Complete integrability of geodesics in toric Sasaki-Einstein spaces

Mihai Visinescu

Department of Theoretical Physics, National Institute for Physics and Nuclear Engineering, Magurele, P.O.Box M.G.-6, Romania

E-mail: `mvisin@theory.nipne.ro`

Abstract. We describe a method for constructing Killing-Yano tensors on toric Sasaki-Einstein manifolds using their geometrical properties. We take advantage of the fact that the metric cones of these spaces are Calabi-Yau manifolds. The complete list of special Killing forms can be extracted making use of the description of the Calabi-Yau manifolds in terms of toric data. This general procedure for toric Sasaki-Einstein manifolds is exemplified in the case of the 5-dimensional spaces $Y^{p,q}$ and $T^{1,1}$. Finally we discuss the integrability of geodesic motion in these spaces.

1. Introduction

Higher order symmetries associated with Killing-Yano (KY) tensors (or Killing forms) and Stäckel-Killing (SK) tensors play an important role in modern gravitational and mathematical physics. Contrary to Killing vectors, these geometrical objects do not have clear meaning as they do not describe continuous symmetries of the space. They are considered as symmetries of the whole phase-space and consequently they are often called *dynamical* or *hidden symmetries*. Their corresponding conserved quantities are polynomial in momenta and are involved in the study of integrability properties of geodesic motions and separation of variables for the Hamilton-Jacobi or quantum Klein-Gordon, Dirac equations.

The generalized Killing equations for SK and KY tensors are quite intricate and to get their solutions by solving the corresponding differential equations is a very difficult task. From this point of view it is of particular importance to find alternative ways for constructing the higher order Killing tensors.

In this paper we investigate the KY and SK tensors on toric Sasaki-Einstein (SE) spaces using their geometrical properties. Recently SE geometry has become of significant interest in various developments in mathematics and theoretical physics [1]. These geometries are believed to play a significant role in studies of consistent string theory compactification and in the context of the AdS/CFT correspondence. CFT duals to the SE manifolds are $\mathcal{N} = 1$ superconformal quiver gauge theories in $3 + 1$ dimensions. Both the geometry and the gauge theory can be characterized by a common data, called a *toric data*. In the case of toric SE manifolds there is an algorithm which constructs the gauge theory from the toric data of the Calabi-Yau (CY) singularity.

The aim of this paper is to present a method to find the conserved quantities for geodesic motions on toric SE spaces. For this purpose we construct all special KY forms on the SE spaces using the fact that these forms are exactly those forms which translate into parallel forms on the CY metric cone [2]. In turn the KY forms on the metric cone are obtained from the toric data of the CY manifolds using the Delzant approach [3] to toric geometries.

The general procedure to get KY tensors on toric SE spaces is exemplified in the case of the 5-dimensional spaces $Y^{p,q}$ and $T^{1,1}$. Having the explicit SK tensors on these spaces we discuss the integrability of geodesic motions in these spaces and show that the systems are completely integrable.

The paper is organized as follows. In the next Section we review the basic definitions of SK and KY tensors. In Section 3 we present some well-known results concerning the special KY forms on SE manifolds and their relations with the parallel forms of the corresponding CY metric cones. In Section 4 it is described the construction of KY parallel forms on CY spaces using the toric data. In Section 5 we exemplify the general scheme in the case of the 5-dimensional spaces $Y^{p,q}$ and $T^{1,1}$ and prove the complete integrability of geodesic motions. The paper ends with conclusions in Section 6.

2. Killing tensors

Let (M, g) be an n-dimensional Riemannian manifold with the metric g and let ∇ be its Levi-Civita connection. A natural generalization of Killing vectors is given by SK tensors of rank $r > 1$ defined as totally symmetric tensor fields $K_{\mu_1\cdots\mu_r}$ satisfying the generalized Killing equation

$$\nabla_{(\mu}K_{\nu_1\cdots\nu_r)} = 0\,, \tag{1}$$

where the round brackets indicate symmetrization over the indices within.

In the presence of a SK tensor the system of a free particle with the Hamiltonian

$$H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu\,, \tag{2}$$

admits the conserved quantity

$$K = K^{\mu_1\cdots\mu_r}p_{\mu_1}\cdots p_{\mu_r}\,, \tag{3}$$

commuting with Hamiltonian (2) in the sense of Poisson brackets. Here p_μ are canonical momenta conjugate to the coordinates x^μ, $p_\mu = g_{\mu\nu}\dot{x}^\nu$ with overdot denoting proper time derivative.

Another important generalization of Killing vector fields is represented by antisymmetric KY tensors which are r-forms obeying the equation [4]

$$\nabla_{(\mu}\Psi_{\nu_1)\nu_2\ldots\nu_r} = 0\,. \tag{4}$$

It turns out that the most part of known interesting KY tensors are the so called special Killing forms satisfying for some constant c the additional equation [2]

$$\nabla_X(d\Psi) = cX^* \wedge \Psi\,, \tag{5}$$

for any vector field X on M, X^* being the 1-form dual to the vector field X.

There is an important connection between these two generalizations of the Killing vectors. To wit, the partially contracted product of two KY tensors $\Psi^{i_1,\ldots,i_r}$ and $\Sigma^{i_1,\ldots,i_r}$ generates a SK tensor of rank 2:

$$K_{ij}^{(\Psi,\Sigma)} = \Psi_{ii_2\ldots i_r}\Sigma_j^{\ i_2\ldots i_r} + \Sigma_{ii_2\ldots i_r}\Psi_j^{\ i_2\ldots i_r}\,. \tag{6}$$

This fact offers a method to generate higher order integrals of motion by identifying the complete set of Killing-Yano tensors.

Let us note that trying to solve straightforwardly the generalized Killing equations (1) or (4) turns out to be very difficult. For this reason in what follows we shall present an approachable way for finding hidden symmetries on SE spaces.

3. Special KY forms on SE spaces

A $(2n-1)$-dimensional manifold M is a *contact manifold* if there exists a 1-form η (called a *contact 1-form*) on M such that:

$$\eta \wedge (d\eta)^{n-1} \neq 0\,. \tag{7}$$

For every choice of contact 1-form η there exists a unique vector field K_η, called the *Reeb vector field*, which satisfies:

$$\eta(K_\eta) = 1 \quad \text{and} \quad K_\eta \lrcorner\, d\eta = 0\,, \tag{8}$$

where $\lrcorner$ is the operator dual to the wedge product.

Let us introduce the metric cone $C(M)$ of the manifold M as the product manifold $M \times \mathbb{R}_{>0}$, with $\dim C(M) = 2n$, endowed with the warped metric

$$\bar{g} := dr^2 + r^2 g\,. \tag{9}$$

Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$.

A compact Riemannian manifold (M, g) is Sasakian if and only if its metric cone $C(M)$ is Kähler and its Kähler 2-form is given by

$$\omega = \frac{1}{2} d(r^2\eta)\,. \tag{10}$$

If the Sasaki manifold M is Einstein

$$\mathrm{Ric}_g = 2(n-1)g\,, \tag{11}$$

then the Kähler metric cone is Ricci-flat ($\mathrm{Ric}_{\bar{g}} = 0$), i.e. CY manifold.

A first set of KY tensors of a SE manifold is given by the special Killing forms

$$\Psi_k = \eta \wedge (d\eta)^k \quad , \quad k = 0, 1, \cdots, n-1\,. \tag{12}$$

These tensors do not exhaust the complete set of KY tensors on SE spaces. In order to find all special KY tensors we must resort to the CY metric cone of a SE manifold. To write explicitly two additional Killing forms we shall express the volume form of the metric cone in terms of the Kähler form ω:

$$d\mathcal{V} = \frac{1}{n!}\omega^n\,. \tag{13}$$

The volume form of a Kähler manifold can be also written as [2, 5]:

$$d\mathcal{V} = \frac{i^n}{2^n}(-1)^{n(n-1)/2}\Omega \wedge \overline{\Omega}, \tag{14}$$

where Ω is the volume holomorphic $(n, 0)$ form of $C(M)$. The additional Killing forms on $C(M)$ are given by the real respectively the imaginary part of the complex volume form.

The last step to finding all special KY tensors of the SE manifold M makes use of the fact that special Killing forms on Sasaki manifolds are exactly those forms which translate into parallel forms on the metric cone. More precisely, for any p-form Ψ on the space M we can define an associated $(p+1)$-form Ψ^C on the cone $C(M)$:

$$\Psi^C := r^p dr \wedge \Psi + \frac{r^{p+1}}{p+1} d\Psi\,. \tag{15}$$

Ψ^C is parallel if and only if Ψ is a special Killing-Yano tensor (5) with constant $c = -(p+1)$ [2].

Therefore the complete list if special KY forms on SE spaces are given by the Killing forms (12) and two additional KY tensors connected with the real and imaginary parts of the complex volume $(n, 0)$ form of $C(M)$ and extracted as in (15).

4. Complex volume form of CY toric manifold

In order to write the complex volume form we need the complex coordinates of the CY cone manifold.

Let us consider that $C(M)$ is toric, the standard n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ acting effectively on $C(M)$

$$\tau : \mathbb{T}^n \to \mathrm{Diff}(C(M), \omega), \tag{16}$$

preserving the Kähler form ω. Let us denote by Φ^i the angular coordinates along the torus action generated by $\partial/\partial\Phi_i$ and write the Reeb vector field as

$$K_\eta = b_i \frac{\partial}{\partial \Phi_i}\,. \tag{17}$$

Associated to $(C(M), \omega, \tau)$ there is a moment map

$$\mu = \frac{1}{2} r^2 \eta\,, \tag{18}$$

and the action coordinates are

$$y^i = \mu\left(\frac{\partial}{\partial \Phi_i}\right)\,. \tag{19}$$

The $\mathbb{T}^n$-invariant Kähler metric on $C(M)$ in the symplectic coordinates (y, Φ) is

$$ds^2 = G_{ij} dy^i dy^j + G^{ij} d\Phi_i d\Phi_j, \tag{20}$$

where G_{ij} is the Hessian of the symplectic potential $G(y)$ in the y coordinates

$$G_{ij} = \frac{\partial^2 G}{\partial y^i \partial y^j} \quad , \quad 1 \leq i, j \leq n, \tag{21}$$

and $G^{ij} = (G_{ij})^{-1}$.

For what follows we need the evaluation of the symplectic potential G. This object is obtained using the Delzant construction [3]. A *Delzant polytope* is a covex polytope such that there are n edges meeting at each vertex, each edge meeting at the vertex is of the form $1 + tu_i$ where $u_i \in \mathbb{Z}^n$, and $\{u_i\}$ can be chosen to form basis in $\mathbb{Z}^n$. This polytope can be described by the inequalities

$$l_a(y) := (y, v_a) \geq 0, \quad \text{for} \quad 1 \leq a \leq d\,, \tag{22}$$

where v_a are the inward pointing normal vectors to the d facets of the polyhedral cone. The set of vectors $\{v_a\}$

$$v_A = v_a^i \left(\frac{\partial}{\partial \Phi_i}\right)\,, \tag{23}$$

is called a *toric data*.

Using the Delzant construction the general symplectic potential has the following form in terms of the toric data [6, 7]:

$$G = G^{can} + G_b + h, \tag{24}$$

where

$$G^{can} = \frac{1}{2} \sum_a l_a(y) \log l_a(y), \tag{25}$$

$$G_b = \frac{1}{2} l_b(y) \log l_b(y) - \frac{1}{2} l_\infty(y) \log l_\infty(y), \tag{26}$$

with $l_b(y) = (b, y)$, $l_\infty(y) = \sum_a (v_a, y)$ and h is a homogeneous degree one function of variables y^i

$$h = \lambda_i y^i + t, \tag{27}$$

λ_i, t being some constants.

The standard complex coordinates of $C(M)$ are w_i on $\mathbb{C}\backslash\{0\}$. Log complex coordinates are $z_i = \log w_i = x_i + \mathrm{i}\Phi_i$, with

$$x_i = \frac{\partial G}{\partial y_i} \,. \tag{28}$$

The final form of the $(n, 0)$ holomorphic form of the Ricci-flat metric on the CY cone turns out to be [8]:

$$\Omega = \exp\left(x_1 + \mathrm{i}\Phi_1\right) dz_1 \wedge \cdots \wedge dz_n \,. \tag{29}$$

5. Applications

We shall exemplify the general procedure in the case of the 5-dimensional spaces $Y^{p,q}$ and $T^{1,1}$. The complete list of special KY tensors allows us to prove the complete integrability of geodesic motions on these spaces.

5.1. $Y^{p,q}$ spaces

A particular interesting class of toric contact structures on $S^2 \times S^3$ have been studied by physicists [9, 10] and denoted by $Y^{p,q}$ where p, q are relative prime integers satisfying $0 \leq q \leq p$.

The explicit local metric of the 5-dimensional $Y^{p,q}$ manifold is given by the line element [11]

$$\begin{aligned} ds^2 = {} & \frac{1 - c\,y}{6}(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9}(d\psi - \cos\theta\, d\phi)^2 \\ & + w(y)\left[d\alpha + \frac{ac - 2y + c\,y^2}{6(a - y^2)}[d\psi - \cos\theta\, d\phi]\right]^2 , \end{aligned} \tag{30}$$

where

$$\begin{aligned} w(y) &= \frac{2(a - y^2)}{1 - cy} \,, \\ q(y) &= \frac{a - 3y^2 + 2cy^3}{a - y^2} \,. \end{aligned} \tag{31}$$

The constant c can be rescaled by a diffeomorphism, so in what follows we take $c = 1$. For $0 < \alpha < 1$ we can take the range of the angular coordinates (θ, ϕ, ψ) to be $0 \leq \theta \leq 2\pi\,, 0 \leq \phi \leq 2\pi\,, 0 \leq \psi \leq 2\pi$. Choosing $0 < a < 1$, the range of the coordinate y is taken between the negative and the smallest positive roots of the cubic equation

$$a - 3y^2 + 2y^3 = 0 \,. \tag{32}$$

The Sasakian 1-form η is

$$\eta = -2y d\alpha + \frac{1 - y}{3}(d\psi - \cos\theta d\phi) \,, \tag{33}$$

and the Reeb vector field is [11]

$$K_\eta = 3\frac{\partial}{\partial\psi} - \frac{1}{2}\frac{\partial}{\partial\alpha} \,. \tag{34}$$

The explicit form of the KY tensor Ψ_1 (12) is

$$\begin{aligned}\Psi_1 = {} & (1-y)^2 \sin\theta \, d\theta \wedge d\phi \wedge d\psi - 6 dy \wedge d\alpha \wedge d\psi \\ & + 6 \cos\theta \, d\phi \wedge dy \wedge d\alpha - 6(1-y) y \sin\theta \, d\theta \wedge d\phi \wedge d\alpha \,.\end{aligned} \tag{35}$$

For the additional KY tensors we shall construct the symplectic potential G (24) using the toric data for $Y^{p,q}$ [11, 12] and finally the holomorphic form Ω (29). From (15) in the case of the 5-dimensional $Y^{p,q}$ spaces we have

$$\Omega := r^3 dr \wedge \Psi + \frac{r^3}{3} d\Psi \,. \tag{36}$$

Decomposing Ψ into its real ($\Re\Psi$) and imaginary ($\Im\Psi$) parts, and ignoring the multiplicative constants, we get the following special real Killing forms [13, 14, 15]:

$$\begin{aligned}\Re\Psi = \sqrt{\frac{1-y}{p(y)}} \Big(& \cos\psi \Big[d\theta \wedge dy + 6p(y) \sin\theta \, d\phi \wedge d\alpha + p(y) \sin\theta \, d\phi \wedge d\psi \Big] \\ & - \sin\psi \Big[\sin\theta \, d\phi \wedge dy - 6p(y) d\theta \wedge d\alpha - p(y) d\theta \wedge d\psi \\ & + p(y) \cos\theta \, d\theta \wedge d\phi \Big] \Big) \,,\end{aligned} \tag{37}$$

$$\begin{aligned}\Im\Psi = \sqrt{\frac{1-y}{p(y)}} \Big(& \sin\psi \Big[d\theta \wedge dy + 6p(y) \sin\theta \, d\phi \wedge d\alpha + p(y) \sin\theta \, d\phi \wedge d\psi \Big] \\ & + \cos\psi \Big[\sin\theta \, d\phi \wedge dy - 6p(y) d\theta \wedge d\alpha - p(y) d\theta \wedge d\psi \\ & + p(y) \cos\theta \, d\theta \wedge d\phi \Big] \Big) \,.\end{aligned} \tag{38}$$

For $Y^{p,q}$ spaces the conjugate momenta to the coordinates $(\theta, \phi, y, \alpha, \psi)$ are [16]:

$$\begin{aligned} p_\theta &= \frac{1-y}{6} \dot\theta \,, \\ p_\phi + \cos\theta p_\psi &= \frac{1-y}{6} \sin^2\theta \dot\phi \,, \\ p_y &= \frac{1}{6p(y)} \dot y \,, \\ p_\alpha &= w(y) \left(\dot\alpha + f(y) \left(\dot\psi - \cos\theta \dot\phi \right) \right) \,, \\ p_\psi &= w(y) f(y) \dot\alpha + \left[\frac{q(y)}{9} + w(y) f^2(y) \right] \left(\dot\psi - \cos\theta \dot\phi \right) \,. \end{aligned} \tag{39}$$

From the isometry $SU(2) \times U(1) \times U(1)$ of the metric (30) we have that the momenta p_ϕ, p_ψ and p_α are conserved. p_ϕ is the third component of the $SU(2)$ angular momentum and p_ψ, p_α are associated to the $U(1)$ factors. In addition, the total $SU(2)$ angular momentum

$$\vec{J}^{\,2} = p_\theta^2 + \frac{1}{\sin^2\theta} \left(p_\phi + \cos\theta p_\psi \right)^2 + p_\psi^2 \,, \tag{40}$$

is also conserved [16, 17].

The next conserved quantities, quadratic in momenta, will be expressed in terms of Stäckel-Killing tensors as in (3). The Stäckel-Killing tensors of rank two on $Y^{p,q}$ will be constructed from Killing-Yano tensors according to (6). For this purpose we shall use the Killing 3-form Ψ_1 (35) and the additional 2-forms (37) and (38).

The first Stäckel-Killing tensor $K^{(1)}_{\mu\nu}$ is constructed according to (6) using the real part of the Killing form Ψ (37):

$$K^{(1)}_{\mu\nu} = (\Re\Psi)_{\mu\lambda}(\Re\Psi)^{\lambda}{}_{\nu}\,. \tag{41}$$

Let us denote by $K^{(1)}$ the corresponding quantity constructed from $K^{(1)}_{\mu\nu}$.

The next SK tensor can be constructed using the imaginary part $\Im\Psi$, but we find that this SK tensor produces the same conserved quantity $K^{(1)}$.

In principle a mixed combination of KY tensors $\Re\Psi$ and $\Im\Psi$ produces a SK tensor (6), but it proves that all components of this tensor are zero.

Finally we construct the Stäckel-Killing tensor from the Killing form Ψ_1:

$$K^{(2)}_{\mu\nu} = (\Psi_1)_{\mu\lambda\sigma}(\Psi_1)^{\lambda\sigma}{}_{\nu}\,, \tag{42}$$

and let us denote by $K^{(2)}$ the corresponding conserved quantity.

In conclusion we have a set of 7 conserved quantities $H, p_\phi, p_\psi, p_\alpha, \vec{J}^{\,2}, K^{(1)}, K^{(2)}$. We can examine if this set constitutes a functionally independent set of constants of motion for the geodesics of $Y^{p,q}$ constructing the Jacobian:

$$\mathcal{J} = \frac{\partial(H, p_\phi, p_\psi, p_\alpha, \vec{J}^{\,2}, K^{(1)}, K^{(2)})}{\partial(\theta, \phi, y, \alpha, \psi, \dot{\theta}, \dot{\phi}, \dot{y}, \dot{\alpha}, \dot{\psi})}\,. \tag{43}$$

The rank of this Jacobian is 5, exactly the number of the degrees of freedom, which means that the system is completely integrable [18] in contrast to the assertion made in [17]. In spite of the presence of the Stäckel-Killing tensors $K^{(1)}$ and $K^{(2)}$, the system is not superintegrable, $K^{(1)}$ and $K^{(2)}$ being a combination of the first integrals $H, p_\phi, p_\psi, p_\alpha, \vec{J}^{\,2}$. Therefore the toric Sasaki-Einstein spaces $Y^{p,q}$ spaces possess several Killing-Yano tensors, but these Killing forms do not generate new Stäckel-Killing tensors, i.e. genuine conserved quantities.

5.2. $T^{1,1}$ space

The homogeneous Sasaki-Einstein metric on $S^2 \times S^3$ is usually referred to as $T^{1,1}$. The $T^{1,1}$ space was considered as the first example of toric Sasaki-Einstein/quiver duality [19].

The isometries of $T^{1,1}$ form the group $SU(2)\times SU(2)\times U(1)$ and the metric of this space may be written down explicitly by utilizing the fact that it is a $U(1)$ bundle over $S^2\times S^2$. Let us denote by (θ_1, ϕ_1) and (θ_2, ϕ_2) the coordinates which parametrize the two sphere in a conventional way, and the angle $\psi \in [0, 4\pi)$ to parametrize the $U(1)$ fiber. Using these parametrizations the $T^{1,1}$ metric may be written as [20, 11]:

$$\begin{aligned} ds^2(T^{1,1}) =& \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ &+ \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\,. \end{aligned} \tag{44}$$

The globally defined contact 1-form η is:

$$\eta = \frac{1}{3}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)\,, \tag{45}$$

and the Reeb vector field K_η has the form:

$$K_\eta = 3\frac{\partial}{\partial\psi}\,. \tag{46}$$

From the contact form η (45), using (12) we construct the Killing 3-form

$$\begin{aligned}\Psi_1 =& \frac{1}{9}(\sin\theta_1 d\psi\wedge d\theta_1\wedge d\phi_1 + \sin\theta_2 d\psi\wedge d\theta_2\wedge d\phi_2\\ &-\cos\theta_1\sin\theta_2 d\theta_2\wedge d\phi_1\wedge d\phi_2\\ &+\cos\theta_2\sin\theta_1 d\theta_1\wedge d\phi_1\wedge d\phi_2)\,.\end{aligned} \tag{47}$$

Again, using the toric data for $T^{1,1}$ we evaluate the symplectic potential G and the complex coordinates on the CY cone manifold. The additional real Killing forms extracted from the holomorphic volume $(3,0)$ form are [21]

$$\begin{aligned}\Re\Psi =& \cos\psi\, d\theta_1\wedge d\theta_2 + \sin\theta_2\sin\psi\, d\theta_1\wedge d\phi_2\\ &-\sin\theta_1\sin\psi\, d\theta_2\wedge d\phi_1\\ &-\sin\theta_1\sin\theta_2\cos\psi\, d\phi_1\wedge d\phi_2\,,\end{aligned} \tag{48}$$

$$\begin{aligned}\Im\Psi =& \sin\psi\, d\theta_1\wedge d\theta_2 - \sin\theta_2\cos\psi\, d\theta_1\wedge d\phi_2\\ &+\sin\theta_1\cos\psi\, d\theta_2\wedge d\phi_1\\ &-\sin\theta_1\sin\theta_2\sin\psi\, d\phi_1\wedge d\phi_2\,.\end{aligned} \tag{49}$$

The conjugate momenta to the coordinates $(\theta_1,\theta_2,\phi_1,\phi_2,\psi)$ are:

$$\begin{aligned}p_{\theta_1} &= \frac{1}{6}\dot\theta_1\,,\\ p_{\theta_2} &= \frac{1}{6}\dot\theta_2\,,\\ p_{\phi_1} &= \frac{1}{6}\sin^2\theta_1\,\dot\phi_1 + \frac{1}{9}\cos^2\theta_1\,\dot\phi_1 + \frac{1}{9}\cos\theta_1\,\dot\psi + \frac{1}{9}\cos\theta_1\cos\theta_2\,\dot\phi_2\,,\\ p_{\phi_2} &= \frac{1}{6}\sin^2\theta_2\,\dot\phi_2 + \frac{1}{9}\cos^2\theta_2\,\dot\phi_2 + \frac{1}{9}\cos\theta_2\,\dot\psi + \frac{1}{9}\cos\theta_1\cos\theta_2\,\dot\phi_1\,,\\ p_\psi &= \frac{1}{9}\,\dot\psi + \frac{1}{9}\cos\theta_1\,\dot\phi_1 + \frac{1}{9}\cos\theta_2\,\dot\phi_2\,.\end{aligned} \tag{50}$$

Taking into account the isometries of $T^{1,1}$, the momenta p_{ϕ_1}, p_{ϕ_2} and p_ψ are conserved. On the other hand two total $SU(2)$ angular momenta are also conserved:

$$\begin{aligned}\vec J_1^{\,2} =& p_{\theta_1}^2 + \frac{1}{\sin^2\theta_1}(p_{\phi_1} - \cos\theta_1 p_\psi)^2 + p_\psi^2\\ =& \frac{1}{36}\left[\dot\theta_1^2 + \sin^2\theta_1\dot\phi_1^2\right] + \frac{1}{81}\Big[\dot\psi^2 + \cos^2\theta_1\dot\phi_1^2 + \cos^2\theta_2\dot\phi_2^2\\ &+2\cos\theta_1\,\dot\phi_1\dot\psi + 2\cos\theta_2\dot\phi_2\dot\psi + 2\cos\theta_1\cos\theta_2\dot\phi_1\dot\phi_2\Big]\,,\\ \vec J_2^{\,2} =& p_{\theta_2}^2 + \frac{1}{\sin^2\theta_2}(p_{\phi_2} - \cos\theta_2 p_\psi)^2 + p_\psi^2\\ =& \frac{1}{36}\left[\dot\theta_2^2 + \sin^2\theta_2\dot\phi_2^2\right] + \frac{1}{81}\Big[\dot\psi^2 + \cos^2\theta_1\dot\phi_1^2 + \cos^2\theta_2\dot\phi_2^2\\ &+2\cos\theta_1\,\dot\phi_1\dot\psi + 2\cos\theta_2\dot\phi_2\dot\psi + 2\cos\theta_1\cos\theta_2\dot\phi_1\dot\phi_2\Big]\,.\end{aligned} \tag{51}$$

As for the $Y^{p,q}$ spaces we construct the conserved quantities associated with the Killing forms (47), (48), (49) and find again that we have only two nontrivial conserved quantities $K^{(1)}$ and $K^{(2)}$ as in the previous case. In fact metric (44) of the $T^{1,1}$ space can be obtained from metric (30) of the $Y^{p,q}$ spaces taking $c = 0$, setting $a = 3$ and making some changes of coordinates [9].

Finally we investigate the integrability of the geodesic motion on $T^{1,1}$ and for this purpose we construct the Jacobian:

$$\mathcal{J} = \frac{\partial(H, p_{\phi_1}, p_{\phi_2}, p_\psi, \vec{J}_1^{\,2}, \vec{J}_2^{\,2}, K^{(1)}, K^{(2)})}{\partial(\theta_1, \theta_2, \phi_1, \phi_2, \psi, \dot{\theta}_1, \dot{\theta}_2, \dot{\phi}_1, \dot{\phi}_2, \dot{\psi})} \,. \tag{52}$$

As in the case of $Y^{p,q}$ spaces, the rank of this Jacobian is 5 implying the complete integrability of the geodesic motion on $T^{1,1}$. Because not all aforesaid constants of motion are functionally independent, from 8 constants we can choose a subset of 5 constants as functionally independent. For $T^{1,1}$ space the expressions of the constants of motion are simpler than in the case of $Y^{p,q}$ spaces and we can write explicitly the relations between them. For example, selecting the subset $(H, p_{\phi_1}, p_{\phi_2}, p_\psi, \vec{J}_1^{\,2})$ as functionally independent constants of motion, the constants $\vec{J}_2^{\,2}, K^{(1)}$ and $K^{(2)}$ can be expressed in terms of the chosen subset:

$$\begin{aligned} \vec{J}_2^{\,2} &= \frac{1}{3}H + \frac{1}{2}p_\psi^2 - \vec{J}_1^{\,2} \,, \\ K^{(1)} &= 72H - 324 p_\psi^2 \,, \\ K^{(2)} &= 16H + 72 p_\psi^2 \,. \end{aligned} \tag{53}$$

6. Conclusions

An important point of interest in physics is to identify constants of motion or conserved quantities of a system.

KY tensors are known to be highly relevant for both mathematics and physics as they can be used to construct higher order integrals of motion. In this paper we present a method to extract Killing forms on a toric SE space using successively its contact form, the symplectic, complex coordinates and holomorphic volume form of its CY metric cone.

A detailed analysis of the KY tensors on 5-dimensional $Y^{p,q}$ and $T^{1,1}$ spaces shows that the number of functionally independent constants of motion is five implying the complete integrability.

Using toric geometry, many other SE manifolds can be investigated and these spaces are a good testing ground for the predictions of the AdS/CFT correspondence.

Acknowledgments

The work of M. V. has been partly supported by the joint Romanian-LIT, JINR, Dubna Research Project, theme no. 05-6-1119-2014/2016 and partly by the program NUCLEU PN-09370102, Romania.

References

[1] Sparks J 2011 Sasaki-Einstein manifolds *Surv. Diff. Geom.* **16** 265 (*Preprint* 1004.2461)

[2] Semmelmann U 2003 Conformal Killing forms on Riemannian manifolds *Math. Z.* **245** 503 (*Preprint* math/0206117)

[3] Delzant T 1988 Hamiltoniens périodiques et images convexes de l'application moment *Bull. Soc. Math. France* **116** 315

[4] Yano K 1952 Some remarks on tensor fields and curvature *Ann. of Math.* **55** 328

[5] Visinescu M and Vîlcu G E 2012 Hidden symmetries of Euclideanised Kerr-NUT-(A)dS metrics in certain scaling limit *SIGMA* **8** 058 (*Preprint* 1205.6036)

[6] Guillemin V 1994 Kähler structures on toric varieties *J. Diff. Geom.* **40** 285

[7] Abreu M 2003 Kähler geometry of toric manifolds in symplectic coordinates *Fields Institute Commun.* **35** 1
[8] Martelli D, Sparks J and Yau S T 2005 The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds *Commun. Math. Phys.* **268** 39 (*Preprint* hep-th/0503183)
[9] Gauntlett J P, Martelli D, Sparks J and Waldram D 2004 Sasaki-Einstein metrics on $S^2 \times S^3$ *Adv. Theor. Math. Phys.* **8** 711 (*Preprint* hep-th/0403002)
[10] Martelli D and Sparks J 2008 Notes on toric Sasaki-Einstein seven-manifolds and AdS_4/CFT_3 *J. High Energy Phys.* **11** 016 (*Preprint* 0808.0904)
[11] Martelli D and Sparks J 2006 Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals *Commun. Math. Phys.* **262** 51 (*Preprint* hep-th/0411238)
[12] Oota Y and Yasui Y 2006 Toric Sasaki-Einstein manifolds and Heun equations *Nucl. Phys.* **B742** 275 (*Preprint* hep-th/0512124)
[13] Visinescu M 2012 Killing forms on the five-dimensional Einstein-Sasaki Y(p,q) spaces *Mod. Phys. Lett. A* **27** 1250217 (*Preprint* 1207.2581)
[14] Slesar V, Visinescu M and Vîlcu G E 2014 Special Killing forms on Sasaki-Einstein manifolds *Phys. Scripta* **89** 125205 (*Preprint* 1403.1015)
[15] Slesar V, Visinescu M and Vîlcu G E 2015 Toric data, Killing and complete integrability of geodesics in Sasaki-Einstein spaces $Y^{p,q}$ *Annals of Physics* **361** 548
[16] Benvenuti S and Kruczenski M 2006 Semiclassical strings in Sasaki-Einstein manifolds and long operators in $\mathcal{N} = 1$ gauge theories *JHEP* **0610** 051 (*Preprint* hep-th/0505046)
[17] Rubín de Celis E and O. P. Santillán O P 2012 Massless geodesics in $AdS_5 \times Y(p,q)$ as a superintegrable system *JHEP* **09** 032 (*Preprint* 1205.3256)
[18] Babalic E M and Visinescu M 2015 Complete integrability of geodesic motion in Sasaki-Einstein toric $Y^{p,q}$ spaces *Mod. Phys. Lett. A* **30** 1550180 (*Preprint* 1505.03976)
[19] Klebanov I R and Witten E 1998 Superconformal field theory on threebranes at a Calabi-Yau singularity *Nucl. Phys.* **B536** 199
[20] Candelas P and de la Ossa X C 1990 Comments on conifolds *Nucl. Phys.* **B342** 246
[21] Slesar V, Visinescu M and Vîlcu G E 2015 Hidden symmetries on toric Sasaki-Einstein spaces *EPL* **110** 31001