{D}^* \tag{119}$$

which, in index form, is equivalent to Eqs.(99 -100). This subgroup is valid for a Hamiltonian describing $q(x_1)$ and $\bar{q}(x_2)$, $q(x_1)$ and $D(x_2)$, $\bar{D}(x_1)$ and $q(x_2)$, $\bar{D}(x_1)$ and $D(x_2)$ interacting through a scalar potential $V = br$ as we have seen earlier.

In general for $\frac{m}{2}$ flavors and $n = \frac{1}{2}m(m+1)$, we have

$$\begin{aligned} Z &= \begin{pmatrix} u_0 M & u_0 B & \mathbf{u}\cdot\mathbf{Q} \\ u_0 B^\dagger & u_0 N & \mathbf{u}\cdot\mathbf{D}^* \\ \epsilon\mathbf{u}^*\cdot\mathbf{Q}^\dagger & \epsilon\mathbf{u}^*\cdot\mathbf{D}^T & u_0^* L \end{pmatrix} \\ &= \begin{pmatrix} m\times m & m\times n & m\times 1 \\ n\times m & n\times n & n\times 1 \\ 1\times m & 1\times n & 1\times 1 \end{pmatrix} \end{aligned} \tag{120}$$

As examples, for 2 flavors, $M = 4\times 4$, $N = 10\times 10$; for 6 flavors (icluding the top quark), $M = 12\times 12$, $N = 78\times 78$.

The automorhism group of this algebra includes $SU(m)\times SU(n)\times SU(3)^c$. If $m = 6$, it includes $SU(6)\times SU(3)^c$.

If $\mathbf{Q}$ is Majorana and $\mathbf{D}$ real, then the group becomes $Osp(n/m)\times SU(3)^c$, with subgroup $Sp(2n/R)\times O(m)\times SU(3)^c$. We shall explore quark models built by use of such groups as well as use of auxiliary octonions of quadratic norm $\frac{1}{2}$ which are related to the split octonion units we used in this paper in a subsequent publication.

We would also like to mention that we have worked out the relativistic formulation through the spin realization of the Wess-Zumino algebra in detail, but due to lack of space we leave the discussion for another publication at this point.

8. Future of Mass formulae and some prospects

Here we have shown that the quark model with potentials derived from QCD, including the quark-diquark model for excited hadrons gives mass formulae in very good agreement with experiment and goes a long way in explaining the approximate symmetries and supesymmetries of the hadronic spectrum, including the symmetry breaking mechanism. For heavy quarks the non relativistic approximation can be used so that the potential models for the spectra of charmonium [19] and the $b\bar{b}$ system [20] are even simpler. In this approach gluons are eliminated leaving quarks interacting through potentials.

It is also possible to take an opposite approach by eliminating quarks as well as gluons, leaving only an effective theory that involves mesons (quark bound states) and baryons as collective excitations (solitons) of the mesonic field. This is the way pioneered by Skyrme [21] and revived in recent years (for a review see [22]). The simplest model uses an $SU(2)\times SU(2)$ nonlinear chiral model involving pi-mesons as originally formulated by Skyrme in 1958. In 1960 Skyrme added a fourth order term to stabilize the soliton, so that it could be interpreted as a nucleon. The model was enriched by the Syracuse school

[23] and by Witten and his collaborators [24]. More recently vector mesons were added, improving the predictions (for review see [25]). In the original model, mass formulae were obtained only for a tower of $I = S$ excitations. Recently $SU(4)$ symmetry [26] and in principle $SU(6)$ symmetry were incorporated and through fermionic quantization of the soliton the tower excitations were chopped off, leading to a more realistic model [27], [28].

However, even in these improved versions the mass relations are not nearly as good as in the potential quark model when the correct pion decay constant is used [29]. There is also an overall positive contribution to hadronic masses that is not easily disposed of, although there are some recent attempts to deal with the problem [30]. In the absolute scale baryon masses are about 20% too high and vector mesons enter the theory on a very different footing than pseudoscalar mesons, making it very difficult to relate their masses.

A skyrme model that can compete with the potential quark model is still hidden in the future.

A better understanding of hadron masses through the QCD theory of quarks and gluons has shifted the mass problem from the hadronic level to the level of the elementary constituent quarks, leptons, weak bosons and Higgs bosons of the standard model. Since we now know that there are three generations of fundamental fermions with light left handed neutrinos, the development of a theory of masses for the finite system has become a burning fundamental problem. What is needed is the equivalent of a Gell-Mann-Okubo formula for the six quarks and for the leptons.

A group theoretical treatment would have to involve a horizontal group associated with the three generations, namely a new $SU(3)$. If E_6 is the right grand unification group for each generation [31] as also suggested by string theory [32] then one has to enlarge it to a group admitting $E_6 \times SU(3)$ as a subgroup. Such a group also arises in string theory is E_8, the ultimate exceptional group [33]. It will also accommodate the $O(10)$ GUT through its $O(10) \times O(6)$ or $O(10) \times SU(3) \times U(1)$ decomposition [11]. An unusual symmetry breaking pattern of E_8 through a subgroup $O(5)$ or $SU(5)$ might isolate the t-quark and give it a high mass while all the other fundamental fermions are massless in this approximation. In order to approach the fundamental mass problem in imitation of hadron masses we would have to introduce new primary particles (preons) that would yield quarks and leptons as bound states. But then it would be very difficult to understand why the fundamental fermions of the standard model are finite in number.

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