On existence of an x-integral for a semi-discrete chain of hyperbolic type

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Abstract. A class of semi-discrete chains of the form $t_{1x} = f(x, t, t_1, t_x)$ is considered. For the given chains easily verifiable conditions for existence of x-integral of minimal order 4 are obtained.

1. Introduction

In the present paper we consider the integrable differential-difference chains of hyperbolic type

$$t_{1x} = f(x, t, t_1, t_x), \tag{1}$$

where the function $t(n, x)$ depends on discrete variable n and continuous variable x. We use the following notations $t_x = \frac{\partial}{\partial x} t$ and $t_1 = t(n + 1, x)$. It is also convenient to denote $t_{[k]} = \frac{\partial^k}{\partial x^k} t$, $k \in \mathbb{N}$ and $t_m = t(n + m, x)$, $m \in \mathbb{Z}$.

The integrability of the chain (1) is understood as Darboux integrability that is existence of so called x- and n-integrals [1, 4]. Let us give the necessary definitions.

Definition 1 *Function $F(x, t, t_1, \ldots, t_k)$ is called an x-integral of the equation (1) if*

$$D_x F(x, t, t_1, \ldots, t_k) = 0$$

for all solutions of (1). The operator D_x is the total derivative with respect to x.

Definition 2 *Function $G(x, t, t_x, \ldots, t_{[m]})$ is called an n-integral of the equation (1) if*

$$DG(x, t, t_x, \ldots, t_{[m]}) = G(x, t, t_x, \ldots, t_{[m]})$$

for all solutions of (1). The operator D is a shift operator.

To show the existence of x- and n-integrals we can use the notion of characteristic ring. The notion of characteristic ring was introduced by Shabat to study hyperbolic systems of exponential type (see [11]). This approach turns out to be very convenient to study and classify the integrable equations of hyperbolic type (see [12] and references there in).

For difference and differential-difference chains the notion of characteristic ring was developed by Habibullin (see [3]-[8]). In particular, in [4] the following theorem was proved

Theorem 3 *(see [4]). A chain (1) admits a non-trivial x-integral if and only if its characteristic x-ring is of finite dimension.*
A chain (1) admits a non-trivial n-integral if and only if its characteristic n-ring is of finite dimension.

For known examples of integrable chains the dimension of the characteristic ring is small. The differential-difference chains with three dimensional characteristic x-ring were considered in [6]. We consider chains with four dimensional characteristic x-ring, such chains admit x-integral of minimal order four. That is we obtain necessary and sufficient conditions for a chain to have a four dimensional characteristic x-ring. This conditions can be easily checked by direct calculations.

Note that if a chain (1) admits a nontrivial x-integral $F(x,t,t_1,\dots t_k)$ and a non trivial n-integral $G(x,t,t_x,\dots,t_{[m]})$ its solutions satisfy two ordinary equations

$$F(x,t,t_1,\dots,t_k)=a(n),$$

$$G(x,t,t_x,\dots,t_{[m]})=b(x)$$

for some functions $a(n)$ and $b(x)$. This allows to solve (1) (see [9]).

The paper is organized as follows. In Section 2 we derive necessary and sufficient conditions on function $f(x,t,t_1,t_x)$ so that the chain (1) has four dimensional characteristic ring and in Section 3 we consider some applications of the derived conditions.

2. Chains admitting four dimensional x-algebra.

Suppose F is an x-integral of the chain (1) then its positive shifts and negative shifts $D^k F$, $k\in\mathbb{Z}$, are also x-integrals. So, looking for an x-integral it is convenient to assume that it depends on positive and negative shits of t.

To express x derivatives of negative shifts we can apply D^{-1} to the chain (1) and obtain

$$t_x=f(x,t_{-1},t,t_x).$$

Solving the above equation for t_{-1x} we get

$$t_{-1x}=g(x,t_{-1},t,t_x).$$

Let $F(x,t,t_1,t_{-1},\dots)$ be an x-integral of the chain (1). Then on solutions of (1) we have

$$D_xF=\frac{\partial F}{\partial x}+t_x\frac{\partial F}{\partial t}+t_{1x}\frac{\partial F}{\partial t_1}+t_{-1x}\frac{\partial F}{\partial t_{-1}}+t_{2x}\frac{\partial F}{\partial t_2}+t_{-2x}\frac{\partial F}{\partial t_{-2}}+\dots=0$$

or

$$D_xF=\frac{\partial F}{\partial x}+t_x\frac{\partial F}{\partial t}+f\frac{\partial F}{\partial t_1}+g\frac{\partial F}{\partial t_{-1}}+Df\frac{\partial F}{\partial t_2}+D^{-1}g\frac{\partial F}{\partial t_{-2}}+\dots=0.$$

Define a vector field

$$K=\frac{\partial}{\partial x}+t_x\frac{\partial}{\partial t}+f\frac{\partial}{\partial t_1}+g\frac{\partial}{\partial t_{-1}}+Df\frac{\partial}{\partial t_2}+D^{-1}g\frac{\partial}{\partial t_{-2}}+\dots, \tag{2}$$

then

$$D_xF=K\,F.$$

Note that F does not depend on t_x but the coefficients of K do depend on t_x. So we introduce a vector field

$$X=\frac{\partial}{\partial t_x} \tag{3}$$

The vector fields K and X generate the characteristic x-ring L_x.

Let us introduce some other vector fields from L_x.

$$C_1 = [X, K] \quad \text{and} \quad C_n = [X, C_{n-1}] \quad n = 2, 3, \ldots \tag{4}$$

and

$$Z_1 = [K, C_1] \quad \text{and} \quad Z_n = [K, Z_{n-1}] \quad n = 2, 3, \ldots \tag{5}$$

Thus

$$C_1 = \frac{\partial}{\partial t} + f_{t_x} \frac{\partial}{\partial t_1} + g_{t_x} \frac{\partial}{\partial t_{-1}} + \ldots$$

$$C_2 = f_{t_x t_x} \frac{\partial}{\partial t_1} + g_{t_x t_x} \frac{\partial}{\partial t_{-1}} + \ldots$$

$$Z_1 = (f_{t_x x} + t_x f_{t_x t} + f f_{t_x t_1} - f_t - f_{t_x} f_{t_1}) \frac{\partial}{\partial t_1} + (g_{t_x x} + t_x g_{t_x t} + g g_{t_x t_1} - g_t - g_{t_x} g_{t_1}) \frac{\partial}{\partial t_{-1}} + \ldots$$

and so on.

It is easy to see that if $f_{t_x t_x} \neq 0$ then the vector fields X, K, C_1 and C_2 are linearly independent and must form a basis of L_x provided $dim L_x = 4$. By Lemma 3.6 in [6], if $f_{t_x t_x} = 0$ and $(f_{t_x x} + t_x f_{t_x t} + f f_{t_x t_1} - f_t - f_{t_x} f_{t_1}) = 0$ then $dim L_x = 3$. So in the case $f_{t_x t_x} = 0$ we may assume $(f_{t_x x} + t_x f_{t_x t} + f f_{t_x t_1} - f_t - f_{t_x} f_{t_1}) \neq 0$. Then the vector fields X, K, C_1 and Z_1 are linearly independent and must form a basis of L_x provided $dim L_x = 4$. We consider this two cases separately.

In the rest of the paper we assume that the characteristic ring L_x is four dimensional.

Remark 4 *It is convenient to check equalities between vector fields using the automorphism $D(\ )D^{-1}$. Direct calculations show that*

$$DXD^{-1} = \frac{1}{f_x} X,$$

$$DKD^{-1} = K - \frac{f_x + t_x f_t + f f_{t_1}}{f_{t_x}} X.$$

The images of other vector fields under this automorphism can be obtained by commuting DXD^{-1} and DKD^{-1}.

2.1. $f(x, t, t_1, t_x)$ is non linear with respect to t_x.

Let $f(x, t, t_1, t_x)$ be non linear with respect to t_x, $f_{t_x t_x} \neq 0$. Then the vector fields X, K, C_1 and C_2 form a basis of L_x. For the algebra L_x to be spanned by X, K, C_1 and C_2 it is enough that C_3 and Z_1 are linear combinations of X, K, C_1 and C_2. From the form of the vector fields it follows that we must have

$$C_3 = \lambda C_2 \quad \text{and} \quad Z_1 = \mu C_2$$

for some functions μ and λ. The conditions for the above equalities to hold are given by the following theorem.

Theorem 5 *The chain (1) with $f_{t_x t_x} \neq 0$ has characteristic ring L_x of dimension four if and only if the following conditions hold*

$$D\left(\frac{f_{t_x t_x t_x}}{f_{t_x t_x}}\right) = \frac{f_{t_x t_x t_x} f_{t_x} - 3 f_{t_x t_x}^2}{f_{t_x t_x} f_{t_x}^2}. \tag{6}$$

$$D\left(\frac{f_{xt_x}+t_xf_{t_xt}+ff_{t_xt_1}-f_t-f_{t_x}f_{t_1}}{f_{t_xt_x}}\right)=$$

$$\frac{f_{xt_x}+t_xf_{t_xt}+ff_{t_xt_1}-f_t-f_{t_x}f_{t_1}}{f_{t_xt_x}}f_{t_x}-(f_x+t_xf_t+f_{t_1}). \tag{7}$$

The characteristic ring is generated by the vector fields X, K, C_1, C_2.

Proof. By Remark 4 we have

$$DC_2D^{-1}=\frac{1}{f_{t_x}^2}C_2-\frac{f_{t_xt_x}}{f_{t_x}^3}C_1+\frac{f_{t_xt_x}f_t}{f_{t_x}^4}X$$

$$DC_3D^{-1}=\frac{1}{f_{t_x}^3}C_2-\frac{3f_{t_xt_x}}{f_{t_x}^4}C_2-\frac{f_{t_xt_xt_x}f_{t_x}-3f_{t_xt_x}^2}{f_{t_x}^5}C_1+f_t\frac{f_{t_xt_xt_x}f_{t_x}-3f_{t_xt_x}^2}{f_{t_x}^6}X$$

$$DZ_1D^{-1}=\frac{1}{f_{t_x}}Z_1-\left(\frac{mf_{t_x}+p}{f_{t_x}^2}\right)\left(C_1-\frac{f_t}{f_{t_x}}X\right),$$

where $p=\dfrac{f_x+t_xf_t+ff_{t_1}}{f_{t_x}}$ and $m=\dfrac{-(f_{xt_x}+t_xf_{t_xt}+ff_{t_xt_1})+f_t+f_{t_x}f_{t_1}}{f_{t_x}}$.

The equality $C_3=\lambda C_2$ implies that

$$DC_3D^{-1}=(D\lambda)\,DC_2D^{-1}. \tag{8}$$

Substituting expressions for DC_2D^{-1} and DC_3D^{-1} into (8) and comparing coefficients of C_1, C_2 and X we obtain that λ satisfies

$$\lambda=f_{t_x}(D\lambda)+\frac{3f_{t_xt_x}}{f_{t_x}}$$

$$(D\lambda)=\frac{f_{t_xt_xt_x}f_{t_x}-3f_{t_xt_x}^2}{f_{t_xt_x}f_{t_x}^2}.$$

We can find λ and $D\lambda$ independently and condition that $D\lambda$ is a shift of λ leads to (6).
The equality $Z_1=\mu C_2$ implies that

$$DZ_1D^{-1}=(D\mu)\,DC_2D^{-1}. \tag{9}$$

Substituting expressions for DC_2D^{-1} and DC_3D^{-1} into (9) and comparing coefficients of C_1, C_2 and X we obtain that μ satisfies

$$\mu-\frac{f_x+t_xf_t+ff_{t_1}}{f_{t_x}}=\frac{(D\mu)}{f_{t_x}}$$

and

$$-(f_{xt_x}+t_xf_{t_xt}+ff_{t_xt_1}-f_t-f_{t_x}f_{t_1})+\frac{f_x+t_xf_t+ff_{t_1}}{f_{t_x}}f_{t_xt_x}=-\frac{f_{t_xt_x}(D\mu)}{f_{t_x}}$$

We can find μ and $D\mu$ independently and condition that $D\mu$ is a shift of μ leads to (7). □

Remark 6 *Let $\dim L_x=4$ and $f_{t_{xx}}\neq 0$. Then the characteristic ring L_x have the following multiplication table*

	X	K	C_1	C_2
X	0	C_1	C_2	μC_2
K	$-C_1$	0	λC_2	ρC_2
$C_1 1$	$-C_2$	$-\lambda C_2$	0	ηC_2
C_2	$-\mu C_2$	$-\rho C_2$	$-\eta C_2$	0

where $\rho = \lambda\mu + X(\lambda)$ *and* $\eta = X(\rho) - K(\mu)$.

Example 7 *Consider the following chain*

$$t_{1x} = \frac{tt_x - \sqrt{t_x^2 - M^2}(t_1 + t)}{t_1}$$

introduced by Habibullin and Zheltukhina [10]. We can easily check that the function

$$f(t, t_1, t_x) = \frac{tt_x - \sqrt{t_x^2 - M^2}(t_1 + t)}{t_1}$$

satisfies the conditions of Theorem 5. Hence the corresponding x*-algebra is four dimensional. The chain has the following* x*-integral*

$$F = \frac{(t_1^2 - t^2)(t_1^2 - t_2^2)}{t_1^2}.$$

2.2. $f(x, t, t_1, t_x)$ *is linear with respect to* t_x*.*
Let $f(x, t, t_1, t_x)$ be linear with respect to t_x, $f_{t_x t_x} = 0$. Then vector fields X, K, C_1 and Z_1 form a basis of L_x. The condition $f_{t_x t_x} = 0$ also implies that the vector field $C_2 = 0$, see [6]. For the algebra L_x to be spanned by X, K, C_1 and Z it is enough that Z_2 is a linear combination of X, K, C_1 and Z_1. From the form of the vector fields it follows that we must have

$$Z_2 = \alpha Z_1$$

for some function α. The conditions for the above equality to hold given by the following theorem.

Theorem 8 *The chain (1) with* $f_{t_x t_x} = 0$ *has the characteristic ring* L_x *of dimension four if and only if the following condition hold*

$$D\left(\frac{K(m)}{m} - m + \frac{f_t}{f_{t_x}}\right) = \frac{K(m)}{m} + m - f_{t_1}. \tag{10}$$

where $m = \dfrac{-(f_{xt_x} + t_x f_{t_x t} + f f_{t_x t_1}) + f_t + f_{t_x} f_{t_1}}{f_{t_x}}$. *The characteristic ring is generated by the vector fields* X, K, C_1, Z_1.

Proof. By Remark 4 we have

$$DZ_1 D^{-1} = \frac{1}{f_{t_x}} Z_1 - \left(\frac{m f_{t_x} + p}{f_{t_x}^2}\right)\left(C_1 - \frac{f_t}{f_{t_x}} X\right),$$

and

$$DZ_2 D^{-1} = \left(K\left(\frac{1}{f_{t_x}}\right) + \frac{\alpha + m}{f_{t_x}}\right) Z_1 + \left(K\left(\frac{m}{f_{t_x}}\right) + \frac{m f_t}{f_{t_x}^2} - pX\left(\frac{m}{f_{t_x}}\right)\right)\left(C_1 - \frac{f_t}{f_{t_x}} X\right)$$

The equality $Z_2 = \alpha Z_1$ implies that

$$DZ_2D^{-1} = (D\alpha)\, DZ_1D^{-1}. \tag{11}$$

Substituting expressions for DZ_1D^{-1} and DZ_2D^{-1} into (11) and comparing coefficients of C_1, Z_1 and X we obtain that α and $D(\alpha)$ satisfy

$$K\left(\frac{1}{f_{t_x}}\right) + \frac{m}{f_{t_x}} + \frac{\alpha}{f_{t_x}} = \frac{D(\alpha)}{f_{t_x}}$$

$$K\left(\frac{m}{f_{t_x}}\right) + \frac{mf_t}{f_{t_x}^2} = \frac{mD(\alpha)}{f_{t_x}}$$

We can find α and $D(\alpha)$ independently and condition that $D(\alpha)$ is a shift of α leads to (10). □

Remark 9 *Let* $\dim L_x = 4$ *and* $f_{t_{xx}} = 0$. *Then the characteristic ring* L_x *have the following multiplication table*

	X	K	C_1	Z_1
X	0	C_1	0	0
K	$-C_1$	0	Z_1	αZ_1
$C1$	0	$-Z_1$	0	$X(\alpha)Z_1$
Z_1	0	$-\alpha Z_1$	$-X(\alpha)Z_1$	0

Example 10 *Consider the following chain*

$$t_{1x} = t_x + e^{\frac{t+t_1}{2}}$$

introduced by Dodd and Bullough [2]. We can easily check that the function

$$f(t, t_1, t_x) = t_x + e^{\frac{t+t_1}{2}}$$

satisfies the conditions of Theorem 8. Hence the corresponding x*-algebra is four dimensional. The chain has the following* x*-integral*

$$F = e^{\frac{t_1-t}{2}} + e^{\frac{t_1-t_2}{2}}$$

3. Applications

The conditions derived in the previous section can be used to determine some restrictions on the form of the function $f(x, t, t_1, t_x)$ in (1).

Lemma 11 *Let the chain (1) have four dimensional characteristic* x*-ring. Then*

$$f = M(x, t, t_x)A(x, t, t_1) + t_xB(x, t, t_1) + C(x, t, t_1), \tag{12}$$

where M, A, B *and* C *are some functions.*

Proof. Let $f_{t_xt_x} \neq 0$ (if $f_{t_xt_x} = 0$ then f obviously has the above form). Since characteristic x-ring has dimension four the condition (6) holds. It is easy to see that (6) implies that $\dfrac{f_{t_xt_xt_x}}{f_{t_xt_x}}$ does not depend on t_1. Hence

$$X(\ln |f_{t_xt_x}|) = M_1(x, t, t_x) \quad \text{and} \quad \ln |f_{t_xt_x}| = M_2(x, t, t_x) + A_1(x, t, t_1).$$

The last equality implies (12). □

We can also put some restrictions on the shifts of the function $f(x, t, t_1, t_x)$ in (1).

Lemma 12 *Let the chain (1) have four dimensional characteristic x-ring and* $f_{t_x t_x} \neq 0$. *Then*

$$Df = -H_1(x,t,t_1,t_2)t_x + H_2(x,t,t_1,t_2)f + H_3(x,t,t_1,t_2), \tag{13}$$

where H_1, H_2 *and* H_3 *are some functions.*

Proof. Note that the shift operator D and the vector field X satisfy

$$DX = \frac{1}{f_{t_x}} XD. \tag{14}$$

The condition (6) can be written as

$$DX(\ln|f_{t_x t_x}|) = \frac{1}{f_{t_x}} X(\ln|f_{t_x t_x}| - \ln|f_{t_x}|^3)$$

Using (14) we get

$$\frac{1}{f_{t_x}} XD(\ln|f_{t_x t_x}|) = \frac{1}{f_{t_x}} X(\ln|f_{t_x t_x}| - \ln|f_{t_x}|^3)$$

which implies that

$$X\left(\ln\left|f_{t_x}^3 \frac{Df_{t_x t_x}}{f_{t_x t_x}}\right|\right) = 0 \quad \text{or} \quad X\left(f_{t_x}^3 \frac{Df_{t_x t_x}}{f_{t_x t_x}}\right) = 0.$$

Thus $Df_{t_x t_x} = H_1(x,t,t_1,t_2)\frac{f_{t_x t_x}}{f_{t_x}^3}$. Since $Df_{t_x t_x} = DX(f_{t_x})$ and $\frac{f_{t_x t_x}}{f_{t_x}^3} = -\frac{1}{f_{t_x}} X(\frac{1}{f_{t_x}})$ we can rewrite previous equality using (14) as

$$X\left(Df_{t_x} + H_1(x,t,t_1,t_2)\frac{1}{f_{t_x}}\right) = 0$$

which implies

$$Df_{t_x} = -H_1(x,t,t_1,t_2)\frac{1}{f_{t_x}} + H_2(x,t,t_1,t_2).$$

Writing

$$DX(f) = -H_1(x,t,t_1,t_2)\frac{1}{f_{t_x}} + H_2(x,t,t_1,t_2)\frac{f_{t_x}}{f_{t_x}}$$

and applying (14) as before we get

$$X(Df + H_1(x,t,t_1,t_2)t_x - H_2(x,t,t_1,t_2)f) = 0.$$

The last equality gives (13). $\square$

Note that the equality (13) can be written as

$$t_{2x} = H_2(x,t,t_1,t_2)t_{1x} - H_1(x,t,t_1,t_2)t_x + H_3(x,t,t_1,t_2).$$

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