Geometric structures, fractal self-similarity, squeezed coherent states and electrodynamics

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Abstract. The formalism describing fractal self-similar structures has been shown in recent works to be isomorphic to the one of a system of damped/amplified oscillators, which is a prototype of a dissipative system and of the environment in which it is embedded, and to squeezed coherent states. Fractal-like structures appear to be generated by coherent quantum condensation processes, and thus they appear as macroscopic quantum systems, as it happens with crystals, ferromagnets, superconductors and like systems characterized by ordered patterns. In this report, by resorting to such results it is shown that in space-time regions where the magnetic field may be approximated to be constant and the electric field is derivable from a harmonic potential, the isomorphism also exists between electrodynamics and fractal-like structures. A link is thus established between self-similarity, dissipation, coherent states and electrodynamics. The relation between quantum dissipation and non-commutative geometry in the plane is also commented upon. The macroscopic appearances (forms) of the fractals seem to emerge out of a process of morphogenesis as the macroscopic manifestation of the underlying dissipative, coherent quantum dynamics.

1. Introduction

In recent works [1]-[5] it has been shown that an isomorphism exists between the formalism describing fractal self-similar structures and squeezed generalized SU(1,1) coherent states in quantum field theory (QFT). This result can be framed in the research line which has led to recognize that systems characterized by observable ordered patterns, such as crystals, superconductors, ferromagnets, provide examples of "macroscopic quantum systems" arising as the effect of homogeneous, coherent boson condensation in the system ground state [6, 7]. These systems are quantum systems not in the trivial sense that they are made of atoms, molecules and other quantum components, as every system is made of, but in the sense that their observable physical properties, such as electrical conductivity, magnetization, crystal structure, critical temperature behavior, etc. cannot be explained without recourse to the study of the dynamics of their quantum components. Similarly, the appearance in these systems, especially in the course of phase transitions, of topologically non trivial 'extended objects', such as crystal dislocations, vortices, domain wall, kinks and other kind of 'defects', is found to be the manifestation of non-homogeneous boson condensation at quantum dynamical level [6, 7]. These results, which rest on a mathematically sound formulation and wide experimental confirmation, have suggested the possibility that also the self-similarity properties of fractal-like structures might arise from the
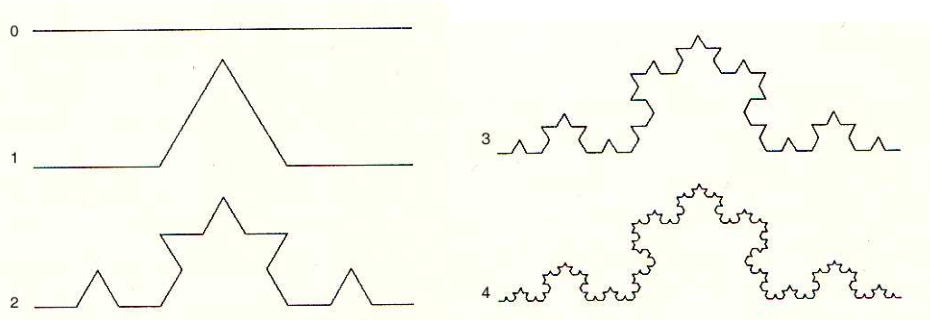
Figure 1. The first five stages of the Koch curve

underlying dynamics of their quantum components involving as well the mechanism of coherent boson condensation in the system ground state. Recent observations seem to confirm such a view since they show that when a crystal is submitted to deforming stress actions the induced deformations in the lattice structure form, at low temperature, self-similar fractal patterns and provide an example of "emergence of fractal dislocation structures" [8] in dissipative, non-equilibrium systems, thus appearing as the result of non-homogeneous coherent phonon condensation. A first conjecture of the relation between fractal self-similarity and coherent states was proposed in [9] and then the study of fractals in the frame of the entire analytical functions was presented in [1] - [5], also with application to the observed brain functional activity [1], [10] - [13], to water molecular structure in the presence of nafion and filtering [14] and in a numerical simulation of pancreatic beta cell clusters (Lagerhans islets) [15]. More recently, it has been shown [16] that the isomorphism between the logarithmic spiral and other fractal structures and the squeezed coherent states representing the system of damped/amplified oscillators, a prototype of a dissipative system and the environment in which it is embedded, may be extended to electrodynamics in space-time regions where the magnetic field may be approximated to be constant and the electric field is derivable from a harmonic potential. The plan of the report is the following. In Section 2 a summary is presented of the isomorphism between the Koch curve, the logarithmic spiral and squeezed coherent states. Section 3 summarizes the isomorphism with electrodynamics. Section 5 is devoted to the conclusions.

2. Geometric structures emerging from coherent dynamics
In order to show how the dynamical description of the geometrical feature of fractal self-similarity is obtained, in this Section I briefly summarize the examples of the Koch curve (Figure 1) and of the logarithmic spiral (Figure 2) [17, 18]. The results can be extended to fractals which are generated iteratively according to a prescribed recipe (deterministic fractals), such as the Sierpinski gasket and carpet, the Cantor set, etc. [17, 19]. The results also extend to the golden spiral and its relation with Fibonacci progression [4].

Denote by $u_0 = 1$ the starting stage and by $u_{n,q}(\alpha)$ the n-th stage of the Koch curve construction, with $\alpha = 4$ and $q = 1/3^d$. One has [1, 2]

$$u_{n,q}(\alpha) = (q\,\alpha)^n = 1, \qquad \text{for any } n. \tag{1}$$

The fractal or self-similarity dimension [17] $d = \ln 4/\ln 3 \approx 1.2619$, is thus obtained. Notice that self-similarity is properly defined only in the $n \to \infty$ limit.

In full generality one may consider the complex α-plane. Then the study of the fractal properties is carried on in the space $\mathcal{F}$ of the entire analytic functions. One restricts at the end the conclusions to real $q\,\alpha$, $q\,\alpha \to Re(q\,\alpha)$. Putting $q = e^{-d\,\theta}$, Eq. (1) is written as $d\,\theta = \ln \alpha$ and the functions $u_{n,q}(\alpha)$ are, apart the normalization factor $1/\sqrt{n!}$, nothing but the restriction
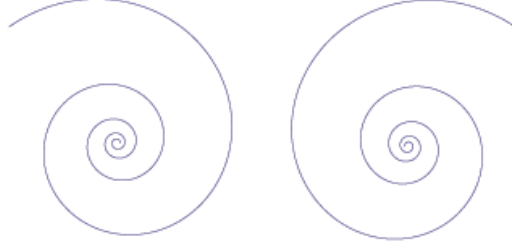
Figure 2. The anti-clockwise and the clockwise logarithmic spiral.

to real $q\alpha$ of the functions which form indeed a basis in the space $\mathcal{F}$ of the entire analytic functions:

$$u_{n,q}(\alpha) = \frac{(q\,\alpha)^n}{\sqrt{n!}}\;, \qquad\qquad n \in \mathcal{N}_+\;, \quad q\,\alpha \in \mathcal{C}\;, \qquad\qquad (2)$$

One then realizes that $\mathcal{F}$ is the so-called Fock-Bargmann representation of the Weyl–Heisenberg algebra [20] where the (Glauber) coherent states are described. The fractal self-similarity properties and coherent states are thus readily recognized to be related. One introduces then the finite difference operator $\mathcal{D}_q$, called the q-derivative operator [21, 22, 23] and the q-deformed algebraic structure, with $q = e^\zeta$, $\zeta \in \mathcal{C}$, is obtained. Denote the coherent state by $|\alpha\rangle$, with $a|\alpha\rangle = \alpha|\alpha\rangle$, and a the annihilator operator. Application of q^N to $|\alpha\rangle$, $N \equiv \alpha\, d/d\alpha$, gives the q-deformed coherent state

$$q^N|\alpha\rangle = |q\alpha\rangle = \exp\left(-\frac{|q\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(q\alpha)^n}{\sqrt{n!}}\;|n\rangle\;. \qquad\qquad (3)$$

$|q\alpha\rangle$ is a squeezed coherent state [21], with $\zeta = \ln q$ the squeezing parameter and q^N acts in $\mathcal{F}$ as the squeezing operator. By applying $(a)^n$ to $|q\alpha\rangle$ and restricting to real $q\alpha$, the n-th iteration stage of the fractal is obtained

$$\langle q\alpha|(a)^n|q\alpha\rangle = (q\alpha)^n = u_{n,q}(\alpha), \qquad q\alpha \to Re(q\alpha). \qquad\qquad (4)$$

Thus, the n-th term, $n = 0, 1, 2, .., \infty$, in the coherent state series Eq. (3) represents, in a one-to-one correspondence, the fractal n-th stage of iteration. $(a)^n$ acts as a "magnifying" lens [1, 2, 19]. q^N is called *the fractal operator* [1, 2].

The self-similarity properties of the Koch curve can be thus described in terms of the coherent state squeezing transformation.

I consider now the logarithmic spiral. It is represented in polar coordinates (r, θ) by

$$r = r_0\, e^{d\,\theta}, \qquad\qquad (5)$$

with r_0 and d arbitrary real constants and $r_0 > 0$. Eq. (5) is represented by the straight line of slope d in a log-log plot with abscissa $\theta = \ln e^\theta$: $d\,\theta = \ln(r/r_0)$.

The self-similarity property consists in the constancy of the angular coefficient $\tan^{-1} d$ and the rescaling $\theta \to n\,\theta$ affects r/r_0 by the power $(r/r_0)^n$.

The associated parametric equations are:

$$\xi \;=\; r(\theta)\cos\theta = r_0\, e^{d\,\theta}\cos\theta \qquad\qquad (6)$$

$$\eta \;=\; r(\theta)\sin\theta = r_0\, e^{d\,\theta}\sin\theta \qquad\qquad (7)$$

The point $z = \xi + i\eta = r_0\, e^{d\theta}\, e^{i\theta}$ on the spiral is fully specified in the complex z-plane by the sign of $d\theta$. Due to the completeness of the (hyperbolic) basis $\{e^{-d\theta}, e^{+d\theta}\}$ one needs to consider both the points $z_1 = r_0\, e^{-d\theta}\, e^{-i\theta}$ and $z_2 = r_0\, e^{+d\theta}\, e^{+i\theta}$. Opposite signs for the imaginary exponent $\pm i\,\theta$ are also considered for convenience. z_1 and z_2 solve the equations

$$m\,\ddot{z}_1 + \gamma\,\dot{z}_1 + \kappa\, z_1 = 0 \tag{8}$$

$$m\,\ddot{z}_2 - \gamma\,\dot{z}_2 + \kappa\, z_2 = 0 \tag{9}$$

respectively, where the parameter t has been introduced, $\theta = \theta(t)$; "dot" denotes derivative with respect to t, and, up to an arbitrary additive constant, it is put $\theta(t) = \Gamma t$, with $\Gamma \equiv \gamma/2m$. m, γ and κ are positive real constants. Thus, $z_1(t) = r_0\, e^{-i\Omega t}\, e^{-\Gamma t}$ and $z_2(t) = r_0\, e^{+i\Omega t}\, e^{+\Gamma t}$ solutions of Eqs. (8) and (9) describe the logarithmic spirals and the parameter t can be interpreted as the time parameter.

$|d\theta/dt| = |\Gamma/d|$ denotes the spiral "angular velocity". The notation $\Omega^2 = (1/m)(\kappa - \gamma^2/4m) = \Gamma^2/d^2$, with $\kappa > \gamma^2/4m$, is used. Note that the time-reversed, *but distinct*, image of the right-handed chirality spiral (indirect spiral, $q \equiv e^{-d\theta} < 1$) is the left-handed chirality spiral (direct spiral, $q > 1$).

By putting $[z_1(t) + z_2^*(-t)]/2 = x(t)$ and $[z_1^*(-t) + z_2(t)]/2 = y(t)$, Eqs. (8) and (9) reduce to

$$m\ddot{x} + \gamma\dot{x} + kx = 0, \tag{10}$$

$$m\ddot{y} - \gamma\dot{y} + ky = 0. \tag{11}$$

which by using [16, 24] $x_1 \equiv (x+y)/\sqrt{2}$ and $x_2 \equiv (x-y)/\sqrt{2}$ become

$$m\ddot{x}_1 + \gamma\dot{x}_2 + kx_1 = 0, \tag{12}$$

$$m\ddot{x}_2 + \gamma\dot{x}_1 + kx_2 = 0. \tag{13}$$

In passing, I observe that such a system of damped/amplified oscillators belongs to the class of deterministic systems à la 't Hooft [25]-[29], which remain deterministic even when described by means of Hilbert space techniques. These oscillators have a quantum representation in terms of squeezed $SU(1,1)$ coherent states. The QFT quantization procedure can be summarized as follows.

One assumes, as customary, the commutators $[x, p_x] = i\hbar = [y, p_y]$, $[x, y] = 0 = [p_x, p_y]$ and the annihilation and creation operators are introduced:

$$a \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}}\left(\frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x\right); \quad a^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}}\left(\frac{p_x}{\sqrt{m}} + i\sqrt{m}\Omega x\right) \tag{14}$$

$$b \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}}\left(\frac{p_y}{\sqrt{m}} - i\sqrt{m}\Omega y\right); \quad b^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}}\left(\frac{p_y}{\sqrt{m}} + i\sqrt{m}\Omega y\right) \tag{15}$$

satisfying the commutation relations $[a, a^\dagger] = 1 = [b, b^\dagger]$, $[a, b] = 0 = [a, b^\dagger]$. Then the Hamiltonian H for the quantum damped/amplified oscillator system is obtained [30] (see also [7, 31, 32])

$$H = H_0 + H_I \tag{16}$$

$$H_0 = \hbar\Omega(A^\dagger A - B^\dagger B), \qquad H_I = i\hbar\Gamma(A^\dagger B^\dagger - AB) \tag{17}$$

where $A \equiv (1/\sqrt{2})(a+b)$, $B \equiv (1/\sqrt{2})(a-b)$.

The vacuum state is $|0\rangle \equiv |n_A = 0, n_B = 0\rangle = |0\rangle \otimes |0\rangle$, where n_A and n_B denote the number of A's and B's and $(A \otimes 1)|0\rangle \otimes |0\rangle \equiv A|0\rangle = 0$; $(1 \otimes B)|0\rangle \otimes |0\rangle \equiv B|0\rangle = 0$. The vacuum time

evolution is controlled by H_I: $|0(t)\rangle = e^{-it\frac{H}{\hbar}}|0\rangle = e^{-it\frac{H_I}{\hbar}}|0\rangle$, with $\langle 0(t)|0(t)\rangle = 1$, $\forall t$. $|0(t)\rangle$ is explicitly given by

$$|0(t)\rangle = \prod_\kappa \frac{1}{\cosh\left(\Gamma_\kappa t\right)} \exp\left(\tanh\left(\Gamma_\kappa t\right) A_\kappa^\dagger B_\kappa^\dagger\right)|0\rangle \tag{18}$$

We have $\lim_{t\to\infty}\langle 0(t)|0\rangle \propto \lim_{t\to\infty}\exp\left(-t\Gamma\right) = 0$. In the infinite volume limit, for $\int d^3\kappa\, \Gamma_\kappa$ finite and positive,

$$\langle 0(t)|0\rangle \to 0 \quad\text{as}\quad V \to \infty \quad \forall t \tag{19}$$

and $\langle 0(t)|0(t')\rangle \to 0$ as $V \to \infty$ $\forall t$ and t', $t' \neq t$. The meaning of these relations is that a representation $\{|0(t)\rangle\}$ of the canonical commutation relations (CCR) is defined at each time t and is unitarily inequivalent to any other representation $\{|0(t')\rangle, \forall t' \neq t\}$ in the infinite volume limit. The system thus evolves in time through unitarily inequivalent representations of CCR [30]. The number of modes A_κ (or B_κ) condensed in $|0(t)\rangle$ is given by

$$\mathcal{N}_{A_\kappa}(t) = \langle 0(t)|A_\kappa^\dagger A_\kappa|0(t)\rangle = \sinh^2 \Gamma_\kappa t \tag{20}$$

The states generated by $B^\dagger$ represent the sink where the energy dissipated by the quantum damped oscillator flows, or, in other words, the B-oscillator represents the reservoir coupled to the A-oscillator [30].

The two mode realization of the algebra $su(1,1)$ is obtained by defining $J_+ \equiv A^\dagger B^\dagger$, $J_- \equiv J_+^\dagger \equiv AB$, $J_3 \equiv (1/2)(A^\dagger A + B^\dagger B + 1)$, then $[J_+, J_-] = -2J_3$, $[J_3, J_\pm] = \pm J_\pm$, with the $SU(1,1)$ Casimir operator $\mathcal{C}$ given by $\mathcal{C}^2 = (1/4)(A^\dagger A - B^\dagger B)^2$. The commutation relation $[H_0, H_I] = 0$ guarantees that the initial condition of positiveness for the eigenvalues of H_0 is protected against transitions to negative energy states.

As said, $|0(t)\rangle$ is thus a two-mode time dependent generalized $SU(1,1)$ coherent state [7, 20, 30, 33]. $|0(t)\rangle$ is also recognized to be a squeezed coherent state characterized by the q-deformation of Lie-Hopf algebra [7, 21, 22, 23]. I also remark that A and B are entangled modes. This entanglement cannot be destroyed by the action of any unitary operator, a feature absent in quantum mechanics.

Finally, I observe that the variations in time of the number of particles condensed in the vacuum gives heat dissipation $dQ = \frac{1}{\beta}dS$ and time evolution is controlled by entropy variations [30, 34], which is consistent with the fact that dissipation implies breaking of time-reversal invariance (*the arrow of time*). The Hamiltonian H turns out to be actually the *fractal free energy* for the coherent boson condensation process out of which the fractal is formed. The system temperature $T = \hbar\Gamma$ is proportional to the background zero point energy: $\hbar\Gamma \propto \hbar\Omega/2$ [7, 27, 28, 29].

It can be shown [4] that also in the case of the Koch curve and other fractals the isomorphism with the system of damped/amplified oscillators and squeezed SU(1,1) coherent states can be established. For brevity, here I do not report the derivation of these results.

In closing this Section, I observe that letting $p_{z_\pm}$ denote the momenta and $v_\pm = \dot{z}_\pm$ the forward in time and backward in time velocities, with the notation $+ \equiv 1$ and $- \equiv 2$, it is:

$$v_\pm = \pm\frac{1}{m}\left(p_{z_\pm} \mp (1/2)\gamma z_\mp\right), \qquad\text{with}\qquad [v_+, v_-] = i\frac{\gamma}{m^2} \tag{21}$$

Conjugate position coordinates (ξ_+, ξ_-) can be defined by putting $\xi_\pm = \mp(m/\gamma)v_\mp$, with

$$[\xi_+, \xi_-] = i\frac{1}{\gamma}. \tag{22}$$

This suggests that the relation between dissipation and noncommutative geometry in the plane can be shown to exist. For brevity I will not discuss further such an issue and the role of the deformed Hopf algebra [16, 35, 36, 37].

A final remark is that the continuous time evolution includes the discrete group of transformations $z_1(m) = r_0(e^{-2\pi d})^m \to z_1(m+1) = r_0(e^{-2\pi d})^{(m+1)} = z_1(m)(e^{-2\pi d})$, with integer $m = 1, 2, 3...$, due to the T integer multiplicity, $\theta(T) = 2\pi$ at $T = 2\pi d/\Gamma$ and, at $t = mT$, $z_1 = r_0(e^{-2\pi d})^m$, $z_2 = r_0(e^{2\pi d})^m$. The isomorphism seems thus to appear as a homomorphism.

3. Self-similarity and electrodynamics

In the previous Section we have established the isomorphism between (fractal) self-similarity and the squeezed SU(1,1) coherent states associated to the damped/amplified oscillator system Eqs. (8) and (9) (or, equivalently Eqs. (10) and (11) or Eqs. (12) and (13)). We now discuss the relation with electrodynamics. Consider the vector potential given by $\vec{A} = (1/2)\vec{B} \times \vec{r}$, $\vec{r} = (x_1, x_2, x_3)$. It is $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{\nabla} \cdot \vec{A} = 0$. Choose the reference frame so that the magnetic field $\vec{B}$ be a constant vector, $\vec{B} = \vec{\nabla} \times \vec{A} = -B\hat{\mathbf{3}}$. Then, $A_3 = 0$ and

$$A_i = \frac{B}{2}\epsilon_{ij}x_j , \qquad i, j = 1, 2. \tag{23}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$; $\epsilon_{ii} = 0$. Let $B \equiv \gamma/e$. The third component, $i = 3$, of $(\vec{v} \times \vec{B})$, $v_i = \dot{x}_i$, vanishes. Moreover, assume that $\vec{E}$ is given by the gradient of the harmonic potential $\Phi \equiv \frac{k}{2e}(x_1{}^2 - x_2{}^2) \equiv \Phi_1 - \Phi_2$, $\vec{E} = -\vec{\nabla}\Phi$; and $E_3 = 0$. Eqs. (12) and (13) are now rewritten as

$$F_e^i = e E^i + e(\vec{v} \times \vec{B})^i \tag{24}$$

$$F_{-e}^i = -e E^i - e(\vec{v} \times \vec{B})^i \tag{25}$$

namely, the Lorentz forces $\vec{F}_e$ and $\vec{F}_{-e}$, acting on two opposite charges with same velocity $\vec{v}$ in the same electric and magnetic fields, $\vec{E}$ and $\vec{B}$. Use of $i = 1$ in Eq. (24) and $i = 2$ in Eq. (25) gives indeed Eqs. (12) and (13). Considering $i = 2$ in Eq. (24) and $i = 1$ in Eq. (25) leads to similar result.

One can show [24] that by using Eqs. (23), Eqs. (12) and (13) can be derived from the Lagrangian

$$L = \frac{1}{2m}(m\dot{x}_1 + e_1 A_1)^2 - \frac{1}{2m}(m\dot{x}_2 + e_2 A_2)^2 - \frac{e^2}{2m}(A_1{}^2 - A_2{}^2) - e\Phi, \tag{26}$$

and the Hamiltonian is

$$H = H_1 - H_2 = \frac{1}{2m}(p_1 - e_1 A_1)^2 + e_1\Phi_1 - \frac{1}{2m}(p_2 + e_2 A_2)^2 + e_2\Phi_2 \tag{27}$$

In the least energy state (where $H = 0$, $H_1 = H_2$) the respective contributions to the energy compensate each other. One of the charged particles (one of the oscillators) may be considered to represent the em field in which the other one is embedded and vice-versa.

I remark that Eqs. (24) and (25) (i.e. Eqs. (12) and (13)) are derived [16] by usual integration over the volume of the equations for the matter part $T_m^{\mu\nu}$ and the em part $T_\gamma^{\mu\nu}$ of the total energy-momentum tensor $T^{\mu\nu}$ in electrodynamics

$$\partial_\mu T_m^{\mu\nu} = e F^{\alpha\nu} J_\alpha \tag{28}$$

$$\partial_\mu T_\gamma^{\mu\nu} = -e F^{\alpha\nu} J_\alpha \tag{29}$$

where J_α denotes the current and as usual $F^{\alpha\beta} = \partial^\beta A^\alpha - \partial^\alpha A^\beta$ ($g^{\mu\nu} = (1, -1, -1, -1)$, $\mu = 0, 1, 2, 3$; $\hbar = 1 = c$). We see that the non-vanishing divergences of $T_m^{\mu\nu}$ and $T_\gamma^{\mu\nu}$

compensate each other, $\partial_\mu T_m^{\mu\nu} = -\partial_\mu T_\gamma^{\mu\nu}$, so that the conservation of the total $T^{\mu\nu}$ holds: $\partial_\mu T^{\mu\nu} = \partial_\mu (T_m^{\mu\nu} + T_\gamma^{\mu\nu}) = 0$.

For $\nu = 0$, volume integration of Eqs. (28) and (29) gives the rate of changes in time of the energy of the matter field and em field, $\mathcal{E}_m$ and $\mathcal{E}_\gamma$, respectively:

$$\partial_0 \mathcal{E}_m = e\,\vec{E} \cdot \vec{v} = -\partial_0 \mathcal{E}_\gamma \qquad (30)$$

For $\nu = i = 1, 2, 3$, integration of Eqs. (28) and (29) over the volume gives, as said, Eqs. (24) and (25).

A representation of the content of Maxwell equations and the associated conservation laws is thus provided, under the conditions specified above, by the considered system of damped/amplified oscillators. As seen, their realization in terms of squeezed coherent states turns out to be isomorphic to fractal self-similarity properties. This establishes the link between electrodynamics and fractal self-similarity.

4. Conclusions

In this paper I have summarized results showing that an isomorphism exists between the formalism describing fractal self-similarity properties and squeezed coherent states in QFT representing a system of damped/amplified oscillators. I have also shown that the isomorphism extends to electrodynamics involving the basic conservation laws implied by Maxwell equation. These conclusions may turn out to be of interest in view of the widely diffused presence in Nature of fractal-like structures and of the ubiquitous occurrence of power laws in physics, in biology and neuroscience. Fractals appear to be macroscopic quantum systems, in the sense specified in the Introduction, arising from a process of morphogenesis as the manifestation of deformations (squeezing) of coherent dissipative quantum dynamics. Such results lead to an integrated vision of Nature resting on the paradigm of coherence and dissipation. Nature appears to be modulated by coherence, rather than being hierarchically layered in isolated compartments, in collections of isolated systems and phenomena.

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