# Representations of quantum SU(2) operators on a local chart

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**Abstract.** Hilbert space representations of quantum SU(2) by multiplication operators on a local chart are constructed, where the local chart is given by tensor products of square integrable functions on a quantum disc and on the classical unit circle. The actions of generators of quantum SU(2), generators of the opposite algebra, and noncommutative partial derivatives are computed on a Hilbert space basis.

## 1. Introduction

In [4], a twisted Dirac operator for quantum SU(2) is constructed by using so-called disc coordinates. The starting point of this construction is a representation of quantum SU(2) by multiplication operators on the tensor product of $L_2(\mathbb{S}^1)$ with the quantum disc algebra, where the inner product on the latter space is defined by a positive weighted trace. The construction of the Dirac operator uses noncommutative first order differential operators satisfying a twisted Leibniz rule. Only this twisted Leibniz rule is needed to prove that the Dirac operator has bounded twisted commutators with differentiable functions, thus avoiding an explicit description of the Hilbert space actions of the involved operators.

On the other hand, an explicit description of the Hilbert space representation of the quantum algebra of differentiable functions and the action of the Dirac operator is crucial for determining the analytic properties of the twisted spectral triple, for instance the spectrum of the Dirac operator, its K-homology class, its eigenvectors, and the C*-closure of the quantum algebra. The purpose of the present work is to describe the actions of all operators occurring in [4] on a Hilbert space basis. Unfortunately, a detailed analysis of the twisted spectral triple from [4] is behind the scope of this paper.

## 2. Quantum SU(2) in disc coordinates

Let $q \in (0, 1)$. The coordinate ring $\mathcal{O}(\mathrm{SU}_q(2))$ of quantum SU(2) is the *-algebra generated by $c$ and $d$ satisfying

$$cd = qdc, \qquad c^*d = qdc^*, \qquad cc^* = c^*c, \tag{1}$$

$$d^*d + q^2cc^* = 1, \qquad dd^* + cc^* = 1. \tag{2}$$

Classically, i. e. $q = 1$, the universal C*-algebra generated by $c$ and $d$ is isomorphic to $C(\mathbb{S}^3)$. The universal C*-closure $C(\mathrm{SU}_q(2))$ of $\mathcal{O}(\mathrm{SU}_q(2))$ has been studied in [6] and [8]. Here we will use the fact that it is generated by the operators $c, d \in B(\ell_2(\mathbb{N}) \bar{\otimes} \ell_2(\mathbb{Z}))$ given by

$$c(e_n \otimes b_k) = q^n e_n \otimes b_{k+1}, \qquad d(e_n \otimes b_k) = \sqrt{1 - q^{2(n+1)}}\, e_{n+1} \otimes b_k, \tag{3}$$

where $\{e_n\}_{n\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are orthonormal bases for $\ell_2(\mathbb{N})$ and $\ell_2(\mathbb{Z})$, respectively. Note that $c$ and $d$ from (3) do indeed satisfy the relations (1) and (2). Moreover, the representation of $\mathcal{O}(\mathrm{SU}_q(2))$ defined by (3) is faithful.

Let $u : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, $u(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t}$, denote the unitary generator of $C(\mathbb{S}^1)$. On the orthonormal basis $\{b_k := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}kt} : k \in \mathbb{Z}\}$ for $L_2(\mathbb{S}^1) \cong \ell_2(\mathbb{Z})$, multiplication by $u$ becomes the bilateral shift

$$u b_k = b_{k+1}, \qquad k \in \mathbb{Z}. \tag{4}$$

Consider the bounded operators $z$ and $z^*$ on $\ell_2(\mathbb{N})$ given by

$$z e_n := \sqrt{1 - q^{2(n+1)}}\, e_{n+1}, \qquad z^* e_n := \sqrt{1 - q^{2n}}\, e_{n-1}. \tag{5}$$

Obviously, $z$ and $z^*$ satisfy the quantum disc relation

$$z^* z - q^2 z z^* = 1 - q^2. \tag{6}$$

The polynomial *-algebra generated by $z$ and $z^*$ will be denoted by $\mathcal{O}(\mathrm{D}_q)$. Here and subsequently, $\mathrm{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\bar{\mathrm{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ stand for the open and closed unit discs, respectively, and $\mathcal{O}(\mathrm{D}_q)$ will be called quantum disc algebra. As shown in [2], the C*-closure $C(\mathrm{D}_q)$ of $\mathcal{O}(\mathrm{D}_q)$ is isomorphic to the Toeplitz algebra. Moreover, (5) determines the unique (up to unitary equivalence) faithful irreducible *-representation of $C(\mathrm{D}_q)$, see e. g. [5].

Defining $y \in B(\ell_2(\mathbb{N}))$ by

$$y e_n := q^n e_n, \quad n \in \mathbb{N}, \tag{7}$$

we have

$$y = \sqrt{1 - zz^*}, \qquad yz = qzy, \qquad z^* y = q y z^*. \tag{8}$$

In particular, $y = \sqrt{1 - zz^*} \in C(\mathrm{D}_q)$. Finally, with $u$, $z$ and $y$ given in (4), (5) and (7), the operators $c$ and $d$ in (3) can be written

$$c = y \otimes u = \sqrt{1 - zz^*} \otimes u, \qquad d = z \otimes 1. \tag{9}$$

The representation (9) corresponds to the classical parametrization

$$\psi : \bar{\mathrm{D}} \times [-\pi, \pi] \longrightarrow \mathbb{S}^3, \qquad \psi(z, t) := (z, \sqrt{1 - z\bar{z}}\, \mathrm{e}^{\mathrm{i}t}),$$

The restriction of $\psi$ to the open set $\mathrm{D} \times (-\pi, \pi)$ defines a dense coordinate chart for $\mathbb{S}^3 \cong \mathrm{SU}(2)$ which is compatible with the standard differential structure on $\mathbb{S}^3$. Moreover, the generators $c$ and $d$ correspond in the classical limit $q \to 1$ to the coordinate functions

$$c(z, \sqrt{1 - z\bar{z}}\, \mathrm{e}^{\mathrm{i}t}) = \sqrt{1 - z\bar{z}}\, \mathrm{e}^{\mathrm{i}t}, \qquad d(z, \sqrt{1 - z\bar{z}}\, \mathrm{e}^{\mathrm{i}t}) = z.$$

We call them disc coordinates because the discs $\psi_t(\bar{\mathrm{D}}) := \{\psi(z, t) : z \in \bar{\mathrm{D}}\}$ for fixed $t \in (-\pi, \pi]$ correspond to the 2-dimensional symplectic leaves associated to a known Poisson group structure on SU(2) [1].

## 3. Differential calculus

One of the differential operators occurring in [4] is the classical partial derivative $-\mathrm{i}\frac{\partial}{\partial t}$ acting on $C^{(1)}(\mathbb{S}^1)$. On the basis vectors $b_k$ from (4), one obviously has

$$-\mathrm{i}\tfrac{\partial}{\partial t} b_k = k\, b_k\,, \qquad k \in \mathbb{Z}. \tag{10}$$

Our next aim is to describe noncommutative partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ on the quantum disc algebra. As in [7], consider the first order differential *-calculus $\mathrm{d} : \mathcal{O}(\mathrm{D}_q) \longrightarrow \Omega(\mathrm{D}_q)$ given by $\Omega(\mathrm{D}_q) = \mathrm{d}z\,\mathcal{O}(\mathrm{D}_q) + \mathrm{d}z^*\mathcal{O}(\mathrm{D}_q)$ with $\mathcal{O}(\mathrm{D}_q)$-bimodule structure

$$z\,\mathrm{d}z = q^{-2}\mathrm{d}z\,z, \quad z^*\,\mathrm{d}z = q^2\mathrm{d}z\,z^*, \quad z\,\mathrm{d}z^* = q^{-2}\mathrm{d}z^*\,z, \quad z^*\mathrm{d}z^* = q^2\mathrm{d}z^*\,z^*,$$

satisfying the Leibniz rule $\mathrm{d}(fg) = \mathrm{d}(f)g + f\,\mathrm{d}(g)$ for all $f, g \in \mathcal{O}(\mathrm{D}_q)$. We define the partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ by

$$d(f) = \mathrm{d}z\,\tfrac{\partial}{\partial z}(f) + \mathrm{d}z^*\,\tfrac{\partial}{\partial \bar z}(f), \qquad f \in \mathcal{O}(\mathrm{D}_q).$$

On monomials, one gets $\mathrm{d}(z^n z^{*k}) = \sum_{j=0}^{n-1} q^{-2j}\mathrm{d}z\,z^{n-1}z^{*k} + \sum_{l=0}^{k-1} q^{-2n+2l}\mathrm{d}z\,z^n z^{*k-1}$, therefore

$$\tfrac{\partial}{\partial z}(z^n z^{*k}) = q^{-2(n-1)}\,\tfrac{1-q^{2n}}{1-q^2}\,z^{n-1}z^{*k}, \qquad \tfrac{\partial}{\partial \bar z}(z^n z^{*k}) = q^{-2n}\,\tfrac{1-q^{2k}}{1-q^2}\,z^n z^{*k-1}.$$

From (6), it follows that the monomials $z^n z^{*k}$ span the linear space $\mathcal{O}(\mathrm{D}_q)$. Using the facts that commutators satisfy the Leibniz rule, $[z,z] = [z^*, z^*] = 0$, $\frac{1}{1-q^2}[z^*, z] = y^2$, and $y^2$ commutes with $z$ and $z^*$ in the same way as $\mathrm{d}z$ and $\mathrm{d}z^*$ do, one readily verifies that

$$\tfrac{\partial}{\partial z}(f) = \tfrac{1}{1-q^2}\,y^{-2}\,[z^*, f], \qquad \tfrac{\partial}{\partial \bar z}(f) = \tfrac{-1}{1-q^2}\,y^{-2}\,[z, f], \qquad f \in \mathcal{O}(\mathrm{D}_q). \tag{11}$$

In order extend the partial derivatives to an algebra of differentiable functions, we first observe that, by (6) and (8), each $f \in \mathcal{O}(\mathrm{D}_q)$ can be written

$$f = \sum_{n=0}^{N} z^n p_n(y^2) + \sum_{n=1}^{M} p_{-n}(y^2)\,z^{*n}, \qquad N, M \in \mathbb{N}, \tag{12}$$

with polynomials $p_n$ in one variable. Furthermore, for all functions $f \in C(\mathrm{spec}\{y^2\})$, the formulas in (11) together with (8) give

$$\tfrac{\partial}{\partial z} f(y^2) = \tfrac{1}{1-q^2}\,y^{-2}\,(z^* f(y^2) - f(y^2)z^*) = -\frac{f(y^2) - f(q^2y^2)}{y^2 - q^2y^2}\,z^* = -\nabla_{q^2} f(y^2)\,z^*,$$

$$\tfrac{\partial}{\partial \bar z} f(y^2) = \tfrac{-1}{1-q^2}\,y^{-2}\,(z f(y^2) - f(y^2)z) = -z\,\frac{f(y^2) - f(q^2y^2)}{y^2 - q^2y^2} = -z\,\nabla_{q^2} f(y^2),$$

where

$$\nabla_{q^2} f(x) := \frac{f(x) - f(q^2x)}{x - q^2x}$$

denotes the $q^2$-difference operator and the operators $f(y^2)$ are defined by the spectral calculus of the self-adjoint operator $y^2$. The reason why we prefer to consider functions $f = f(y^2)$ instead of $f = f(y)$ is because $z \mapsto 1 - z\bar z$ is a differentiable function on $\bar{\mathrm{D}}$ but $z \mapsto \sqrt{1 - z\bar z}$ is not. Note that

$$\mathrm{spec}(y^2) = \{q^{2n} : n \in \mathbb{N}\} \cup \{0\}$$

and $\lim_{k\to\infty}\frac{f(q^{2k})-f(q^{2k+2})}{q^{2k}-q^{2k+2}} = \lim_{k\to\infty}\big(\frac{1}{1-q^2}\frac{f(q^{2k})-f(0)}{q^{2k}} - \frac{q^2}{1-q^2}\frac{f(q^{2k+2})-f(0)}{q^{2k+2}}\big) = f'(0)$ if the derivative of $f$ in 0 exists. In that case, $\nabla_{q^2}\psi(y^2)$ defines a bounded operator on $\ell_2(\mathbb{N})$. This together with (12) motivates the following definition of an algebra of differentiable functions on the quantum disc:

$$\mathcal{F}^{(1)}(\mathrm{D}_q) := \left\{ \sum_{n=0}^{N} z^n f_n(y^2) + \sum_{n=1}^{M} f_{-n}(y^2)\, z^{*n} \;:\; M, N \in \mathbb{N},\;\; f_j \in C^{(1)}(\mathrm{spec}(y^2)) \right\}. \tag{13}$$

Observe that differentiability of $f \in C^{(1)}(\mathrm{spec}(y^2))$ amounts to the differentiability of $f$ in 0 which is the only accumulation point of $\mathrm{spec}(y^2)$. Using that $f(y) \in \mathcal{F}^{(1)}(\mathrm{D}_q)$ if and only if $f(q^k y) \in \mathcal{F}^{(1)}(\mathrm{D}_q)$ for any $k \in \mathbb{Z}$, it is easily seen that $\mathcal{F}^{(1)}(\mathrm{D}_q)$ is a *-algebra of bounded operators on $\ell_2(\mathbb{N})$ containing $\mathcal{O}(\mathrm{D}_q)$. It follows from the (twisted) Leibniz rules

$$y^{-2}[\zeta, z^n f(y)] = y^{-2}[\zeta, z^n] f(y) + q^{-2n} z^n y^{-2}[\zeta, f(y)],$$
$$y^{-2}[\zeta, f(y) z^{*n}] = y^{-2}[\zeta, f(y)] z^{*n} + f(y) y^{-2}[\zeta, z^{*n}], \quad f \in C^{(1)}(\mathrm{spec}(y^2)),\;\; \zeta \in \{z, z^*\},\;\; n \in \mathbb{N},$$

that the actions of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ given by the expressions in (11) are well defined on $\mathcal{F}^{(1)}(\mathrm{D}_q)$.

## 4. Noncommutative integration

Let $s$ denote the unilateral shift operator acting on $\ell_2(\mathbb{N})$ by $s e_e = e_{n+1}$. Using the relations

$$z = s\sqrt{1-q^2y^2}, \quad z^* = s^*\sqrt{1-y^2}, \quad f(y)\, s = s\, f(qy), \quad s^* f(y) = f(qy)\, s^*, \tag{14}$$

where $f \in L_\infty(\mathrm{spec}(y)) := \{g : \mathrm{spec}(y) \to \mathbb{C} : \text{bounded}\}$, one sees that each $\psi \in \mathcal{F}^{(1)}(\mathrm{D}_q)$ can be written $\psi = \sum_{n=0}^{N} s^n g_n(y^2) + \sum_{n=1}^{M} g_{-n}(y^2)\, s^{*n}$ with continuous (but not necessarily differentiable) functions $g_j \in C(\mathrm{spec}(y^2))$. For a noncommutative integration theory on the quantum disc, it happens to be more convenient to work with the *-algebra of bounded operators

$$\mathcal{F}(\mathrm{D}_q) := \left\{ \sum_{n=0}^{N} s^n g_n(y) + \sum_{n=1}^{M} g_{-n}(y)\, s^{*n} \;:\; M, N \in \mathbb{N},\;\; g_j \in L_\infty(\mathrm{spec}(y)) \right\}. \tag{15}$$

According to the above remark, $\mathcal{F}^{(1)}(\mathrm{D}_q) \subset \mathcal{F}(\mathrm{D}_q)$.

We follow [4,5,7] and define for $\alpha > 0$

$$\int_{\mathrm{D}_q}^{\alpha} \;:\; \mathcal{F}(\mathrm{D}_q) \longrightarrow \mathbb{C}, \quad \int_{\mathrm{D}_q}^{\alpha} \psi := (1-q)\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(\psi\, y^\alpha). \tag{16}$$

Since $y^\alpha$ is a positive compact operator for all $\alpha > 0$, the weighted trace $\int_{\mathrm{D}_q}^{\alpha}$ is a well-defined positive functional on $\mathcal{F}(\mathrm{D}_q)$. As the trace over weighted shift operators vanishes, we have

$$\int_{\mathrm{D}_q}^{\alpha} \psi = \int_{\mathrm{D}_q} \left( \sum_{n=0}^{N} s^n g_n(y) + \sum_{n=1}^{M} g_{-n}(y)\, s^{*n} \right) = (1-q) \sum_{n\in\mathbb{N}} g_0(q^n) q^{\alpha n}. \tag{17}$$

In terms of the Jackson integral $\int_0^1 f(y)\mathrm{d}_q y = (1-q)\sum_{n\in\mathbb{N}} f(q^n)q^n$, the functional $\int_{\mathrm{D}_q}^{\alpha}$ may be written

$$\int_{\mathrm{D}_q}^{\alpha} \left( \sum_{n=0}^{N} s^n g_n(y) + \sum_{n=1}^{M} g_{-n}(y) s^{*n} \right) = \int_0^1 \int_{-\pi}^{\pi} \left( \sum_{n=0}^{N} \mathrm{e}^{\mathrm{i}n\theta} g_n(y) + \sum_{n=1}^{M} g_{-n}(y) \mathrm{e}^{-\mathrm{i}n\theta} \right) \mathrm{d}\theta\, y^{\alpha-1} \mathrm{d}_q y.$$

By (14), the commutation relation between $y^\alpha$ and functions from $\mathcal{F}(\mathrm{D}_q)$ can be expressed by the automorphism $\sigma^\alpha : \mathcal{F}(\mathrm{D}_q) \longrightarrow \mathcal{F}(\mathrm{D}_q)$,

$$\sigma^\alpha\Big(\sum_{n=0}^{N} s^n g_n(y) + \sum_{n=1}^{M} g_{-n}(y) s^{*n}\Big) = \sum_{n=0}^{N} (q^{-\alpha}s)^n g_n(y) + \sum_{n=1}^{M} g_{-n}(y)\,(q^\alpha s)^{*n}. \tag{18}$$

Then $hy^\alpha = y^\alpha \sigma^\alpha(h)$ and hence, by the trace property,

$$\int_{\mathrm{D}_q}^{\alpha} gh = (1-q)\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(ghy^\alpha) = (1-q)\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(\sigma^\alpha(h) g y^\alpha) = \int_{\mathrm{D}_q}^{\alpha} \sigma^\alpha(h) g \tag{19}$$

for all $f, g \in \mathcal{F}(\mathrm{D}_q)$. Note also that

$$(\sigma^\alpha(h))^* = \sigma^{-\alpha}(h^*), \qquad h \in \mathcal{F}(\mathrm{D}_q),$$

where $\sigma^{-\alpha}$ denotes the inverse of $\sigma^\alpha$.

Using $\mathrm{Tr}_{\ell_2(\mathbb{N})}(s^n s^{*k} f(y) y) = 0$ if $k \neq n$, one easily verifies that $\int_{\mathrm{D}_q}$ is faithful. Therefore

$$\langle f, g\rangle := \int_{\mathrm{D}_q}^{\alpha} f^* g, \qquad f, g \in \mathcal{F}(\mathrm{D}_q), \tag{20}$$

defines an inner product on $\mathcal{F}(\mathrm{D}_q)$. Its Hilbert space closure of will be denoted by $L_2(\mathrm{D}_q)$.

**5. Representations on the quantum disc**

The left multiplication of $\mathcal{O}(\mathrm{D}_q)$ on $\mathcal{F}(\mathrm{D}_q)$ defines a bounded *-representation of $\mathcal{O}(\mathrm{D}_q)$ on $L_2(\mathrm{D}_q)$ since

$$\langle xf, g\rangle = \int_{\mathrm{D}_q}^{\alpha} f^* x^* g = \langle f, x^* g\rangle, \qquad f, g \in \mathcal{F}(\mathrm{D}_q), \quad x \in \mathcal{O}(\mathrm{D}_q).$$

In [4], representations of the opposite algebra $\mathcal{O}(\mathrm{D}_q)^{\mathrm{op}}$ play also a crucial role in the definition of the Dirac operator. Recall that the opposite algebra $\mathcal{A}^{\mathrm{op}}$ of an algebra $\mathcal{A}$ is the vector space $\mathcal{A}$ with the opposite multiplication $a^{\mathrm{op}}\, b^{\mathrm{op}} = (ba)^{\mathrm{op}}$. $\mathcal{A}^{\mathrm{op}}$ remains to be a *-algebra if $\mathcal{A}$ is one. The right multiplication of $\mathcal{O}(\mathrm{D}_q)$ on $\mathcal{F}(\mathrm{D}_q)$ defines a representation of $\mathcal{O}(\mathrm{D}_q)^{\mathrm{op}}$ which is not a *-representation because

$$\langle x^{\mathrm{op}} f, g\rangle = \langle fx, g\rangle = \int_{\mathrm{D}_q}^{\alpha} x^* f^* g = \int_{\mathrm{D}_q}^{\alpha} f^* g\, \sigma^{-\alpha}(x^*) = \langle f, \sigma^{-\alpha}(x^*)^{\mathrm{op}} g\rangle \tag{21}$$

by (19).

Our next aim is to construct an orthonormal basis for $L_2(\mathrm{D}_q)$. To this end, we introduce the following notation: As customary, the symbol $\delta_{nm}$ denotes the Kronecker delta. For $k \in \mathbb{Z}$, let $\delta_{q^k} : \mathbb{R} \longrightarrow \{0, 1\}$ be defined by

$$\delta_{q^k}(t) := \begin{cases} 1, & t = q^k, \\ 0, & t \neq q^k. \end{cases}$$

If $a$ is a densely defined operator, we set

$$a^{\#k} := \begin{cases} a^k, & k \geq 0, \\ a^{*k}, & k < 0. \end{cases} \tag{22}$$

Note that $\delta_{q^n}$ is continuous on $\mathrm{spec}(y)$ and $\delta_{q^n}(y)$ is the orthogonal projection onto the one-dimensional subspace $\mathrm{span}\{e_n\} \subset \ell_2(\mathbb{N})$. For any (measurable) function $f : \mathrm{spec}(y) \longrightarrow \mathbb{C}$, we have

$$f(y)\,\delta_{q^n}(y) = \delta_{q^n}(y)\,f(y) = f(q^n)\,\delta_{q^n}(y). \tag{23}$$

In particular, $\delta_{q^m}(y)\delta_{q^n}(y) = \delta_{mn}\,\delta_{q^n}(y)$. Furthermore, it follows from (14) that

$$s\,\delta_{q^n}(y) = \delta_{q^n}(q^{-1}y)\,s = \delta_{q^{n+1}}(y)\,s, \quad s^*\,\delta_{q^n}(y) = \delta_{q^n}(qy)\,s^* = \delta_{q^{n-1}}(y)\,s^*. \tag{24}$$

As a consequence, for all $k, n \in \mathbb{N}$ such that $n < k$,

$$s^{*k}\,\delta_{q^n}(y) = \delta_{q^{n-k}}(y)\,s^{*k} = 0, \quad \delta_{q^n}(y)\,s^k = s^k\,\delta_{q^{n-k}}(y) = 0, \tag{25}$$

since $\delta_{q^{n-k}}(t) = 0$ on $\mathrm{spec}(y)$. From $ss^* = 1 - \delta_{q^0}(y)$, we get

$$ss^{*k}\,\delta_{q^n}(y) = (1-\delta_{q^0}(y))s^{*k-1}\,\delta_{q^n}(y) = s^{*k-1}\,(1-\delta_{q^{k-1}}(y))\,\delta_{q^n}(y) = s^{*k-1}\,\delta_{q^n}(y) \tag{26}$$

as we may assume that $n > k-1$ by (25). Note that $s^{\#k}\,s^{\#l} = s^{\#k+l}$ does not hold in general, for instance $s^{\#1}\,s^{\#-1} = 1 - \delta_{q^0}(y) \neq s^{\#0}$. However, from (26) and $s^*s = 1$, we conclude that

$$s^{\#l}s^{\#k}\,\delta_{q^n}(y) = s^{\#l+k}\,\delta_{q^n}(y) \quad \text{for all} \;\; k, l \in \mathbb{Z}, \quad n \in \mathbb{N}. \tag{27}$$

Using (20), (27), (19), (23) and (17), we compute

$$\langle s^{\#k}\,\delta_{q^n}(y), s^{\#l}\,\delta_{q^m}(y)\rangle = \int_{\mathrm{D}_q}^{\alpha} \delta_{q^n}(y)\,s^{\#-k}\,s^{\#l}\delta_{q^m}(y) = \int_{\mathrm{D}_q}^{\alpha} s^{\#l-k}\,\delta_{q^m}(y)\,\delta_{q^n}(y) = (1-q)q^{\alpha n}\delta_{nm}\delta_{kl}. \tag{28}$$

The last equation is the key to constructing an orthonormal basis for $L_2(\mathrm{D}_q)$.

**Proposition 1.** *For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $k \geq -n$, define*

$$\eta_{nk} := \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\,s^{\#k}\,\delta_{q^n}(y). \tag{29}$$

*Then $\{\eta_{nk} : n \in \mathbb{N},\;\; k \in \mathbb{Z},\;\; k \geq -n\}$ is an orthonormal basis for $L_2(\mathrm{D}_q)$.*

*Proof.* It follows from (28) that the set defined in the proposition is orthonormal. Therefore it only remains to show that it is complete. As $L_2(\mathrm{D}_q)$ is the closure of $\mathcal{F}(\mathrm{D}_q)$, it suffices to show that the elements $s^k\,f(y)$ and $f(y)\,s^{*k}$ can be expanded in this basis, where $f \in L_\infty(\mathrm{spec}(y))$ and $k \in \mathbb{N}$. Since $\delta_{q^n}(y)$ is the orthogonal projection onto the eigenspace corresponding to the eigenvalue $q^n$ of the self-adjoint operator $y$, the spectral theorem gives $f(y) = \sum_{n\in\mathbb{N}} f(q^n)\,\delta_{q^n}(y)$. Thus $s^k\,f(y) = \sqrt{1-q}\,\sum_{n\in\mathbb{N}} q^{\alpha n/2}\,f(q^n)\,\eta_{nk}$ and the series converges in $L_2(\mathrm{D}_q)$ since $f$ is bounded and $\{q^{\alpha n/2}\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$. Similarly, by (25),

$$f(y)\,s^{*k} = \sum_{n\in\mathbb{N}} f(q^n)\,\delta_{q^n}(y)\,s^{*k} = \sum_{n\in\mathbb{N}} f(q^n)\,s^{*k}\,\delta_{q^{n+k}}(y) = \sqrt{1-q}\,q^{\alpha k/2} \sum_{n\in\mathbb{N}} q^{\alpha n/2}\,f(q^n)\,\eta_{n+k,k}\,,$$

where the sequence of coefficients $\{q^{\alpha n/2}\,f(q^n)\}_{n\in\mathbb{N}}$ belongs to $\in \ell_2(\mathbb{N})$. □

We will now compute the actions of our noncommutative multiplication and partial differential operators on this basis. To begin,

$$y\,\eta_{nk} = \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\,y\,s^{\#k}\,\delta_{q^n}(y) = q^k\,\tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\,s^{\#k}\,y\,\delta_{q^n}(y) = q^{n+k}\,\tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\,s^{\#k}\,\delta_{q^n}(y) = q^{n+k}\,\eta_{nk}\,, \tag{30}$$

where we used (14) in the second equality and (23) in the third. Similarly,

$$y^{\mathrm{op}}\eta_{nk} = \eta_{nk}\, y = \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\#k}\,\delta_{q^n}(y)\, y = q^n\, \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\#k}\,\delta_{q^n}(y) = q^n\,\eta_{nk}\,. \tag{31}$$

Writing $z = s\sqrt{1-q^2y^2}$, it follows from (30) and (27) that

$$\begin{aligned} z\,\eta_{nk} &= \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s\sqrt{1-q^2y^2}\, s^{\#k}\,\delta_{q^n}(y) = \sqrt{1-q^{2(n+k+1)}}\,\tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\#k+1}\,\delta_{q^n}(y) \\ &= \sqrt{1-q^{2(n+k+1)}}\,\eta_{n,k+1}\,, \end{aligned} \tag{32}$$

and analogously, for $z^* = s^*\sqrt{1-y^2}$,

$$\begin{aligned} z^*\,\eta_{nk} &= \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^*\sqrt{1-y^2}\, s^{\#k}\,\delta_{q^n}(y) = \sqrt{1-q^{2(n+k)}}\,\tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\#k-1}\,\delta_{q^n}(y) \\ &= \sqrt{1-q^{2(n+k)}}\,\eta_{n,k-1}\,, \end{aligned} \tag{33}$$

The actions of $z^{\mathrm{op}}$ and $z^{*\mathrm{op}}$ are calculated by applying (24), (23) and (29):

$$\begin{aligned} z^{\mathrm{op}}\eta_{nk} = \eta_{nk}\, z &= \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\#k}\,\delta_{q^n}(y)\, s\sqrt{1-q^2y^2} = \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\#k+1}\,\delta_{q^{n-1}}(y)\sqrt{1-q^2y^2} \\ &= q^{-\alpha/2}\sqrt{1-q^{2n}}\,\tfrac{q^{-\alpha(n-1)/2}}{\sqrt{1-q}}\, s^{\#k+1}\,\delta_{q^{n-1}}(y) \\ &= q^{-\alpha/2}\sqrt{1-q^{2n}}\,\eta_{n-1,k+1}\,, \end{aligned} \tag{34}$$

$$\begin{aligned} z^{*\mathrm{op}}\eta_{nk} = \eta_{nk}\, z^* &= \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\#k}\,\delta_{q^n}(y)\, s^*\sqrt{1-y^2} = \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\#k-1}\,\delta_{q^{n+1}}(y)\sqrt{1-y^2} \\ &= q^{\alpha/2}\sqrt{1-q^{2(n+1)}}\,\tfrac{q^{-\alpha(n+1)/2}}{\sqrt{1-q}}\, s^{\#k+1}\,\delta_{q^{n+1}}(y) \\ &= q^{\alpha/2}\sqrt{1-q^{2(n+1)}}\,\eta_{n+1,k-1}\,. \end{aligned} \tag{35}$$

Note that $(z^{\mathrm{op}})^* = q^{-\alpha}z^{*\mathrm{op}} = (\sigma^{-\alpha}(z^*))^{\mathrm{op}}$ as observed in (21).

By Equation (13) and the remark following it, the basis vectors $\eta_{nk}$ belong to $\mathcal{F}^{(1)}(\mathrm{D}_q)$. As eigenvectors of the self-adjoint operator $y$, they also belong to the domain of the unbounded operator $y^{-2}$. We will use the formulas in (11) to compute the actions of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ on $\eta_{nk}$. From (30), (33) and (35), it follows that

$$\begin{aligned} \tfrac{\partial}{\partial z}\eta_{nk} &= \tfrac{1}{1-q^2}\, y^{-2}\,(z^*\eta_{nk} - \eta_{nk}\, z^*) = \tfrac{1}{1-q^2}\, y^{-2}\,(z^*\eta_{nk} - z^{*\mathrm{op}}\eta_{nk}) \\ &= \tfrac{q^{-2(n+k)}}{1-q^2}\big(q^2\sqrt{1-q^{2(n+k)}}\,\eta_{n,k-1} - q^{\alpha/2}\sqrt{1-q^{2(n+1)}}\,\eta_{n+1,k-1}\big). \end{aligned} \tag{36}$$

Further, by (30), (32) and (34),

$$\begin{aligned} \tfrac{\partial}{\partial \bar z}\eta_{nk} &= \tfrac{-1}{1-q^2}\, y^{-2}\,(z\,\eta_{nk} - \eta_{nk}\, z) = \tfrac{-1}{1-q^2}\, y^{-2}\,(z\,\eta_{nk} - z^{\mathrm{op}}\eta_{nk}) \\ &= -\tfrac{q^{-2(n+k)}}{1-q^2}\big(q^{-2}\sqrt{1-q^{2(n+k+1)}}\,\eta_{n,k+1} - q^{-\alpha/2}\sqrt{1-q^{2n}}\,\eta_{n-1,k+1}\big). \end{aligned} \tag{37}$$

To compare the representations of $z$ and $z^*$ with the irreducible ones from (5), it is convenient to change the basis. For $n,k\in\mathbb{N}$, set $e_{nk} := \eta_{n,k-n}$. As an immediate consequence of Proposition 1, $\{e_{nk} : n,k\in\mathbb{N}\}$ is an orthonormal basis for $L_2(\mathrm{D}_q)$ and the change of basis is given by the unitary operator

$$U : L_2(\mathrm{D}_q) \longrightarrow L_2(\mathrm{D}_q), \quad U\, e_{nk} = \eta_{n,k-n}, \quad U^*\eta_{nk} = e_{n,k+n}\,.$$

In the new basis, we have

$$y\, e_{nk} = U^* \, y \, \eta_{n,k-n} = q^k U^* \eta_{n,k-n} = q^k e_{nk} \,,$$

$$z\, e_{nk} = U^* \, z \, \eta_{n,k-n} = \sqrt{1-q^{2(k+1)}}\, U^* \eta_{n,k-n+1} = \sqrt{1-q^{2(k+1)}}\, e_{n,k+1} \,, \tag{38}$$

$$z^* e_{nk} = U^* z^* \eta_{n,k-n} = \sqrt{1-q^{2k}}\, U^* \eta_{n,k-n-1} = \sqrt{1-q^{2k}}\, e_{n,k-1} \,. \tag{39}$$

In particular, on each Hilbert space

$$\mathcal{H}_n := \operatorname{span}\{e_{nk} : k \in \mathbb{N}\} \cong \ell_2(\mathbb{N}), \tag{40}$$

we recover the unique irreducible *-representations (5) of the quantum disc $\mathcal{O}(\mathrm{D}_q)$. Further, for the opposite operators, one gets

$$y^{\mathrm{op}} e_{nk} = U^* y^{\mathrm{op}} \eta_{n,k-n} = q^n U^* \eta_{n,k-n} = q^n e_{nk} \,, \tag{41}$$

$$z^{\mathrm{op}} e_{nk} = U^* z^{\mathrm{op}} \eta_{n,k-n} = q^{-\alpha/2} \sqrt{1-q^{2n}}\, U^* \eta_{n-1,k-n+1} = q^{-\alpha/2} \sqrt{1-q^{2n}}\, e_{n-1,k} \,, \tag{42}$$

$$z^{*\mathrm{op}} e_{nk} = U^* z^{*\mathrm{op}} \eta_{n,k-n} = q^{\alpha/2} \sqrt{1-q^{2(n+1)}}\, U^* \eta_{n+1,k-n-1} = q^{\alpha/2} \sqrt{1-q^{2(n+1)}}\, e_{n+1,k} \,. \tag{43}$$

Now, setting $\zeta^{\mathrm{op}} := q^{\alpha/2} z^{\mathrm{op}}$, $\zeta^{*\mathrm{op}} := q^{-\alpha/2} z^{*\mathrm{op}}$ and

$$\mathcal{H}_k^{\mathrm{op}} := \operatorname{span}\{e_{nk} : n \in \mathbb{N}\} \cong \ell_2(\mathbb{N}), \tag{44}$$

we obtain an irreducible *-representation of $\mathcal{O}(\mathrm{D}_q)^{\mathrm{op}}$ on $\mathcal{H}_k^{\mathrm{op}}$, that is, $(\zeta^{\mathrm{op}})^* = \zeta^{*\mathrm{op}}$ and

$$\zeta^{\mathrm{op}} \zeta^{*\mathrm{op}} - q^2 \zeta^{*\mathrm{op}} \zeta^{\mathrm{op}} = 1 - q^2. \tag{45}$$

Note that we have of course

$$[f, g^{\mathrm{op}}] = f\, g^{\mathrm{op}} - g^{\mathrm{op}} f = 0 \quad \text{for all} \quad f \in \mathcal{O}(\mathrm{D}_q), \quad g^{\mathrm{op}} \in \mathcal{O}(\mathrm{D}_q)^{\mathrm{op}}, \tag{46}$$

since the elements of $\mathcal{O}(\mathrm{D}_q)$ act only on the second index of $e_{nk}$ and the elements of $\mathcal{O}(\mathrm{D}_q)^{\mathrm{op}}$ act only on the first.

We compute the actions of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ on $e_{nk}$ by using the commutator representation (11) again. Thus

$$\tfrac{\partial}{\partial z} e_{nk} = \tfrac{1}{1-q^2}\, y^{-2} (z^* e_{nk} - z^{*\mathrm{op}} e_{nk}) = \tfrac{q^{-2k}}{1-q^2} \big( q^2 \sqrt{1-q^{2k}}\, e_{n,k-1} - q^{\alpha/2} \sqrt{1-q^{2(n+1)}}\, e_{n+1,k} \big),$$

$$\tfrac{\partial}{\partial \bar z} e_{nk} = \tfrac{-1}{1-q^2}\, y^{-2} (z\, e_{nk} - z^{\mathrm{op}} e_{nk}) = -\tfrac{q^{-2k}}{1-q^2} \big( q^{-2} \sqrt{1-q^{2(k+1)}}\, e_{n,k+1} - q^{-\alpha/2} \sqrt{1-q^{2n}}\, e_{n-1,k} \big).$$

Finally recall that (3) defines a faithful *-representation of $C(\mathrm{SU}_q(2))$, and consider the Hilbert space $L_2(\mathrm{D}_q) \bar\otimes L_2(\mathbb{S}^1)$ with orthonormal basis $\{e_{nkl} : n,k \in \mathbb{N},\ l \in \mathbb{Z}\}$, where

$$e_{nkl} := e_{nk} \otimes b_l = \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\# k-n} \delta_{q^n}(y) \otimes \tfrac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i} l t}. \tag{47}$$

As in (9), we define a *-representation of $\mathcal{O}(\mathrm{SU}_q(2))$ on $L_2(\mathrm{D}_q) \bar\otimes L_2(\mathbb{S}^1)$ by setting

$$c := y \otimes u = \sqrt{1 - zz^*} \otimes u, \qquad d := z \otimes 1. \tag{48}$$

It follows from (38)–(40) that $L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$ decomposes into the direct sum

$$L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1) = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n\,\bar{\otimes}\,L_2(\mathbb{S}^1) \cong \bigoplus_{n\in\mathbb{N}} \ell_2(\mathbb{N})\,\bar{\otimes}\,\ell_2(\mathbb{Z}),$$

and on each copy of $\ell_2(\mathbb{N})\,\bar{\otimes}\,\ell_2(\mathbb{Z})$, we recover the representation from (3). Hence the representation (48) decomposes into the infinite orthogonal sum of *-representations which are all unitarily equivalent to (3). In particular, (48) defines a faithful *-representation of $\mathcal{O}(\mathrm{SU}_q(2))$ and the C*-closure of the *-algebra generated by $c$ and $d$ from (48) is isomorphic to $C(\mathrm{SU}_q(2))$.

To obtain *-representations of the opposite algebras $\mathcal{O}(\mathrm{SU}_q(2))^{\mathrm{op}}$ and $C(\mathrm{SU}_q(2))^{\mathrm{op}}$, the generators $c^{\mathrm{op}}$ and $d^{\mathrm{op}}$ have to satisfy the opposite relations of (1) and (2). Analogous to $d = z\otimes 1$ and $c = \sqrt{1-zz^*}\otimes u$ with $z$ satisfying (6) and $u$ being a unitary generator of $C(\mathbb{S}^1)$, a *-representation of $\mathcal{O}(\mathrm{SU}_q(2))^{\mathrm{op}}$ is given by replacing $z$ by $\zeta^{\mathrm{op}}$ satisfying (45) (the opposite relation of (6)) and $u$ by a unitary operator $u^{\mathrm{op}}$ with the same properties as $u$. For $u^{\mathrm{op}}$ we may take $u^*$ which generates the same C*-algebra $C(\mathbb{S}^1)$ as $u$ does. The adjoint of $u$ is chosen for consistency because then $c$ and $d$ act as forward shifts, and $c^{\mathrm{op}}$ and $d^{\mathrm{op}}$ act as backward shifts. For $\zeta^{\mathrm{op}} = q^{\alpha/2}z^{\mathrm{op}}$ and $\zeta^{*\mathrm{op}} = q^{-\alpha/2}z^{*\mathrm{op}}$, as defined in the paragraph after Equation (43), we get $\sqrt{1-\zeta^{\mathrm{op}}\zeta^{*\mathrm{op}}} = \sqrt{1-z^{\mathrm{op}}z^{*\mathrm{op}}} = y^{\mathrm{op}}$. Thus, setting

$$c^{\mathrm{op}} := y^{\mathrm{op}}\otimes u^* = \sqrt{1-\zeta^{\mathrm{op}}\zeta^{*\mathrm{op}}}\otimes u^*, \qquad d^{\mathrm{op}} := \zeta^{\mathrm{op}}\otimes 1, \tag{49}$$

yields a *-representation of $\mathcal{O}(\mathrm{SU}_q(2))^{\mathrm{op}}$ on $L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$. Moreover, with $\mathcal{H}_k^{\mathrm{op}}$ defined in (44), we have the Hilbert space decomposition

$$L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1) = \bigoplus_{k\in\mathbb{N}} \mathcal{H}_k^{\mathrm{op}}\,\bar{\otimes}\,L_2(\mathbb{S}^1) \cong \bigoplus_{k\in\mathbb{N}} \ell_2(\mathbb{N})\,\bar{\otimes}\,\ell_2(\mathbb{Z}),$$

and, by (41)–(43) and (4), $c^{\mathrm{op}}$ and $d^{\mathrm{op}}$ act on each $\mathcal{H}_{k_0}^{\mathrm{op}}\,\bar{\otimes}\,L_2(\mathbb{S}^1) = \mathrm{span}\{e_{nk_0}\otimes b_l : n\in\mathbb{N},\ l\in\mathbb{Z}\}$ by

$$c^{\mathrm{op}}(e_{nk_0}\otimes b_l) = q^n\, e_{nk_0}\otimes b_{l-1}, \qquad d^{\mathrm{op}}(e_{nk_0}\otimes b_l) = \sqrt{1-q^{2(n-1)}}\, e_{n-1,k_0}\otimes b_l\,. \tag{50}$$

Note that (50) are the adjoint relations of (3), and that taking the opposite relations of (1) and (2) amounts to interchanging $c$ and $d$ with their adjoints $c^*$ and $d^*$. Therefore, as much as (3) defines a faithful *-representation of $\mathcal{O}(\mathrm{SU}_q(2))$ and $C(\mathrm{SU}_q(2))$, (50) yields a faithful *-representation of $\mathcal{O}(\mathrm{SU}_q(2))^{\mathrm{op}}$ and $C(\mathrm{SU}_q(2))^{\mathrm{op}}$ and so does their direct sum representation on $L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$. Finally we remark that the operators $c$ and $d$ from (48) commute with the operators $c^{\mathrm{op}}$ and $d^{\mathrm{op}}$ from (49) since $u$ and $u^*$ belong to the commutative C*-algebra $C(\mathbb{S}^1)$, and the operators in the left factor of the tensor products commute by (46). As a consequence, the representations of $C(\mathrm{SU}_q(2))$ and $C(\mathrm{SU}_q(2))^{\mathrm{op}}$ on $L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$ commute.

For the convenience of the reader, we finish the paper by collecting the most important formulas in a final theorem.

**Theorem 2.** *Let $z, z^*\in B(\ell_2(\mathbb{N}))$ denote the generators of the quantum disc $\mathcal{O}(\mathrm{D}_q)$ given in* (5) *and let $y = \sqrt{1-zz^*}$. With $\mathcal{F}(\mathrm{D}_q)$ defined in* (15)*, the *-algebra $\mathcal{F}(\mathrm{D}_q)\otimes L_\infty(\mathbb{S}^1)$ is considered an algebra of bounded functions on a local chart for quantum SU(2) in the following sense: Define an inner product on $\mathcal{F}(\mathrm{D}_q)\otimes L_\infty(\mathbb{S}^1)$ by*

$$\langle f\otimes\phi, g\otimes\psi\rangle := \int_{\mathrm{D}_q}^{\alpha} f^*g \int_{\mathbb{S}^1}\bar{\phi}\psi\,\mathrm{d}\lambda = (1-q)\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(f^*gy^\alpha)\int_{\mathbb{S}^1}\bar{\phi}\psi\,\mathrm{d}\lambda, \quad \alpha>0,$$

*where $\lambda$ stands for the Lebesgue measure on $\mathbb{S}^1$. The Hilbert space completion of $\mathcal{F}(\mathrm{D}_q)\otimes L_\infty(\mathbb{S}^1)$ will be denoted by $L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$. Furthermore, with $\mathcal{F}^{(1)}(\mathrm{D}_q)$ described in* (13)*, the *-subalgebra $\mathcal{F}^{(1)}(\mathrm{D}_q)\otimes C^{(1)}(\mathbb{S}^1)$ of $\mathcal{F}(\mathrm{D}_q)\otimes L_\infty(\mathbb{S}^1)$ is viewed as an algebra of differentiable functions* *on the local chart. The Hilbert space $L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$ has an orthonormal basis of differentiable functions $\{e_{nkl} : n,k\in\mathbb{N},\ l\in\mathbb{Z}\}\subset \mathcal{F}^{(1)}(\mathrm{D}_q)\otimes C^{(1)}(\mathbb{S}^1)$, where*

$$e_{nkl} = \tfrac{q^{-\alpha n/2}}{\sqrt{1-q}}\, s^{\#k-n}\,\delta_{q^n}(y)\otimes \tfrac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}lt}.$$

*The left multiplication $x\,(f\otimes\psi) := xf\otimes\psi$, $x\in\mathcal{O}(\mathrm{D}_q)$, $f\otimes\psi\in\mathcal{F}(\mathrm{D}_q)\otimes L_\infty(\mathbb{S}^1)$, defines a bounded *-representation of $\mathcal{O}(\mathrm{D}_q)$ on $L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$. On the above orthonormal basis, the operators $z$, $z^*$ and $y$ act by*

$$z\,e_{nkl} = \sqrt{1-q^{2(k+1)}}\,e_{n,k+1,l}\,,\quad z^*\,e_{nkl} = \sqrt{1-q^{2k}}\,e_{n,k-1,l}\,,\quad y\,e_{nkl} = q^k\,e_{nkl}\,.$$

*The right multiplication $x^{\mathrm{op}}\,(f\otimes\psi) := fx\otimes\psi$, $x\in\mathcal{O}(\mathrm{D}_q)^{\mathrm{op}}$, $f\otimes\psi\in\mathcal{F}(\mathrm{D}_q)\otimes L_\infty(\mathbb{S}^1)$, defines a bounded representation of $\mathcal{O}(\mathrm{D}_q)^{\mathrm{op}}$ on $L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$ which is not a *-representation. The actions of $z^{\mathrm{op}}$, $z^{*\mathrm{op}} = q^{\alpha}(z^{\mathrm{op}})^*$ and $y^{\mathrm{op}} = (y^{\mathrm{op}})^*$ are given by*

$$z^{\mathrm{op}}\,e_{nkl} = q^{-\alpha/2}\sqrt{1-q^{2n}}\,e_{n-1,kl}\,,\quad z^{*\mathrm{op}}\,e_{nkl} = q^{\alpha/2}\sqrt{1-q^{2(n+1)}}\,e_{n+1,kl}\,,\quad y^{\mathrm{op}}\,e_{nkl} = q^n\,e_{nkl}\,.$$

*Let $u\in C(\mathbb{S}^1)$, $u(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t}$, denote the unitary generator of $C(\mathbb{S}^1)$. Setting*

$$c := y\otimes u = \sqrt{1-zz^*}\otimes u, \qquad d := z\otimes 1,$$
$$c^{\mathrm{op}} := y^{\mathrm{op}}\otimes u^* = \sqrt{1-z^{\mathrm{op}}z^{*\mathrm{op}}}\otimes u^*, \qquad d^{\mathrm{op}} := q^{\alpha/2}z^{\mathrm{op}}\otimes 1,$$

*yields commuting *-representations of $C(\mathrm{SU}_q(2))$ and $C(\mathrm{SU}_q(2))^{\mathrm{op}}$ on $L_2(\mathrm{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$ given by left and right multiplication, respectively. On basis vectors, these representations read*

$$c\,e_{nkl} = q^k\,e_{nk,l+1}\,, \qquad d\,e_{nkl} = \sqrt{1-q^{2(k+1)}}\,e_{n,k+1,l}\,,$$
$$c^{\mathrm{op}}\,e_{nkl} = q^n\,e_{nk,l-1}\,, \qquad d^{\mathrm{op}}\,e_{nkl} = \sqrt{1-q^{2n}}\,e_{n-1,kl}\,.$$

*The partial derivatives $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar z}$ and $\frac{\partial}{\partial t}$ act on differentiable functions $g\otimes\phi\in\mathcal{F}^{(1)}(\mathrm{D}_q)\otimes C^{(1)}(\mathbb{S}^1)$ by $\frac{\partial}{\partial z}(g\otimes\phi) = \frac{\partial}{\partial z}g\otimes\phi$, $\frac{\partial}{\partial \bar z}(g\otimes\phi) = \frac{\partial}{\partial \bar z}g\otimes\phi$ and $\frac{\partial}{\partial t}(g\otimes\phi) = g\otimes\frac{\partial}{\partial t}\phi$. On basis elements, one obtains*

$$\tfrac{\partial}{\partial z}\,e_{nkl} = \tfrac{q^{-2k}}{1-q^2}\big(q^2\sqrt{1-q^{2k}}\,\eta_{n,k-1,l} - q^{\alpha/2}\sqrt{1-q^{2(n+1)}}\,e_{n+1,kl}\big),$$
$$\tfrac{\partial}{\partial \bar z}\,e_{nkl} = -\tfrac{q^{-2k}}{1-q^2}\big(q^{-2}\sqrt{1-q^{2(k+1)}}\,\eta_{n,k+1,l} - q^{-\alpha/2}\sqrt{1-q^{2n}}\,e_{n-1,kl}\big),$$
$$\tfrac{\partial}{\partial t}\,e_{nkl} = \mathrm{i}l\,e_{nkl}\,.$$

Note that the representations of $C(\mathrm{SU}_q(2))$ and $C(\mathrm{SU}_q(2))^{\mathrm{op}}$ do not depend on $\alpha$ but the actions of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ do so.

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