Do the Kontsevich tetrahedral flows preserve or destroy the space of Poisson bi-vectors ?

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From the paper "Formality Conjecture" (see Ref. [1]):

I am aware of only one such a class, it corresponds to simplest good graph, the complete graph with 4 *vertices* (*and* 6 *edges*)*. This class gives a remarkable vector field on the space of bi-vector fields on* $\mathbb{R}^d$*. The evolution with respect to the time* t *is described by the following non-linear partial differential equation*:

$$\frac{d\alpha}{dt} = \sum_{i,j=1}^{n} \left(\sum_{k,\ell,m,k',\ell',m'=1}^{n} \frac{\partial^3 \alpha_{ij}}{\partial x_k \partial x_\ell \partial x_m} \frac{\partial \alpha_{kk'}}{\partial x_{\ell'}} \frac{\partial \alpha_{\ell\ell'}}{\partial x_{m'}} \frac{\partial \alpha_{mm'}}{\partial x_{k'}} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where $\alpha = \sum_{i,j} \alpha_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ *is a bi-vector field on* $\mathbb{R}^d$*. ...*

It follows from general properties of cohomology that 1) ***this evolution preserves the class of*** (***real-analytic***) ***Poisson structures****, ...*

In fact, I cheated a little bit. In the formula for the vector field on the space of bivector fields which one get from the tetrahedron graph, an additional term is present. This term is equal (*up to a numerical factor*) *to*

$$\sum_{i,m=1}^{n} \left(\sum_{j,k,\ell,k',\ell',m'=1}^{n} \frac{\partial^2 \alpha_{ij}}{\partial x_k \partial x_\ell} \frac{\partial^2 \alpha_{km}}{\partial x_{k'} \partial x_{\ell'}} \frac{\partial \alpha_{k'\ell}}{\partial x_{m'}} \frac{\partial \alpha_{m'\ell'}}{\partial x_j} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_m}.$$

It is possible to prove formally that ***if*** α ***is a Poisson bracket, i.e. if*** $[\alpha, \alpha] = 0 \in T^2(\mathbb{R}^d)$***, then the additional term shown above vanishes.***

By using twelve Poisson structures with high-degree polynomial coefficients as explicit counterexamples, we show in [2] that both the above claims are false: neither does the first flow preserve the property of bi-vectors to be Poisson nor does the second flow vanish identically at the Poisson bi-vectors. The counterexamples at hand themselves suggest a correction to the formula for the "exotic" flow on the space of Poisson bi-vectors; in fact, this flow is encoded by the balanced sum involving both the Kontsevich tetrahedral graphs (that give rise to the flows mentioned above). We reveal that it is only the balance (1 : 6) for which the flow does preserve the space of Poisson bi-vectors.

[1] *Kontsevich M.* (1997) Formality conjecture. *Deformation theory and symplectic geometry* (Ascona 1996), Math. Phys. Stud. **20**, Kluwer Acad. Publ., Dordrecht, 139–156.

[2] *Bouisaghouane A., Kiselev A.* (2016) Do the Kontsevich tetrahedral flows preserve or destroy the space of Poisson bi-vectors ? *Preprint* IHÉS/M/16/12 (Bures-sur-Yvette, France), 12 p.