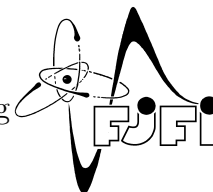


CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering



Discrete Schrödinger Operator with a Complex Step Potential

Diskrétní Schrödingerův operátor s komplexním schodovitým potenciálem

Master's thesis

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II. ÚDAJE K DIPLOMOVÉ PRÁCI

Název diplomové práce:

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Název diplomové práce anglicky:

Discrete Schrödinger operator with complex step potential

Pokyny pro vypracování:

- 1) Shrňte dosavadní výsledky a doplňte analýzu spektrálních vlastností diskrétního Schrödingerova operátoru s komplexním schodovitým potenciálem pro obecnou vazebnou konstantu.
- 2) Najděte spodní a horní odhady normy rezolventy studovaného operátoru. Diskutujte a numericky ilustруйте důsledky pro pseudospektrum.
- 3) V závislosti na vazebné konstantě diskutujte existenci slabě vázaných stavů, nebo spektrální stabilitu vzhledem k malým poruchám potenciálu.
- 4) V závislosti na vazebné konstantě se pokuste najít explicitní podmínky na normu potenciálu garantující spektrální stabilitu.
- 5) Pokuste se o analýzu existence vlastních hodnot vložených v esenciálním spektru.

Seznam doporučené literatury:

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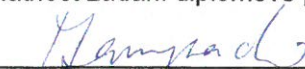
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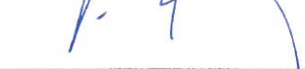
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III. PŘEVZETÍ ZADÁNÍ

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Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v diplomové práci.

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Datum převzetí zadání

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Podpis studenta

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Author's declaration:

I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

I further declare that AI tools were used in full compliance with the guidelines established by CTU in Prague.

Prague, January 6, 2025

Bc. Vojtěch Bartoš

A handwritten signature in black ink, appearing to read 'Bartoš', with a long horizontal line extending from the end of the word.

Název práce:

Diskrétní Schrödingerův operátor s komplexním schodovitým potenciálem

Autor: Bc. Vojtěch Bartoš

Obor: Matematické inženýrství

Druh práce: Diplomová práce

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Abstrakt: V této diplomové práci byly zkoumány spektrální vlastnosti diskrétního Schrödingerova operátoru s komplexním schodovitým potenciálem H_α na $\ell^2(\mathbb{Z})$, kde vazebná konstanta α je komplexním parametrem. Pseudospektrum tohoto operátoru bylo přibližně lokalizováno pomocí horního a spodního odhadu normy rezolventního operátoru. Dále bylo odvozeno asymptotické chování pseudospektra v blízkosti spektra. Pomocí Stoneovy formule byla explicitně popsána spektrální míra operátoru H_α . Na základě Birmanova–Schwingerova principu byla studována existence a jednoznačnost diskrétních vlastních hodnot porušeného operátoru $H_\alpha + V$, kde V je diagonální komplexní potenciál. Na potenciál V je kladen předpoklad, že diagonála je dostatečně malá v ℓ^1 normě vážené kvadrátem indexu. Pro $\alpha = 0$ bylo ukázáno, že $H_\alpha + V$ má lokálně jednoznačné vlastní hodnoty. Naproti tomu pro $\alpha \neq 0$ byla globálně a stejnoměrně odhadnuta norma Birmanova–Schwingerova operátoru a na základě toho bylo ukázáno, že $H_\alpha + V$ má stejné spektrum jako H_α včetně jeho částí, neboli H_α vykazuje úplnou spektrální stabilitu.

Klíčová slova: Birmanův–Schwingerův princip, diskrétní Schrödingerův operátor, nesamosdruženost, pseudospektrum, schodovitý potenciál, slabé vazby, spektrální míra, spektrální stabilita

Title: **Discrete Schrödinger Operator with a Complex Step Potential**

Author: Bc. Vojtěch Bartoš

Abstract: This diploma thesis investigates the spectral properties of the discrete Schrödinger operator with a complex step potential H_α on $\ell^2(\mathbb{Z})$, where the coupling constant α is a complex parameter. The pseudospectrum of this operator was approximately localized using upper and lower estimates of the resolvent operator norm. Furthermore, the asymptotic behavior of the pseudospectrum near the spectrum was derived. By means of Stone's formula, the spectral measure of the operator H_α was explicitly described. Using the Birman–Schwinger principle, the existence and uniqueness of discrete eigenvalues of the perturbed operator $H_\alpha + V$ were studied, where V is a diagonal complex potential. The assumption on V is that its diagonal sequence is sufficiently small in the ℓ^1 norm weighted by the square of the index. For $\alpha = 0$, it was shown that $H_\alpha + V$ has locally unique eigenvalues. On the other hand, for $\alpha \neq 0$, the norm of the Birman–Schwinger operator was globally, uniformly estimated, and on this basis, it was shown that $H_\alpha + V$ has the same spectrum as H_α , including all its parts, i.e. H_α exhibits complete spectral stability.

Key words: Birman–Schwinger principle, discrete Schrödinger operator, non-self-adjointness, pseudospectrum, step-like potential, weak coupling, spectral measure, spectral stability

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Introduction

Consider the continuous one-dimensional Laplace operator

$$\Delta = \frac{d^2}{dx^2},$$

defined on a suitable domain. It is a prominent operator in various areas of mathematics, notably spectral analysis, as it frequently appears in publications in this field. If we replace derivatives in the common definition of the continuous Laplace operator with differences, we arrive at the discrete Laplace operator. The action of this operator is given by

$$\forall x \in \ell^2(\mathbb{Z}) \quad : \quad (\Delta x)_n := -x_{n+1} + 2x_n - x_{n-1}.$$

Much like its continuous counterpart, the discrete Laplace operator plays a fundamental role in spectral analysis, where it serves as a key tool for understanding the behavior of eigenvalues and the structure of functions defined on discrete spaces.

For our purposes, we will define the discrete Laplacian H_0 as the sum of the forward shift and the backward shift, i.e.

$$\forall x \in \ell^2(\mathbb{Z}) \quad : \quad (H_0 x)_n := x_{n+1} + x_{n-1}.$$

There is a simple relation between the two operators $H_0 = -\Delta + 2I$. These elementary operations are insignificant in view of spectral analysis as they shift the spectrum in the complex plane or mirror it along the imaginary axis. The discrete Schrödinger operator with a complex step potential is defined as

$$\forall x \in \ell^2(\mathbb{Z}) \quad : \quad (H_\alpha x)_n := \begin{cases} x_{n-1} + x_{n+1} & n < 0, \\ x_{n-1} + \alpha x_n + x_{n+1} & n \geq 0, \end{cases}$$

where α is a complex parameter. The potential of this Schrödinger operator can be understood as the discrete Heaviside function multiplied by the complex coupling constant α . Such an operator is non-self-adjoint for any $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

The study of self-adjoint operators has enjoyed great attention from researchers since the birth of quantum mechanics. However, the study of non-self-adjoint operators as toy models has become prevalent only in the past couple of decades. Although self-adjoint operators are indeed fundamental in quantum mechanics, where they guarantee real spectra, there are many compelling reasons to investigate non-self-adjoint operators. Moreover, from a purely mathematical perspective, non-self-adjoint operators exhibit intriguing spectral behavior such as non-trivial pseudospectra that can differ significantly from their self-adjoint counterparts.

Our main motivation for considering H_α stems from [10]. Therein, the authors considered the operator

$$H := -\frac{d^2}{dx^2} + \text{isgn}(x),$$

defined on a suitable domain. The operator $H_{2i} - iI$ can be viewed as a discrete counterpart to H . As the shift of the operator by a multiple of the identity operator is insignificant with respect to spectral analysis, H_α generalizes H in one complex parameter. In many ways, H and H_α exhibit similar behavior. A comparison was performed in [2]. Later, in [7], the operator H was further studied. In another relevant publication, [5], the authors consider a similar type of potential added to the Dirac operator.

Firstly, we introduce the notation used in this thesis and state several standard results from functional analysis. A prominent part of the first chapter is the introduction of the Joukowski transform, which serves an important role in the analysis of H_α . We derive several properties of this transform including its bijective nature, its inverse, and asymptotic formulas.

The aim of the second chapter is to analyze spectral properties of H_α . Firstly, we describe the spectrum of H_α and its parts, and the Green's kernel. Then, for $\text{Im } \alpha \neq 0$, when H_α is non-self-adjoint, we approximately localize the pseudospectrum of H_α . We also provide asymptotic formulas for these estimates as the spectral parameter approaches the spectrum. We conclude this chapter by explicitly deriving the spectral measure of H_α for $\text{Im } \alpha = 0$.

In the third chapter, we investigate how small but complex perturbations of H_α change the spectrum. In other words, using the Birman–Schwinger principle we investigate the spectrum of $H_\alpha + V$ where V is a diagonal complex potential. We complement the classical results for the continuous case obtained by Simon in [19] and the discrete case in [12] by allowing the potential V to be complex. Depending on the coupling constant α we show whether $H_\alpha + V$ has a nonempty point spectrum or exhibits spectral stability and under what assumptions on the potential V .

Chapter 1

Preliminaries

1.1 Notation

Notation 1.1: We denote a general Hilbert space by \mathcal{H} . The norm on \mathcal{H} and the operator norm are both denoted by $\|\cdot\|$ and we use the symbol $\langle \cdot, \cdot \rangle$ for the inner product, which is linear in the second argument. We denote the space of bounded operators on \mathcal{H} as $\mathcal{B}(\mathcal{H})$ and the space of compact operators on \mathcal{H} as $\mathcal{K}(\mathcal{H})$.

Notation 1.2: We denote the identity operator by I . Let $A \in \mathcal{B}(\mathcal{H})$. For $a \in \mathbb{C}$ we abuse notation by

$$A + a := A + aI.$$

The *spectrum* of A is defined as the complement of the *resolvent set*, i.e. $\sigma(A) := \mathbb{C} \setminus \rho(A)$, where

$$\rho(A) := \{\lambda \in \mathbb{C} \mid (A - \lambda)^{-1} \in \mathcal{B}(\mathcal{H})\}.$$

We decompose the spectrum into the *point spectrum*, *continuous spectrum*, and *residual spectrum*:

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A),$$

where

$$\sigma_p(A) := \{\lambda \in \mathbb{C} \mid A - \lambda \text{ is not injective}\},$$

$$\sigma_c(A) := \{\lambda \in \mathbb{C} \mid A - \lambda \text{ is injective but not surjective and } \text{Ran}(A - \lambda) \text{ is dense in } \mathcal{H}\},$$

$$\sigma_r(A) := \{\lambda \in \mathbb{C} \mid A - \lambda \text{ is injective and } \text{Ran}(A - \lambda) \text{ is not dense in } \mathcal{H}\}.$$

The *algebraic multiplicity* of an isolated eigenvalue of A is defined by

$$\nu_a(\lambda) := \dim(\text{Ran}(P_\lambda)),$$

where

$$P_\lambda := \frac{1}{2\pi i} \oint_{\partial B(\lambda, \varepsilon)} (A - z)^{-1} dz.$$

The radius ε is chosen such that $\overline{B(\lambda, \varepsilon)} \cap \sigma(A) = \emptyset$. The *discrete spectrum* and *essential spectrum* of A as defined by

$$\sigma_d(A) := \{\lambda \in \sigma_p(A) \mid \lambda \text{ is isolated and } \nu_a(\lambda) < \infty\},$$

$$\sigma_{\text{ess}}(A) := \sigma(A) \setminus \sigma_d(A).$$

Notation 1.3: The linear span of the set S is denoted as $\text{span}(S)$.

Notation 1.4: Let $a, b \in \mathbb{C}$. We define the line segment between these two complex numbers as the set of all convex combinations:

$$[a, b] := \{at + b(1 - t) \mid t \in [0, 1]\}.$$

Notation 1.5: Let $v = \{v_n\}_{n \in \mathbb{Z}}$ be a complex-valued sequence. We define the diagonal matrix with generating sequence v as

$$\text{diag}(v) := \{v_n \delta_{m,n}\}_{m,n \in \mathbb{Z}},$$

where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise.

Notation 1.6: Let $v = \{v_n\}_{n \in \mathbb{Z}}$ be a complex-valued sequence. We define

$$\begin{aligned} |v| &:= \{|v_n|\}_{n \in \mathbb{Z}}, & V &:= \text{diag}(v), \\ |v|^{1/2} &:= \{\sqrt{|v_n|}\}_{n \in \mathbb{Z}}, & |V| &:= \text{diag}(|v|), \\ v_{1/2} &:= \{\sqrt{|v_n|} \text{sgn } v_n\}_{n \in \mathbb{Z}}, & |V|^{1/2} &:= \text{diag}(|v|^{1/2}), \\ & & V_{1/2} &:= \text{diag}(v_{1/2}), \end{aligned}$$

where the signum function for complex arguments is given by

$$\text{sgn } z := \begin{cases} \frac{z}{|z|} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Notation 1.7: Let $v \in \ell^1(\mathbb{Z})$ and $k \in \mathbb{N}$. We define the weighted ℓ^1 norm of v by

$$\|v\|_{\ell^1(\mathbb{Z}, m^k)} := |v_0| + \sum_{m \in \mathbb{Z}} |m|^k |v_m|.$$

We also define subspaces of $\ell^1(\mathbb{Z})$ by

$$\ell^1(\mathbb{Z}, m^k) := \{v \in \ell^1(\mathbb{Z}) \mid \|v\|_{\ell^1(\mathbb{Z}, m^k)} < \infty\}.$$

One can verify that $\|\cdot\|_{\ell^1(\mathbb{Z}, m^k)}$ satisfies the properties of a norm. Moreover, it is straightforward to observe that $\|v\|_{\ell^1(\mathbb{Z})} \leq \|v\|_{\ell^1(\mathbb{Z}, m^k)} \leq \|v\|_{\ell^1(\mathbb{Z}, m^l)}$, whenever $k \leq l$.

1.2 Standard results from functional analysis

Theorem 1.8 (Riesz–Schauder): Let $A \in \mathcal{K}(\mathcal{H})$. Then

1. $0 \in \sigma(A)$,
2. every non-zero element of the spectrum is an eigenvalue,
3. the set of all eigenvalues is at most countable,
4. every non-zero eigenvalue has a finite multiplicity,
5. the spectrum has at most one accumulation point $\lambda = 0$.

Theorem 1.9 (Weyl's theorem): Let $A \in \mathcal{B}(\mathcal{H})$ and $V \in \mathcal{K}(\mathcal{H})$. Assume the following to hold.

- The interior of $\sigma(A)$ is empty in the topology of \mathbb{C} .
- Each connected component of $\rho(A)$ contains a point in $\rho(A + V)$.

Then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A + V)$.

The proof of Theorem 1.9 can be found in [17].

Lemma 1.10: Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}.$$

Proof. Choose any $\lambda \notin \sigma(AB) \cup \{0\}$, then there exists a bounded inverse operator to $(AB - \lambda)$. Then set

$$C := \frac{1}{\lambda}(-I + B(AB - \lambda)^{-1}A),$$

Clearly, C is a bounded operator. One can easily check that C is the inverse to $(BA - \lambda)$. We get that $\rho(AB) \setminus \{0\} \subset \rho(BA)$. This inclusion still holds if we remove zero from the right hand side. By interchanging A and B we get the opposite inclusion. If we take the complement of this set equality we arrive at the assertion of the lemma. \square

Theorem 1.11 (Birman–Schwinger principle): Let $H, V \in \mathcal{B}(\mathcal{H})$, $\lambda \in \rho(H)$. Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $V = AB$ and define the Birman–Schwinger operator by

$$K(\lambda) := B(H - \lambda)^{-1}A.$$

Then

1. $\lambda \in \sigma_{\text{p}}(H + V) \implies -1 \in \sigma_{\text{p}}(K(\lambda)),$
2. $V \in \mathcal{K}(\mathcal{H}) \text{ and } -1 \in \sigma_{\text{p}}(K(\lambda)) \implies \lambda \in \sigma_{\text{p}}(H + V).$

Proof. For the first statement:

Suppose $\lambda \in \sigma_{\text{p}}(H + V)$. Then there exists a non-zero $\psi \in \mathcal{H}$ such that

$$(H + V)\psi = \lambda\psi \iff (H - \lambda)\psi = -V\psi. \quad (1.2.1)$$

Since $\lambda \in \rho(H)$, the operator $(H - \lambda)$ has a bounded inverse. Applying $(H - \lambda)^{-1}$, we find

$$-\psi = (H - \lambda)^{-1}V\psi = (H - \lambda)^{-1}AB\psi.$$

Next, applying B to both sides and letting $\phi := B\psi$, we derive

$$-\phi = B(H - \lambda)^{-1}A\phi = K(\lambda)\phi. \quad (1.2.2)$$

Since $\phi \neq 0$ (as $\phi = 0$ leads to a contradiction with $\lambda \in \sigma_{\text{p}}(H)$), it follows that -1 is an eigenvalue of $K(\lambda)$.

Next, we move to the second statement. If -1 is an eigenvalue of $K(\lambda)$ and in turn an element of the spectrum, then by Lemma 1.10 we have that $-1 \in \sigma((H - \lambda)^{-1}AB)$. Since V is

a compact operator, $(H - \lambda)^{-1}V$ is also compact. Then by Theorem 1.8 it follows that -1 is an eigenvalue of $(H - \lambda)^{-1}V$ and so

$$\exists \psi \neq 0 \quad : \quad (H - \lambda)^{-1}V\psi = -\psi.$$

If we apply $(H - \lambda)$ on the equation from the left, we arrive at

$$\exists \psi \neq 0 \quad : \quad (H + V)\psi = \lambda\psi,$$

that is $\lambda \in \sigma_p(H + V)$. □

In this paper, we specialize Theorem 1.11 to the Hilbert space $\ell^2(\mathbb{Z})$ with additional assumptions on the potential V . Specifically, we assume that V is a diagonal operator, enabling the use of Notation 1.6.

A common choice for decomposing the diagonal operator V in the Birman–Schwinger principle is to set

$$A = |V|^{1/2}, \quad B = V_{1/2}.$$

Additionally, we impose the assumption that the sequence v defining V is summable in absolute value, i.e. $v \in \ell^1(\mathbb{Z})$. This summability condition ensures that $\lim_{n \rightarrow \pm\infty} v_n = 0$. One easily checks that

$$V \in \mathcal{K}(\mathcal{H}) \quad \iff \quad \lim_{n \rightarrow \pm\infty} v_n = 0.$$

Then it follows that V is a compact operator. Under these assumptions, the Birman–Schwinger principle establishes an equivalence formulated in the next theorem.

Theorem 1.12: Let $H \in \mathcal{B}(\ell^2(\mathbb{Z}))$, $\lambda \in \rho(H)$, $v \in \ell^1(\mathbb{Z})$. If we denote the Birman–Schwinger operator

$$K(\lambda) = V_{1/2}(H - \lambda)^{-1}|V|^{1/2},$$

then

$$\lambda \in \sigma_p(H + V) \quad \iff \quad -1 \in \sigma_p(K(\lambda)).$$

Theorem 1.13 (Schur test): Let $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ be a doubly infinite matrix of complex entries. If there exists a positive number p_j for all $j \in \mathbb{Z}$ such that

$$\sup_{j \in \mathbb{Z}} \frac{1}{p_j} \sum_{i \in \mathbb{Z}} |a_{i,j}| p_i =: \alpha < \infty \quad \text{and} \quad \sup_{i \in \mathbb{Z}} \frac{1}{p_i} \sum_{j \in \mathbb{Z}} |a_{i,j}| p_j =: \beta < \infty,$$

then $A \in \mathcal{B}(\ell^2(\mathbb{Z}))$ and $\|A\| \leq \sqrt{\alpha\beta}$.

Proof. Choose an arbitrary $\psi \in \text{span}\{e_n \mid n \in \mathbb{Z}\}$, then

$$\begin{aligned} \|A\psi\|^2 &= \sum_{i \in \mathbb{Z}} \left| \sum_{j \in \mathbb{Z}} a_{i,j} \psi_j \right|^2 \leq \sum_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \sqrt{|a_{i,j}| p_j} \sqrt{\frac{|a_{i,j}|}{p_j}} |\psi_j| \right)^2 \\ &\stackrel{\text{C.-S.}}{\leq} \underbrace{\sum_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} |a_{i,j}| p_j \right)}_{\leq \alpha p_i, \forall i} \left(\sum_{j \in \mathbb{Z}} \frac{|a_{i,j}|}{p_j} |\psi_j|^2 \right) \stackrel{*}{\leq} \alpha \sum_{j \in \mathbb{Z}} \underbrace{\left(\frac{1}{p_j} \sum_{i \in \mathbb{Z}} |a_{i,j}| p_i \right)}_{\leq \beta} |\psi_j|^2 \leq \alpha\beta \|\psi\|^2. \end{aligned}$$

C.-S. signifies the Cauchy–Schwarz inequality and after the starred inequality, the interchange of summation is justified by Tonelli’s theorem. □

Remark. A few observations about this operator norm inequality:

1. If the given doubly infinite matrix A is symmetric in terms of the absolute value of its entries, then

$$\alpha = \beta \quad \text{and} \quad \|A\| \leq \sup_{j \in \mathbb{Z}} \frac{1}{p_j} \sum_{i \in \mathbb{Z}} |a_{i,j}| p_i.$$

2. In some cases, we may obtain simpler estimates by setting $p_j = 1$ for all $j \in \mathbb{Z}$. While this approach might yield rougher bounds, it can simplify computations significantly in practical applications.

Weyl's criterion can be formulated in various ways, we present the version best suited for our case. Typically, Weyl's criterion is stated for normal operators, where the criterion asserts an equivalence. However, since we intend to apply the theorem to a non-normal operator, we restrict our statement to a single implication that does not rely on this assumption.

Theorem 1.14 (Weyl's criterion): Let $A \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then

$$\exists \{x_n\}_{n=1}^{\infty} \subset \mathcal{H} \setminus \{0\} : \lim_{n \rightarrow \infty} \frac{\|(A - \lambda)x_n\|}{\|x_n\|} = 0 \implies \lambda \in \sigma(A).$$

Proof. We shall prove the contrapositive implication,

$$\lambda \in \rho(A) \implies \exists K > 0, \quad \forall \{x_n\}_{n=1}^{\infty} \subset \mathcal{H} \setminus \{0\} : \lim_{n \rightarrow \infty} \frac{\|(A - \lambda)x_n\|}{\|x_n\|} \geq K.$$

Choose $\lambda \in \rho(A)$ and $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H} \setminus \{0\}$ arbitrarily. Then for any $n \in \mathbb{N}$, we have

$$\|x_n\| = \|(A - \lambda)^{-1}(A - \lambda)x_n\| \leq \|(A - \lambda)^{-1}\| \|(A - \lambda)x_n\|.$$

Since $\lambda \in \rho(A)$, we have $\|(A - \lambda)^{-1}\| \in (0, \infty)$. This allows us to set

$$K := \frac{1}{\|(A - \lambda)^{-1}\|}.$$

And because $\forall n \in \mathbb{N} : x_n \neq 0$ we can estimate

$$K \leq \frac{\|(A - \lambda)x_n\|}{\|x_n\|}.$$

To conclude the proof, take the limit as $n \rightarrow \infty$. □

Theorem 1.15 (Theorem XIII.106 in [17]): Let A be a trace class operator. Then

$$\det(I + A) = \prod_{j=1}^{N(A)} (1 + \lambda_j(A))$$

where $\{\lambda_j(A)\}_{j=1}^{N(A)}$ are the eigenvalues of A counted with algebraic multiplicity.

The two following theorems are standard results from complex analysis and can be found in [4].

Theorem 1.16 (Rouché's theorem): Let C be a simple closed contour. Suppose that

- two functions f and g are holomorphic inside and on C ,
- at each point z on C $|f(z)| < |g(z)|$.

Then g and $g + f$ have the same number of zeros, counting multiplicities, inside C .

Theorem 1.17 (Maximum modulus principle): Suppose that a function f is continuous on a closed bounded region R and that is holomorphic and not constant in the interior of R . Then the maximum value of $|f|$ in R which is always reached, occurs on the boundary of R and never in the interior

1.3 The Joukowski transform

A key mathematical tool in spectral analysis of the operator H_α , which will be defined later, see (2.0.1), is the *Joukowski transform*. It relates the spectral parameter λ to a new variable k through the equation

$$\lambda = k + k^{-1}. \quad (1.3.1)$$

The inverse of this transform, $k = k(\lambda)$, will be derived later. When analyzing the operator H_0 , we will denote the Joukowski transform parameter by k to pay homage to [11] and to differentiate it from the case of H_α for $\alpha \neq 0$, where we use two instances of Joukowski transform

$$\lambda = \xi + \xi^{-1}, \quad \lambda - \alpha = \eta + \eta^{-1}. \quad (1.3.2)$$

Let us state some properties of this transform.

Proposition 1.18: Joukowski transform given by (1.3.1) is a bijective map between the sets

$$\{k \in \mathbb{C} \mid 0 < |k| < 1\} \quad \longleftrightarrow \quad \mathbb{C} \setminus [-2, 2].$$

Furthermore, it maps $\{k \in \mathbb{C} \mid 0 < |k| \leq 1\}$ onto the whole complex plane, though not injectively.

Proof. Let us parameterize the set

$$\{k \in \mathbb{C} \mid 0 < |k| < 1\} = \{k = re^{i\phi} \mid r \in (0, 1), \phi \in (-\pi, \pi)\}.$$

The elements k of the punctured unit disk get transformed by the transform as

$$\lambda = re^{i\phi} + \frac{1}{r}e^{-i\phi} = \left(r + \frac{1}{r}\right) \cos \phi + i\left(r - \frac{1}{r}\right) \sin \phi. \quad (1.3.3)$$

If we fix some value of r and consider λ as a function of ϕ , it traces ellipses in the complex plane with foci at -2 and 2 . If we vary the value of r , we clearly see that we arrive at a parametrization of

$$\mathbb{C} \setminus [-2, 2] = \left\{ \left(r + \frac{1}{r}\right) \cos \phi + i\left(r - \frac{1}{r}\right) \sin \phi \mid r \in (0, 1), \phi \in (-\pi, \pi) \right\}.$$

Furthermore, if we set $r = 1$ and vary ϕ we get a parametrization of the interval $[-2, 2]$. Clearly, in this case, the bijective nature is no longer present. \square

Proposition 1.19: Let $\lambda \in \mathbb{C} \setminus [-2, 2]$. The inverse Joukowski transform $k = k(\lambda)$ is given by

$$k(\lambda) = \begin{cases} (\lambda + \sqrt{\lambda^2 - 4})/2 & \text{Re } \lambda < 0 \text{ or } \text{Re } \lambda = 0, \text{Im } \lambda \leq 0, \\ (\lambda - \sqrt{\lambda^2 - 4})/2 & \text{Re } \lambda > 0 \text{ or } \text{Re } \lambda = 0, \text{Im } \lambda \geq 0, \end{cases}$$

where $\sqrt{\cdot}$ is assumed to be its principal branch. Furthermore, the reciprocal value of k is given by

$$(k(\lambda))^{-1} = \begin{cases} (\lambda - \sqrt{\lambda^2 - 4})/2 & \text{Re } \lambda < 0 \text{ or } \text{Re } \lambda = 0, \text{Im } \lambda \leq 0, \\ (\lambda + \sqrt{\lambda^2 - 4})/2 & \text{Re } \lambda > 0 \text{ or } \text{Re } \lambda = 0, \text{Im } \lambda \geq 0. \end{cases}$$

Proof. Let us break down the proof of this proposition into three parts.

1. Form of k :

If we multiply (1.3.1) by a non-zero k , we obtain a quadratic equation $k^2 - \lambda k + 1 = 0$, the solutions of which read

$$k_{\pm} = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

2. Reciprocal value of k :

Since the constant term in a quadratic equation is the product of its roots and the constant term in the studied equation is 1, we have $k_+ k_- = 1$; hence,

$$k_{\pm}^{-1} = k_{\mp} = \frac{\lambda \mp \sqrt{\lambda^2 - 4}}{2}.$$

From this follows the equivalence

$$|k_+| \geq 1 \quad \iff \quad |k_-| \leq 1.$$

3. Piece-wise nature of k :

It suffices to show the following

$$\text{Re } \lambda > 0 \text{ or } (\text{Re } \lambda = 0 \text{ and } \text{Im } \lambda \geq 0) \quad \implies \quad |k_-| \leq |k_+|. \quad (1.3.4)$$

- Let $\text{Re } \lambda \geq 0$, then one may write $\lambda = x + iy$, where $x \geq 0$ and $y \in \mathbb{R}$. Considering the principal square root, we know that $\forall z \in \mathbb{C} : \text{Re } \sqrt{z} \geq 0$. This allows us to write $\sqrt{\lambda^2 - 4} = u + iv$, where also $u \geq 0$ and $v \in \mathbb{R}$. Let us rewrite the inequality in (1.3.4)

$$\begin{aligned} |x + iy - u - iv| &\leq |x + iy + u + iv| \\ (x - u)^2 + (y - v)^2 &\leq (x + u)^2 + (y + v)^2 \\ 0 &\leq 4xu + 4yv. \end{aligned}$$

The assumption above and said property of complex square root implies that $4xu \geq 0$. To conclude this proof we need to show that $\text{sgn } y = \text{sgn } v$, i.e. $\text{sgn } \text{Im } \lambda = \text{sgn } \text{Im } \sqrt{\lambda^2 - 4}$. Applying the formula for the square root of a complex number

$$\sqrt{a + ib} = \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} + i \text{sgn}(b) \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}$$

on $\sqrt{\lambda^2 - 4}$ we get

$$\begin{aligned} \operatorname{Im} \sqrt{\lambda^2 - 4} &= \operatorname{Im} \sqrt{(x^2 - y^2 - 4) + i(2xy)} \\ &= \operatorname{sgn}(2xy) \sqrt{\frac{\sqrt{(x^2 - y^2 - 4)^2 + (2xy)^2} - x^2 + y^2 + 4}{2}}. \end{aligned}$$

From this one easily sees

$$\operatorname{sgn}(\operatorname{Im} \sqrt{\lambda^2 - 4}) = \operatorname{sgn}(2 \underbrace{\operatorname{Re} \lambda}_{\geq 0} \operatorname{Im} \lambda) = \operatorname{sgn}(\operatorname{Im} \lambda).$$

- Let $\operatorname{Re} \lambda = 0, \operatorname{Im} \lambda \geq 0$, then one may write $\lambda = ic$, where $c \geq 0$. A simple estimation yields (1.3.4); we have

$$|\lambda - \sqrt{\lambda^2 - 4}| = |ic - \sqrt{-c^2 - 4}| \leq c + \sqrt{c^2 + 4} = |ic + \sqrt{-c^2 - 4}| = |\lambda + \sqrt{\lambda^2 - 4}|$$

Similarly, one would show the other implication

$$\operatorname{Re} \lambda < 0 \text{ or } \operatorname{Re} \lambda = 0, \operatorname{Im} \lambda \leq 0 \quad \implies \quad |k_-| \geq |k_+|. \quad \square$$

When deriving the Green's kernel in Section 2.1.3 a prominent expression arises, the Wronskian, denoted by $w = \xi - \eta^{-1}$. A property of the Wronskian of paramount importance is that for $\alpha \neq 0$ it never reaches 0 in absolute value.

Lemma 1.20: Let $\alpha \neq 0$. Then for the Joukowski parameters ξ and η of λ given by (1.3.2) we have

$$\exists K > 0 \forall \lambda \in \mathbb{C} : |\xi(\lambda) - \eta(\lambda)^{-1}| > K.$$

Proof. We will prove this lemma in two steps. First, we show that $\forall \lambda \in \mathbb{C} : \xi(\lambda) \neq \eta(\lambda)^{-1}$. After this, we show that $\lim_{\lambda \rightarrow \infty} |\xi(\lambda) - \eta(\lambda)^{-1}| \neq 0$. Combining these two statements one can clearly see that the expression is nowhere zero and also does not approach zero in the limit; hence, there has to exist a constant K with the desired property.

If we plug in the expressions of ξ and η^{-1} from Proposition 1.19 into $|\xi - \eta^{-1}|$ and multiply by 2, we arrive at

$$2(\xi - \eta^{-1}) = \begin{cases} \alpha + \sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4}, \\ \alpha + \sqrt{\lambda^2 - 4} - \sqrt{(\lambda - \alpha)^2 - 4}, \\ \alpha - \sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4}, \\ \alpha - \sqrt{\lambda^2 - 4} - \sqrt{(\lambda - \alpha)^2 - 4}, \end{cases}$$

where the piece-wise nature of the function depends on the real and imaginary parts of λ and $(\lambda - \alpha)$ but for finding the root of this function this distinction is inconsequential. As illustrated below, once we take squares, the piecewise details no longer affect the outcome, so we omit them. If we take one option, we get

$$\begin{aligned} \alpha + \sqrt{\lambda^2 - 4} &= \sqrt{(\lambda - \alpha)^2 - 4} \\ \alpha^2 + 2\alpha\sqrt{\lambda^2 - 4} + \alpha^2 - 4 &= \lambda^2 - 2\alpha\lambda + \alpha^2 - 4 \\ \sqrt{\lambda^2 - 4} &= \lambda \\ \lambda^2 - 4 &= \lambda^2 \\ -4 &= 0. \end{aligned}$$

There is clearly no solution to this equation.

In the limit $\lambda \rightarrow \infty$ both parameters ξ and η approach 0. Therefore, the term $\xi + \eta^{-1}$ not only does not vanish, it blows up. \square

Remark. The aforementioned estimate works for every non-zero α . In later sections of the text, we will be mainly using the lower bound for the Wronskian. The aforementioned lemma allows us to define a constant dependent on α in the following way

$$\forall \alpha \neq 0 : C(\alpha) := \sup_{\lambda \in \mathbb{C}} \frac{1}{|\xi - \eta^{-1}|}. \quad (1.3.5)$$

Lemma 1.20 yields no quantitative information about the value of $C(\alpha)$. However, when α is real, Lemma 1.21 provides an exact value. For this case, a plot of $C(\alpha)$ is shown in Figure 1.1.

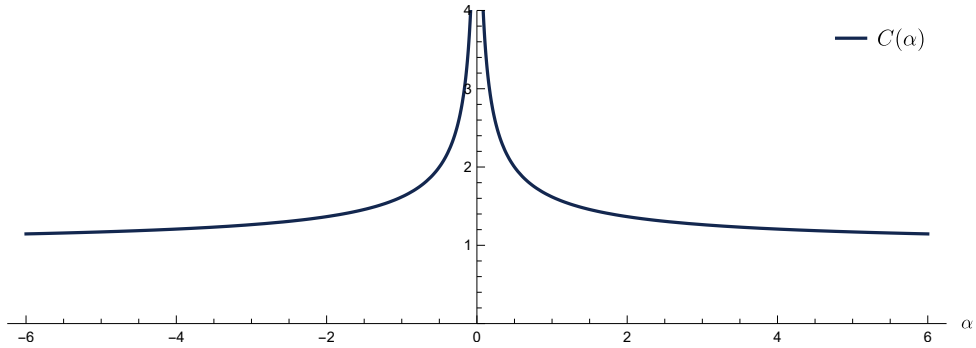


Figure 1.1: Plot of $C(\alpha)$ for real α .

Lemma 1.21: Let $\alpha \in \mathbb{R} \setminus \{0\}$, then

$$C(\alpha) = \left(|\alpha| \left(1 + \frac{|\alpha|}{2} - \sqrt{\frac{|\alpha|^2}{4} + |\alpha|} \right) \right)^{-\frac{1}{2}}.$$

Proof. Since the problem exhibits symmetry about $\alpha = 0$, we may assume without the loss of generality that $\alpha > 0$. The objective is to determine

$$\inf_{\lambda \in \mathbb{C}} |\xi - \eta^{-1}|.$$

The function $\xi - \eta^{-1}$ is holomorphic on the domain $\mathbb{C} \setminus ([-2, 2] \cup [-2 + \alpha, 2 + \alpha])$. However, it is not continuous up to the boundary, as $\xi - \eta^{-1}$ may approach different values depending on the side from which the boundary segments are approached. This arises because the set excludes two, or in some cases only one, line segment(s) in the complex plane, depending on the parameter α . Consequently, care must be taken when applying the maximum modulus principle, Theorem 1.17.

To address this, we decompose the complex plane into the upper and lower half-planes,

$$\mathbb{C}^{+i} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}, \quad \mathbb{C}^{-i} = \{z \in \mathbb{C} \mid \text{Im } z < 0\}.$$

Although these regions are unbounded, we pass to the Joukowski transform parameters ξ and η , which map them to the bounded regions

$$\mathbb{D}^{+i} = \{z \in \mathbb{C} \mid \text{Im } z > 0 \text{ and } |z| < 1\}, \quad \mathbb{D}^{-i} = \{z \in \mathbb{C} \mid \text{Im } z < 0 \text{ and } |z| < 1\}.$$

With this established, the maximum modulus principle ensures that the infimum of $|\xi - \eta^{-1}|$ is attained on the boundary of \mathbb{D}^{+i} or \mathbb{D}^{-i} .

This naturally leads to three, not necessarily mutually exclusive cases, where A: $|\xi| = 1$, B: $|\eta| = 1$, or C: $\xi, \eta \in (-1, 1)$.

A: Let $|\eta| = 1$, then we can write $\eta = e^{i\theta}$, where $\theta \in (-\pi, \pi]$ and we have

$$\xi + \xi^{-1} = \eta + \eta^{-1} + \alpha = 2 \cos \theta + \alpha \begin{cases} \in [-2, 2] & \dots \text{A1,} \\ \notin [-2, 2] & \dots \text{A2.} \end{cases}$$

Now we discuss these sub-cases.

A1: Under the assumption $\alpha > 0$ the condition A1 reduces to $2 \cos \theta + \alpha \leq 2$ and can occur only if $\alpha < 4$. Together with condition A both ξ and η are equal to one in absolute value; hence, there is a function $\theta \mapsto \phi(\theta)$ such that $\xi = e^{i\phi(\theta)}$. Then the equation relating ξ and η reads

$$2 \cos \phi(\theta) = 2 \cos \theta + \alpha \tag{1.3.6}$$

and one can easily take the arc-cosine function to describe the function ϕ explicitly, but there is no need. Let us continue,

$$\begin{aligned} |\xi - \eta^{-1}|^2 &= |e^{i\phi(\theta)} - e^{-i\theta}|^2 = |e^{i(\phi(\theta)+\theta)} - 1|^2 \\ &= (\cos(\phi(\theta) + \theta) - 1)^2 + \sin^2(\phi(\theta) + \theta) \\ &= 2 - 2 \cos(\phi(\theta) + \theta) \\ &= 2 - 2 \cos \theta \cos \phi + 2 \sin \theta \sin \phi. \end{aligned}$$

Next, substitute $t = \cos \theta$, we then have $\cos \phi = t + \frac{\alpha}{2}$, we express the sine functions in terms of t , and we arrive at

$$|\xi - \eta^{-1}|^2 = 2 - 2t \left(t + \frac{\alpha}{2} \right) + 2\sqrt{1-t^2} \sqrt{1 - \left(t + \frac{\alpha}{2} \right)^2} =: 2f(t).$$

The substitution we have utilized allows for $t \in [-1, 1]$. The condition A1 restricts $t \leq 1 - \frac{\alpha}{2}$. The function f is symmetric about $t = -\frac{\alpha}{4}$. This point is also the only critical point of f ; furthermore, it is a global maximum of the function. Hence The minimum of f is attained at the endpoints. For case A1, we have

$$\min_{\text{A1}} |\xi - \eta^{-1}|^2 = 2f(-1) = \alpha.$$

A2: Under the assumption $\alpha > 0$ the condition A2 reduces to $2 \cos \theta + \alpha > 2$ and occurs for any α . In this case, ξ is a real number given by

$$\xi + \xi^{-1} = 2 \cos \theta + \alpha. \tag{1.3.7}$$

The objective function then reads

$$|\xi - \eta^{-1}|^2 = \left| \xi - e^{-i\theta} \right|^2 = (\xi - \cos \theta)^2 + \sin^2 \theta = \xi^2 - 2\xi \cos \theta + 1 = \alpha\xi.$$

Lemma 1.19 yields

$$\xi = \cos \theta + \frac{\alpha}{2} - \sqrt{\left(\cos \theta + \frac{\alpha}{2}\right)^2 - 1},$$

and so

$$\begin{aligned} |\xi - \eta^{-1}|^2 &= \alpha \left(\cos \theta + \frac{\alpha}{2} - \sqrt{\left(\cos \theta + \frac{\alpha}{2}\right)^2 - 1} \right) \\ &= \alpha \left(t + \frac{\alpha}{2} - \sqrt{\left(t + \frac{\alpha}{2}\right)^2 - 1} \right), \end{aligned}$$

where we again substituted $t = \cos \theta \in [-1, 1]$ and the condition A2 restricts $t > 1 - \frac{\alpha}{2}$. The function above is decreasing as a function of t , so the minimum is attained at $t = 1$. For case A2, we have

$$\min_{A2} |\xi - \eta^{-1}|^2 = \alpha \left(1 + \frac{\alpha}{2} - \sqrt{\left(1 + \frac{\alpha}{2}\right)^2 - 1} \right).$$

Since $\left(1 + \frac{\alpha}{2} - \sqrt{\left(1 + \frac{\alpha}{2}\right)^2 - 1}\right)$ is some positive ξ which is less than one, we have

$$\min_A |\xi - \eta^{-1}|^2 = \min_{A2} |\xi - \eta^{-1}|^2.$$

B: Let $|\xi| = 1$, then we can write $\xi = e^{i\phi}$, where $\phi \in (-\pi, \pi]$ and we have

$$\eta + \eta^{-1} = 2 \cos \phi - \alpha \begin{cases} \in [-2, 2] & \dots \text{B1,} \\ \notin [-2, 2] & \dots \text{B2.} \end{cases}$$

B1: This case is largely the same as A1. Here we have $2 \cos \phi \geq \alpha - 2$. $|\eta| = 1$ and so there is a function $\phi \mapsto \theta(\phi)$ such that $\eta = e^{i\theta(\phi)}$. With the substitution $t = \cos \phi$, we get

$$|\xi - \eta^{-1}|^2 = \dots = 2 - 2t \left(t - \frac{\alpha}{2}\right) + 2\sqrt{1-t^2} \sqrt{1 - \left(t - \frac{\alpha}{2}\right)^2} := 2f(t).$$

Condition B1 restricts $t \geq \frac{\alpha}{2} - 1$ and the result for this case reads

$$\min_{B1} |\xi - \eta^{-1}|^2 = \alpha.$$

B2: For $\alpha > 0$ the condition B2 reads $2 \cos \phi < \alpha - 2$. η is a real number given by

$$\eta + \eta^{-1} = 2 \cos \phi - \alpha.$$

Then we have

$$|\xi - \eta^{-1}|^2 = \left| e^{i\phi} - \eta^{-1} \right|^2 = (\cos \phi - \eta^{-1})^2 + \sin^2 \phi = \eta^2 - 2\eta \cos \phi + 1 = -\alpha\eta.$$

Lemma 1.19 yields

$$\eta^{-1} = \cos \phi - \frac{\alpha}{2} - \sqrt{\left(\cos \phi - \frac{\alpha}{2}\right)^2 - 1}.$$

After substituting $t = \cos \phi$, we arrive at

$$|\xi - \eta^{-1}|^2 = -\alpha \left(t - \frac{\alpha}{2} - \sqrt{\left(t - \frac{\alpha}{2}\right)^2 - 1} \right) =: -\alpha g(t).$$

From the substitution $t \in [-1, 1]$ but the condition B2 restricts $t \leq \frac{\alpha}{2} - 1$ so we have

$$t \in \begin{cases} [-1, 1] & \text{if } \alpha \geq 4, \\ [-1, \frac{\alpha}{2} - 1) & \text{if } \alpha \in (0, 4). \end{cases}$$

The function g is increasing and has no critical point, so g will attain its maximum or approach its supremum at the right endpoint. For this final case, we have

$$\inf_{\text{B2}} |\xi - \eta^{-1}|^2 = -\alpha \sup_t g(t) = \begin{cases} -\alpha g\left(\frac{\alpha}{2} - 1\right) = \alpha & \alpha \in (0, 4), \\ -\alpha g(1) = -\alpha \left(1 - \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \alpha}\right) & \alpha \geq 4. \end{cases}$$

Since $\left(1 - \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \alpha}\right)$ is some η^{-1} which is less than -1, we have

$$\min_{\text{B}} |\xi - \eta^{-1}|^2 = \alpha.$$

C: The last case for the discussion to be exhaustive is when both ξ and $\eta \in (-1, 1)$, this corresponds to $\lambda \in \mathbb{R} \setminus ([-2, 2] \cup [-2 + \alpha, 2 + \alpha])$. Depending on the value of α , there are two or three connected components of this set. Let us discuss these three sub-cases separately.

C1: Let $\lambda < -2$. Then $\xi, \eta < 0$ and we have

$$2|\xi - \eta^{-1}| = 2(\xi - \eta^{-1}) = \alpha + \sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4} \Big|_{\lambda=-2} = \alpha + \sqrt{(\alpha + 2)^2 - 4}.$$

C2: Let $\lambda \in (2, \alpha - 2)$, possible only for $\alpha > 4$. Then $\xi > 0$ and $\eta < 0$. We have

$$\begin{aligned} 2|\xi - \eta^{-1}| &= 2(\xi - \eta^{-1}) = \alpha - \sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4} \\ &= \begin{cases} \alpha + \sqrt{(2 - \alpha)^2 - 4} & \text{if } \lambda = 2, \\ \alpha - \sqrt{(2 - \alpha)^2 - 4} & \text{if } \lambda = \alpha - 2. \end{cases} \end{aligned}$$

C3: Let $\lambda > \alpha + 2$. Then $\xi, \eta > 0$ and

$$2|\xi - \eta^{-1}| = 2(\xi - \eta^{-1}) = -\alpha + \sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4} \Big|_{\lambda=\alpha+2} = -\alpha + \sqrt{(\alpha + 2)^2 - 4}.$$

For each of these sub-cases, the following argumentation applies. The Joukowski inverse comes from Lemma 1.19. We used the fact that $\xi < \eta^{-1}$. None of these functions in terms of λ have any critical points; thus, we evaluate the functions at the endpoints. A simple calculation would show that the value for case C3 is minimal. Lastly, if we were to square this minimal value, divide by 4, and rearrange the terms, we would arrive at the expression for the minimal value for case A.

If we repeat the same argument that finalized case A, we get

$$\min |\xi - \eta^{-1}|^2 = \alpha \left(1 + \frac{\alpha}{2} - \sqrt{\left(1 + \frac{\alpha}{2}\right)^2 - 1} \right).$$

Taking the reciprocal value of the square root of this expression we arrive at $C(\alpha)$. \square

Lemma 1.22: Let $\lambda \in (-2, 2)$ and $\tilde{\lambda} = \lambda + i\varepsilon$. The Joukowski parameter k and its reciprocal, associated with $\tilde{\lambda}$, follow the asymptotic expansions

$$k = e^{i\phi} \left(1 + \frac{\varepsilon}{2 \sin \phi} \right) + \mathcal{O}(\varepsilon^2), \quad k^{-1} = e^{-i\phi} \left(1 - \frac{\varepsilon}{2 \sin \phi} \right) + \mathcal{O}(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0_+,$$

where $\phi = -\arccos(\lambda/2) \in (-\pi, 0)$.

Proof. The unit circle $\{e^{i\phi} \mid \phi \in (-\pi, \pi]\}$ is mapped non-injectively onto the interval $[-2, 2]$ by the Joukowski transform. This is clear if we plug this parametrization into (1.3.1); moreover, we have $\lambda = 2 \cos \phi$ and we can express ϕ in terms of λ . There are two options to do so, $\phi = \pm \arccos(\lambda/2)$; this follows from the non-injective nature of the mapping. We chose the negative sign because the lower half of the punctured unit disk is mapped onto the upper complex half-plane, where $\tilde{\lambda}$ resides, by the Joukowski transform. One can easily see this in (1.3.3). By this choice and the assumption $\lambda \in (-2, 2)$, we have $\phi \in (-\pi, 0)$. We obtain the exact expression for the Joukowski parameter of $\tilde{\lambda} = 2 \cos \phi + i\varepsilon$ from Lemma 1.19, which prescribes which of \pm we need to choose

$$k = \frac{2 \cos \phi + i\varepsilon \pm \sqrt{(2 \cos \phi + i\varepsilon)^2 - 4}}{2}.$$

But the assumption of the lemma is that $\lambda \notin [-2, 2]$. Since this assumption is not met for $\varepsilon = 0$, choosing the negative branch ensures k at $\varepsilon = 0$ maps correctly under the transform:

$$k \Big|_{\varepsilon=0} = \frac{2 \cos \phi - \sqrt{(2 \cos \phi)^2 - 4}}{2} = \cos \phi - \sqrt{-\sin^2 \phi} = \cos \phi - i|\sin \phi| = e^{i\phi},$$

where for $\phi \in (-\pi, 0)$ we have $\sin \phi < 0$. To capture the first-order behavior in ε , we differentiate k with respect to ε and then evaluate at $\varepsilon = 0$:

$$\begin{aligned} \frac{dk}{d\varepsilon} &= \frac{i}{2} \left(1 - \frac{i\varepsilon + 2 \cos \phi}{\sqrt{(2 \cos \phi + i\varepsilon)^2 - 4}} \right), \\ \frac{dk}{d\varepsilon} \Big|_{\varepsilon=0} &= \frac{i}{2} \left(1 - \frac{2 \cos \phi}{\sqrt{(2 \cos \phi)^2 - 4}} \right) = \frac{i}{2} \left(1 + \frac{\cos \phi}{i \sin \phi} \right) = \frac{1}{2 \sin \phi} e^{i\phi}. \end{aligned}$$

Combining these two calculations we get

$$k = e^{i\phi} \left(1 + \frac{\varepsilon}{2 \sin \phi} \right) + \mathcal{O}(\varepsilon^2).$$

To check the expansion for the reciprocal value we multiply the known expansion for k with the proposed expansion for k^{-1} and arrive at $kk^{-1} = 1 + \mathcal{O}(\varepsilon)$. \square

Remark. The asymptotic expansions of k when approaching λ from the other side is similar. One needs to take $\phi = \arccos(\lambda/2)$ and replace ε with $-\varepsilon$ in the asymptotic expression.

Corollary 1.23: Let $\lambda \in (-2, 2)$ and $\tilde{\lambda} = \lambda + i\varepsilon$. Then

$$\begin{aligned} |k|^2 &= 1 + \frac{\varepsilon}{\sin \phi} + \mathcal{O}(\varepsilon^2), & |k| &= 1 + \frac{\varepsilon}{2 \sin \phi} + \mathcal{O}(\varepsilon^2), & \text{as } \varepsilon \rightarrow 0_+, \\ |k - k^{-1}|^2 &= 4 \sin^2 \phi + \mathcal{O}(\varepsilon^2), & |k - k^{-1}| &= -2 \sin \phi + \mathcal{O}(\varepsilon^2), & \text{as } \varepsilon \rightarrow 0_+, \end{aligned}$$

where $\phi = -\arccos(\lambda/2) \in (-\pi, 0)$.

Proof. If we take the asymptotic expansion for k and k^{-1} from the previous lemma, a simple calculation yields

$$\begin{aligned} |k|^2 &= \left(1 + \frac{\varepsilon}{2 \sin \phi} + \mathcal{O}(\varepsilon^2)\right)^2 = 1 + \frac{\varepsilon}{\sin \phi} + \mathcal{O}(\varepsilon^2), \\ |k - k^{-1}|^2 &= \left|e^{i\phi} - e^{-i\phi} + \frac{\varepsilon}{2 \sin \phi} (e^{i\phi} + e^{-i\phi}) + \mathcal{O}(\varepsilon^2)\right|^2 = 4 \sin^2 \phi + \mathcal{O}(\varepsilon^2). \end{aligned}$$

as $\varepsilon \rightarrow 0_+$. If we apply the asymptotic expansion $\sqrt{1+x} = 1 + \frac{x}{2} + \mathcal{O}(x^2)$ as $x \rightarrow 0_+$, we arrive at the asymptotic expansions for the absolute values of the desired expression. Note that since $\phi \in (-\pi, 0)$ we have $|\sin \phi| = -\sin \phi$. \square

Lemma 1.24: Let $\tilde{\lambda} = 2 + i\varepsilon$. Then the Joukowski parameter k and its reciprocal, associated with $\tilde{\lambda}$, follow the asymptotic expansions

$$k = 1 - \sqrt{i\varepsilon} + \mathcal{O}(\varepsilon), \quad k^{-1} = 1 + \sqrt{i\varepsilon} + \mathcal{O}(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0_+.$$

Similarly, for $\lambda = -2 + i\varepsilon$, we have

$$k = -1 + \sqrt{-i\varepsilon} + \mathcal{O}(\varepsilon), \quad k^{-1} = -1 - \sqrt{-i\varepsilon} + \mathcal{O}(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0_+.$$

Proof. We will show proof only for $\tilde{\lambda} = 2 + i\varepsilon$ as the other case is analogous. Since $\forall \varepsilon > 0 : \operatorname{Re} \tilde{\lambda} > 0$, Lemma 1.19 states that

$$k = \frac{2 + i\varepsilon - \sqrt{(2 + i\varepsilon)^2 - 4}}{2}.$$

Then after some manipulation of the expression we apply the Taylor expansion $\sqrt{1-x} = 1 + \mathcal{O}(x)$, as $x \rightarrow 0_+$, and arrive at the desired asymptotic formula

$$k = 1 + \frac{i}{2}\varepsilon - \sqrt{i\varepsilon} \sqrt{1 - \frac{\varepsilon}{4i}} = 1 + \frac{i}{2}\varepsilon - \sqrt{i\varepsilon}(1 + \mathcal{O}(\varepsilon)) = 1 - \sqrt{i\varepsilon} + \frac{i}{2}\varepsilon + \mathcal{O}(\varepsilon^{3/2}) = 1 - \sqrt{i\varepsilon} + \mathcal{O}(\varepsilon),$$

as $\varepsilon \rightarrow 0_+$. To check the expansion for the reciprocal value we multiply the known expansion for k with the proposed expansion for k^{-1} and arrive at $kk^{-1} = 1 + \mathcal{O}(\varepsilon)$, as $\varepsilon \rightarrow 0_+$. \square

Corollary 1.25: Let $\lambda \in \{-2, 2\}$ and $\tilde{\lambda} = \lambda + i\varepsilon$. Then

$$\begin{aligned} |k|^2 &= 1 - \sqrt{2\varepsilon} + \mathcal{O}(\varepsilon), & |k| &= 1 - \frac{\sqrt{\varepsilon}}{\sqrt{2}} + \mathcal{O}(\varepsilon), \\ |k - k^{-1}|^2 &= 4\varepsilon + \mathcal{O}(\varepsilon^{3/2}), & |k - k^{-1}| &= 2\sqrt{\varepsilon} + \mathcal{O}(\varepsilon), \end{aligned} \quad \text{as } \varepsilon \rightarrow 0_+.$$

Proof. Again, let us provide the calculations for $\lambda = 2$ as the other case is analogous. Take the asymptotic expansion for k from the previous lemma and calculate

$$\begin{aligned} |k|^2 &= \left| 1 - \sqrt{i\varepsilon} + \mathcal{O}(\varepsilon) \right|^2 = \left(1 - \frac{1}{\sqrt{2}}\sqrt{\varepsilon} \right)^2 + \left(\frac{1}{\sqrt{2}}\sqrt{\varepsilon} \right)^2 + \mathcal{O}(\varepsilon^2) = 1 - \sqrt{2\varepsilon} + \mathcal{O}(\varepsilon), \\ |k - k^{-1}|^2 &= \left| 1 - \sqrt{i\varepsilon} - 1 - \sqrt{i\varepsilon} + \mathcal{O}(\varepsilon) \right|^2 = \left| -\sqrt{2\varepsilon} - i\sqrt{2\varepsilon} + \mathcal{O}(\varepsilon) \right|^2 = 4\varepsilon + \mathcal{O}(\varepsilon^{3/2}), \end{aligned}$$

as $\varepsilon \rightarrow 0_+$. As was done previously, if we apply the asymptotic expansion $\sqrt{1+x} = 1 + \frac{x}{2} + \mathcal{O}(x^2)$ as $x \rightarrow 0_+$, we arrive at the asymptotic expansions for the absolute values of the desired expressions. \square

Chapter 2

Spectral properties of H_α

The discrete nature of the operator arises from the underlying Hilbert space. It is the space of square summable doubly infinite sequences

$$\ell^2(\mathbb{Z}) = \left\{ \{x_n\}_{n=-\infty}^{\infty} \mid \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty \right\}.$$

In this space we define an orthogonal basis

$$\forall n \in \mathbb{Z} : \quad e_n := \{\delta_{k,n}\}_{k=-\infty}^{\infty}.$$

We now formally introduce the operator under study. The *discrete Schrödinger operator with complex step potential* will be denoted by H_α . It is dependent on one free complex parameter α . Thus, we are considering a single-parameter class of operators $\{H_\alpha\}_{\alpha \in \mathbb{C}}$. The operator is defined by its action on $\forall x \equiv \{x_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{Z})$ by

$$(H_\alpha x)_n = \begin{cases} x_{n-1} + x_{n+1} & n < 0, \\ x_{n-1} + \alpha x_n + x_{n+1} & n \geq 0. \end{cases} \quad (2.0.1)$$

The matrix representation of H_α reads

$$H_\alpha = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & \alpha & 1 & & \\ & & & 1 & \alpha & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Since $H_\alpha^* = H_{\bar{\alpha}}$, the operator H_α is not self-adjoint if $\text{Im} \alpha \neq 0$. Furthermore, with this restriction on the parameter α H_α is not normal as is evident by

$$(H_\alpha H_\alpha^*)_{-1,0} = \bar{\alpha}, \quad (H_\alpha^* H_\alpha)_{-1,0} = \alpha.$$

The operator H_α is bounded. To show this we decompose

$$H_\alpha = S + \alpha D + S^*, \quad (2.0.2)$$

where

$$\forall n \in \mathbb{Z} : \quad S e_n := e_{n-1}, \quad S^* e_n = e_{n+1},$$

$$\forall n \geq 0 : \quad De_n := e_n, \quad \forall n < 0 : \quad De_n := 0.$$

The following clearly holds,

$$\|S\| = \|D\| = \|S^*\| = 1 \quad \implies \quad \|H_\alpha\| \leq \|S\| + |\alpha|\|D\| + \|S^*\| = 2 + |\alpha|.$$

2.1 Spectral analysis of H_α

The spectrum of H_α is purely continuous and coincides with $[-2, 2] \cup [-2 + \alpha, 2 + \alpha]$. While we originally proved this result in [1], we include a detailed derivation here for the sake of completeness. It is well known that the spectrum of H_0 is purely continuous and coincides with the interval $[-2, 2]$. For this reason, we assume $\alpha \neq 0$. The proof is organized into four distinct steps.

1. Prove that the point spectrum is empty.
2. Prove that the residual spectrum is empty.
3. Identify a suitable subset of the resolvent set of H_α , denoted by X .
4. Demonstrate that the complement of X is a subset of the spectrum.

2.1.1 Point spectrum

A complex number λ is an eigenvalue of H_α if and only if there exists a non-zero vector $x \in \ell^2(\mathbb{Z})$ such that it satisfies the eigenvalue equation $H_\alpha x = \lambda x$. In its expanded form the equation reads

$$\begin{aligned} \lambda x_n &= x_{n-1} + x_{n+1} & n < 0, \\ (\lambda - \alpha)x_n &= x_{n-1} + x_{n+1} & n \geq 0. \end{aligned}$$

If we define the sequence

$$\kappa_n := \begin{cases} \lambda & n < 0, \\ \lambda - \alpha & n \geq 0, \end{cases}$$

we can rewrite the equation above in a more concise form

$$\forall n \in \mathbb{Z} \quad : \quad \kappa_n x_n = x_{n+1} + x_{n-1}. \quad (2.1.1)$$

This is a difference equation of second order with a two-dimensional set of solutions. To find the solution, we apply the two Joukowski transforms

$$\lambda = \xi + \xi^{-1}, \quad \lambda - \alpha = \eta + \eta^{-1}.$$

This transformation of λ enables us to express the solution in a simple form. Assume that $\xi \neq \pm 1$ and $\eta \neq \pm 1$, as these special cases will be addressed later. For $n \leq -2$, the solutions to (2.1.1) are given by $x_n = \xi^n$ and $x_n = \xi^{-n}$, while for $n \geq 1$, the solutions are $x_n = \eta^{-n}$ and $x_n = \eta^n$. Indeed,

$$\begin{aligned} \lambda x_n &= (\xi + \xi^{-1})\xi^n = \xi^{n+1} + \xi^{n-1} = x_{n+1} + x_{n-1} \\ \lambda x_n &= (\xi + \xi^{-1})\xi^{-n} = \xi^{-n+1} + \xi^{-n-1} = x_{n+1} + x_{n-1} \end{aligned} \quad n \leq -2,$$

$$\begin{aligned} (\lambda - \alpha)x_n &= (\eta + \eta^{-1})\eta^n = \eta^{n+1} + \eta^{n-1} = x_{n+1} + x_{n-1} \\ (\lambda - \alpha)x_n &= (\eta + \eta^{-1})\eta^{-n} = \eta^{-n+1} + \eta^{-n-1} = x_{n+1} + x_{n-1} \end{aligned} \quad n \geq 1.$$

The general solution to (2.1.1) takes the form

$$\begin{aligned} x_n &= A\xi^n + B\xi^{-n} & n < 0, \\ x_n &= C\eta^n + D\eta^{-n} & n \geq 0. \end{aligned} \quad (2.1.2)$$

Since the space of solutions has dimension two, two out of the four constants A, B, C, D are dependent on the other two. We can determine them by plugging (2.1.2) into the difference equation (2.1.1) at the indices $n \in \{-1, 0\}$

$$\begin{aligned} n = -1 & : A\xi^{-2} + B\xi^2 + C + D = (\xi + \xi^{-1})(A\xi^{-1} + B\xi) \\ & A + B = C + D, \end{aligned} \quad (2.1.3)$$

$$\begin{aligned} n = 0 & : A\xi^{-1} + B\xi + C\eta + D\eta^{-1} = (\eta + \eta^{-1})(C + D) \\ & A\xi^{-1} + B\xi = C\eta^{-1} + D\eta. \end{aligned} \quad (2.1.4)$$

Under the standing assumption that $\xi \neq \pm 1$ and $\eta \neq \pm 1$ we can derive

$$\begin{aligned} A &= C \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} + D \frac{\eta - \xi}{\xi^{-1} - \xi}, & B &= C \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} + D \frac{\xi - \eta^{-1}}{\xi - \xi^{-1}}, \\ C &= A \frac{\xi^{-1} - \eta}{\eta^{-1} - \eta} + B \frac{\xi - \eta}{\eta^{-1} - \eta}, & D &= A \frac{\xi^{-1} - \eta^{-1}}{\eta - \eta^{-1}} + B \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}}. \end{aligned}$$

Let us define two auxiliary sets of doubly infinite sequences

$$\begin{aligned} \ell^2(+\infty) &:= \left\{ \{x_n\}_{n \in \mathbb{Z}} \subset \mathbb{C} \mid \sum_{n=0}^{+\infty} |x_n|^2 < \infty \right\}, \\ \ell^2(-\infty) &:= \left\{ \{x_n\}_{n \in \mathbb{Z}} \subset \mathbb{C} \mid \sum_{n=-\infty}^0 |x_n|^2 < \infty \right\}. \end{aligned}$$

By choosing the values of two independent constants we get two specific solutions to the eigenvalue equation, let us denote them y and z

$$y_n = \begin{vmatrix} A = 0 \\ B = 1 \end{vmatrix} = \begin{cases} \xi^{-n} & n < 0, \\ \frac{\xi - \eta}{\eta^{-1} - \eta} \eta^n + \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}} \eta^{-n} & n \geq 0, \end{cases} \quad (2.1.5)$$

$$z_n = \begin{vmatrix} C = 1 \\ D = 0 \end{vmatrix} = \begin{cases} \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} \xi^n + \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} \xi^{-n} & n < 0, \\ \eta^n & n \geq 0. \end{cases} \quad (2.1.6)$$

One can clearly see that

$$\begin{aligned} |\xi| < 1 &\iff y \in \ell^2(-\infty), \\ |\eta| < 1 &\iff z \in \ell^2(+\infty). \end{aligned}$$

Let us verify that they are linearly independent,

$$\begin{aligned} \begin{vmatrix} y_0 & z_0 \\ y_1 & z_1 \end{vmatrix} &= \begin{vmatrix} \frac{\xi-\eta}{\eta^{-1}-\eta} + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}} & 1 \\ \frac{\xi-\eta}{\eta^{-1}-\eta}\eta + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{-1} & \eta \end{vmatrix} \\ &= \frac{\xi-\eta}{\eta^{-1}-\eta}\eta + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta - \frac{\xi-\eta}{\eta^{-1}-\eta}\eta - \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{-1} = \xi - \eta^{-1} \neq 0. \end{aligned}$$

Vectors y and z are linearly independent if and only if $\eta^{-1} \neq \xi$ which holds as long as $\alpha \neq 0$. One can easily verify this from (1.3.2). A general solution to the eigenvalue equation can be written as a linear combination of y and z . The last step is to show that no such linear combination is an element of $\ell^2(\mathbb{Z})$.

Choose any $a, b \in \mathbb{C}$ and set $x = ay + bz$. Then

$$x_n = \begin{cases} a\xi^{-n} + b\left(\frac{\eta^{-1}-\xi}{\xi^{-1}-\xi}\xi^n + \frac{\eta^{-1}-\xi^{-1}}{\xi-\xi^{-1}}\xi^{-n}\right) & n < 0, \\ a\left(\frac{\xi-\eta}{\eta^{-1}-\eta}\eta^n + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{-n}\right) + b\eta^n & n \geq 0, \end{cases}$$

Suppose $x \in \ell^2(\mathbb{Z})$ and from the definition of the Joukowski transform we assert that $|\xi| \leq 1, |\eta| \leq 1$. Then

$$x \in \ell^2(\mathbb{Z}) \implies x \in \ell^2(-\infty) \quad \text{and} \quad x \in \ell^2(+\infty).$$

For $|\xi| = 1$ or $|\eta| = 1$ and $a, b \neq 0$, the necessary condition for series convergence is not met for

$$\sum_{n=0}^{\infty} |x_n|^2, \quad \text{or} \quad \sum_{n=-\infty}^{-1} |x_n|^2,$$

respectively. This leads to a contradiction with $x \in \ell^2(\mathbb{Z})$. Assuming $|\xi| < 1$ and $|\eta| < 1$, the term ξ^n diverges for negative indices. To satisfy the condition $x \in \ell^2(-\infty)$, this term must vanish. For this to hold, one of the following must be true:

$$b = 0, \quad \text{or} \quad \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} = 0.$$

The latter condition necessitates that $\eta^{-1} = \xi$, which is not possible under the assumption $\alpha \neq 0$. Thus, we conclude that $b = 0$.

In the sum spanning the positive indices, there is a divergent term η^{-n} . For this term to vanish, one of the following conditions must be satisfied:

$$a = 0, \quad \text{or} \quad \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}} = 0.$$

Once again, we disregard the latter condition.

Consequently, we arrive at:

$$x \in \ell^2(\mathbb{Z}) \iff a = 0 \quad \text{and} \quad b = 0 \iff \forall n \in \mathbb{Z} : x_n = 0.$$

Let us finally discuss the special cases $\xi = \pm 1, \eta = \pm 1$ and their combinations. First we choose $\xi = -1$ and $\eta \neq \pm 1$. Then for indices $n \leq -2$ the solutions to (2.1.1) read $x_n = (-1)^n$

and $x_n = n(-1)^n$ as is evident by

$$\begin{aligned}\lambda x_n &= -2(-1)^n = (-1)^{n+1} + (-1)^{n-1} = x_{n+1} + x_{n-1} \\ \lambda x_n &= -2n(-1)^{-n} = -(2n-1+1)(-1)^n & n \leq -2. \\ &= (n+1)(-1)^{n+1} + (n-1)(-1)^{n-1} = x_{n+1} + x_{n-1}\end{aligned}$$

If we consider $n \geq 1$ the solutions read $x_n = \eta^{-n}$ and $x_n = \eta^n$. The general solution is in the form

$$\begin{aligned}x_n &= A(-1)^n + Bn(-1)^{-n} & n < 0, \\ x_n &= C\eta^n + D\eta^{-n} & n \geq 0.\end{aligned}\tag{2.1.7}$$

To assert $x \in \ell^2(\mathbb{Z})$ we assume $A, B = 0$. If $|\eta| = 1$ and $C, D \neq 0$, then the necessary condition of series convergence is not met for positive indices. If $|\eta| < 1$ and x is a non-zero element of $\ell^2(\mathbb{Z})$, then $A, B, D = 0$ and $C \neq 0$. This results in a contradiction with a condition posed upon A, B, C, D , which we get by plugging (2.1.7) in (2.1.1) for $n = -1$

$$\begin{aligned}A - 2B + C\eta^0 + D\eta^0 &= -2(A - B) \\ A - 4B + C + D &= 0.\end{aligned}$$

Now we choose $\xi = -1$ and $\eta = 1$. Then the solution to the eigenvalue equation reads

$$\begin{aligned}x_n &= A(-1)^n + Bn(-1)^{-n} & n < 0, \\ x_n &= C + Dn & n \geq 0.\end{aligned}$$

If $x \in \ell^2(\mathbb{Z})$, the form of x necessitates that it is a trivial solution. The other special cases and their combination would be shown analogously.

Proposition 2.1: Let $\alpha \in \mathbb{C}$, then

$$\sigma_p(H_\alpha) = \emptyset.$$

Proof. If $\alpha \neq 0$, the discussion above showed that there is no non-trivial solution to the eigenvalue equation that would lie in $\ell^2(\mathbb{Z})$. In the case of $\alpha = 0$, the lack of eigenvalues is well known. \square

2.1.2 Residual spectrum

We first establish a general result that connects the absence of a point spectrum to the absence of a residual spectrum under certain assumptions. This foundational theorem, presented below, will subsequently enable us to demonstrate that the residual spectrum of H_α is indeed empty.

Theorem 2.2: Let $A \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be self-adjoint and $B \in \mathcal{B}(\ell^2(\mathbb{Z}))$ a diagonal operator. Then

$$\sigma_p(A + B) = \emptyset \quad \implies \quad \sigma_r(A + B) = \emptyset.$$

Proof. For any bounded operator T we have $\text{Ker}(T^*) = \text{Ran}(T)^\perp$. Let $\lambda \in \sigma_r(T)$, then

$$\text{Ker}(T^* - \bar{\lambda}) = \text{Ran}(T - \lambda)^\perp \neq \{0\}.$$

Hence, $\bar{\lambda} \in \sigma_p(T^*)$. This allows us state a general characterization of the residual spectrum:

$$\sigma_r(T) = \{\lambda \notin \sigma_p(T) \mid \bar{\lambda} \in \sigma_p(T^*)\}.\tag{2.1.8}$$

This idea was borrowed from Chapter 5 of [14]. If substitute $T = A + B$ into (2.1.8) and apply the assumptions, the expression reduces to

$$\sigma_r(A + B) = \{\lambda \in \mathbb{C} \mid \bar{\lambda} \in \sigma_p(A + \bar{B})\}.$$

Now, let $\bar{\lambda} \in \sigma_p(A + \bar{B})$. By definition, there exists a non-zero $\psi \in \ell^2(\mathbb{Z})$ such that $(A + \bar{B})\psi = \bar{\lambda}\psi$. Applying element-wise complex conjugation on this equation yields

$$(A + B)\bar{\psi} = \lambda\bar{\psi}.$$

This implies that $\lambda \in \sigma_p(A + B)$. However, this contradicts the assumption that $\sigma_p(A + B) = \emptyset$. Therefore, $\sigma_p(A + \bar{B}) = \emptyset$. By the characterization of the residual spectrum, this directly implies that $\sigma_r(A + B) = \emptyset$ as well. \square

Proposition 2.3: Let $\alpha \in \mathbb{C}$, then

$$\sigma_r(H_\alpha) = \emptyset.$$

Proof. In Theorem 2.2 we set $A = H_0$ and $B = \alpha D$, recall the decomposition (2.0.2). The operator H_0 is self-adjoint and its point spectrum is empty as was shown in Proposition 2.1. The operator αD is diagonal. Thus, by Theorem 2.2 we get that the residual spectrum of H_α is empty. \square

2.1.3 Continuous spectrum

This section is more extensive than the previous one; hence, it is imperative to first discuss our approach. In those sections we proved that the point and residual parts of the spectrum are empty, i.e. the spectrum of H_α is purely continuous. The remaining question is what the spectrum is. First, we will define a suitable set, denoted by X . Then we will show that X is a subset of the resolvent set using Green's kernel approach. Lastly, we will show the opposite inclusion utilizing Weyl's criterion.

Definition of the set X

Let us recall the solution of the eigenvalue difference equation y (2.1.5) and z (2.1.6), which depend on ξ and η . If both ξ and η are within absolute values less than 1, we can uniquely assign them to λ according to Proposition 1.18. Furthermore, we know that

$$\begin{aligned} |\xi| < 1 &\iff y \in \ell^2(-\infty), \\ |\eta| < 1 &\iff z \in \ell^2(+\infty). \end{aligned}$$

If both conditions $|\xi| < 1$ and $|\eta| < 1$ are satisfied, we will have two vectors suitable for use in Green's kernel theorem. Ensuring these two conditions are met defines the set X , a candidate for the resolvent set of H_α . We have

$$\begin{aligned} |\xi| < 1 &\implies \lambda = \xi + \xi^{-1} \in \mathbb{C} \setminus [-2, 2], \\ |\eta| < 1 &\implies \lambda = \eta + \eta^{-1} + \alpha \in \mathbb{C} \setminus [-2 + \alpha, 2 + \alpha]. \end{aligned}$$

We define

$$X := \mathbb{C} \setminus ([-2, 2] \cup [-2 + \alpha, 2 + \alpha]). \quad (2.1.9)$$

Green's kernel

In this section, we draw upon insights from [13]. This reference discusses the spectral properties of Jacobi operators, i.e. operators with tridiagonal matrix representations. Here, H_α represents a Jacobi operator. The Green's kernel theorem as stated in [13] assumes the self-adjointness of the operator. However, it is evident that for a choice of parameter $\alpha \notin \mathbb{R}$, the operator H_α does not satisfy this assumption. We will proceed by adapting the construction of the Green's kernel and show that it indeed is the resolvent operator of H_α in our specific case. This requires demonstrating that it is a bounded operator and is the inverse of $(H_\alpha - \lambda)$. In doing so, we will establish that $X \subset \rho(H_\alpha)$. This construction depends on the vectors y and z , see (2.1.5). and (2.1.6), respectively. The Green's kernel $G(\lambda)$ takes values for indices $m, n \in \mathbb{Z}$ as follows

$$G_{m,n}(\lambda) = \frac{1}{w_k(y, z)} \begin{cases} z_n y_m & m \leq n, \\ z_m y_n & m > n, \end{cases}$$

where $w \equiv w_k(y, z) = z_{k+1}y_k - z_k y_{k+1}$, $k \in \mathbb{Z}$. Here, w is known as the Wronskian, and is non-zero if and only if y and z are linearly independent. This condition has already been verified. Notably, we denote G as dependent on λ ; this is indeed the case, but this dependency is embedded within the vectors y and z , which are themselves defined in terms of ξ and η , uniquely determined by $\lambda \in X$. The Wronskian also depends on λ .

An important property of the Wronskian $w_k(y, z)$ is that it is independent of k . To see this, choose arbitrary $k \in \mathbb{Z}$ and recall the eigenvalue equation (2.1.1), from which we derive the relation $x_{n+1} = \kappa_n x_n - x_{n-1}$, then

$$\begin{aligned} w_k(y, z) &= z_{k+1}y_k - z_k y_{k+1} = (\kappa_k z_k - z_{k-1})y_k - z_k(\kappa_k y_k - y_{k-1}) \\ &= \kappa_k z_k y_k - z_{k-1}y_k - z_k \kappa_k y_k + z_k y_{k-1} = z_k y_{k-1} - z_{k-1}y_k = w_{k-1}(y, z). \end{aligned}$$

This allows us to omit the subscript on the Wronskian. For clarity in subsequent discussions, let us define

$$\tilde{G} := wG. \tag{2.1.10}$$

We provide two equivalent matrix representations of $G(\lambda)$. One of these formulations may be less cumbersome to work with in certain scenarios than the other and vice versa. The matrix representations of $G(\lambda)$ read

$$G_{m,n}(\lambda) = \frac{1}{w} \begin{cases} \frac{\xi - \eta}{\eta^{-1} - \eta} \eta^{m+n} + \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}} \eta^{|m-n|} & m, n \geq 0, \\ \eta^m \xi^{-n} & m \geq 0, n < 0, \\ \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} \xi^{|m-n|} + \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} \xi^{-m-n} & m, n < 0, \\ \eta^n \xi^{-m} & m < 0, n \geq 0, \end{cases} \quad (2.1.11)$$

$$G_{m,n}(\lambda) = \begin{cases} \frac{\eta^{m+n} - \eta^{|m-n|}}{\eta^{-1} - \eta} + \frac{\eta^{m+n}}{w} & m, n \geq 0, \\ \frac{1}{w} \eta^m \xi^{-n} & m \geq 0, n < 0, \\ \frac{\xi^{|m-n|} - \xi^{-m-n}}{\xi - \xi^{-1}} + \frac{\xi^{-m-n}}{w} & m, n < 0, \\ \frac{1}{w} \eta^n \xi^{-m} & m < 0, n \geq 0. \end{cases} \quad (2.1.12)$$

Boundedness of $G(\lambda)$

Schur test, Theorem 1.13, will be used to obtain the upper bound. We set $p_j = 1$ for all $j \in \mathbb{Z}$. This way we get a readily available, albeit possibly rough, estimate. Let $\lambda \in X$ be arbitrary. In this setting, the Schur test implies

$$\|G(\lambda)\| \leq \sup_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |G_{m,n}(\lambda)|.$$

Finding the actual supremum proves to be quite difficult; therefore, we will be taking an estimate of $\sum_{n \in \mathbb{Z}} |G_{m,n}(\lambda)|$ from above while ridding it of the dependence on m . Recall the definition of $C(\alpha)$ from (1.3.5). Due to the form that $G(\lambda)$ takes, we need to investigate two branches:

- Let $m \geq 0$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} |G_{m,n}(\lambda)| &= \frac{1}{|w|} \sum_{n=0}^{\infty} \left| \frac{\xi - \eta}{\eta^{-1} - \eta} \eta^{m+n} + \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}} \eta^{|m-n|} \right| + \frac{|\eta|^m}{|w|} \sum_{n=1}^{\infty} |\xi|^n \\ &\leq \frac{1}{|\eta^{-1} - \eta|} \left(\frac{|\xi - \eta|}{|w|} |\eta|^m \sum_{n=0}^{\infty} |\eta|^n + \sum_{n=-m}^{\infty} |\eta|^{|n|} \right) + \frac{|\eta|^m}{|w|} \sum_{n=1}^{\infty} |\xi|^n \\ &= \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} \left(\frac{|\xi - \eta|}{|w|} |\eta|^m + 1 + |\eta| - |\eta|^{m+1} \right) + \frac{|\eta|^m |\xi|}{|w| (1 - |\xi|)} \\ &\leq \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} \left(\frac{|\xi - \eta|}{|w|} + 1 + |\eta| \right) + \frac{|\xi|}{|w| (1 - |\xi|)} =: \tilde{U}_1(\lambda) \\ &\leq \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} (2C(\alpha) + 2) + \frac{C(\alpha)}{1 - |\xi|} =: U_1(\lambda). \end{aligned}$$

- Let $m < 0$. Then

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} |G_{m,n}(\lambda)| &= \frac{1}{|w|} \sum_{n=1}^{\infty} \left| \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} \xi^{|m-n|} + \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} \xi^{m+n} \right| + \frac{|\xi|^m}{|w|} \sum_{n=0}^{\infty} |\eta|^n \\
&\leq \frac{1}{|\xi^{-1} - \xi|} \left(\sum_{n=-m+1}^{\infty} |\xi|^{|n|} + \frac{|\xi^{-1} - \eta^{-1}|}{|w|} |\xi|^m \sum_{n=1}^{\infty} |\xi|^n \right) + \frac{|\xi|^m}{|w|} \sum_{n=0}^{\infty} |\eta|^n \\
&= \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} \left(1 + |\xi| - |\xi|^m + \frac{|\xi^{-1} - \eta^{-1}|}{|w|} |\xi|^m \right) + \frac{|\eta|^m}{|w|(1 - |\eta|)} \\
&\leq \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} \left(1 + |\xi| + \frac{|\xi^{-1} - \eta^{-1}|}{|w|} |\xi| \right) + \frac{|\eta|}{|w|(1 - |\eta|)} =: \tilde{U}_2(\lambda) \\
&\leq \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} ((|\xi|^{-1} + |\eta|^{-1})C(\alpha) + 2) + \frac{C(\alpha)}{1 - |\eta|} =: U_2(\lambda).
\end{aligned}$$

In the analysis of both branches, the second-to-last estimate

$$\tilde{U}(\lambda) := \max\{\tilde{U}_1(\lambda), \tilde{U}_2(\lambda)\} \quad (2.1.13)$$

is somewhat less crude and will be utilized in the study of the pseudospectra. To demonstrate the boundedness of the resolvent operator, we employ the estimate

$$U(\lambda) := \max\{U_1(\lambda), U_2(\lambda)\}.$$

While this estimate is rougher, it provides a clearer argument for boundedness.

Proposition 2.4: Let $\alpha \neq 0$. Then

$$\forall \lambda \in X : \|(H_\alpha - \lambda)^{-1}\| \leq \max\{U_1(\lambda), U_2(\lambda)\} =: U(\lambda)$$

Proposition 2.5: Let $\alpha \in \mathbb{C}$. Then

$$\forall \lambda \in X : G(\lambda) \in \mathcal{B}(\ell^2(\mathbb{Z})).$$

Proof. For $\alpha = 0$, we have $X = \mathbb{C} \setminus [-2, 2]$, and the boundedness of the resolvent operator is well known. On the other hand, if $\alpha \neq 0$ and we utilize the upper bound from Proposition 2.5, it follows that for $\lambda \in X$, the upper bound $U(\lambda)$ is a finite number since $|\xi|, |\eta| \in (0, 1)$. \square

$G(\lambda)$ is an inverse of $(H_\alpha - \lambda)$

As one may expect, here we need to show $(H_\alpha - \lambda)G = G(H_\alpha - \lambda) = I$. We will prove this by showing $\forall x \in \ell^2(\mathbb{Z}) : (H_\alpha - \lambda)Gx = x$. Recall that $wG = \tilde{G}$ where w is the aforementioned Wronskian. If we use the notation from (2.1.1), we have

$$\begin{aligned}
w((H_\alpha - \lambda)Gx)_m &= \sum_{n=-\infty}^{\infty} \tilde{G}_{m,n-1}x_n - \kappa_n \tilde{G}_{m,n}x_n + \tilde{G}_{m,n+1}x_n \\
&= \sum_{n=-\infty}^{m-1} z_m \underbrace{(y_{n-1} - \kappa_n y_n + y_{n+1})}_{=0} x_n + \sum_{n=m+1}^{\infty} y_m \underbrace{(z_{n-1} - \kappa_n z_n + z_{n+1})}_{=0} x_n \\
&\quad + (z_m y_{m-1} - \kappa_m z_m y_m + z_{m+1} y_m) x_m \stackrel{*}{=} (z_m y_{m-1} - y_{m-1} z_m - z_m y_{m+1} + z_{m+1} y_m) x_m \\
&= (z_{m+1} y_m - z_m y_{m+1}) x_m = w_m(y, z) x_m = w x_m.
\end{aligned}$$

Then we divide the equation by the non-zero w and arrive at the desired equation. When calculating, we used the fact that y and z are solutions of (2.1.1). If we were to interchange $(H_\alpha - \lambda)$ and G , we would arrive at the same result.

Proposition 2.6: Let $\alpha \in \mathbb{C}$. Then

$$\forall \lambda \in X \quad : \quad (H_\alpha - \lambda)G = G(H_\alpha - \lambda) = I.$$

Proposition 2.7: Let $\alpha \in \mathbb{C}$. Then

$$\sigma(H_\alpha) \subset [-2, 2] \cup [-2 + \alpha, 2 + \alpha]$$

Proof. By Proposition 2.5 and by Proposition 2.6 we have

$$\lambda \in X \quad \implies \quad \lambda \in \rho(H_\alpha),$$

in other words

$$X \subset \rho(H_\alpha).$$

Transitioning to the complement we get the asserted inclusion. □

The inclusion $\rho(H_\alpha) \subset X$

In this section, we use one implication of Weyl's criterion as stated in Theorem 1.14. This particular formulation was chosen deliberately, as it addresses the needs of our analysis. The operator H_α is generally non-normal, and the specific implication that holds under these conditions suffices for our purposes.

For a given $\lambda \in \mathbb{C}$, suppose there exists a sequence $\{\psi^{(n)}\}_{n=1}^\infty \subset \ell^2(\mathbb{Z}) \setminus \{0\}$ such that

$$\lim_{n \rightarrow \infty} \frac{\|(H_\alpha - \lambda)\psi^{(n)}\|}{\|\psi^{(n)}\|} = 0.$$

Then $\lambda \in \sigma(H_\alpha)$. Since it is already established that $X \subset \rho(H_\alpha)$, we need not test this condition for all $\lambda \in \mathbb{C}$. Instead, we restrict λ to X^c and demonstrate that $\lambda \in \sigma(H_\alpha)$. This proves the inclusion $\rho(H_\alpha) \subset X$. Taking $\lambda \in X^c$ we need to consider two cases $|\xi| = 1$ and $|\eta| = 1$, see (2.1.9).

For $\lambda \in X^c$ such that $|\eta| = 1$ we define $\forall n \in \mathbb{N}$

$$(\zeta^{(n)})_k := \begin{cases} \eta^k & k \in \{0, 1, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $\forall n \in \mathbb{N} : \zeta^{(n)} \in \ell^2(\mathbb{Z})$ since $\|\zeta^{(n)}\| = \sqrt{n}$. A simple calculation yields

$$((H_\alpha - \lambda)\zeta^{(n)})_m = \begin{cases} 1 & m = -1, \\ -\eta^{-1} & m = 0, \\ -\eta^{n+1} & m = n, \\ \eta^n & m = n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to take the norm, $\|(H_\alpha - \lambda)\zeta^{(n)}\| = 2$, and evaluate the limit

$$\frac{\|(H_\alpha - \lambda)\zeta^{(n)}\|}{\|\zeta^{(n)}\|} = \frac{2}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0.$$

The set $\{\eta \mid |\eta| = 1\}$ corresponds with $[-2 + \alpha, 2 + \alpha]$ in the λ -plane. By applying Weyl's criterion we get $[-2 + \alpha, 2 + \alpha] \subset \sigma(H_\alpha)$

For $\lambda \in X^c$ such that $|\xi| = 1$ we define $\forall n \in \mathbb{N}$

$$(\chi^{(n)})_k := \begin{cases} \xi^{-k} & k \in \{-n, \dots, 1, 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Similar reasoning as above yields that $[-2, 2] \subset \sigma(H_\alpha)$.

Proposition 2.8: Let $\alpha \in \mathbb{C}$. Then

$$\sigma(H_\alpha) \subset [-2, 2] \cup [-2 + \alpha, 2 + \alpha].$$

Theorem 2.9: Let $\alpha \in \mathbb{C}$ and consider the operator H_α as given by (2.0.1). Then the spectrum of H_α is purely continuous and coincides with two line segments in the complex plane. More precisely

$$\sigma(H_\alpha) = \sigma_c(H_\alpha) = [-2, 2] \cup [-2 + \alpha, 2 + \alpha] = [-2, 2] + \alpha\{0, 1\}.$$

Proof. In Proposition 2.1 we showed that the point spectrum is empty and similarly in Proposition 2.3 we showed that the residual spectrum is empty. Hence, the spectrum of H_α is purely continuous. In Propositions 2.7 and 2.8 were shown the two inclusions implying the set equality of spectrum in the theorem. \square

The spectrum of H_α is depicted in Figure 2.1.

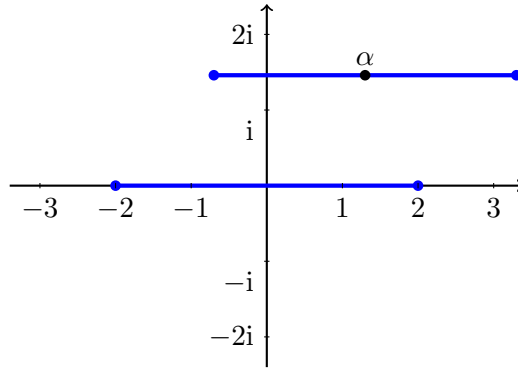


Figure 2.1: Spectrum of the operator H_α .

To conclude this section, we derive the numerical range of the operator H_α . Numerical range of an operator A is a subset of the complex plane \mathbb{C} given by

$$\text{Num}(A) = \{ \langle x, Ax \rangle \mid x \in \ell^2(\mathbb{Z}), \|x\| = 1 \}.$$

The numerical range is always a convex subset of \mathbb{C} and the spectrum of A is contained in its closure.

Theorem 2.10: Let $\alpha \in \mathbb{C}$. Then the numerical range of H_α is given by

$$\text{Num}(H_\alpha) = (-2, 2) + \alpha[0, 1].$$

Proof. Let us define a two-parameter class of vectors from $\ell^2(\mathbb{Z})$ by

$$\tilde{x}_n(q, t) := \begin{cases} \sqrt{t} q^n & n > 0, \\ 0 & n = 0, \\ \sqrt{1-t} q^{-n} & n < 0, \end{cases}$$

where $t \in [0, 1]$ and $q \in (-1, 1)$. Clearly, $\tilde{x}(q, t)$ is an $\ell^2(\mathbb{Z})$ sequence. Its squared norm is given by

$$\|\tilde{x}(q, t)\|^2 = \frac{q^2}{1-q^2}.$$

To construct a unit vector we define $x(q, t)$ as $\tilde{x}(q, t)$ normalized by its norm. Recall the decomposition of H_α (2.0.2) and plug it into $\langle x, H_\alpha x \rangle$. A simple calculation yields

$$\langle x, H_\alpha x \rangle = 2\text{Re} \langle x, Sx \rangle + \alpha \langle x, Dx \rangle = 2q + \alpha t. \quad (2.1.14)$$

Thus, it follows that

$$\{2q + \alpha t \mid q \in (-1, 1), t \in [0, 1]\} \subset \text{Num}(H_\alpha). \quad (2.1.15)$$

In the remainder of the proof, we show that no other complex number may be in $\text{Num}(H_\alpha)$.

Now, let $x \in \ell^2(\mathbb{Z})$ be an arbitrary unit vector. The inner product, $\langle x, Dx \rangle = \sum_{n=0}^{\infty} |x_n|^2$, represents the squared norm of $\ell^2(\mathbb{N}_0)$ part of x . Since x is a unit vector in $\ell^2(\mathbb{Z})$ and norms are non-negative, it follows that $\langle x, Dx \rangle \in [0, 1]$. Consequently, we have

$$\text{Im} \langle x, H_\alpha x \rangle = \langle x, Dx \rangle \text{Im} \alpha \in \text{Im} \alpha [0, 1].$$

Thus, it follows that

$$\{2q + \alpha t \mid q \in \mathbb{R}, t \notin [0, 1]\} \cap \text{Num}(H_\alpha) = \emptyset. \quad (2.1.16)$$

Suppose there exists $x \in \ell^2(\mathbb{Z})$, $\|x\| = 1$, such that $\text{Re} \langle x, Sx \rangle = \pm 1$. The Cauchy–Schwarz inequality yields

$$1 = |\text{Re} \langle x, Sx \rangle| \leq |\langle x, Sx \rangle| \leq \|x\| \|Sx\| = \|x\|^2 = 1. \quad (2.1.17)$$

Since equality holds throughout, it follows that x and Sx are linearly dependent, i.e. $Sx = \beta x$ for some $\beta \in \mathbb{C}$. Furthermore, the equality $\text{Re} \langle x, Sx \rangle = \pm 1$ implies $\langle x, Sx \rangle = \pm 1$. From this, we determine $\beta = \pm 1$, leading to $Sx = \pm x$. The operator S shifts the indices of x by 1. Therefore, all elements of x must be equal in absolute value. However, such a sequence cannot lie in $\ell^2(\mathbb{Z})$, as it would violate the square-summability condition unless it is a trivial sequence, in which case $\|x\| = 1$ does not hold. Thus, $\forall x \in \ell^2(\mathbb{Z})$, $\|x\| = 1 : \text{Re} \langle x, Sx \rangle \neq \pm 1$.

Moreover, from (2.1.17) immediately follows that $\forall x \in \ell^2(\mathbb{Z})$, $\|x\| = 1 : \text{Re} \langle x, Sx \rangle \in [-1, 1]$. Thus, from two arguments it follows that

$$\{2q + \alpha t \mid q \notin (-1, 1), t \in [0, 1]\} \cap \text{Num}(H_\alpha) = \emptyset. \quad (2.1.18)$$

The union of the sets on the left-hand sides of (2.1.15), (2.1.16), and (2.1.18) covers the entire complex plane. Therefore, the subset relation in (2.1.15) is, in fact, an equality:

$$\text{Num}(H_\alpha) = \{2q + \alpha t \mid q \in (-1, 1), t \in [0, 1]\} = (-2, 2) + \alpha[0, 1].$$

□

The numerical range of H_α is depicted in Figure 2.2.

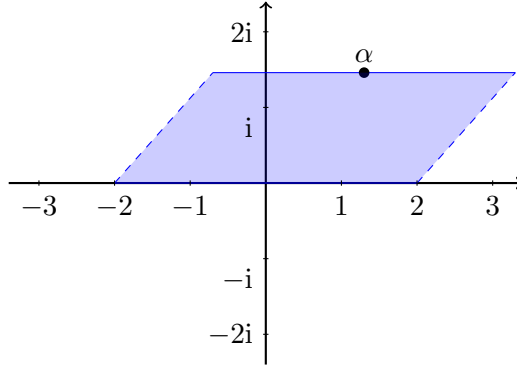


Figure 2.2: Numerical range of the operator H_α .

2.2 Pseudospectrum

In this chapter, we will almost solely rely on the book *Spectra and pseudospectra* by Lloyd Nicholas Trefethen and Mark Embree, see [20]. We will define what pseudospectra are and state their basic properties before we commence pseudospectral analysis of the studied operator H_α . Since these properties are not crucial for this paper and serve only to familiarize us with this new notion, we will not provide proofs; one may find them in [20].

Definition 2.11: Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$ be arbitrary. The ε -pseudospectrum of operator A is defined as the set

$$\sigma_\varepsilon(A) := \sigma(A) \cup \{\lambda \in \mathbb{C} \setminus \sigma(A) \mid \|(A - \lambda)^{-1}\| \geq \varepsilon^{-1}\}.$$

Theorem 2.12: Given $A \in \mathcal{B}(\mathcal{H})$, the pseudospectra $\{\sigma_\varepsilon(A)\}_{\varepsilon > 0}$ have the following properties.

- Each $\sigma_\varepsilon(A)$ is a nonempty subset of \mathbb{C} .
- Any bounded connected component of $\sigma_\varepsilon(A)$ has a nonempty intersection with $\sigma(A)$.
- The pseudospectra are strictly nested supersets of the spectrum.

Describing the pseudospectra of self-adjoint operators is a relatively trivial endeavor because we can describe the norm of the resolvent in terms of the distance between its spectral parameter and the spectrum of the operator.

Theorem 2.13: Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator and $\lambda \in \varrho(A)$, then

$$\|(A - \lambda)^{-1}\| = \frac{1}{\text{dist}(\sigma(A), \lambda)}.$$

Therefore, given a bounded self-adjoint operator with a known spectrum, one can explicitly describe the ε -pseudospectra as the ε -neighborhood of $\sigma(A)$.

Corollary 2.14: Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator, then

$$\varepsilon > 0 : \quad \sigma_\varepsilon(A) = \sigma(A) \cup \{\lambda \in \mathbb{C} \setminus \sigma(A) \mid \text{dist}(\sigma(A), \lambda) \leq \varepsilon\} = \sigma(A) + D(\varepsilon),$$

where $D(\varepsilon)$ denotes a disc in the complex plane centered at the origin with radius ε .

Let us now apply these results to operator H_α . The following proposition immediately follows from the corollary above.

Proposition 2.15: Let α be a real number. Then

$$\varepsilon > 0 : \quad \sigma_\varepsilon(H_\alpha) = [-2, 2] \cup [-2 + \alpha, 2 + \alpha] + D(\varepsilon).$$

For the choice of $\alpha = 0$, the pseudospectrum of H_α is depicted in figure 2.3.

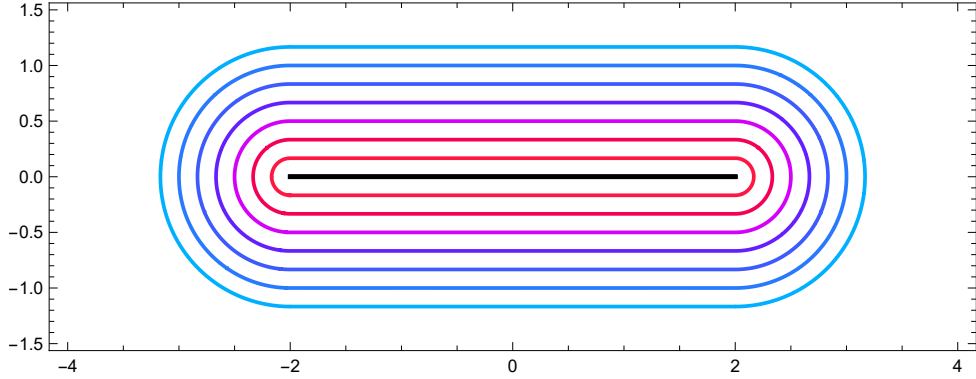


Figure 2.3: ε -pseudospectrum of H_0 for $\varepsilon \in \{\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}, \frac{7}{6}\}$.

Setting aside the trivial case for a real parameter α , we will discuss the general case of α being any complex number. Describing the pseudospectra of non-self-adjoint operators is no more a trivial endeavor. Still, we can get an explicit expression for the pseudospectrum intersected with a certain set. This is justified by the following propositions, which is a standard result from functional analysis.

Proposition 2.16: Let $A \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \rho(A)$. Then

$$\frac{1}{\text{dist}(\lambda, \sigma(A))} \leq \|(A - \lambda)^{-1}\|.$$

A lower bound can also be obtained simply from the definition of operator norm. Let $A \in \mathcal{B}(\mathcal{H})$, then

$$\forall y \in \mathcal{H}, \|y\| = 1 : \quad \|A\| = \sup_{\|x\|=1} \|Ax\| \geq \|Ay\|.$$

However, in our case, Proposition 2.16 gives a more suitable lower bound since we did not find a suitable test vector.

Proposition 2.17 ([18, Proposition 2.8]): Let A be a closed operator on \mathcal{H} . Let U be a connected open subset of $\mathbb{C} \setminus \text{Num}(A)$. If there exists a number $\lambda_0 \in U$ which is contained in $\rho(A)$, then $U \subset \rho(A)$. Moreover,

$$\forall \lambda \in U : \quad \|(A - \lambda)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, \text{Num}(A))}.$$

Lemma 2.18: Let $K \subset \mathbb{C}$ be bounded closed convex set, then its complement, $\mathbb{C} \setminus K$, is connected.

Proof. Let us establish notation for lines, rays, and line segments:

$$\begin{aligned} \langle a, b \rangle &:= \{(1-t)a + tb \mid t \in \mathbb{R}\} & (a, b) &:= \{(1-t)a + tb \mid t \in (0, +\infty)\} \\ [a, b] &:= \{(1-t)a + tb \mid t \in [0, 1]\} & [a, b] &:= \{(1-t)a + tb \mid t \in [0, 1]\} \end{aligned}$$

We will show that $\mathbb{C} \setminus K$ is path-connected; hence, connected. We need to construct a path connecting any two different points $v_1, v_2 \in \mathbb{C} \setminus K$. The set K is bounded; therefore, we can find $r > 0$ such that $K \subset B(0, r)$. If $\langle v_1, v_2 \rangle \cap K = \emptyset$, we connect v_1 and v_2 with the line segment $[v_1, v_2]$. If the line $\langle v_1, v_2 \rangle$ has a non-empty intersection with K , then we denote $\langle v_1, v_2 \rangle \cap K = [k_1, k_2]$, where k_1 and k_2 may equal each other if the intersection is a singleton. Moreover, $\langle v_1, v_2 \rangle \cap K$ is necessarily a convex set. Therefore, the line $\langle v_1, v_2 \rangle$ decomposes into exactly three disjoint parts which read

$$\langle v_1, v_2 \rangle = \langle v_1, k_1 \rangle \cup [k_1, k_2] \cup \langle k_2, v_2 \rangle.$$

Let us denote $\{b_1\} = \partial B(0, r) \cap [k_1, v_1)$ and $\{b_2\} = \partial B(0, r) \cap [k_2, v_2)$. Now we can connect $v_1, v_2 \in \mathbb{C} \setminus K$ with either one of the paths given by the set $[v_1, b_1) \cup \partial B(0, r) \cup (b_2, v_2]$. \square

If an operator is bounded the closure of its numerical range is convex; therefore $\mathbb{C} \setminus \overline{\text{Num}(A)}$ is connected. The restriction to bounded operators simplifies Proposition 2.17 in such a way that the inequality holds for all $\lambda \in \mathbb{C} \setminus \overline{\text{Num}(A)}$.

Recall the numeric range of H_α is given by $\text{Num}(H_\alpha) = (-2, 2) + \alpha[0, 1]$, see Theorem 2.10. Let us define

$$\Omega(\alpha) := \{z \in \mathbb{C} \mid \text{dist}(\lambda, \text{Num}(H_\alpha)) = \text{dist}(\lambda, \sigma(H_\alpha))\}.$$

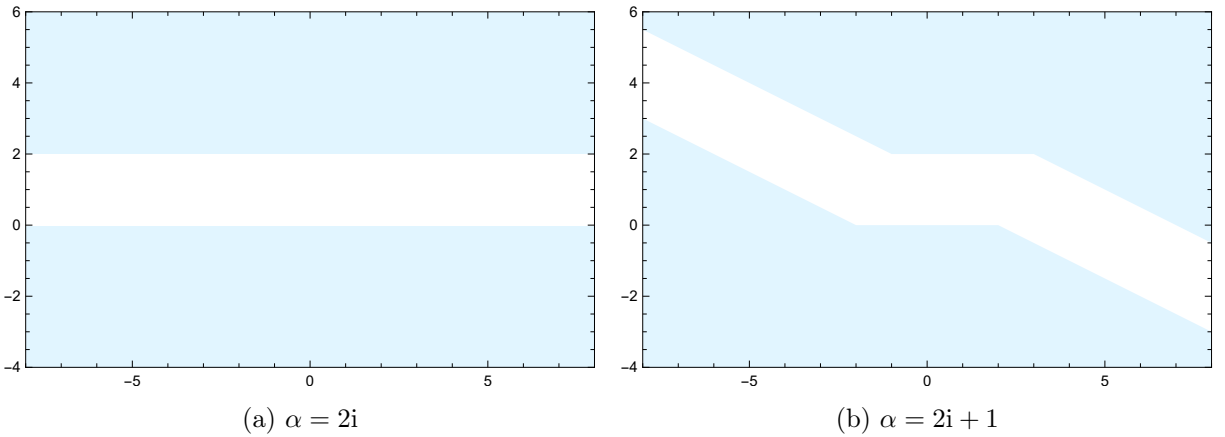


Figure 2.4: Depictions of the set $\Omega(\alpha)$ for certain choices of α .

In Fig. 2.4, we have depicted the set $\Omega(\alpha)$ for certain values of α . For all $\lambda \in \Omega(\alpha)$ the upper and lower estimate of $\|(A - \lambda)^{-1}\|$ equal each other on; therefore, we can describe the ε -spectrum explicitly as

$$\Omega(\alpha) \cap \sigma_\varepsilon(H_\alpha) = \Omega(\alpha) \cap (\sigma(H_\alpha) + D(\varepsilon)).$$

To our knowledge, there is no way to describe the pseudospectra of H_α explicitly outside of $\Omega(\alpha)$; therefore, our task is to obtain a superset and a subset of each ε -pseudospectrum. This

way we get a region where the boundary of the ε -pseudospectrum must reside. We will obtain these supersets and subsets by estimating the norm of the resolvent above and below. As stated above we obtain a lower bound by virtue of Proposition 2.16. These lower bounds are depicted in Figure 2.5. The tool we will use for obtaining upper estimates of an operator's norm is the Schur test, see Theorem 1.13. The Schur test has already been applied in Proposition 2.4, as already stated before, we shall be using the estimate as given by (2.1.13), i.e.

$$\forall \lambda \in \rho(H_\alpha) : \quad \|(H_\alpha - \lambda)^{-1}\| \leq \tilde{U}(\lambda).$$

These upper bounds are shown in Figure 2.6.

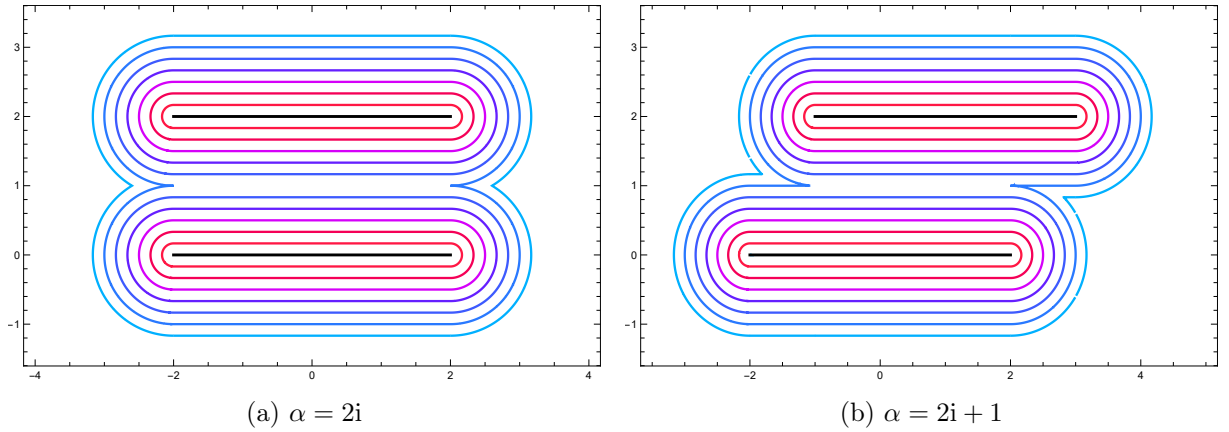


Figure 2.5: Depictions of $\text{dist}(\lambda, \sigma(H_\alpha)) = \varepsilon^{-1}$ for $\varepsilon \in \left\{ \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}, \frac{7}{6}, \right\}$.

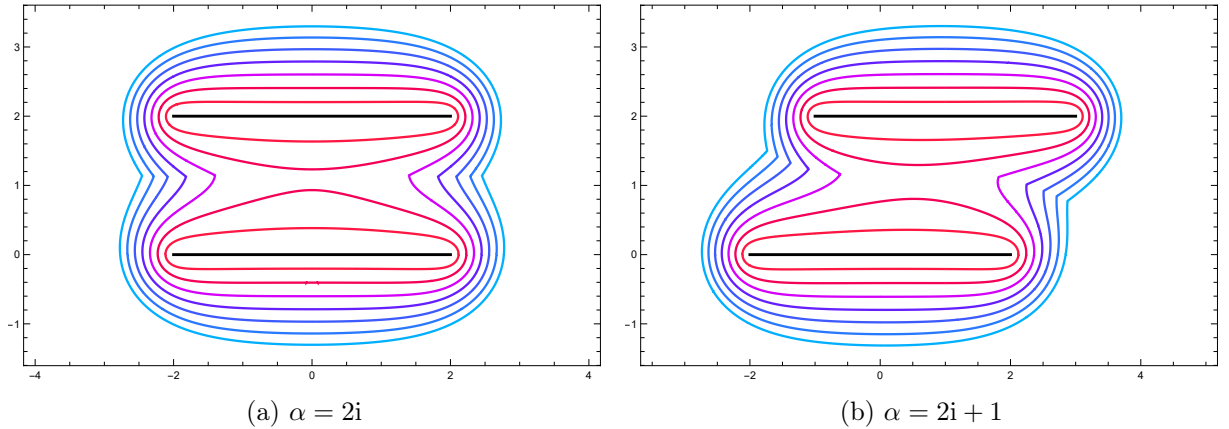


Figure 2.6: Depictions of $\tilde{U}(\lambda) = \varepsilon^{-1}$ for $\varepsilon \in \left\{ \frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \frac{7}{12}, \right\}$.

Theorem 2.19: Let $\text{Im } \alpha \neq 0$. Then the ε -pseudospectra of H_α follow these inclusions:

$$\left\{ \lambda \in \mathbb{C} \mid \frac{1}{\text{dist}(\lambda, \sigma(H_\alpha))} \geq \varepsilon^{-1} \right\} \subset \sigma_\varepsilon(H_\alpha) \subset \left\{ \lambda \in \mathbb{C} \mid \tilde{U}(\lambda) \geq \varepsilon^{-1} \right\}.$$

These estimates are used when we do not have an explicit expression for the resolvent operator's norm, i.e. on $\mathbb{C} \setminus \Omega(\alpha)$.

2.2.1 Asymptotic behavior

In this section we describe the behavior of pseudospectra near the spectrum of H_α . For $\lambda \in \sigma(H_\alpha)$ we derive asymptotic expansions of the upper and lower bounds of $\|(H_\alpha - \lambda - i\varepsilon)^{-1}\|$ as $\varepsilon \rightarrow 0_+$.

As stated above, if $\text{Im } \alpha = 0$ the pseudospectrum is trivially described. We assume the opposite, i.e. $\text{Im } \alpha \neq 0$. In this case, the spectrum is comprised of two connected components. To ensure $\lambda + i\varepsilon$ is not an element of the other connected component of the spectrum we assume $\varepsilon < |\text{Im } \alpha|$.

A trick we can use while evaluating asymptotic pseudospectra is to separate the quadrants of the matrix representation of the resolvent operator

$$G(\lambda) = \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix} = \begin{pmatrix} G^{--} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & G^{-+} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ G^{+-} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & G^{++} \end{pmatrix}. \quad (2.2.1)$$

Taking the norm of the resolvent operator and estimating it with triangle inequality may be too rough of an estimate for global estimation but if we show that all but one term are asymptotically insignificant, we can make the asymptotic formula simpler. The quadrants are separated as in (2.1.11).

Since it is not ideal to work with matrices whose indices are on the negative half-line, we define matrices $\widehat{G}_{m,n}^{--} := G_{-m,-n}^{--}$, $\widehat{G}_{m,n}^{+-} := G_{m,-n}^{+-}$, and $\widehat{G}_{m,n}^{-+} := G_{-m,n}^{-+}$.

In [1] we showed that

$$\begin{aligned} \left\| \begin{pmatrix} G^{--} & 0 \\ 0 & 0 \end{pmatrix} \right\|_{\ell^2(\mathbb{Z})} &= \|\widehat{G}^{--}\|_{\ell^2(\mathbb{N})}, & \left\| \begin{pmatrix} 0 & G^{-+} \\ 0 & 0 \end{pmatrix} \right\|_{\ell^2(\mathbb{Z})} &= \|\widehat{G}^{-+}\|_{\ell^2(\mathbb{N})}, \\ \left\| \begin{pmatrix} 0 & 0 \\ G^{+-} & 0 \end{pmatrix} \right\|_{\ell^2(\mathbb{Z})} &= \|\widehat{G}^{+-}\|_{\ell^2(\mathbb{N})}, & \left\| \begin{pmatrix} 0 & 0 \\ 0 & G^{++} \end{pmatrix} \right\|_{\ell^2(\mathbb{Z})} &= \|G^{++}\|_{\ell^2(\mathbb{N}_0)}. \end{aligned}$$

With this established let us evaluate or estimate the norm of these operators.

The norms of G^{+-} and G^{-+} can be evaluated exactly by applying the following lemma.

Lemma 2.20: Let $p, q \in \mathbb{C}$, $|p|, |q| < 1$. We define an operator $A = A(p, q)$ on $\ell^2(\mathbb{N})$ whose matrix representation reads

$$\forall m, n \in \mathbb{N} \quad : \quad A_{m,n} := p^m q^n.$$

Then $A \in \mathcal{B}(\ell^2(\mathbb{N}))$. Moreover, we can calculate its norm

$$\|A\| = \sqrt{\frac{|p|^2}{1 - |p|^2} \frac{|q|^2}{1 - |q|^2}}.$$

Proof. Choose an arbitrary vector $\psi \in \ell^2(\mathbb{N})$, then the m th element of $A\psi$ reads

$$(A\psi)_m = \sum_{n=1}^{\infty} A_{m,n} \psi_n = \sum_{n=1}^{\infty} p^m q^n \psi_n = p^m \sum_{n=1}^{\infty} q^n \psi_n.$$

Next, we estimate the square of its norm

$$\begin{aligned} \|A\psi\|^2 &= \sum_{m=1}^{\infty} |(A\psi)_m|^2 = \sum_{m=1}^{\infty} |p|^{2m} \left| \sum_{n=1}^{\infty} q^n \psi_n \right|^2 \leq \sum_{m=1}^{\infty} |p|^{2m} \left(\sum_{n=1}^{\infty} |q|^n |\psi_n| \right)^2 \\ &\stackrel{\text{C.-S.}}{\leq} \sum_{m=1}^{\infty} |p|^{2m} \sum_{n=1}^{\infty} |q|^{2n} \sum_{k=1}^{\infty} |\psi_k|^2 = \frac{|p|^2}{1 - |p|^2} \frac{|q|^2}{1 - |q|^2} \|\psi\|^2, \end{aligned}$$

where the marked inequality denotes the Cauchy–Schwarz inequality. This calculation proves the assertion that A is a bounded operator on $\ell^2(\mathbb{N})$ and provides an upper bound for the norm of A . For the lower bound estimation let us choose a test vector $\psi_n := (\bar{q})^n$, then

$$\begin{aligned} \|A\psi\|^2 &= \sum_{m=1}^{\infty} |(A\psi)_m|^2 = \sum_{m=1}^{\infty} |p|^{2m} \left| \sum_{n=1}^{\infty} q^n (\bar{q})^n \right|^2 = \sum_{m=1}^{\infty} |p|^{2m} \left(\sum_{n=1}^{\infty} |q|^{2n} \right)^2 \\ &= \sum_{m=1}^{\infty} |p|^{2m} \sum_{n=1}^{\infty} |q|^{2n} \sum_{k=1}^{\infty} |\bar{q}|^{2k} = \sum_{m=1}^{\infty} |p|^{2m} \sum_{n=1}^{\infty} |q|^{2n} \sum_{k=1}^{\infty} |\psi_k|^2 = \frac{|p|^2}{1-|p|^2} \frac{|q|^2}{1-|q|^2} \|\psi\|^2. \end{aligned}$$

Since these two estimates are equal, we know it is the norm of A itself. \square

Lemma 2.21: Assume $\alpha \neq 0$. Let $\lambda \in \sigma(H_\alpha)$. Then

$$\|G^{+-}(\lambda + i\varepsilon)\| = \|G^{-+}(\lambda + i\varepsilon)\| = \mathcal{O}\left(\varepsilon^{-1/2}\right) \quad \text{as } \varepsilon \rightarrow 0_+.$$

Proof. As a result of Lemma 2.20 we get

$$\|\widehat{G}^{+-}\| = \|\widehat{G}^{-+}\| = \frac{1}{|w|} \frac{1}{\sqrt{1-|\eta|^2}} \frac{|\xi|}{\sqrt{1-|\xi|^2}}. \quad (2.2.2)$$

We provide proof only for $\lambda \in [-2, 2]$, the argument for the second connected component of $\sigma(H_\alpha)$ is the same. Since we assume $\text{Im } \alpha \neq 0$, we have $|\eta| \rightarrow c \in (0, 1)$. We need to further differentiate two cases.

- Let $\lambda \in \{-2, 2\}$. In this case we obtain the asymptotic expansion of $|\xi|^2$ from Corollary 1.25. Then

$$\|G^{+-}(\lambda + i\varepsilon)\| = \frac{1}{|w|_{\varepsilon=0}} \frac{1}{\sqrt{1-c^2}} \frac{1 + \mathcal{O}(\varepsilon^{1/2})}{\sqrt{\sqrt{2\varepsilon} + \mathcal{O}(\varepsilon)}} = \mathcal{O}\left(\varepsilon^{-1/4}\right) = \mathcal{O}\left(\varepsilon^{-1/2}\right), \quad \text{as } \varepsilon \rightarrow 0_+$$

since $w = \mathcal{O}(1)$ as $\varepsilon \rightarrow 0_+$.

- Let $\lambda \in (-2, 2)$. In this case we obtain the asymptotic expansion of $|\xi|^2$ from Corollary 1.23. Then

$$\|G^{+-}(\lambda + i\varepsilon)\| = \frac{1}{|w|_{\varepsilon=0}} \frac{1}{\sqrt{1-c^2}} \frac{1 + \mathcal{O}(\varepsilon)}{\sqrt{\frac{\varepsilon}{\sin \phi} + \mathcal{O}(\varepsilon^2)}} = \mathcal{O}\left(\varepsilon^{-1/2}\right), \quad \text{as } \varepsilon \rightarrow 0_+$$

where $\phi = -\arccos(\lambda/2)$. \square

The norms of the other quadrants are more complicated. We were not able to evaluate them exactly; hence, we give estimates. Multiple estimation techniques were used to take the upper bound, though not one yielded a better asymptotic formula than the others; therefore, we will show the most conspicuous one – the Schur test. We will get the lower bound by virtue of Lemma 2.16.

Proposition 2.22: Let $\lambda \in \rho(H_\alpha)$. Let us denote

$$U^-(\lambda) := \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} \left(2 + C(\alpha) |\xi^{-1} - \xi| \right),$$

$$U^+(\lambda) := \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} \left(C(\alpha) |\eta^{-1} - \eta| - |\eta| + 3 \right).$$

The upper bounds of the norm of \widehat{G}^{--} and G^{++} take the forms

$$\|\widehat{G}^{--}\| \leq U^-, \quad \|G^{++}\| \leq U^+,$$

where $C(\alpha)$ is the uniform upper bound of $|w|^{-1} = |\xi - \eta^{-1}|^{-1}$ from (1.3.5).

Proof. We utilize the formulation of G as given by (2.1.11). First, we establish the upper bounds for the norms using the Schur test. Let $m \geq 0$, then

$$\begin{aligned} \sum_{n=0}^{\infty} |G_{m,n}^{++}(\lambda)| &= \frac{1}{|w|} \sum_{n=0}^{\infty} \left| \frac{\xi - \eta}{\eta^{-1} - \eta} \eta^{m+n} + \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}} \eta^{|m-n|} \right| \\ &\leq \frac{1}{|\eta^{-1} - \eta|} \left(\frac{|\xi - \eta|}{|w|} |\eta|^m \sum_{n=0}^{\infty} |\eta|^n + \sum_{n=-m}^{\infty} |\eta|^{|n|} \right) \\ &= \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} \left(\frac{|\xi - \eta|}{|w|} |\eta|^m + 1 + |\eta| - |\eta|^{m+1} \right) \\ &\leq \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} \left((C(\alpha) |\eta^{-1} - \eta| + 1 - |\eta|) |\eta|^m + 1 + |\eta| \right) \\ &\leq \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} \left(C(\alpha) |\eta^{-1} - \eta| - |\eta| + 3 \right), \end{aligned}$$

where we used

$$\frac{\xi - \eta}{w} = 1 - \frac{\eta^{-1} - \eta}{w}$$

and estimated its absolute value with triangle inequality and with (1.3.5). The last inequality is justified because $(C(\alpha) |\eta^{-1} - \eta| + 1 - |\eta|) \geq 0$. Since the dependence on m has been eliminated from the expression, the last estimate serves as an upper bound for $\|G^{++}(\lambda)\|$. A comparable method is applied to analyze G^{--} . Let $m \geq 1$, then

$$\begin{aligned} \sum_{n=1}^{\infty} |\widehat{G}_{m,n}^{--}(\lambda)| &= \frac{1}{|w|} \sum_{n=1}^{\infty} \left| \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} \xi^{|m-n|} + \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} \xi^{m+n} \right| \\ &\leq \frac{1}{|\xi^{-1} - \xi|} \left(\sum_{n=-m+1}^{\infty} |\xi|^{|n|} + \frac{|\xi^{-1} - \eta^{-1}|}{|w|} |\xi|^m \sum_{n=1}^{\infty} |\xi|^n \right) \\ &= \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} \left(1 + |\xi| - |\xi|^m + \frac{|\xi^{-1} - \eta^{-1}|}{|w|} |\xi|^m \right) \\ &= \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} \left(1 + |\xi| + |\xi|^m C(\alpha) |\xi^{-1} - \xi| \right) \\ &\leq \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} \left(2 + C(\alpha) |\xi^{-1} - \xi| \right). \end{aligned}$$

□

Lemma 2.23: Assume $\alpha \neq 0$. Then

$$\begin{aligned} \lambda \in [-2, 2] & : & U^+(\lambda + i\varepsilon) &\leq \frac{2}{\varepsilon} + \mathcal{O}(\varepsilon^{-1/2}), & U^-(\lambda + i\varepsilon) &= \mathcal{O}(1), \\ \lambda \in [-2 + \alpha, 2 + \alpha] & : & U^-(\lambda + i\varepsilon) &\leq \frac{2}{\varepsilon} + \mathcal{O}(\varepsilon^{-1/2}), & U^+(\lambda + i\varepsilon) &= \mathcal{O}(1), \end{aligned}$$

as $\varepsilon \rightarrow 0_+$.

Proof. We show the proof only for $\lambda \in [-2, 2]$, the method for the other connected component of the spectrum is the same. Let $\lambda \in (-2, 2)$ and substitute the asymptotic formulas from Corollary 1.23, we get

$$U^+(\lambda + i\varepsilon) = \frac{2}{\varepsilon} + \mathcal{O}(1), \quad U^-(\lambda + i\varepsilon) = \mathcal{O}(1)$$

as $\varepsilon \rightarrow 0_+$. However, for $\lambda \in \{-2, 2\}$ applying asymptotic formulas from Corollary 1.25 yields

$$U^+(\lambda + i\varepsilon) = \frac{\sqrt{2}}{\varepsilon} + \mathcal{O}(\varepsilon^{-1/2}), \quad U^-(\lambda + i\varepsilon) = \mathcal{O}(1),$$

as $\varepsilon \rightarrow 0_+$. To obtain a more concise result we estimate $\sqrt{2} \leq 2$ for $\lambda \in \{-2, 2\}$ and we replace the asymptotic order $\mathcal{O}(1)$ with the stronger asymptotic order $\mathcal{O}(\varepsilon^{-1/2})$ for $\lambda \in (-2, 2)$. \square

Theorem 2.24: Let $\alpha \neq 0$ and $\lambda \in \sigma(H_\alpha)$. Then

$$\frac{1}{\varepsilon} \leq \|(H_\alpha - \lambda - i\varepsilon)^{-1}\| \leq \frac{2}{\varepsilon} + \mathcal{O}(\varepsilon^{-1/2}) \quad \text{as } \varepsilon \rightarrow 0_+.$$

Proof. For the lower estimate we have an exact expression from Proposition 2.16 which reads

$$\|(H_\alpha - \lambda - i\varepsilon)^{-1}\| \geq \frac{1}{\text{dist}(\lambda + i\varepsilon, \sigma(H_\alpha))} = \frac{1}{\varepsilon}.$$

We obtain the upper estimate by taking the triangle inequality of the decomposed resolvent operator, see (2.2.1), if we then estimate further using Proposition 2.22, we get

$$\|(H_\alpha - \lambda - i\varepsilon)^{-1}\| \leq \|\widehat{G}^{--}\| + \|\widehat{G}^{+-}\| + \|\widehat{G}^{-+}\| + \|G^{++}\| \leq U^-(\lambda + i\varepsilon) + 2\|\widehat{G}^{-+}\| + U^+(\lambda + i\varepsilon).$$

We substitute the asymptotic formulas from Lemmas 2.21 and 2.23 and see that all but one terms are asymptotically insignificant. \square

2.3 Spectral measure

In this section, we derive the spectral measure associated with the operator H_α , more precisely we derive the Radon–Nikodym derivative of the spectral measure with respect to the Lebesgue measure, the *spectral density*. To ensure the spectral measure is well-defined, we restrict our analysis to $\alpha \in \mathbb{R}$, in which case H_α is a self-adjoint operator. The spectral density of the discrete operator H_α is matrix-valued, and its entries are determined by the Stone’s formula

$$\frac{d\mu_{m,n}}{d\lambda}(\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} (G_{m,n}(\lambda + i\varepsilon) - G_{m,n}(\lambda - i\varepsilon)),$$

where $\lambda \in \sigma(H_\alpha)$. If need be, we denote the spectral density as $d\mu_{m,n}$.

Let $z = \lambda + i\varepsilon$. To calculate the spectral density one must take the limit as z approaches $\lambda \in \sigma(A)$ from each side. By using the Joukowsky transform, the analysis of the spectral density simplifies significantly. Instead of taking limits as $z \rightarrow \lambda \in (-2, 2)$ from either side of the cut in the λ -plane, we exploit the fact that the interval $(-2, 2)$ is the image of the unit circle $\partial\mathbb{D}$ under the transform. Specifically, each point $\lambda \in (-2, 2)$ corresponds to two distinct pre-images on $\partial\mathbb{D}$:

$$k_1 = e^{i\theta}, \quad k_2 = e^{-i\theta} = \overline{k_1}, \quad \phi \in (-\pi, 0).$$

One can easily see this in Lemma 1.22. Thus, the discontinuity of the Green's kernel at $\lambda \in (-2, 2)$ is captured by evaluating G at these two pre-images in the k -plane. This approach avoids the need to compute explicit limits in the λ -plane, providing a more direct and elegant analysis.

This applies directly to analyzing H_0 . When using this argument on H_α one must take into account the parameters ξ and η .

When depicting the spectral measure we shall show only four particular entries. As was shown in [16], the spectral measure of a self-adjoint Jacobi operator is related to a 2×2 matrix

$$d\mu = \begin{pmatrix} d\mu_{0,0} & d\mu_{0,1} \\ d\mu_{1,0} & d\mu_{1,1} \end{pmatrix}$$

with $d\mu_{0,1} = d\mu_{1,0}$. We also note that $d\mu_{0,0}$ and $d\mu_{1,1}$ are both positive probability measures but $d\mu_{1,0}$ is a signed measure.

2.3.1 Spectral measure of H_0

If we set $\alpha = 0$, the analysis is quite straightforward. The resolvent operator for H_0 reads

$$G_{m,n}(\lambda) = \frac{k^{|m-n|}}{k - k^{-1}}.$$

For $\lambda \in (-2, 2)$ we have corresponding $\theta \in (0, \pi)$ by $\lambda = 2 \cos \theta$. Due to the argument above, we have $k = e^{i\theta}$ and

$$\lim_{\substack{z \rightarrow 2 \cos \theta \\ \text{Im } z > 0}} (G_{m,n}(z) - G_{m,n}(\bar{z})) = G_{m,n}(k) - G_{m,n}(\bar{k}).$$

We abuse the notation by writing $G(\lambda) = G(k)$, where the dependence on λ is implicit through the Joukowsky transform.

We set $N = |m - n|$ and evaluate

$$\frac{k^N}{k - k^{-1}} - \frac{\bar{k}^N}{\bar{k} - \bar{k}^{-1}} = \frac{e^{-iN\theta}}{e^{-i\theta} - e^{i\theta}} - \frac{e^{iN\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{e^{iN\theta} + e^{-iN\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{i \cos(N\theta)}{\sin \theta}.$$

This results in the spectral density of H_0 taking the form

$$\frac{d\mu_{m,n}}{d\lambda} = \frac{1}{2\pi} \frac{\cos(|m - n|\theta)}{\sin \theta}.$$

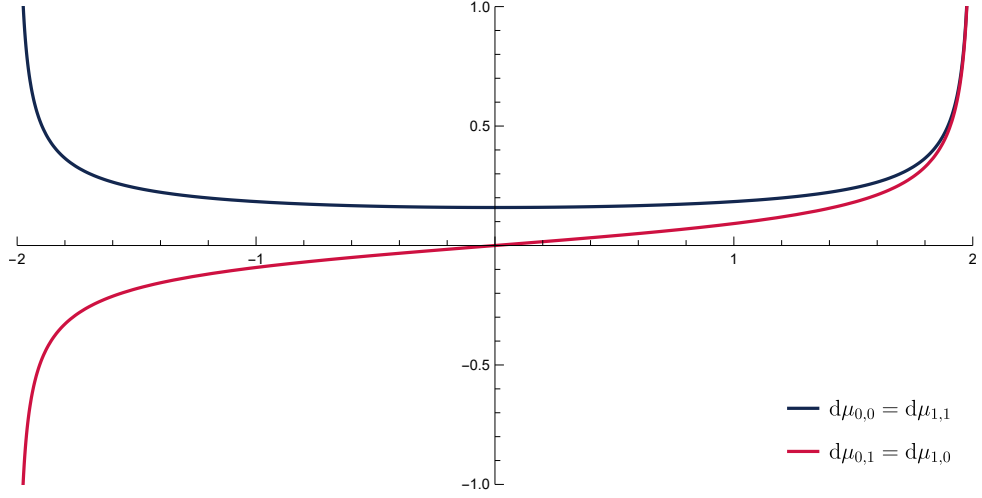


Figure 2.7: Spectral density of H_0 .

2.3.2 Spectral measure of H_α

When considering $\alpha \neq 0$ real, the situation becomes quite tedious. There is one or two connected components of the spectrum. We need to differentiate which connected component, or line segment, we are approaching and whether we are approaching their intersection. In this section, we will be using the (2.1.12) formulation of the resolvent operator.

- Let $\lambda \in (2, 2) \cap (-2 + \alpha, 2 + \alpha)$. Of course, this set is non-empty if and only if $\alpha \in (-4, 4)$. Since λ resides in both line segments simultaneously, both Joukowski parameters ξ and η are equal to 1 in absolute value; therefore, we can write $\xi = e^{i\phi}$ and $\eta = e^{i\theta}$, where $\phi(\theta) = \arccos(\cos \theta + \frac{\alpha}{2})$ from (1.3.6). Once again we abuse the notation and write $G(\lambda) = G(\xi, \eta)$. For each quadrant of the matrix $G(\lambda)$ we calculate the spectral density separately.

- Let $m \geq 0$ and $n < 0$. Then

$$\begin{aligned} G_{m,n}(\xi, \eta) - G_{m,n}(\bar{\xi}, \bar{\eta}) &= \frac{\eta^m \xi^n}{\xi - \eta^{-1}} - \frac{\bar{\eta}^m \bar{\xi}^n}{\bar{\xi} - \bar{\eta}^{-1}} = \frac{e^{im\theta} e^{-in\phi}}{e^{i\phi} - e^{-i\theta}} - \frac{e^{-im\theta} e^{in\phi}}{e^{-i\phi} - e^{i\theta}} \\ &= i \frac{\sin(m\theta - n\phi - \phi) - \sin(m\theta - n\phi + \theta)}{1 - \cos(\phi + \theta)}, \end{aligned}$$

$$\frac{d\mu_{m,n}}{d\lambda} = \frac{1}{2\pi} \frac{\sin(m\theta - (n+1)\phi) - \sin((m+1)\theta - n\phi)}{1 - \cos(\phi + \theta)}.$$

- Let $m < 0$ and $n \geq 0$. Then similarly we have

$$\frac{d\mu_{m,n}}{d\lambda} = \frac{1}{2\pi} \frac{\sin(n\theta - (m+1)\phi) - \sin((n+1)\theta - m\phi)}{1 - \cos(\phi + \theta)}.$$

– Let $m \geq 0$ and $n \geq 0$. Then

$$\begin{aligned} \frac{d\mu_{m,n}}{d\lambda} &= \frac{1}{2\pi} \frac{\cos((m+n)\theta) - \cos(|m-n|\theta)}{\sin \theta} \\ &+ \frac{1}{2\pi} \frac{\sin((m+n)\theta - \phi) - \sin((m+n+1)\theta)}{1 - \cos(\phi + \theta)}. \end{aligned}$$

– Let $m < 0$ and $n < 0$. Then

$$\begin{aligned} \frac{d\mu_{m,n}}{d\lambda} &= \frac{1}{2\pi} \frac{\cos((m+n)\phi) - \cos(|m-n|\phi)}{\sin \phi} \\ &+ \frac{1}{2\pi} \frac{\sin((m+n)\phi - \theta) - \sin((m+n+1)\phi)}{1 - \cos(\phi + \theta)}. \end{aligned}$$

- Let $\lambda \in (-2, 2)$ and $\lambda \notin (-2 + \alpha, 2 + \alpha)$. In this case, we again have $|\xi| = 1$ so $\xi = e^{i\phi}$. The parameter η is no longer equal to 1 in absolute value and so it is uniquely determined by λ . Since $\alpha \in \mathbb{R}$, $\eta \in (-1, 1)$.

– Let $m \geq 0$ and $n < 0$. Then

$$\begin{aligned} G_{m,n}(\xi, \eta) - G_{m,n}(\bar{\xi}, \eta) &= \frac{\eta^m e^{-in\phi}}{e^{in\phi} - \eta^{-1}} - \frac{\eta^m e^{+in\phi}}{e^{-in\phi} - \eta^{-1}} \\ &= 2i\eta^m \frac{\sin(n\phi)\eta^{-1} - \sin((n+1)\phi)}{1 - 2\eta^{-1} \cos \phi + \eta^{-2}}, \end{aligned}$$

$$\frac{d\mu_{m,n}}{d\lambda} = \frac{\eta^m}{\pi} \frac{\sin(n\phi)\eta^{-1} - \sin((n+1)\phi)}{1 - 2\eta^{-1} \cos \phi + \eta^{-2}}.$$

– Let $m < 0$ and $n \geq 0$. Then

$$\frac{d\mu_{m,n}}{d\lambda} = \frac{\eta^n}{\pi} \frac{\sin(m\phi)\eta^{-1} - \sin((m+1)\phi)}{1 - 2\eta^{-1} \cos \phi + \eta^{-2}}.$$

– Let $m \geq 0$ and $n \geq 0$. Then

$$\frac{d\mu_{m,n}}{d\lambda} = -\frac{\eta^{m+n}}{\pi} \frac{\sin(\phi)}{1 - 2\eta^{-1} \cos \phi + \eta^{-2}}.$$

– Let $m < 0$ and $n < 0$. Then

$$\begin{aligned} \frac{d\mu_{m,n}}{d\lambda} &= \frac{1}{2\pi} \frac{\cos((m+n)\phi) - \cos(|m-n|\phi)}{\sin \phi} \\ &+ \frac{1}{\pi} \frac{\eta^{-1} \sin((m+n)\phi) - \sin((m+n+1)\phi)}{1 - 2\eta^{-1} \cos \phi + \eta^{-2}}. \end{aligned}$$

- Let $\lambda \notin (-2, 2)$ and $\lambda \in (-2 + \alpha, 2 + \alpha)$. In this case, we again have $|\eta| = 1$ so $\eta = e^{i\theta}$. This time the parameter ξ is no longer equal to 1 in absolute value and so it is uniquely determined by λ .

– Let $m \geq 0$ and $n < 0$. Then

$$\frac{d\mu_{m,n}}{d\lambda} = \frac{\xi^{-n}}{\pi} \frac{\xi \sin(m\theta) - \sin((m+1)\theta)}{\xi^2 - 2\xi \cos \theta + 1}.$$

– Let $m < 0$ and $n \geq 0$. Then

$$\frac{d\mu_{m,n}}{d\lambda} = \frac{\xi^{-m} \xi \sin(n\theta) - \sin((n+1)\theta)}{\pi \xi^2 - 2\xi \cos \theta + 1}.$$

– Let $m \geq 0$ and $n \geq 0$. Then

$$\begin{aligned} \frac{d\mu_{m,n}}{d\lambda} &= \frac{1}{2\pi} \frac{\cos((m+n)\theta) - \cos(|m-n|\theta)}{\sin \theta} \\ &\quad + \frac{1}{\pi} \frac{\xi \sin((m+n)\theta) - \sin((m+n+1)\theta)}{\xi^2 - 2\xi \cos \theta + 1}. \end{aligned}$$

– Let $m < 0$ and $n < 0$. Then

$$\frac{d\mu_{m,n}}{d\lambda} = -\frac{\xi^{-m-n}}{\pi} \frac{\sin(\theta)}{\xi^2 - 2\xi \cos \theta + 1}.$$

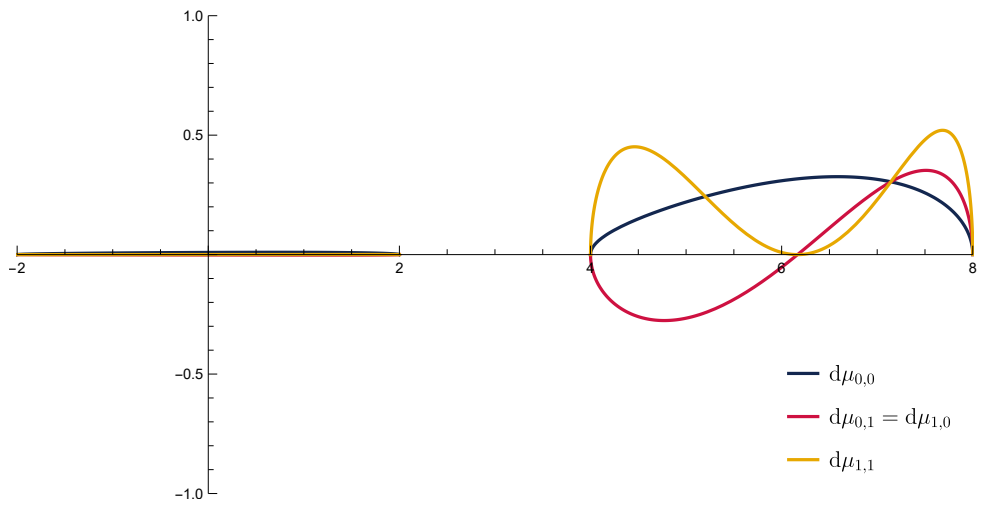
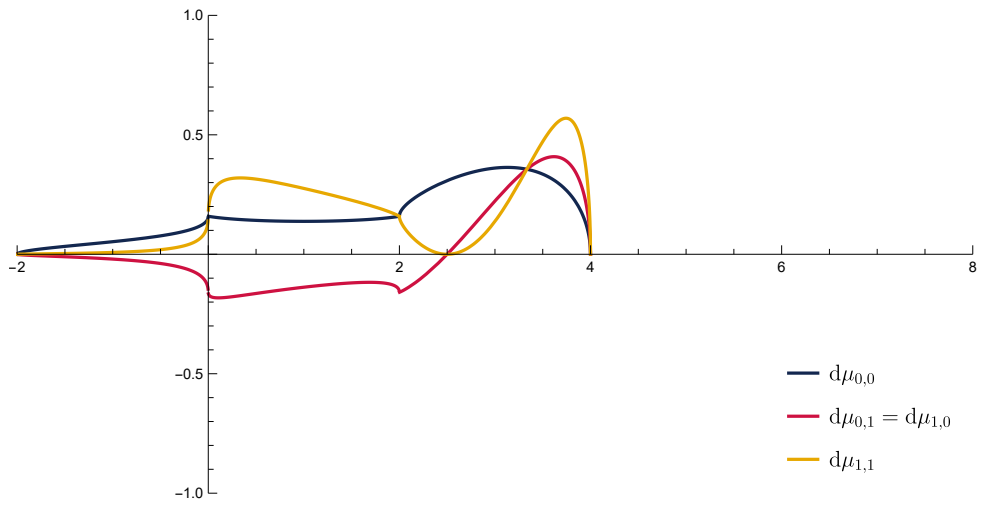


Figure 2.8: Spectral density of H_α for two values of α .

Chapter 3

Spectral properties of H_α under small complex perturbations

In this section, we investigate the existence, potential uniqueness, and asymptotic behavior of eigenvalues that arise when a small potential V is added to a given operator. The presence of such a perturbation can, in some cases, lead to the emergence of new eigenvalues outside the essential spectrum (called *bound states*). However, for certain classes of operators, the addition of a small potential does not give rise to any eigenvalues, and the spectrum remains unchanged. When this invariance of the spectrum occurs under small perturbations, we say that the operator exhibits *spectral stability*.

3.1 Weak coupling of H_0

Suppose some potential V , in the weak coupling regime we analyze discrete eigenvalues of the operator $H_\varepsilon := H_0 + \varepsilon V$ when ε is small. This section is motivated by the classical result from [19] by Simon. For a pedagogical text see also [15]. Simon showed that if the potential V is real-valued and the function $|V(x)|(1+x^2)$ is integrable, then the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + \varepsilon V$ on $L^2(\mathbb{R})$ has a bound state for all small positive ε if and only if the integral of V is non-positive. A discrete generalization was proven in [12] for a more general class of convolution-type operators, although the potential is still assumed to be real-valued. We complement these results by generalizing the analysis to allow for complex-valued potentials.

Consider the discrete Laplace operator, i.e. discrete Schrödinger operator with zero potential, or H_0 . Its action on $x \in \ell^2(\mathbb{Z})$ is given by

$$(H_0 x)_n = x_{n-1} + x_{n+1} \quad n \in \mathbb{Z}.$$

As discussed in Section 1.3, the Joukowski transform of the spectral parameter λ will be denoted by k , i.e. $\lambda = k + k^{-1}$. The operator H_0 is self-adjoint. As a well-known fact, we state that the spectrum of H_0 coincides with the interval $[-2, 2]$. Green's kernel of H_0 reads

$$(H_0 - \lambda)_{m,n}^{-1} = (H_0 - k - k^{-1})_{m,n}^{-1} = \frac{k^{|m-n|}}{k - k^{-1}}, \quad m, n \in \mathbb{Z}, \quad 0 < |k| < 1,$$

is consistent with (2.1.12), if we set $\alpha = 0$.

Since the discrete version of the Laplacian is bounded, there arises a significant difference to the continuous setting. The spectrum of the continuous Laplacian is $[0, \infty)$; therefore, one

needs to consider only the left neighborhood of zero when looking for bound states. In the case of the discrete Laplace operator, we need to investigate both the neighborhood of 2 and the neighborhood of -2 . However, the following proposition allows us to investigate, in fact, only one of these cases because the behavior is equivalent.

Proposition 3.1: Consider the Laplace operator H_0 and let $V = \text{diag}(\{v_n\}_{n \in \mathbb{Z}})$ be a bounded potential. Then the spectrum of the Schrödinger operator $H_V := H_0 + V$ satisfies

$$\sigma(H_V) = -\sigma(H_{-V}).$$

Proof. Let $x = \{x_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ and consider the unitary operator U given by $(Ux)_n = (-1)^n x_n$. Then

$$\begin{aligned} U^{-1}H_V Ux &= UH_V Ux = UH_V Ux = UH_V U\{x_n\} = UH_V\{(-1)^n x_n\} \\ &= U\{(-1)^{n-1}x_{n-1} + (-1)^n v_n x_n + (-1)^{n-1}x_{n-1}\} = \{-x_{n-1} + v_n x_n - x_{n+1}\} = -H_{-V}x. \end{aligned}$$

Since this operation does not change the spectrum and scaling the operator by -1 also scales the spectrum as a subset of \mathbb{C} by -1 we have

$$\sigma(H_V) = \sigma(-H_{-V}) = -\sigma(H_{-V}).$$

□

Before we move further, let us reiterate the Birman–Schwinger principle (Theorem 1.12) which states that, whenever $v \in \ell^1(\mathbb{Z})$, any $\lambda \in \rho(H_0)$ is an eigenvalue of $H_0 + V$ if and only if -1 is an eigenvalue of the Birman–Schwinger operator $K(\lambda)$. The entries of its matrix representation read

$$K_{m,n}(\lambda) = \sqrt{|v_m|} \frac{k^{|m-n|}}{k - k^{-1}} \sqrt{|v_n|} \text{sgn } v_n, \quad m, n \in \mathbb{Z}.$$

Later, we will be scaling the potential V by some small $\varepsilon > 0$, for this purpose, we will denote $H_\varepsilon := H_0 + \varepsilon V$. The Birman–Schwinger principle still holds in the following sense. Any $\lambda \in \rho(H_0)$ is an eigenvalue of H_ε if and only if -1 is an eigenvalue of $\varepsilon K(\lambda)$.

The trick is to decompose the Birman–Schwinger operator into the sum of a well-behaved operator and a rank-one operator as follows

$$K(\lambda) = L(\lambda) + M(\lambda),$$

where the matrix entries of these operators read

$$\begin{aligned} L_{m,n}(\lambda) &:= \sqrt{|v_m|} \frac{1}{k - k^{-1}} \sqrt{|v_n|} \text{sgn } v_n, \\ M_{m,n}(\lambda) &:= \sqrt{|v_m|} \frac{k^{|m-n|} - 1}{k - k^{-1}} \sqrt{|v_n|} \text{sgn } v_n. \end{aligned}$$

The operator $L(\lambda)$ is rank-one. Indeed, if we take $\psi \in \ell^2(\mathbb{Z})$, we have for $m \in \mathbb{Z}$

$$\begin{aligned} (L(\lambda)\psi)_m &= \sum_{n \in \mathbb{Z}} L_{m,n}(\lambda)\psi_n = \sum_{n \in \mathbb{Z}} \sqrt{|v_m|} \frac{1}{k - k^{-1}} \sqrt{|v_n|} \text{sgn } v_n \psi_n \\ &= \frac{1}{k - k^{-1}} \left(\sum_{n \in \mathbb{Z}} \sqrt{|v_n|} \text{sgn } v_n \psi_n \right) \sqrt{|v_m|} = \frac{1}{k - k^{-1}} \langle v_{1/2}, \psi \rangle \sqrt{|v_m|}. \end{aligned}$$

Lemma 3.2: Let $N \geq 0$. The complex function f given by

$$f(z) := \frac{z^N - 1}{z^{-1} - z}$$

satisfies

$$\sup_{z \in \mathbb{D}^+} |f(z)| \leq \frac{N}{\sqrt{2}},$$

where $\mathbb{D}^+ := \{z \in \mathbb{C} \mid |z| \leq 1, \operatorname{Re} z > 0\}$.

Proof. Let N be an even number, then there is some k such that $N = 2k$. Then we rewrite

$$f(z) = -z \frac{1 - (z^2)^k}{1 - z^2} = -z \sum_{i=0}^{k-1} z^{2i}$$

and estimate its absolute value on the unit disk

$$|f(z)| \leq |z| \sum_{i=0}^{k-1} |z|^{2i} \leq k = \frac{N}{2} \leq \frac{N}{\sqrt{2}}.$$

Though the last estimate is unnecessarily crude, we still take it since it leads to a consistent result with N being odd.

If, on the other hand, N is an odd number the situation is more complex. We write $N = 2k+1$. The function f then takes the form

$$f(z) = -z \frac{1 - z^{2k+1}}{1 - z^2} = -\frac{z}{1+z} \frac{1 - z^{2k+1}}{1-z} = -\frac{z}{1+z} \sum_{i=0}^{2k} z^i.$$

Let us define

$$g(z) := \frac{z}{1+z}.$$

Since g is holomorphic on \mathbb{D}^+ and continuous up to the boundary, we shall use the maximum modulus principle to determine

$$\sup_{z \in \mathbb{D}^+} |g(z)|.$$

We need only to evaluate the supremum on the boundary, which we decompose into a line segment and a half circle.

A Let $z = it$, where $t \in (-1, 1)$. Then we have

$$\begin{aligned} g(it) &= \frac{it}{1+it} = t \frac{i+t}{1+t^2}, \\ |g(it)|^2 &= t^2 \frac{1+t^2}{(1+t^2)^2} = \frac{t^2}{1+t^2}, \\ \frac{d}{dt} |g(it)|^2 &= \frac{2t}{(1+t^2)^2}. \end{aligned}$$

The only root of the derivative is at $t = t$ where a minimum is attained. Thus, this yields

$$\sup_{t \in (-1, 1)} |g(it)| = |g(i)| = \frac{1}{\sqrt{2}}.$$

B Let $z = e^{i\theta}$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then we have

$$\begin{aligned} g(e^{i\theta}) &= \frac{e^{i\theta}}{1 + e^{i\theta}}, \\ |g(e^{i\theta})|^2 &= \frac{(\cos^2 \theta + \cos \theta - \sin^2 \theta)^2 + (\sin \theta + 2 \cos \theta \sin \theta)^2}{(2 + 2 \cos \theta)^2} = \frac{1}{2 + 2 \cos \theta}, \\ \frac{d}{dt} |g(e^{i\theta})|^2 &= \frac{\sin \theta}{2(1 + \cos \theta)}. \end{aligned}$$

Again the only root of the derivative is at $\theta = 0$ where the minimum is attained. Thus, the supremum is at the endpoint

$$\sup_{\theta \in (-\pi/2, \pi/2)} |g(e^{i\theta})| = |g(e^{i\pi/2})| = \frac{1}{\sqrt{2}}.$$

With this established, we can estimate the absolute value of f on the set \mathbb{D}^+

$$|f(z)| = |g(z)| \left| \sum_{i=0}^{2k} z^i \right| \leq \frac{1}{\sqrt{2}} \sum_{i=0}^{2k} |z|^i \leq \frac{1}{\sqrt{2}} \sum_{i=0}^{2k} 1 = \frac{1}{\sqrt{2}} (2k + 1) = \frac{N}{\sqrt{2}}. \quad \square$$

Remark. The matrix entries of $M(\lambda)$ as functions of λ are not defined at $\lambda = 0, 2$, corresponding to $k = 0, 1$, as was shown in the proof of Lemma 3.2, these are, in fact, removable singularities. By continuous extensions, we may consider M to be defined on $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$.

Proposition 3.3: Let $v \in \ell^1(\mathbb{Z}, m^2)$, then

$$\forall \lambda \in \mathbb{C}, \operatorname{Re} \lambda \geq 0 : \|M(\lambda)\| \leq \sqrt{2} \|v\|_{\ell^1(\mathbb{Z}, m^2)}.$$

Proof. We estimate the standard operator norm by the Hilbert–Schmidt norm, then we estimate using Lemma 3.2:

$$\begin{aligned} \|M(\lambda)\|^2 &\leq \|M(\lambda)\|_{HS}^2 = \sum_{m, n \in \mathbb{Z}} |M_{m, n}(\lambda)|^2 = \sum_{m, n \in \mathbb{Z}} |v_m| \left(\frac{k^{|m-n|} - 1}{k - k^{-1}} \right)^2 |v_n| \\ &\leq \sum_{m, n \in \mathbb{Z}} |v_m| \frac{|m-n|^2}{2} |v_n| \leq \sum_{m, n \in \mathbb{Z}} |v_m| (m^2 + n^2) |v_n| \\ &= \sum_{m, n \in \mathbb{Z}} m^2 |v_m| |v_n| + \sum_{m, n \in \mathbb{Z}} n^2 |v_m| |v_n| = 2 \sum_{m, n \in \mathbb{Z}} m^2 |v_m| |v_n| \\ &= 2 \sum_{m \in \mathbb{Z}} m^2 |v_m| \sum_{n \in \mathbb{Z}} |v_n| \leq 2 \left(|v_0| + \sum_{n \in \mathbb{Z}} |v_n| n^2 \right)^2 = 2 \|v\|_{\ell^1(\mathbb{Z}, m^2)}^2. \quad \square \end{aligned}$$

Lemma 3.4: Let $v \in \ell^1(\mathbb{Z}, m^2)$. Then there exists ε_1 for which all $\varepsilon \in (0, \varepsilon_1)$ satisfy

$$\lambda = k + k^{-1} \in \sigma_p(H_\varepsilon) \iff k - k^{-1} + \varepsilon \langle \overline{v_{1/2}}, (I + \varepsilon M(\lambda))^{-1} |v\rangle^{1/2} \rangle = 0,$$

where $H_\varepsilon := H_0 + \varepsilon V$.

Proof. To ensure the invertibility of $(I + \varepsilon M(\lambda))$ we use Lemma 3.3 and choose $\varepsilon_1 > 0$ such that any $\varepsilon < \varepsilon_1$ satisfies $\varepsilon\sqrt{2}\|v\|_{\ell^1(\mathbb{Z}, m^2)} < 1$. Then we can write

$$I + \varepsilon K(\lambda) = (I + \varepsilon M(\lambda))(I + (I + \varepsilon M(\lambda))^{-1}\varepsilon L(\lambda)).$$

With all of this stated, we arrive at three equivalent statements

$$-1 \in \sigma_p(\varepsilon K(\lambda)) \iff \lambda \in \sigma_d(H_\varepsilon) \iff \det(I + (I + \varepsilon M(\lambda))^{-1}\varepsilon L(\lambda)) = 0.$$

If we denote

$$\begin{aligned} X_m &:= \sum_{n \in \mathbb{Z}} (I + M(\lambda))_{m,n}^{-1} |v_n|^{1/2}, \\ Y_m &:= (v_{1/2})_m, \end{aligned}$$

we can write

$$(I + M(\lambda))^{-1}\varepsilon L(\lambda) = \frac{\varepsilon}{k - k^{-1}} XY^T$$

and easily find the spectrum of this rank-one operator

$$\sigma\left(\frac{\varepsilon}{k - k^{-1}} XY^T\right) = \left\{ \frac{\varepsilon}{k - k^{-1}} X^T Y, 0 \right\}.$$

Since a rank-one operator is a trace class operator, we can use Theorem 1.15,

$$\det(I + A) = \prod_{\lambda \in \sigma(A)} (1 + \lambda),$$

to get

$$\det(I + (I + \varepsilon M(\lambda))^{-1}\varepsilon L(\lambda)) = 1 + \frac{\varepsilon}{k - k^{-1}} X^T Y = 1 + \frac{\varepsilon}{k - k^{-1}} \langle \overline{v_{1/2}}, (I + \varepsilon M(\lambda))^{-1} |v|^{1/2} \rangle.$$

From this, the assertion of the lemma clearly follows. \square

Theorem 3.5: Let $v \in \ell^1(\mathbb{Z}, m^2)$ and denote $U := \sum_{n \in \mathbb{Z}} v_n$. Then $\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$ we have

$$\begin{aligned} \operatorname{Re} U > 0 &: \sigma_d(H_\varepsilon) \cap B_{\varepsilon_0}(2) = \{\lambda_\varepsilon\}, \\ \operatorname{Re} U < 0 &: \sigma_d(H_\varepsilon) \cap B_{\varepsilon_0}(-2) = \{\lambda_\varepsilon\}, \end{aligned}$$

and the following asymptotic formula holds

$$\lambda_\varepsilon = 2 \operatorname{sgn}(\operatorname{Re} U) + \frac{\varepsilon^2 U^2}{4} + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0_+.$$

Proof. Unless stated otherwise, the proof will show only the case $\operatorname{Re} U > 0$ as the other case is analogous per Proposition 3.1. Choose $\varepsilon_1 > 0$ such that any $\varepsilon < \varepsilon_1$ satisfies $\varepsilon\sqrt{2}\|v\|_{\ell^1(\mathbb{Z}, m^2)} < 1$ and apply Lemma 3.4. Then we need only show that the equation

$$k - k^{-1} + \varepsilon \langle \overline{v_{1/2}}, (I + \varepsilon M(\lambda))^{-1} |v|^{1/2} \rangle = 0 \tag{3.1.1}$$

has a solution. Let us investigate the inner product more closely

$$\begin{aligned} \langle \overline{v_{1/2}}, (I + \varepsilon M(\lambda))^{-1} |v|^{1/2} \rangle &= \sum_{n=0}^{\infty} (-1)^n \varepsilon^n \langle \overline{v_{1/2}}, M^n(\lambda) |v|^{1/2} \rangle \\ &= U + \sum_{n=1}^{\infty} (-1)^n \varepsilon^n \langle \overline{v_{1/2}}, M^n(\lambda) |v|^{1/2} \rangle \\ &= U + \tilde{U}, \end{aligned}$$

where \tilde{U} is dependent on λ and ε . Now let us estimate the absolute value of \tilde{U} and in doing so evaluate its asymptotic behavior:

$$|\tilde{U}| \leq \|v\|_{\ell^1} \sum_{n=1}^{\infty} \varepsilon^n \|M(\lambda)\|^n = \|v\|_{\ell^1} \frac{\varepsilon \sqrt{2} \|v\|_{\ell^1(\mathbb{Z}, m^2)}}{1 - \varepsilon \sqrt{2} \|v\|_{\ell^1(\mathbb{Z}, m^2)}} \leq C\varepsilon = \mathcal{O}(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0_+,$$

where $C > 0$ is a constant independent on λ and ε . Now let us reparametrize k in terms of a new variable z , i.e. $z \mapsto k(z)$, like so

$$k - k^{-1} = -\varepsilon U(1 + z).$$

This reparametrization is a bijective map between

$$\mathbb{D} \setminus \{0\} \quad \longleftrightarrow \quad \mathbb{C} \setminus \left[\frac{2i}{\varepsilon U} - 1, -\frac{2i}{\varepsilon U} - 1 \right].$$

By taking the convex combination of the two endpoints of the line segment we can easily show that 0 is not an element of the line segment if and only if $\operatorname{Re} U \neq 0$ and so z may be 0. This parametrization greatly simplifies equation (3.1.1) like so

$$\begin{aligned} -\varepsilon U(1 + z) + \varepsilon(U + \tilde{U}) &= 0 \\ -zU + \tilde{U} &= 0. \end{aligned}$$

For later use of Rouché's theorem, we define two function

$$\begin{aligned} f(z) &:= -zU + \tilde{U}, \\ g(z) &:= -zU. \end{aligned}$$

These are clearly holomorphic functions on the domain of z . Let us estimate the absolute value of their difference,

$$|f(z) - g(z)| = |\tilde{U}| < \varepsilon C = \delta |U| = |g(z)|,$$

for $z \in \partial B_\delta(0)$ where $\delta := \frac{\varepsilon C}{|U|}$. To ensure the holomorphic property of f and g , $B_\delta(0)$ needs to lie within the domain of z , i.e.

$$\left[\frac{2i}{\varepsilon U} - 1, -\frac{2i}{\varepsilon U} - 1 \right] \cap B_\delta(0) = \emptyset.$$

To achieve this, we further need to restrict ε . Let $\varepsilon_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$

$$\delta < \operatorname{dist}\left(0, \left[\frac{2i}{\varepsilon U} - 1, -\frac{2i}{\varepsilon U} - 1 \right]\right).$$

This is possible since it was shown that the interval does not intersect the origin whenever $\operatorname{Re} U \neq 0$. Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. From Rouché's theorem, Theorem 1.16, then follows that $\forall \varepsilon \in (0, \varepsilon_1)$ the functions f and g have the same number of zeros including multiplicity in $B_\delta(0)$. Clearly the function g has exactly one zero in this set and that is the origin. Let us denote the unique simple zero of the function f as $z_0 = z_0(\varepsilon)$. Clearly as ε approaches zero, δ also approaches zero, and so $z_0(\varepsilon) = o(1)$ as $\varepsilon \rightarrow 0_+$. From the bijective nature of the reparametrization we get a unique k_0 such that

$$k_0 - k_0^{-1} = -\varepsilon U(1 + z_0(\varepsilon)).$$

The right-hand side of the equation tends to zero as $\varepsilon \rightarrow 0_+$ and so $k_0 = k_0(\varepsilon) \in \mathbb{D} \setminus \{0\}$ is either in some neighborhood of 1, H_1 , or some neighborhood of -1, H_{-1} , depending on the sign of $\operatorname{Re} U$. Taking the real part of the equation

$$\operatorname{Re}(k_0 - k_0^{-1}) = -\varepsilon \operatorname{Re} U + o(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0_+,$$

we can clearly see the dependence on the sign of $\operatorname{Re} U$

$$\begin{aligned} \operatorname{Re} U > 0 & : & k_0(\varepsilon) \in \mathbb{D} \cap H_1 \\ \operatorname{Re} U < 0 & : & k_0(\varepsilon) \in \mathbb{D} \cap H_{-1}. \end{aligned}$$

Let us finish the proof by deriving the asymptotic formula. Again we assume that $\operatorname{Re} U > 0$ the other case is analogous. Under this assumption $k_0(\varepsilon)$ is in some neighborhood of 1 and so

$$\lambda_\varepsilon = k_0(\varepsilon) + k_0^{-1}(\varepsilon) \in H_2 \setminus [-2, 2].$$

The asymptotic formula then follows from

$$\begin{aligned} k_0(\varepsilon) - k_0^{-1}(\varepsilon) &= -\varepsilon U(1 + o(\varepsilon)) \\ \lambda_\varepsilon^2 &= 4 + \varepsilon^2 U^2 + o(\varepsilon^2) \\ \lambda_\varepsilon &= 2 + \frac{\varepsilon^2 U^2}{4} + o(\varepsilon^2), \end{aligned}$$

where we squared the first equation and expressed λ_ε from $k_0(\varepsilon)$ as $\lambda^2 - 4 = (k - k^{-1})^2$, then we took the square root of the second equation and expanded the square root to the first degree. \square

Remark. The approach using the Rouché's theorem allows us to assert only the local uniqueness of the eigenvalue. For a given potential with a positive sum, there exists a neighborhood of 2 within which there is exactly one eigenvalue. This approach is weaker in its assertion when compared to [19] but relaxes the assumption on the potential V which is no longer restricted to real values. The reason is the lack of variational min-max principle for non-self-adjoint operators.

3.2 Spectral stability of H_α

In this section, we prove that the operator H_α exhibits spectral stability for all $\alpha \neq 0$. This means that if we apply a V , under certain assumptions the spectrum remains the same. Moreover, there are no eigenvalues embedded in the essential spectrum, i.e. the purely continuous spectrum of H_α remains purely continuous for $H_\alpha + V$.

First, we need to establish some auxiliary results.

Lemma 3.6: Let $m, n \geq 0$. The complex function f given by

$$f(z) := \frac{z^{m+n} - z^{|m-n|}}{z^{-1} - z}$$

satisfies

$$\sup_{|z| \leq 1} |f(z)| = \frac{m+n - |m-n|}{2}.$$

Proof. The function f is not well defined at $z \in \{0, \pm 1\}$, where removable singularities exist. We can rewrite f as

$$\frac{z^{m+n} - z^{|m-n|}}{z^{-1} - z} = -z^{|m-n|+1} \frac{1 - z^{m+n-|m-n|}}{1 - z^2} \stackrel{*}{=} -z^{|m-n|+1} \frac{1 - z^{2k}}{1 - z^2} = -z^{|m-n|+1} \sum_{j=0}^{k-1} z^{2j}.$$

The marked equality holds because $(m+n)$ and $|m-n|$ have the same parity, meaning their difference is even. Thus, we can write $2k = m+n - |m-n|$ for some integer k . Taking the absolute value of f in this form and applying the triangle inequality, we obtain $|f(z)| \leq k$, for $|z| \leq 1$. Finally, we confirm that this bound is achieved at $z = 1$. Substituting $z = 1$ into the expression for $f(z)$ yields

$$f(1) = \frac{m+n - |m-n|}{2}.$$

□

Next, we use this lemma to derive a uniform, global estimate for the absolute values of the Green's kernel of H_α . Recall the global estimate $|w|^{-1} \leq C(\alpha)$, see (1.3.5).

Lemma 3.7: Let us define a doubly infinite matrix M by setting

$$M_{m,n} := \begin{cases} \frac{m+n-|m-n|}{2} + C(\alpha) & \text{if } m, n \geq 0, \\ C(\alpha) & \text{if } m \geq 0, n < 0, \\ \frac{-m-n-|m-n|}{2} + C(\alpha) & \text{if } m, n < 0, \\ C(\alpha) & \text{if } m < 0, n \geq 0. \end{cases}$$

Then

$$\forall \lambda \in \rho(H_\alpha), \forall m, n \in \mathbb{Z} : |G_{m,n}(\lambda)| \leq M_{m,n}.$$

Proof. If we estimate the absolute value of the entries of $G(\lambda)$ as expressed in (2.1.12) by applying triangle inequality, using Lemmas 3.6 and 1.20, we obtain the matrix M as an upper bound. □

With the estimate of the Green's kernel established, we can estimate the norm of the Birman–Schwinger operator. Recall the notation for weighted ℓ^1 -spaces, see Notation 1.7.

Proposition 3.8: Let $\alpha \neq 0$ and $v \in \ell^1(\mathbb{Z}, m^2)$. Then the following estimate holds

$$\forall \lambda \in \rho(H_\alpha) : \|K(\lambda)\| \leq (1 + C(\alpha)) \|v\|_{\ell^1(\mathbb{Z}, m^2)}.$$

Proof. Before we move to the calculation itself, let us first show some estimates that we will be using. For $m, n \geq 0$ we have

$$m + n - |m - n| \leq m + n \quad \text{and} \quad \frac{(m + n - |m - n|)^2}{4} \leq \frac{(m + n)^2}{4} \leq \frac{m^2 + n^2}{2}.$$

Similarly, for $m, n < 0$ we estimate

$$-m - n - |m - n| \leq -m - n \quad \text{and} \quad \frac{(-m - n - |m - n|)^2}{4} \leq \frac{(-m - n)^2}{4} \leq \frac{m^2 + n^2}{2}.$$

We will be calculating the Hilbert–Schmidt norm as it dominates the operator norm and it is easier to calculate. Further on, we will use the matrix M from Lemma 3.7 to estimate the absolute value of matrix entries of $G(\lambda)$. With this established, we calculate

$$\begin{aligned} \|K(\lambda)\|^2 &\leq \|K(\lambda)\|_{HS}^2 = \sum_{m,n \in \mathbb{Z}} \left| \sqrt{|v_m|} G_{m,n}(\lambda) \sqrt{|v_n|} \operatorname{sgn} v_n \right|^2 \\ &= \sum_{m,n \in \mathbb{Z}} |v_m| |G_{m,n}|^2 |v_n| \leq \sum_{m,n \in \mathbb{Z}} |v_m| M_{m,n}^2 |v_n| =: \sum_{m,n \in \mathbb{Z}} \widetilde{M}_{m,n} \\ \|K(\lambda)\|^2 &\leq \underbrace{\sum_{\substack{m \geq 0 \\ n \geq 0}} \widetilde{M}_{m,n}}_A + \underbrace{\sum_{\substack{m \geq 0 \\ n < 0}} \widetilde{M}_{m,n}}_B + \underbrace{\sum_{\substack{m < 0 \\ n < 0}} \widetilde{M}_{m,n}}_C + \underbrace{\sum_{\substack{m < 0 \\ n \geq 0}} \widetilde{M}_{m,n}}_D. \end{aligned}$$

We estimate each of the sums separately:

$$\begin{aligned} A &= \sum_{\substack{m \geq 0 \\ n \geq 0}} |v_m| \left(\frac{m + n - |m - n|}{2} + C(\alpha) \right)^2 |v_n| \\ &= \underbrace{\sum_{\substack{m \geq 0 \\ n \geq 0}} \frac{(m + n - |m - n|)^2}{4} |v_m| |v_n|}_{A_1} + \underbrace{\sum_{\substack{m \geq 0 \\ n \geq 0}} C(\alpha)(m + n - |m - n|) |v_m| |v_n|}_{A_2} \\ &\quad + \underbrace{C(\alpha)^2 \sum_{m \geq 0} |v_m| \sum_{n \geq 0} |v_n|}_{A_3}, \\ A_1 &\leq \frac{1}{2} \sum_{\substack{m \geq 0 \\ n \geq 0}} (m^2 + n^2) |v_m| |v_n| = \frac{1}{2} \left(\sum_{m \geq 0} m^2 |v_m| \sum_{n \geq 0} |v_n| + \sum_{m \geq 0} |v_m| \sum_{n \geq 0} n^2 |v_n| \right) \\ &= \sum_{m \geq 0} m^2 |v_m| \sum_{n \geq 0} |v_n| \\ A_2 &\leq C(\alpha) \sum_{\substack{m \geq 0 \\ n \geq 0}} (m + n) |v_m| |v_n| = C(\alpha) \left(\sum_{m \geq 0} m |v_m| \sum_{n \geq 0} |v_n| + \sum_{m \geq 0} |v_m| \sum_{n \geq 0} n |v_n| \right) \\ &= 2C(\alpha) \sum_{m \geq 0} m |v_m| \sum_{n \geq 0} |v_n|, \end{aligned}$$

$$\begin{aligned}
C &= \sum_{\substack{m < 0 \\ n < 0}} |v_m| \left(\frac{-m - n - |m - n|}{2} + C(\alpha) \right)^2 |v_m| = \underbrace{\sum_{\substack{m < 0 \\ n < 0}} \frac{(-m - n + |m - n|)^2}{4} |v_m| |v_n|}_{C_1} \\
&\quad + \underbrace{\sum_{\substack{m < 0 \\ n < 0}} C(\alpha) (-m - n - |m - n|) |v_m| |v_n|}_{C_2} + \underbrace{C(\alpha)^2 \sum_{m < 0} |v_m| \sum_{n < 0} |v_n|}_{C_3},
\end{aligned}$$

$$\begin{aligned}
C_1 &\leq \frac{1}{2} \sum_{\substack{m < 0 \\ n < 0}} (m^2 + n^2) |v_m| |v_n| = \frac{1}{2} \left(\sum_{m < 0} m^2 |v_m| \sum_{n < 0} |v_n| + \sum_{m < 0} |v_m| \sum_{n < 0} n^2 |v_n| \right) \\
&= \sum_{m < 0} m^2 |v_m| \sum_{n < 0} |v_n| \\
C_2 &\leq C(\alpha) \sum_{\substack{m < 0 \\ n < 0}} (-m - n) |v_m| |v_n| = C(\alpha) \left(\sum_{m < 0} |m| |v_m| \sum_{n < 0} |v_n| + \sum_{m < 0} |v_m| \sum_{n < 0} |n| |v_n| \right) \\
&= 2C(\alpha) \sum_{m < 0} |m| |v_m| \sum_{n < 0} |v_n|,
\end{aligned}$$

$$B = \sum_{\substack{m \geq 0 \\ n < 0}} |v_m| C(\alpha)^2 |v_n| = C(\alpha)^2 \sum_{m \geq 0} |v_m| \sum_{n < 0} |v_n|$$

$$D = \sum_{\substack{m < 0 \\ n \geq 0}} |v_m| C(\alpha)^2 |v_n| = C(\alpha)^2 \sum_{m < 0} |v_m| \sum_{n \geq 0} |v_n|,$$

$$\begin{aligned}
A_3 + B + C_3 + D &= C(\alpha)^2 \left(\sum_{m \geq 0} |v_m| \sum_{n \geq 0} |v_n| + \sum_{m \geq 0} |v_m| \sum_{n < 0} |v_n| + \sum_{m < 0} |v_m| \sum_{n < 0} |v_n| + \sum_{m < 0} |v_m| \sum_{n \geq 0} |v_n| \right) \\
&= C(\alpha)^2 \left(\sum_{m \geq 0} |v_m| \sum_{n \in \mathbb{Z}} |v_n| + \sum_{m < 0} |v_m| \sum_{n \in \mathbb{Z}} |v_n| \right) = C(\alpha)^2 \sum_{n \in \mathbb{Z}} |v_n| \sum_{m \in \mathbb{Z}} |v_m| = C(\alpha)^2 \|v\|_{\ell^1(\mathbb{Z})}^2,
\end{aligned}$$

$$\begin{aligned}
A_1 + C_1 &\leq \sum_{m \geq 0} m^2 |v_m| \sum_{n \geq 0} |v_n| + \sum_{m < 0} m^2 |v_m| \sum_{n < 0} |v_n| \\
&\leq \sum_{m \geq 0} m^2 |v_m| \sum_{n \in \mathbb{Z}} |v_n| + \sum_{m < 0} m^2 |v_m| \sum_{n \in \mathbb{Z}} |v_n| = \sum_{m \in \mathbb{Z}} m^2 |v_m| \sum_{n \in \mathbb{Z}} |v_n| = \|v\|_{\ell^1(\mathbb{Z}, m^2)} \|v\|_{\ell^1(\mathbb{Z})},
\end{aligned}$$

$$\begin{aligned}
A_2 + C_2 &\leq 2C(\alpha) \left(\sum_{m \geq 0} m |v_m| \sum_{n \geq 0} |v_n| + \sum_{m < 0} |m| |v_m| \sum_{n < 0} |v_n| \right) \\
&\leq 2C(\alpha) \left(\sum_{m \geq 0} m |v_m| \sum_{n \in \mathbb{Z}} |v_n| + \sum_{m < 0} |m| |v_m| \sum_{n \in \mathbb{Z}} |v_n| \right) \\
&= 2C(\alpha) \sum_{m \in \mathbb{Z}} |m| |v_m| \sum_{n \in \mathbb{Z}} |v_n| = 2C(\alpha) \|v\|_{\ell^1(\mathbb{Z}, m^1)} \|v\|_{\ell^1(\mathbb{Z})}.
\end{aligned}$$

Adding everything together we get the assertion of the proposition.

$$\begin{aligned}
\|K(\lambda)\|^2 &\leq C(\alpha)^2 \|v\|_{\ell^1(\mathbb{Z})}^2 + \|v\|_{\ell^1(\mathbb{Z}, m^2)} \|v\|_{\ell^1(\mathbb{Z})} + 2C(\alpha) \|v\|_{\ell^1(\mathbb{Z}, m^1)} \|v\|_{\ell^1(\mathbb{Z})} \\
&= \|v\|_{\ell^1(\mathbb{Z})} \left(C(\alpha)^2 \|v\|_{\ell^1(\mathbb{Z})} + \|v\|_{\ell^1(\mathbb{Z}, m^2)} + 2C(\alpha) \|v\|_{\ell^1(\mathbb{Z}, m^1)} \right) \\
&\leq \|v\|_{\ell^1(\mathbb{Z}, m^2)} \left(C(\alpha)^2 \|v\|_{\ell^1(\mathbb{Z}, m^2)} + \|v\|_{\ell^1(\mathbb{Z}, m^2)} + 2C(\alpha) \|v\|_{\ell^1(\mathbb{Z}, m^1)} \right) \\
&= (1 + C(\alpha))^2 \|v\|_{\ell^1(\mathbb{Z}, m^2)}^2
\end{aligned}$$

The last estimate is unnecessarily rough but at this point, we prefer simpler expressions to finer though more complicated upper bounds. \square

The following result will be used to prove that there are no embedded eigenvalues in the essential spectrum. The idea is borrowed from [9].

Proposition 3.9: Assume $\alpha \neq 0$ and $v \in \ell^1(\mathbb{Z})$. Let $\lambda \in \sigma(H_\alpha)$ be an eigenvalue of $H_\alpha + V$, i.e. there exists a non-zero vector $u \in \ell^2(\mathbb{Z})$ satisfying $(H_\alpha + V)u = \lambda u$. Then

$$\forall m \in \mathbb{Z} \quad : \quad ((I + K(\lambda + i\varepsilon))V_{1/2}u)_m \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0_+.$$

Proof. Choose an $m \in \mathbb{Z}$ fixed. The eigenvalue equation implies $v_m u_m = \lambda u_m - (H_\alpha u)_m$. Then we have

$$\begin{aligned}
(K(\lambda + i\varepsilon)V_{1/2}u)_m &= (V_{1/2}(H_\alpha - \lambda - i\varepsilon)^{-1}Vu)_m = \operatorname{sgn} v_m \sqrt{|v_m|} \sum_{n \in \mathbb{Z}} G_{m,n}(\lambda + i\varepsilon) v_n u_n \\
&= \operatorname{sgn} v_m \sqrt{|v_m|} \sum_{n \in \mathbb{Z}} G_{m,n}(\lambda + i\varepsilon) (\lambda u_m - (H_\alpha u)_m).
\end{aligned}$$

Let us investigate only a part of the previous expression

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} G_{m,n}(\lambda + i\varepsilon) v_n u_n &= \sum_{n \in \mathbb{Z}} G_{m,n}(\lambda + i\varepsilon) (\lambda u_m + i\varepsilon u_m - (H_\alpha u)_m) - i\varepsilon \sum_{n \in \mathbb{Z}} G_{m,n}(\lambda + i\varepsilon) u_m \\
&= \left\langle \overline{G_{m,\cdot}(\lambda + i\varepsilon)}, (\lambda + i\varepsilon - H_\alpha)u \right\rangle - i\varepsilon \sum_{n \in \mathbb{Z}} G_{m,n}(\lambda + i\varepsilon) u_m \\
&= \left\langle \overline{(H_\alpha - \lambda - i\varepsilon)G_{m,\cdot}(\lambda + i\varepsilon)}, u \right\rangle - i\varepsilon \sum_{n \in \mathbb{Z}} G_{m,n}(\lambda + i\varepsilon) u_m \\
&= -\langle e_m, u \rangle - i\varepsilon \sum_{n \in \mathbb{Z}} G_{m,n}(\lambda + i\varepsilon) u_m \\
&= -u_m - i\varepsilon \left\langle \overline{G_{m,\cdot}(\lambda + i\varepsilon)}, u \right\rangle.
\end{aligned}$$

Now we plug this result back into the previous expression and obtain

$$((I + K(\lambda + i\varepsilon))V_{1/2}u)_m = -i\varepsilon \operatorname{sgn} v_m \sqrt{|v_m|} \left\langle \overline{G_{m,\cdot}(\lambda + i\varepsilon)}, u \right\rangle.$$

We show that the right hand side vanishes as $\varepsilon \rightarrow 0_+$. To this end we estimate the absolute value of the right hand side of the expression above using the Cauchy–Schwarz inequality

$$\left| -i\varepsilon \operatorname{sgn} v_m \sqrt{|v_m|} \left\langle \overline{G_{m,\cdot}(\lambda + i\varepsilon)}, u \right\rangle \right| \leq \varepsilon \sqrt{|v_m|} \|G_{m,\cdot}(\lambda + i\varepsilon)\| \|u\|.$$

The proof is complete if we show that

$$\varepsilon \|G_{m,\cdot}(\lambda + i\varepsilon)\| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0_+. \quad (3.2.1)$$

Recall that we denoted $\tilde{G} = wG$ and that from (1.3.5) we have $|w|^{-1} \leq C(\alpha)$. For the calculation of the norm $\|G_{m,\cdot}(\lambda + i\varepsilon)\|$ the formulation (2.1.11) is the most suitable. ξ and η are the Joukowski parameters of $\lambda + i\varepsilon$. To ensure that $\lambda + i\varepsilon$ is always in the resolvent set of H_α we assume that $\varepsilon < |\text{Im } \alpha|$. This assumption necessitates that ξ and η are unique and less than 1 in absolute value. We have

$$\begin{aligned} \|\tilde{G}_{m,\cdot}(\lambda + i\varepsilon)\|^2 &= \sum_{n \in \mathbb{Z}} |\tilde{G}_{m,n}(\lambda + i\varepsilon)|^2 \\ &= \begin{cases} \sum_{n=0}^{\infty} \left| \frac{\xi - \eta}{\eta^{-1} - \eta} \eta^{m+n} - \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}} \eta^{m-n} \right|^2 + \sum_{n=1}^{\infty} |\eta^m \xi^n|^2 & m \geq 0, \\ \sum_{n=1}^{\infty} |\eta^n \xi^{-m}|^2 + \sum_{n=1}^{\infty} \left| \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} \xi^{m+n} - \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} \eta^{n-m} \right|^2 & m < 0. \end{cases} \end{aligned}$$

Let us estimate both expressions from above. For $m \geq 0$ we have

$$\begin{aligned} \|\tilde{G}_{m,\cdot}(\lambda + i\varepsilon)\|^2 &\leq 2|\eta|^{2m} \frac{|\xi - \eta|^2}{|\eta^{-1} - \eta|^2} \sum_{n=0}^{\infty} |\eta|^{2n} + 2 \frac{|\xi - \eta^{-1}|^2}{|\eta - \eta^{-1}|^2} \sum_{n=-m}^{\infty} |\eta|^{2|n|} + |\eta|^{2m} \sum_{n=1}^{\infty} |\xi|^{2n} \\ &\leq 2 \frac{4}{|\eta^{-1} - \eta|^2} \frac{1}{1 - |\eta|^2} + 2 \frac{2 + 2|\eta|^{-2}}{|\eta - \eta^{-1}|^2} \frac{2}{1 - |\eta|^2} + \frac{1}{1 - |\xi|^2}. \end{aligned} \quad (3.2.2)$$

And for $m < 0$ we estimate

$$\begin{aligned} \|\tilde{G}_{m,\cdot}(\lambda + i\varepsilon)\|^2 &\leq |\xi|^{-2m} \sum_{n=1}^{\infty} |\eta|^{2n} + 2 \frac{|\eta^{-1} - \xi|^2}{|\xi^{-1} - \xi|^2} \sum_{n=1+m}^{\infty} \xi^{2|n|} + 2|\eta|^{-2m} \frac{|\eta^{-1} - \xi^{-1}|^2}{|\xi - \xi^{-1}|^2} \sum_{n=1}^{\infty} |\eta|^{2n} \\ &\leq \frac{1}{1 - |\eta|^2} + 2 \frac{2|\eta|^{-2} + 2}{|\xi^{-1} - \xi|^2} \frac{2}{1 - |\xi|^2} + 2 \frac{2|\eta|^{-2} + 2|\xi|^{-2}}{|\xi - \xi^{-1}|^2} \frac{1}{1 - |\xi|^2}. \end{aligned} \quad (3.2.3)$$

For the remaining expression we need to derive asymptotic formulas as $\varepsilon \rightarrow 0_+$.

Recall the asymptotic expansion of the Joukowski parameter from Lemmas 1.22 and 1.24. Corollaries 1.23 and 1.25 that followed yielded the asymptotic expansions for $|k|^2$ and $|k - k^{-1}|^2$. We apply these corollaries separately on ξ and η .

Let $\lambda \in (-2, 2)$, then

$$|\xi|^{-2} = \mathcal{O}(1), \quad \frac{1}{1 - |\xi|^2} = \mathcal{O}(\varepsilon^{-1}), \quad \frac{1}{|\xi - \xi^{-1}|^2} = \mathcal{O}(1) \quad \text{as } \varepsilon \rightarrow 0_+.$$

Let $\lambda \in (-2 + \alpha, 2 + \alpha)$, then

$$|\eta|^{-2} = \mathcal{O}(1), \quad \frac{1}{1 - |\eta|^2} = \mathcal{O}(\varepsilon^{-1}), \quad \frac{1}{|\eta - \eta^{-1}|^2} = \mathcal{O}(1) \quad \text{as } \varepsilon \rightarrow 0_+.$$

For the endpoints we have

$$\begin{aligned} \lambda \in \{2, -2\} & : |\xi|^{-2} = \mathcal{O}(1), \quad \frac{1}{1 - |\xi|^2} = \mathcal{O}(\varepsilon^{-1/2}), \quad \frac{1}{|\xi - \xi^{-1}|^2} = \mathcal{O}(\varepsilon^{-1}), \\ \lambda \in \{2 + \alpha, 2 + \alpha\} & : |\eta|^{-2} = \mathcal{O}(1), \quad \frac{1}{1 - |\eta|^2} = \mathcal{O}(\varepsilon^{-1/2}), \quad \frac{1}{|\eta - \eta^{-1}|^2} = \mathcal{O}(\varepsilon^{-1}). \end{aligned}$$

as $\varepsilon \rightarrow 0_+$. The remaining cases obey

$$\begin{aligned} \lambda \notin [-2, 2] & : \quad \xi \rightarrow c_1 \quad \text{as } \varepsilon \rightarrow 0_+, \quad \text{where } |c_1| \in (0, 1), \\ \lambda \notin [-2 + \alpha, 2 + \alpha] & : \quad \eta \rightarrow c_2 \quad \text{as } \varepsilon \rightarrow 0_+, \quad \text{where } |c_2| \in (0, 1). \end{aligned}$$

The desired expressions follow

$$\begin{aligned} \lambda \notin [-2, 2] & : \quad |\xi|^{-2} = \mathcal{O}(1), & \frac{1}{1 - |\xi|^2} = \mathcal{O}(1), & \frac{1}{|\xi - \xi^{-1}|^2} = \mathcal{O}(1), \\ \lambda \notin [-2 + \alpha, 2 + \alpha] & : \quad |\eta|^{-2} = \mathcal{O}(1), & \frac{1}{1 - |\eta|^2} = \mathcal{O}(1), & \frac{1}{|\eta - \eta^{-1}|^2} = \mathcal{O}(1), \end{aligned}$$

as $\varepsilon \rightarrow 0_+$.

Recall that $wG = \tilde{G}$, and so $\|G(\lambda + i\varepsilon)\| \leq C(\alpha)\|\tilde{G}(\lambda + i\varepsilon)\| = \mathcal{O}(1)\|\tilde{G}(\lambda + i\varepsilon)\|$ as $\varepsilon \rightarrow 0_+$. By line segments we mean the two line segments $[-2, 2]$ and $[-2 + \alpha, 2 + \alpha]$. Even though that in the case that $\alpha \in [-4, 4]$ they combine into one large line segment, it is useful to view them as two separate line segments. In the remaining part of the proof, we break down every case we need to properly apply the asymptotic expansions and evaluate the asymptotic behavior of $\|G_{m,\cdot}(\lambda + i\varepsilon)\|$. For $m \in \mathbb{Z}$ we have

$$\begin{aligned} A & : \quad \|G_{m,\cdot}(\lambda + i\varepsilon)\| = \mathcal{O}\left(\varepsilon^{-1/2}\right), \\ B & : \quad \|G_{m,\cdot}(\lambda + i\varepsilon)\| = \mathcal{O}\left(\varepsilon^{-3/4}\right), \end{aligned} \tag{3.2.4}$$

where

A: λ is not an endpoint of either line segment. We need to identify three distinct sub-cases

$$\begin{aligned} \lambda & \in (-2, 2) \cup (-2 + \alpha, 2 + \alpha), \\ \lambda & \in (-2, 2) \setminus [-2 + \alpha, 2 + \alpha], \\ \lambda & \in (-2 + \alpha, 2 + \alpha) \setminus [-2, 2]. \end{aligned}$$

B: λ is an endpoint of at least one of the line segments. Again we identify certain sub-cases

$$\begin{aligned} \lambda & \in \{2, -2 + \alpha\} & \text{for } \alpha \in (0, 4), \\ \lambda & \in \{-2, 2 + \alpha\} & \text{for } \alpha \in (-4, 0), \\ \lambda & = \pm 4 & \text{for } \alpha = \pm 4, \\ \lambda & \in \{-2, 2, -2 + \alpha, 2 + \alpha\} & \text{for } \alpha \notin [-4, 4]. \end{aligned}$$

Each of these sub-cases corresponds with a certain set of asymptotic formulas for the expressions. We plug them into (3.2.3) and (3.2.2) and obtain (3.2.4) regardless of m . This concludes the proof because it implies (3.2.1). \square

Now we have everything necessary established to formulate and prove the main result.

Theorem 3.10: Let $\alpha \neq 0$ and $v \in \ell^1(\mathbb{Z}, m^2)$. If we take the sequence v such that

$$(1 + C(\alpha))\|v\|_{\ell^1(\mathbb{Z}, m^2)} < 1,$$

then the potential V does not change the spectrum and there are no embedded eigenvalues, i.e.

$$\sigma(H_\alpha) = \sigma(H_\alpha + V) = \sigma_c(H_\alpha + V).$$

Proof. The interior of $\sigma(H_\alpha)$ is clearly empty in the topology of \mathbb{C} and $\rho(H_\alpha)$ has one connected component which has a non-empty intersection with $\rho(H_\alpha + V)$ since $H_\alpha + V$ is bounded. The operator V is compact. This satisfies the assumptions of Theorem 1.9 and it states that

$$\sigma_{\text{ess}}(H_\alpha) = \sigma_{\text{ess}}(H_\alpha + V). \quad (3.2.5)$$

Denote $(1 + C(\alpha))\|v\|_{\ell^1(\mathbb{Z}, m^2)} =: q < 1$. If we estimate the norm of the Birman–Schwinger operator $K(\lambda)$ for $\lambda \in \rho(H_\alpha)$ according to Lemma 3.8, we get

$$\|K(\lambda)\| \leq q < 1. \quad (3.2.6)$$

This rules out the possibility of -1 being an eigenvalue of $K(\lambda)$ in view of the Birman–Schwinger principle. Therefore, $\lambda \in \rho(H_\alpha) \implies \lambda \notin \sigma_{\text{p}}(H_\alpha + V)$. From (3.2.5) follows that in $\rho(H_\alpha)$ the operator $H_\alpha + V$ can have only eigenvalues. We then conclude that

$$\sigma(H_\alpha + V) \cap \rho(H_\alpha) = \emptyset.$$

Thus $\sigma(H_\alpha) = \sigma(H_\alpha + V)$ since the essential spectra are the same.

To show that the spectrum of $H_\alpha + V$ remain purely continuous, we use Proposition 3.9. Let $\lambda \in \sigma(H_\alpha)$. Assume $V \neq 0$ as the theorem clearly holds for $V = 0$. For a contradiction, suppose $\lambda \in \sigma_{\text{p}}(H_\alpha + V)$. By the definition of an eigenvalue, there exists a non-zero vector $u \in \ell^2(\mathbb{Z})$ such that $(H_\alpha + V)u = \lambda u$. Then Proposition 3.9 states that

$$\forall m \in \mathbb{Z} \quad : \quad \langle e_n, (I + K(\lambda + i\varepsilon))V_{1/2}u \rangle \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0_+.$$

Hence,

$$\forall x \in \ell^2(\mathbb{Z}) \quad : \quad \langle x, K(\lambda + i\varepsilon)V_{1/2}u \rangle \longrightarrow -\langle x, V_{1/2}u \rangle \quad \text{as } \varepsilon \rightarrow 0_+.$$

Notice that $V_{1/2}u \neq 0$ otherwise $\lambda \in \sigma_{\text{p}}(H_\alpha) = \emptyset$. If we plug in $x = V_{1/2}u$ and take the absolute value, we get

$$\lim_{\varepsilon \rightarrow 0_+} |\langle V_{1/2}u, K(\lambda + i\varepsilon)V_{1/2}u \rangle| = \|V_{1/2}u\|^2.$$

On the other hand, if we take the Cauchy–Schwarz inequality first and then plug in the estimate (3.2.6), we arrive at

$$\lim_{\varepsilon \rightarrow 0_+} |\langle V_{1/2}u, K(\lambda + i\varepsilon)V_{1/2}u \rangle| \leq \lim_{\varepsilon \rightarrow 0_+} \|K(\lambda + i\varepsilon)\| \|V_{1/2}u\|^2 \leq q \|V_{1/2}u\|^2.$$

This is a contradiction since $q < 1$. Therefore, $\lambda \in \sigma(H_\alpha) \implies \lambda \notin \sigma_{\text{p}}(H_\alpha + V)$. Hence, the point spectrum of $H_\alpha + V$ is empty. From (3.2.5) it follows that $\lambda \in \sigma(H_\alpha) \implies \lambda \in \sigma_{\text{c}}(H_\alpha + V) \cup \sigma_{\text{r}}(H_\alpha + V)$.

Recall the decomposition (2.0.2). Since the point spectrum of $H_\alpha + V$ is empty, Theorem 2.2 asserts that the residual spectrum of $H_\alpha + V$ is empty. In Theorem 2.2, we set $A = H_0$ and $B = \alpha D + V$ which satisfy the assumptions of the theorem.

In conclusion, the potential V introduces no eigenvalues and the essential spectrum remains the same and purely continuous. \square

Conclusion

This master's thesis is a continuation of the author's Bachelor's degree project [1], and Research project [2]. We built upon the results obtained in those works. We introduced the Joukowski transform and further extended and proved additional assertions. The content of the first chapter consisted also of some standard results from functional analysis and spectral theory.

In the second chapter, we review the basic spectral analysis of H_α worked out in [1], that is the spectrum of H_α is purely continuous and coincides with $[-2, 2] \cup [-2 + \alpha, 2 + \alpha]$. We extended these results by describing the numerical range of H_α for $\alpha \in \mathbb{C}$ and the spectral measure of H_α for $\text{Im } \alpha = 0$. The bulk of this chapter was devoted to the pseudospectral analysis of H_α for $\text{Im } \alpha \neq 0$. On the subset of the complex plane where the distance from the numerical range and from the spectrum are equal, one can describe the pseudospectrum exactly. However, on the complement of this set, it seems impossible to describe the pseudospectrum explicitly; thus, we resorted to its approximate localization. We derived estimates for resolvent operator norm from above and below. The Schur test served as the primary tool for obtaining upper bounds. The lower estimate was obtained by a general estimation technique utilizing the distance from the spectrum. Using these estimates we constructed a subset and a superset of the boundary of the pseudospectrum; thus, approximately localizing it. To describe the asymptotic behavior of the pseudospectrum we derived asymptotic expansions of the resolvent operator norm.

The last chapter contains the main results of this thesis, describing the spectrum of the perturbed operator $H_\alpha + V$. The most important tool used in this chapter is the Birman–Schwinger principle. In [19], the authors consider the continuous Laplace operator Δ . They proved that there exists a unique eigenvalue of $\Delta + \varepsilon V$ if and only if the integral of V is non-positive under the assumptions that ε is small enough, V is a real-valued function, and the function $|V(x)|(1 + x^2)$ is integrable. Subsequently, in [12], this result was extended to the discrete case for a more general class of operators, note that the potential V is also real-valued. We complement these known results by allowing V to be complex-valued. For $\alpha = 0$ we have the following result. We showed that The operator $H_0 + \varepsilon V$ has a locally unique eigenvalue if ε is small enough and the diagonal sequence of V denoted by v satisfies that the sum of v has non-zero real part and that v weighted by the square of the index is small enough. Due to the lack of variational principles in the complex setting we were only able to prove local uniqueness of the eigenvalue.

For context, the authors of [10] considered the continuous operator $H = -\Delta + \text{isgn}(x)$. They showed that there may be eigenvalues of $H + \varepsilon V$ under some assumptions on V . However, these eigenvalues are contained within the numerical range of H and escape to infinity as ε approaches 0. So in this continuous setting the operator H does not exhibit complete spectral stability. If $\alpha \neq 0$, the behavior of H_α differs significantly from H_0 . We assume the sequence on the diagonal of v to be small in the ℓ^1 norm weighted by the square of the index. We were able to take a global uniform estimate of the norm of the Birman–Schwinger operator from above. This allowed us to rule out the existence of eigenvalues of $H_\alpha + V$ in $\rho(H_\alpha)$. Moreover, we showed that the residual spectrum of $H_\alpha + V$ is empty and there are no embedded eigenvalues in the essential spectrum, i.e. the operator H_α exhibits complete spectral stability.

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