Self-adjointness and the Casimir effect with confined quantized spinor matter

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Abstract. A generalization of the MIT bag boundary condition for spinor matter is proposed basing on the requirement that the Dirac hamiltonian operator be self-adjoint. An influence of a background magnetic field on the vacuum of charged spinor matter confined between two parallel material plates is studied. Employing the most general set of boundary conditions at the plates in the case of the uniform magnetic field directed orthogonally to the plates, we find the pressure from the vacuum onto the plates. In physically plausible situations, the Casimir effect is shown to be repulsive, independently of a choice of boundary conditions and of a distance between the plates.

1. Introduction

The self-adjointness of operators of physical observables in quantum mechanics is required by general principles of comprehensibility and mathematical consistency, see, e.g., [1, 2]. To put it simply, a multiple action is well defined for a self-adjoint operator only, allowing for the construction of functions of the operator, such as resolvent, evolution, heat kernel and zeta-function operators, with further implications upon second quantization.

The mathematical demand for the self-adjointness of a basic operator acting on functions defined in a bounded spatial region is a somewhat more general than the physical demand for the confinement of quantized matter fields inside this region. The concept of confined matter fields is quite familiar in the context of condensed matter physics: collective excitations (e.g., spin waves and phonons) exist only inside material objects and do not spread outside. In the context of quantum electrodynamics, if one is interested in the effect of a classical background magnetic field on the vacuum of the quantized electron-positron matter, then the latter should be considered as confined to the spatial region between the sources of the magnetic field, as long as collective quasielectronic excitations inside a magnetized material differ from electronic excitations in the vacuum. It should be noted in this respect that the effect of the background electromagnetic field on the vacuum of quantized charged matter was studied for eight decades almost, see [3, 4] and a review in [5]. However, the concern was for the case of a background field filling the whole (infinite) space, that is hard to be regarded as realistic. The case of both the background and quantized fields confined to a bounded spatial region with boundaries serving as sources of the background field looks much more physically plausible, it can even be regarded as realizable in laboratory. Moreover, there is no way to detect the energy density that is induced in the vacuum in the first case, whereas the pressure from the vacuum onto the boundaries, resulting in the second case, is in principle detectable.

In view of the above, an issue of a choice of boundary conditions ensuring the confinement of the quantized spinor matter gains a crucial significance. It seems that a quest for such boundary conditions was initiated in the context of a model description of hadrons as composites of quarks and gluons [6, 7]. If an hadron is an extended object occupying spatial region Ω bounded by surface $\partial\Omega$, then the condition that the quark matter field be confined inside the hadron is formulated as

$$\mathbf{n}\cdot\mathbf{J}(\mathbf{r})|_{\mathbf{r}\in\partial\Omega}=0, \tag{1}$$

where $\mathbf{n}$ is the unit normal to the boundary surface, and $\mathbf{J}(\mathbf{r})=\psi^\dagger(\mathbf{r})\boldsymbol{\alpha}\psi(\mathbf{r})$, $\mathbf{r}\in\Omega$, with $\psi(\mathbf{r})$ being the quark matter field (an appropriate condition is also formulated for the gluon matter field). Condition (1) should be resolved to take the form of a boundary condition that is linear in $\psi(\mathbf{r})$, and an immediate way of such a resolution is known as the MIT bag boundary condition [8],

$$[I+\mathrm{i}\beta(\mathbf{n}\cdot\boldsymbol{\alpha})]\psi(\mathbf{r})|_{\mathbf{r}\in\partial\Omega}=0 \tag{2}$$

(α^1, α^2, α^3 and β are the generating elements of the Dirac-Clifford algebra), but it is needless to say that this way is not a unique one. The most general boundary condition that is linear in $\psi(\mathbf{r})$ in the case of a simply-connected boundary involves four arbitrary parameters [9], and the explicit form of this boundary condition has been given [10]; the condition is compatible with the self-adjointness of the Dirac hamiltonian operator, and its four parameters can be interpreted as the self-adjoint extension parameters. In the present work, I follow the lines of works [9, 10] by proposing a different, embracing more cases, form of the four-parameter generalization of the MIT bag boundary condition.

Thus, let us consider in general the quantized spinor matter field that is confined to the three-dimensional spatial region Ω bounded by the two-dimensional surface $\partial\Omega$. To study a response of the vacuum to the background magnetic field, we restrict ourselves to the case of the boundary consisting of two parallel planes; the magnetic field is assumed to be uniform and orthogonal to the planes. Such a spatial geometry is typical for the remarkable macroscopic quantum phenomenon which yields the attraction (negative pressure) between two neutral plates and which is known as the Casimir effect [11], see review in [12]. The conventional Casimir effect is due to vacuum fluctuations of the quantized electromagnetic field obeying certain boundary conditions at the bounding plates, and a choice of boundary conditions is physically motivated by material properties of the plates (for instance, metallic or dielectric, see, e.g., [12]). Such a motivation is lacking for the case of vacuum fluctuations of the quantized spinor matter field. That is why there is a necessity in the last case to start from the most general set of mathematically acceptable (i.e. compatible with the self-adjointness) boundary conditions. Further follow physical constraints that the spinor matter be confined within the plates and that the spectrum of the wave number vector in the direction which is orthogonal to the plates be unambiguously (although implicitly) determined. Employing these mathematical and physical restrictions, I consider the generalized Casimir effect which is due to vacuum fluctuations of the quantized spinor matter field in the presence of the background magnetic field; the pressure from the vacuum onto the bounding plates will be found.

2. Self-adjointness and boundary conditions

Defining a scalar product as $(\tilde\chi,\chi)=\int\limits_\Omega \mathrm{d}^3r\,\tilde\chi^\dagger\chi$, we get, using integration by parts,

$$(\tilde\chi,H\chi)=(H^\dagger\tilde\chi,\chi)-\mathrm{i}\int\limits_{\partial\Omega}\mathrm{d}\mathbf{s}\cdot\tilde\chi^\dagger\boldsymbol{\alpha}\chi, \tag{3}$$

where

$$H = H^\dagger = -\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m \tag{4}$$

is the formal expression for the Dirac hamiltonian operator and $\boldsymbol{\nabla}$ is the covariant derivative involving both the affine and bundle connections (natural units $\hbar = c = 1$ are used). Operator H is Hermitian (or symmetric in mathematical parlance),

$$(\tilde{\chi}, H\chi) = (H^\dagger\tilde{\chi}, \chi), \tag{5}$$

if

$$\int_{\partial\Omega} \mathrm{d}\mathbf{s}\cdot\tilde{\chi}^\dagger\boldsymbol{\alpha}\chi = 0. \tag{6}$$

The latter condition can be satisfied in various ways by imposing different boundary conditions for χ and $\tilde{\chi}$. However, among the whole variety, there may exist a possibility that a boundary condition for $\tilde{\chi}$ is the same as that for χ; then the domain of definition of $H^\dagger$ (set of functions $\tilde{\chi}$) coincides with that of H (set of functions χ), and operator H is self-adjoint. The action of a self-adjoint operator results in functions belonging to its domain of definition only, and, therefore, a multiple action and functions of such an operator can be consistently defined.

Condition (6) is certainly fulfilled when the integrand in (6) vanishes, i.e.

$$\tilde{\chi}^\dagger(\mathbf{n}\cdot\boldsymbol{\alpha})\chi|_{\mathbf{r}\in\partial\Omega} = 0. \tag{7}$$

To fulfill the latter condition, we impose the same boundary condition for χ and $\tilde{\chi}$ in the form

$$\chi|_{\mathbf{r}\in\partial\Omega} = K\chi|_{\mathbf{r}\in\partial\Omega}, \quad \tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = K\tilde{\chi}|_{\mathbf{r}\in\partial\Omega}, \tag{8}$$

where K is a matrix (element of the Dirac-Clifford algebra) which is determined by two conditions:

$$K^2 = I \tag{9}$$

and

$$K^\dagger(\mathbf{n}\cdot\boldsymbol{\alpha})K = -\mathbf{n}\cdot\boldsymbol{\alpha}. \tag{10}$$

It should be noted that, in addition to (7), the following combination of χ and $\tilde{\chi}$ is also vanishing at the boundary:

$$\tilde{\chi}^\dagger(\mathbf{n}\cdot\boldsymbol{\alpha})K\chi|_{\mathbf{r}\in\partial\Omega} = \tilde{\chi}^\dagger K^\dagger(\mathbf{n}\cdot\boldsymbol{\alpha})\chi|_{\mathbf{r}\in\partial\Omega} = 0. \tag{11}$$

Using the standard representation for the Dirac matrices,

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \tag{12}$$

(σ^1, σ^2 and σ^3 are the Pauli matrices), one can get

$$K = \begin{pmatrix} 0 & \varrho^{-1} \\ \varrho & 0 \end{pmatrix}, \tag{13}$$

where condition

$$(\mathbf{n}\cdot\boldsymbol{\sigma})\varrho = -\varrho^\dagger(\mathbf{n}\cdot\boldsymbol{\sigma}) \tag{14}$$

defines ϱ as a rank-2 matrix depending on four arbitrary parameters [9]. An explicit form for matrix K is [10]

$$K = \frac{(1+u^2-v^2-\mathbf{t}^2)\beta + (1-u^2+v^2+\mathbf{t}^2)I}{2\mathrm{i}(u^2-v^2-\mathbf{t}^2)}(u\mathbf{n}\cdot\boldsymbol{\alpha} + v\beta\gamma^5 - \mathrm{i}\mathbf{t}\cdot\boldsymbol{\alpha}), \tag{15}$$

where $\gamma^5 = \mathrm{i}\alpha^1\alpha^2\alpha^3$, and $\mathbf{t} = (t^1, t^2)$ is a two-dimensional vector which is tangential to the boundary, $\mathbf{t}\cdot\mathbf{n} = 0$. Matrix K is Hermitian in two cases only when it takes forms

$$K_+ = -\mathrm{i}\beta(\mathbf{n}\cdot\boldsymbol{\alpha}) \quad (u = 1, \quad v = 0, \quad \mathbf{t} = 0) \tag{16}$$

and

$$K_- = \mathrm{i}v\beta\gamma^5 + \mathbf{t}\cdot\boldsymbol{\alpha} \quad (u = 0, \quad v^2 + \mathbf{t}^2 = 1). \tag{17}$$

Matrix K_+ (16) corresponds to the choice of the standard MIT bag boundary condition [8], cf. (2),

$$(I - K_+)\chi|_{\mathbf{r}\in\partial\Omega} = (I - K_+)\tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = 0, \tag{18}$$

when relation (11) takes form

$$\tilde{\chi}^\dagger\beta\chi|_{\mathbf{r}\in\partial\Omega} = 0. \tag{19}$$

It is instructive to go over from off-diagonal matrix K (15) to Hermitian matrix $\tilde{K}$, presenting boundary condition (8) as

$$\chi|_{\mathbf{r}\in\partial\Omega} = \tilde{K}\chi|_{\mathbf{r}\in\partial\Omega}, \quad \tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = \tilde{K}\tilde{\chi}|_{\mathbf{r}\in\partial\Omega}, \tag{20}$$

with $\tilde{K} = \tilde{K}^\dagger$ determined by conditions

$$\tilde{K}^2 = I \tag{21}$$

and

$$[\tilde{K}, \mathbf{n}\cdot\boldsymbol{\alpha}]_+ = 0. \tag{22}$$

This transition is implemented with the use of the block-diagonal Hermitian matrix,

$$N = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \qquad \nu_1^\dagger = \nu_1, \qquad \nu_2^\dagger = \nu_2, \tag{23}$$

obeying condition

$$(I - N)K = K^\dagger(I - N); \tag{24}$$

namely, the result is

$$\tilde{K} = (I - N)K + N. \tag{25}$$

Using parametrization

$$u = -\frac{\sin\tilde{\varphi}}{\cos\varphi\cos\theta + \cos\tilde{\varphi}}, \quad v = \frac{\sin\varphi\cos\theta}{\cos\varphi\cos\theta + \cos\tilde{\varphi}},$$

$$t^1 = \frac{\sin\theta\cos\eta}{\cos\varphi\cos\theta + \cos\tilde{\varphi}}, \quad t^2 = \frac{\sin\theta\sin\eta}{\cos\varphi\cos\theta + \cos\tilde{\varphi}},$$

$$-\pi/2 < \varphi \le \pi/2, \quad -\pi/2 \le \tilde{\varphi} < \pi/2, \quad 0 \le \theta < \pi, \quad 0 \le \eta < 2\pi, \tag{26}$$

one gets

$$K = \mathrm{i}\frac{\beta\cos\varphi\cos\theta + I\cos\tilde{\varphi}}{\cos^2\varphi\cos^2\theta - \cos^2\tilde{\varphi}}[\mathbf{n}\cdot\boldsymbol{\alpha}\sin\tilde{\varphi} - \beta\gamma^5\sin\varphi\cos\theta + \mathrm{i}(\alpha^1\cos\eta + \alpha^2\sin\eta)\sin\theta)], \tag{27}$$

where

$$[\mathbf{n}\cdot\boldsymbol{\alpha},\, \alpha^1]_+ = [\mathbf{n}\cdot\boldsymbol{\alpha},\, \alpha^2]_+ = [\alpha^1, \alpha^2]_+ = 0. \tag{28}$$

Then matrix N takes form

$$N = \beta \cos\varphi \cos\tilde{\varphi} \cos\theta - \beta\gamma^5 (\mathbf{n}\cdot\boldsymbol{\alpha}) \sin\varphi \sin\tilde{\varphi} \cos\theta + \mathrm{i}(\alpha^1 \cos\eta + \alpha^2 \sin\eta)(\mathbf{n}\cdot\boldsymbol{\alpha}) \sin\tilde{\varphi} \sin\theta, \quad (29)$$

and one gets

$$\tilde{K} = [\beta \mathrm{e}^{\mathrm{i}\varphi\gamma^5} \cos\theta + (\alpha^1 \cos\eta + \alpha^2 \sin\eta) \sin\theta] \mathrm{e}^{\mathrm{i}\tilde{\varphi}\mathbf{n}\cdot\boldsymbol{\alpha}}; \quad (30)$$

in particular,

$$K_+ = \tilde{K}|_{\varphi=0,\,\tilde{\varphi}=-\pi/2,\,\theta=0}, \quad K_- = \tilde{K}|_{\varphi=\pi/2,\,\tilde{\varphi}=0}. \quad (31)$$

Thus, the explicit form of the boundary condition ensuring the self-adjointness of operator H (4) is

$$\left\{I - [\beta \mathrm{e}^{\mathrm{i}\varphi\gamma^5} \cos\theta + (\alpha^1 \cos\eta + \alpha^2 \sin\eta) \sin\theta] \mathrm{e}^{\mathrm{i}\tilde{\varphi}\mathbf{n}\cdot\boldsymbol{\alpha}}\right\} \chi|_{\mathbf{r}\in\partial\Omega} = 0 \quad (32)$$

(the same condition is for $\tilde{\chi}$), and relation (11) takes form

$$\tilde{\chi}^\dagger [\beta \mathrm{e}^{\mathrm{i}\varphi\gamma^5} \cos\theta + (\alpha^1 \cos\eta + \alpha^2 \sin\eta) \sin\theta] \mathrm{e}^{\mathrm{i}(\tilde{\varphi}+\pi/2)\mathbf{n}\cdot\boldsymbol{\alpha}} \chi|_{\mathbf{r}\in\partial\Omega} = 0. \quad (33)$$

Four parameters of boundary condition (32), $\varphi, \tilde{\varphi}, \theta$ and η, can be interpreted as the self-adjoint extension parameters. It should be emphasized that the values of these parameters vary in general from point to point of the boundary. In this respect the "number" of self-adjoint extension parameters is in fact infinite, moreover, it is not countable but is of power of a continuum. This distinguishes the case of an extended boundary from the case of an excluded point (contact interaction), when the number of self-adjoint extension parameters is finite, being equal to n^2 for the deficiency index equal to $\{n, n\}$ (see, e.g., [1]).

In the context of the Casimir effect, one usually considers spatial region Ω with a disconnected boundary consisting of two connected components, $\partial\Omega = \partial\Omega^{(+)} \bigcup \partial\Omega^{(-)}$. Choosing coordinates $\mathbf{r} = (x, y, z)$ in such a way that x and y are tangential to the boundary, while z is normal to it, we identify the position of $\partial\Omega^{(\pm)}$ with, say, $z = \pm a/2$. In general, there are 8 self-adjoint extension parameters: φ_+, $\tilde{\varphi}_+$, θ_+ and η_+ corresponding to $\partial\Omega^{(+)}$ and φ_-, $\tilde{\varphi}_-$, θ_- and η_- corresponding to $\partial\Omega^{(-)}$. However, if some symmetry is present, then the number of self-adjoint extension parameters is diminished. For instance, if the boundary consists of two parallel planes, then the cases differing by the values of η_+ or η_- are physically indistinguishable, since they are related by a rotation around a normal to the boundary. To avoid this unphysical degeneracy, one has to fix

$$\theta_+ = \theta_- = 0, \quad (34)$$

and there remains 4 self-adjoint extension parameters: φ_+, $\tilde{\varphi}_+$, φ_- and $\tilde{\varphi}_-$. Operator H (4) acting on functions which are defined in the region bounded by two parallel planes is self-adjoint, if the following condition holds:

$$\left\{I - \beta \exp[\mathrm{i}(\varphi_\pm \gamma^5 \pm \tilde{\varphi}_\pm \alpha^z)]\right\} \chi|_{z=\pm a/2} = 0 \quad (35)$$

(the same condition holds for $\tilde{\chi}$). The latter ensures the fulfilment of constraints

$$\tilde{\chi}^\dagger \alpha^z \chi|_{z=\pm a/2} = 0 \quad (36)$$

and

$$\tilde{\chi}^\dagger \beta \exp\left\{\mathrm{i}[\varphi_\pm \gamma^5 \pm (\tilde{\varphi}_\pm + \pi/2)\alpha^z]\right\} \chi|_{z=\pm a/2} = 0. \quad (37)$$

3. Induced vacuum energy in the magnetic field background

The operator of a spinor field which is quantized in an ultrastatic background is presented in the form

$$\hat{\Psi}(t,\mathbf{r}) = \sum\!\!\!\!\!\!\int_{E_\lambda>0} \mathrm{e}^{-\mathrm{i}E_\lambda t}\psi_\lambda(\mathbf{r})\hat{a}_\lambda + \sum\!\!\!\!\!\!\int_{E_\lambda<0} \mathrm{e}^{-\mathrm{i}E_\lambda t}\psi_\lambda(\mathbf{r})\hat{b}^\dagger_\lambda, \tag{38}$$

where $\hat{a}^\dagger_\lambda$ and $\hat{a}_\lambda$ ($\hat{b}^\dagger_\lambda$ and $\hat{b}_\lambda$) are the spinor particle (antiparticle) creation and destruction operators, satisfying anticommutation relations $[\hat{a}_\lambda, \hat{a}^\dagger_{\lambda'}]_+ = [\hat{b}_\lambda, \hat{b}^\dagger_{\lambda'}]_+ = \langle\lambda|\lambda'\rangle$, wave functions $\psi_\lambda(\mathbf{r})$ form a complete set of solutions to the stationary Dirac equation

$$H\psi_\lambda(\mathbf{r}) = E_\lambda\psi_\lambda(\mathbf{r}); \tag{39}$$

λ is the set of parameters (quantum numbers) specifying a one-particle state with energy E_λ; symbol $\sum\!\!\!\!\!\int$ denotes summation over discrete and integration (with a certain measure) over continuous values of λ. Ground state $|\mathtt{vac}>$ is defined by condition $\hat{a}_\lambda|\mathtt{vac}>= \hat{b}_\lambda|\mathtt{vac}>= 0$. The temporal component of the operator of the energy-momentum tensor is given by expression

$$\hat{T}^{00} = \frac{\mathrm{i}}{4}[\hat{\Psi}^\dagger(\partial_0\hat{\Psi}) - (\partial_0\hat{\Psi}^T)\hat{\Psi}^{\dagger T} - (\partial_0\hat{\Psi}^\dagger)\hat{\Psi} + \hat{\Psi}^T(\partial_0\hat{\Psi}^{\dagger T})], \tag{40}$$

where superscript T denotes a transposed spinor. Consequently, the formal expression for the vacuum expectation value of the energy density is

$$\varepsilon =< \mathtt{vac}|\hat{T}^{00}|\mathtt{vac} >= -\frac{1}{2}\sum\!\!\!\!\!\!\int |E_\lambda|\psi^\dagger_\lambda(\mathbf{r})\psi_\lambda(\mathbf{r}). \tag{41}$$

Let us consider the quantized charged massive spinor field in the background of a static uniform magnetic field; then $\boldsymbol{\nabla} = \boldsymbol{\partial} - \mathrm{i}e\mathbf{A}$ and the connection can be chosen as $\mathbf{A} = (-yB, 0, 0)$, where B is the value of the magnetic field strength which is directed along the z-axis in Cartesian coordinates $\mathbf{r} = (x, y, z)$, $\mathbf{B} = (0, 0, B)$. The one-particle energy spectrum is

$$E_{nk} = \pm\omega_{nk}, \tag{42}$$

where

$$\omega_{nk} = \sqrt{2n|eB| + k^2 + m^2},\ -\infty < k < \infty,\ n = 0, 1, 2, \ldots, \tag{43}$$

k is the value of the wave number vector along the z-axis, and n labels the Landau levels. Using the explicit form of the complete set of solutions to the Dirac equation, one can get that expression (41) takes form

$$\varepsilon^\infty = -\frac{|eB|}{2\pi^2}\int\limits_{-\infty}^{\infty} \mathrm{d}k \sum_{n=0}^{\infty} \iota_n\omega_{nk}, \tag{44}$$

where the superscript on the left-hand side indicates that the magnetic field fills the whole (infinite) space; the appearance of factor $\iota_n = 1 - \frac{1}{2}\delta_{n0}$ on the right-hand side is due to the fact that there is one solution for the lowest Landau level, $\psi^{(0)}_{q0k}(\mathbf{r})$ (q is the value of the wave number vector along the x-axis, $-\infty < q < \infty$), and there are two solutions otherwise, $\psi^{(j)}_{qnk}(\mathbf{r})$ ($j = 1, 2$), $n \geq 1$. The integral and the sum in (44) are divergent and require regularization and renormalization. This problem has been solved long ago by Heisenberg and Euler [3] (see also [4]), and we just list here their result

$$\varepsilon^\infty_{\mathrm{ren}} = \frac{1}{8\pi^2}\int\limits_0^\infty \frac{\mathrm{d}\tau}{\tau}\mathrm{e}^{-\tau}\left[\frac{eBm^2}{\tau}\coth\left(\frac{eB\tau}{m^2}\right) - \frac{m^4}{\tau^2} - \frac{1}{3}e^2B^2\right]; \tag{45}$$

note that the renormalization procedure involves subtraction at $B = 0$ and renormalization of the charge.

Let us turn now to the quantized charged massive spinor field in the background of a static uniform magnetic field in spatial region Ω bounded by two parallel planes $\partial\Omega^{(+)}$ and $\partial\Omega^{(-)}$; the position of $\partial\Omega^{(\pm)}$ is identified with $z = \pm a/2$, and the magnetic field is orthogonal to the boundary. The solution to (39) in region Ω is chosen as a superposition of two plane waves propagating in opposite directions along the z-axis,

$$\psi^{(j)}_{qnl}(\mathbf{r}) = \psi^{(j)}_{qnk_l}(\mathbf{r}) + \psi^{(j)}_{qn-k_l}(\mathbf{r}), \quad j = 0, 1, 2, \tag{46}$$

where the values of wave number vector k_l ($l = 0, \pm1, \pm2, ...$) are determined from the boundary condition, see (35):

$$\left\{I - \beta \exp[\mathrm{i}(\varphi_\pm \gamma^5 \pm \tilde{\varphi}_\pm \alpha^z)]\right\} \psi^{(j)}_{qnl}(\mathbf{r})|_{z=\pm a/2} = 0, \quad (j = 1, 2) \quad n \geq 1 \tag{47}$$

and

$$\left[I + \frac{\beta}{2}\left(\pm\alpha^z\gamma^5 - 1\right)\mathrm{e}^{\mathrm{i}(\varphi_\pm - \tilde{\varphi}_\pm)\gamma^5}\Theta(\pm eB) - \frac{\beta}{2}\left(\pm\alpha^z\gamma^5 + 1\right)\mathrm{e}^{\mathrm{i}(\varphi_\pm + \tilde{\varphi}_\pm)\gamma^5}\Theta(\mp eB)\right]\psi^{(0)}_{q0l}(\mathbf{r})|_{z=\pm a/2} = 0; \tag{48}$$

the step function is introduced as $\Theta(u) = 1$ at $u > 0$ and $\Theta(u) = 0$ at $u < 0$. This boundary condition ensures that the normal component of current $\mathbf{J}_{qnlj}(\mathbf{r}) = \psi^{(j)\dagger}_{qnl}(\mathbf{r})\boldsymbol{\alpha}\psi^{(j)}_{qnl}(\mathbf{r})$ ($j = 0, 1, 2$) vanishes at the boundary, see (36),

$$J^z_{qnlj}(\mathbf{r})|_{z=\pm a/2} = 0, \tag{49}$$

which, cf. (1), signifies that the quantized matter is confined within the boundaries.

The boundary condition depends on four self-adjoint extension parameters, $\varphi_+, \tilde{\varphi}_+, \varphi_-$ and $\tilde{\varphi}_-$, in the case of $n \geq 1$, see (47), and on two self-adjoint extension parameters, $\varphi_+ - \tilde{\varphi}_+$ and $\varphi_- + \tilde{\varphi}_-$ ($eB > 0$), or $\varphi_+ + \tilde{\varphi}_+$ and $\varphi_- - \tilde{\varphi}_-$ ($eB < 0$), in the case of $n = 0$, see (48). As was mentioned in the previous section, the values of these self-adjoint extension parameters may vary arbitrarily from point to point of the bounding planes. However, in the context of the Casimir effect, such a generality seems to be excessive, lacking physical motivation and, moreover, being impermissible, as long as boundary condition (47)-(48) is to be regarded as the one determining the spectrum of the wave number vector in the z-direction. Therefore, we shall assume in the following that the self-adjoint extension parameters are independent of coordinates x and y.

It should be noted that value $k_l = 0$ is allowed for special cases only. Really, we have in the case of $k_l = 0$:

$$\psi^{(j)}_{qnl}(\mathbf{r})|_{z=a/2} = \psi^{(j)}_{qnl}(\mathbf{r})|_{z=-a/2}, \tag{50}$$

and boundary condition (47)-(48) can be presented in the form

$$R\,\psi^{(j)}_{qnl}(\mathbf{r})|_{k_l=0} = 0, \tag{51}$$

where

$$\left\{\begin{array}{l} R_{11} = \sin\frac{\varphi_+ - \tilde{\varphi}_+}{2}, \quad R_{12} = 0, \quad R_{13} = \mathrm{i}\cos\frac{\varphi_+ - \tilde{\varphi}_+}{2}, \quad R_{14} = 0, \\ R_{21} = 0, \quad R_{22} = \sin\frac{\varphi_+ + \tilde{\varphi}_+}{2}, \quad R_{23} = 0, \quad R_{24} = \mathrm{i}\cos\frac{\varphi_+ + \tilde{\varphi}_+}{2}, \\ R_{31} = \sin\frac{\varphi_- + \tilde{\varphi}_-}{2}, \quad R_{32} = 0, \quad R_{33} = \mathrm{i}\cos\frac{\varphi_- + \tilde{\varphi}_-}{2}, \quad R_{34} = 0, \\ R_{41} = 0, \quad R_{42} = \sin\frac{\varphi_- - \tilde{\varphi}_-}{2}, \quad R_{43} = 0, \quad R_{44} = \mathrm{i}\cos\frac{\varphi_- - \tilde{\varphi}_-}{2} \end{array}\right\}. \tag{52}$$

The determinant of matrix R is:

$$\det R = -\sin\frac{\varphi_+ - \varphi_- + \tilde{\varphi}_+ + \tilde{\varphi}_-}{2}\sin\frac{\varphi_+ - \varphi_- - \tilde{\varphi}_+ - \tilde{\varphi}_-}{2}. \tag{53}$$

The necessary and sufficient condition for value $k_l = 0$ to be admissible is $\det R = 0$. Otherwise, $\det R \neq 0$ and value $k_l = 0$ is excluded from the spectrum, because equation (51) then allows for the trivial solution only, $\psi^{(j)}_{qnl}(\mathbf{r})|_{k_l=0} \equiv 0$.

The spectrum of k_l is determined from a transcendental equation which in general possesses two branches and allows for complex values of k_l (details will be published elsewhere). It is not clear which of the branches should be chosen, and, therefore, we restrict ourselves to boundary conditions corresponding to the case of a single branch. The latter is ensured by imposing constraint

$$\varphi_+ = \varphi_- = \varphi, \quad \tilde{\varphi}_+ = \tilde{\varphi}_- = \tilde{\varphi}. \tag{54}$$

Then relations (35) and (37) take forms

$$\left\{ I - \beta \exp[\mathrm{i}(\varphi\gamma^5 \pm \tilde{\varphi}\alpha^z)] \right\} \chi|_{z=\pm a/2} = 0 \tag{55}$$

and

$$\tilde{\chi}^\dagger \beta \exp\left\{ \mathrm{i}[\varphi\gamma^5 \pm (\tilde{\varphi} + \pi/2)\alpha^z] \right\} \chi|_{z=\pm a/2} = 0. \tag{56}$$

respectively, while the equation determining the spectrum of k_l takes form

$$\cos(k_l a) + \frac{\omega_{nk_l}\, \mathrm{sgn}(E_{nk_l}) \cos\tilde{\varphi} - m\cos\varphi}{k_l \sin\tilde{\varphi}} \sin(k_l a) = 0, \tag{57}$$

where $\mathrm{sgn}(u) = \Theta(u) - \Theta(-u)$ is the sign function; note that the spectrum is real, consisting of values of the same sign, say, $k_l > 0$ (values of the opposite sign ($k_l < 0$) should be excluded to avoid double counting).

In the case of $\tilde{\varphi} = -\pi/2$, the spectrum of k_l is independent of the number of the Landau level, n, and of the sign of the one-particle energy, $\mathrm{sgn}(E_{nk_l})$; it is determined from equation

$$\cos(k_l a) + \frac{m\cos\varphi}{k_l} \sin(k_l a) = 0. \tag{58}$$

In the case of $\tilde{\varphi} = 0$, the k_l-spectrum is also independent of n and of $\mathrm{sgn}(E_{nk_l})$; moreover, it is independent of φ, since the determining equation takes form

$$\sin(k_l a) = 0; \tag{59}$$

note that value $k_l = 0$ is admissible in this case, see (53)-(54). In what follows, we shall consider the most general case of two self-adjoint extension parameters, φ and $\tilde{\varphi}$, when the k_l-spectrum depends on n and on $\mathrm{sgn}(E_{nk_l})$, see (57).

Wave functions $\psi^{(j)}_{qnl}(\mathbf{r})$ $(j = 0, 1, 2)$ satisfy the requirements of orthonormality

$$\int_\Omega \mathrm{d}^3 r\, \psi^{(j)\dagger}_{qnl}(\mathbf{r}) \psi^{(j')}_{q'n'l'}(\mathbf{r}) = \delta_{jj'}\delta_{nn'}\delta_{ll'}\delta(q - q'), \quad j, j' = 0, 1, 2 \tag{60}$$

and completeness

$$\sum_{\mathrm{sgn}(E_{nk_l})} \int_{-\infty}^{\infty} \mathrm{d}q \sum_l \left[\psi^{(0)}_{q0l}(\mathbf{r}) \psi^{(0)\dagger}_{q0l}(\mathbf{r}') + \sum_{n=1}^{\infty} \sum_{j=1,2} \psi^{(j)}_{qnl}(\mathbf{r}) \psi^{(j)\dagger}_{qnl}(\mathbf{r}') \right] = I\delta(\mathbf{r} - \mathbf{r}'). \tag{61}$$

Consequently, we obtain the following expression for the vacuum expectation value of the energy per unit area of the boundary surface

$$\frac{E}{S} \equiv \int_{-a/2}^{a/2} \mathrm{d}z\, \varepsilon = -\frac{|eB|}{2\pi} \sum_{\mathrm{sgn}(E_{nk_l})} \sum_l \sum_{n=0}^{\infty} \iota_n \omega_{nk_l}. \tag{62}$$

4. Casimir energy and force

Expression (62) can be regarded as purely formal, since it is ill-defined due to the divergence of infinite sums over l and n. To tame the divergence, a factor containing a regularization parameter should be inserted in (62). A summation over values $k_l \geq 0$, which are determined by (57), can be performed with the use of the Abel-Plana formula and its generalizations. In the cases of $\tilde{\varphi} = 0$ and of $\varphi = -\tilde{\varphi} = \pi/2$, the well-known versions of the Abel-Plana formula (see, e.g., [12]),

$$\sum_{\mathrm{sgn}(E_{nk_l})} \sum_{k_l \geq 0} f(k_l^2) \Bigg|_{\sin(k_l a)=0} = \frac{a}{\pi} \int_{-\infty}^{\infty} \mathrm{d}k f(k^2) - \frac{2\mathrm{i}a}{\pi} \int_0^{\infty} \mathrm{d}\kappa \frac{f[(-\mathrm{i}\kappa)^2] - f[(\mathrm{i}\kappa)^2]}{\mathrm{e}^{2\kappa a} - 1} + f(0) \tag{63}$$

and

$$\sum_{\mathrm{sgn}(E_{nk_l})} \sum_{k_l > 0} f(k_l^2) \Bigg|_{\cos(k_l a)=0} = \frac{a}{\pi} \int_{-\infty}^{\infty} \mathrm{d}k f(k^2) + \frac{2\mathrm{i}a}{\pi} \int_0^{\infty} \mathrm{d}\kappa \frac{f[(-\mathrm{i}\kappa)^2] - f[(\mathrm{i}\kappa)^2]}{\mathrm{e}^{2\kappa a} + 1}, \tag{64}$$

are used, respectively. Otherwise, we use the following version of the Abel-Plana formula:

$$\sum_{\mathrm{sgn}(E_{nk_l})} \sum_{k_l > 0} f(k_l^2) = \frac{a}{\pi} \int_{-\infty}^{\infty} \mathrm{d}k f(k^2) + \frac{2\mathrm{i}a}{\pi} \int_0^{\infty} \mathrm{d}\kappa \Lambda(\kappa) \{ f[(-\mathrm{i}\kappa)^2] - f[(\mathrm{i}\kappa)^2] \}$$

$$-f(0) - \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}k f(k^2) \frac{m \cos\varphi \sin\tilde{\varphi} [k^2 - \mu_n(\varphi, \tilde{\varphi})]}{[k^2 + \mu_n(\varphi, \tilde{\varphi})]^2 + 4k^2 m^2 \cos^2\varphi \sin^2\tilde{\varphi}}, \tag{65}$$

where

$$\Lambda(\kappa) = \Bigg(-[\kappa^2 \cos 2\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})] \mathrm{e}^{2\kappa a} + \kappa^2 + 2\kappa m \cos\varphi \sin\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})$$

$$+ \frac{\sin\tilde{\varphi}}{a} \Big\{ -\kappa^2 m \cos\varphi (\cos 2\tilde{\varphi} \mathrm{e}^{2\kappa a} - 1) + [(2\kappa \sin\tilde{\varphi} - m \cos\varphi) \mathrm{e}^{2\kappa a}$$

$$+ m \cos\varphi] \mu_n(\varphi, \tilde{\varphi}) \Big\} [\kappa^2 - 2\kappa m \cos\varphi \sin\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})]^{-1} \Bigg)$$

$$\times \Big\{ [\kappa^2 - 2\kappa m \cos\varphi \sin\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})] \mathrm{e}^{4\kappa a}$$

$$-2[\kappa^2 \cos 2\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})] \mathrm{e}^{2\kappa a} + \kappa^2 + 2\kappa m \cos\varphi \sin\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi}) \Big\}^{-1} \tag{66}$$

and

$$\mu_n(\varphi, \tilde{\varphi}) = 2n|eB| \cos^2\tilde{\varphi} + m^2 \sin(\varphi + \tilde{\varphi}) \sin(\varphi - \tilde{\varphi}). \tag{67}$$

In (63)-(65), $f(u^2)$ as a function of complex variable u is assumed to decrease sufficiently fast at large distances from the origin of the complex u-plane, and this decrease is due to the use of some kind of regularization for (62). However, the regularization in the second integral on the right-hand side of (63)-(65) can be removed; then

$$\mathrm{i}\{ f[(-\mathrm{i}\kappa)^2] - f[(\mathrm{i}\kappa)^2] \} = -\frac{|eB|}{\pi} \sum_{n=0}^{\infty} \iota_n \sqrt{\kappa^2 - \omega_{n0}^2} \tag{68}$$

with the range of κ restricted to $\kappa > \omega_{n0}$ for the corresponding terms; here, recalling (43), $\omega_{n0} = \sqrt{2n|eB| + m^2}$. As to the first integral on the right-hand side of (63)-(65), one immediately recognizes that it is equal to ε^∞ (44) multiplied by a. Hence, if one ignores for a moment the last term of (63), as well as the terms in the last line of (65), then the problem of regularization and removal of the divergency in expression (62) is the same as that in the case of no boundaries, when the magnetic field fills the whole space. Defining the Casimir energy as the vacuum energy per unit area of the boundary surface, which is renormalized in the same way as in the case of no boundaries, we obtain

$$\frac{E_{\rm ren}}{S} = a\varepsilon^\infty_{\rm ren} - \frac{2|eB|}{\pi^2} a \sum_{n=0}^{\infty} \iota_n \int\limits_{\omega_{n0}}^{\infty} {\rm d}\kappa \Lambda(\kappa) \sqrt{\kappa^2 - \omega_{n0}^2}$$

$$+\frac{|eB|}{2\pi} \sum_{n=0}^{\infty} \iota_n \omega_{n0} + \frac{|eB|}{2\pi^2} \int\limits_{-\infty}^{\infty} {\rm d}k \sum_{n=0}^{\infty} \iota_n \sqrt{k^2 + \omega_{n0}^2} \frac{m\cos\varphi \sin\tilde{\varphi}[k^2 - \mu_n(\varphi,\tilde{\varphi})]}{[k^2 + \mu_n(\varphi,\tilde{\varphi})]^2 + 4k^2 m^2 \cos^2\varphi \sin^2\tilde{\varphi}}, \qquad (69)$$

$\varepsilon^\infty_{\rm ren}$ is given by (45). The sums and the integral in the last line of (69) (which are due to the terms in the last line of (65) and which can be interpreted as describing the proper energies of the boundary planes containing the sources of the magnetic field) are divergent, but this divergency is of no concern for us, because it has no physical consequences. Rather than the Casimir energy, a physically relevant quantity is the Casimir force per unit area of the boundary surface, i.e. pressure, which is defined as

$$F = -\frac{\partial}{\partial a} \frac{E_{\rm ren}}{S}, \qquad (70)$$

and which is free from divergencies. We obtain

$$F = -\varepsilon^\infty_{\rm ren} - \frac{2|eB|}{\pi^2} \sum_{n=0}^{\infty} \iota_n \int\limits_{\omega_{n0}}^{\infty} {\rm d}\kappa \Upsilon(\kappa) \sqrt{\kappa^2 - \omega_{n0}^2}, \qquad (71)$$

where

$$\Upsilon(\kappa) \equiv -\frac{\partial}{\partial a} a\Lambda(\kappa) = \Big[\upsilon_1(\kappa){\rm e}^{6\kappa a} + \upsilon_2(\kappa){\rm e}^{4\kappa a} + \upsilon_3(\kappa){\rm e}^{2\kappa a} + \upsilon_4(\kappa)\Big]$$
$$\times \Big\{[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]{\rm e}^{4\kappa a}$$
$$-2[\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]{\rm e}^{2\kappa a} + \kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})\Big\}^{-2} \qquad (72)$$

and

$$\upsilon_1(\kappa) = -(2\kappa a - 1)[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})][\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]$$
$$-2[\kappa^2 m\cos\varphi\cos 2\tilde{\varphi} - (2\kappa\sin\tilde{\varphi} - m\cos\varphi)\mu_n(\varphi,\tilde{\varphi})]\kappa\sin\tilde{\varphi}, \qquad (73)$$
$$\upsilon_2(\kappa) = (4\kappa a - 3)\Big\{[\kappa^2 - \mu_n(\varphi,\tilde{\varphi})]^2 - 4\kappa^2 m^2\cos^2\varphi\sin^2\tilde{\varphi}\Big\}$$
$$+8\kappa^2[\kappa^2\cos^2\tilde{\varphi} - m^2\cos^2\varphi - \mu_n(\varphi,\tilde{\varphi})]\sin^2\tilde{\varphi} + 4[\kappa^2 + \mu_n(\varphi,\tilde{\varphi})]\kappa m\cos\varphi\sin\tilde{\varphi}, \qquad (74)$$
$$\upsilon_3(\kappa) = -(2\kappa a - 3)[\kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})][\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]$$
$$-2[\kappa^2 m\cos\varphi\cos 2\tilde{\varphi} + (2\kappa\sin\tilde{\varphi} + m\cos\varphi)\mu_n(\varphi,\tilde{\varphi})]\kappa\sin\tilde{\varphi}, \qquad (75)$$
$$\upsilon_4(\kappa) = -[\kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]^2. \qquad (76)$$

In the case of $\tilde{\varphi} = -\pi/2$, relations (72)-(76) are simplified:

$$\Upsilon(\kappa)|_{\tilde{\varphi}=-\pi/2} = \frac{\left[(2\kappa a - 1)\left(\kappa^2 - m^2\cos^2\varphi\right) - 2\kappa m\cos\varphi\right]{\rm e}^{2\kappa a} - (\kappa - m\cos\varphi)^2}{\left[(\kappa + m\cos\varphi)\,{\rm e}^{2\kappa a} + \kappa - m\cos\varphi\right]^2}. \qquad (77)$$

This case was exhaustively studied earlier [10] in a different parametrization corresponding to substitution $\cos\varphi \to 1/\cosh\vartheta$; we remind here that the case of the MIT bag boundary condition is obtainable at $\varphi = 0$ or $\vartheta = 0$, respectively.

In the case of $\tilde{\varphi} = 0$, we obtain

$$\Upsilon(\kappa)|_{\tilde{\varphi}=0} = -\frac{(2\kappa a - 1)\,\mathrm{e}^{2\kappa a} + 1}{(\mathrm{e}^{2\kappa a} - 1)^2}. \tag{78}$$

The spectrum of k_l in this case is explicitly given by $k_l = \frac{\pi}{a} l$ $(l = 0, 1, 2, ...)$, and the Casimir pressure takes form

$$F|_{\tilde{\varphi}=0} = -\varepsilon_{\text{ren}}^{\infty} + \frac{2|eB|}{\pi^2} \sum_{n=0}^{\infty} \iota_n \int_{\omega_{n0}}^{\infty} \frac{\mathrm{d}\kappa}{\mathrm{e}^{2\kappa a} - 1} \frac{\kappa^2}{\sqrt{\kappa^2 - \omega_{n0}^2}}, \tag{79}$$

or, in the alternative representation,

$$F|_{\tilde{\varphi}=0} = -\varepsilon_{\text{ren}}^{\infty} + \frac{2|eB|}{\pi^2} \sum_{n=0}^{\infty} \iota_n \omega_{n0}^2 \sum_{j=1}^{\infty} \left[K_0(2j\omega_{n0}a) + \frac{1}{2j\omega_{n0}a} K_1(2j\omega_{n0}a) \right], \tag{80}$$

where $K_\rho(s)$ is the Macdonald function of order ρ.

It is instructive to consider also the case of $\varphi = \pi/2$, when relations (72)-(76) are reduced to

$$\begin{aligned}
\Upsilon(\kappa)|_{\varphi=\pi/2} = -\Bigg\{ & \left[(2\kappa a - 1)\left(1 - \frac{2\kappa^2 \sin^2\tilde{\varphi}}{\kappa^2 - \omega_{n0}^2 \cos^2\tilde{\varphi}}\right) - \frac{\kappa^2 \omega_{n0}^2 \sin^2 2\tilde{\varphi}}{\left(\kappa^2 - \omega_{n0}^2 \cos^2\tilde{\varphi}\right)^2} \right] \mathrm{e}^{6\kappa a} \\
& - \left[4\kappa a - 3 + \frac{2\kappa^2\left(\kappa^2 - \omega_{n0}^2\right)\sin^2 2\tilde{\varphi}}{\left(\kappa^2 - \omega_{n0}^2\cos^2\tilde{\varphi}\right)^2} \right] \mathrm{e}^{4\kappa a} \\
& + \left[(2\kappa a - 3)\left(1 - \frac{2\kappa^2 \sin^2\tilde{\varphi}}{\kappa^2 - \omega_{n0}^2 \cos^2\tilde{\varphi}}\right) + \frac{\kappa^2 \omega_{n0}^2 \sin^2 2\tilde{\varphi}}{\left(\kappa^2 - \omega_{n0}^2 \cos^2\tilde{\varphi}\right)^2} \right] \mathrm{e}^{2\kappa a} + 1 \Bigg\} \\
& \times \left[\left(\mathrm{e}^{2\kappa a} - 1\right)^2 + \frac{4\kappa^2 \sin^2\tilde{\varphi}}{\kappa^2 - \omega_{n0}^2 \cos^2\tilde{\varphi}} \mathrm{e}^{2\kappa a} \right]^{-2}.
\end{aligned} \tag{81}$$

This case interpolates between the case of spectrum $k_l = \frac{\pi}{a} l$ $(l = 0, 1, 2, ...)$, see (78)-(80), and the case of spectrum $k_l = \frac{\pi}{a}(l + \frac{1}{2})$ $(l = 0, 1, 2, ...)$ with

$$\Upsilon(\kappa)|_{\varphi=-\tilde{\varphi}=\pi/2} = \frac{(2\kappa a - 1)\,\mathrm{e}^{2\kappa a} - 1}{(\mathrm{e}^{2\kappa a} + 1)^2} \tag{82}$$

and

$$F|_{\varphi=-\tilde{\varphi}=\pi/2} = -\varepsilon_{\text{ren}}^{\infty} - \frac{2|eB|}{\pi^2} \sum_{n=0}^{\infty} \iota_n \int_{\omega_{n0}}^{\infty} \frac{\mathrm{d}\kappa}{\mathrm{e}^{2\kappa a} + 1} \frac{\kappa^2}{\sqrt{\kappa^2 - \omega_{n0}^2}}, \tag{83}$$

or, alternatively,

$$F|_{\varphi=-\tilde{\varphi}=\pi/2} = -\varepsilon_{\text{ren}}^{\infty} - \frac{2|eB|}{\pi^2} \sum_{n=0}^{\infty} \iota_n \omega_{n0}^2 \sum_{j=1}^{\infty} (-1)^{j-1} \left[K_0(2j\omega_{n0}a) + \frac{1}{2j\omega_{n0}a} K_1(2j\omega_{n0}a) \right]. \tag{84}$$

5. Asymptotics at small and large separations of plates

The expression for the Casimir pressure, see (71), can be presented as

$$F = -\varepsilon_{\rm ren}^{\infty} + \Delta_{\varphi,\tilde\varphi}(a), \tag{85}$$

where the first term is equal to minus the vacuum energy density which is induced by the magnetic field in unbounded space, see (45), whereas the second term which is given by the sum over n and the integral over κ in (71) depends on the distance between bounding plates and on a choice of boundary conditions at the plates.

In the case of a weak magnetic field, $B \ll m^2|e|^{-1}$, substituting the sum by integral $\int\limits_0^\infty {\rm d}n$ and changing the integration variable, we get

$$\Delta_{\varphi,\tilde\varphi}(a) = -\frac{1}{\pi^2}\int\limits_m^\infty {\rm d}\kappa(\kappa^2 - m^2)^{3/2}\int\limits_0^1 {\rm d}v\sqrt{1-v}\,\tilde\Upsilon(\kappa, v), \quad |eB| \ll m^2, \tag{86}$$

where $\tilde\Upsilon(\kappa, v)$ is obtained from $\Upsilon(\kappa)$ (72) by substitution $\mu_n(\varphi,\tilde\varphi) \to \tilde\mu_{v,\kappa^2}(\varphi,\tilde\varphi)$ with

$$\tilde\mu_{v,\kappa^2}(\varphi,\tilde\varphi) = v(\kappa^2 - m^2)\cos^2\tilde\varphi + m^2\sin(\varphi+\tilde\varphi)\sin(\varphi-\tilde\varphi). \tag{87}$$

In the limit of small distances between the plates, $ma \ll 1$, (86) becomes independent of the φ-parameter:

$$\Delta_{\varphi,\tilde\varphi}(a) = \frac{1}{4a^4}\biggl\{\frac{\pi^2}{30} - \int\limits_0^1 {\rm d}v\,\rho_{\tilde\varphi}(v)\biggl(1 - \frac{|\rho_{\tilde\varphi}(v)|}{\pi}\biggr)\biggl[\frac{3}{2}\sqrt{1-v}\,\rho_{\tilde\varphi}(v)\biggl(1 - \frac{|\rho_{\tilde\varphi}(v)|}{\pi}\biggr)$$

$$+\frac{v\sin 2\tilde\varphi}{1 - v\cos^2\tilde\varphi}\biggl(\frac{1}{2} - \frac{|\rho_{\tilde\varphi}(v)|}{\pi}\biggr)\biggr]\biggr\}, \quad \sqrt{|eB|}a \ll ma \ll 1, \tag{88}$$

where

$$\rho_{\tilde\varphi}(v) = \arcsin\biggl(\frac{\sin\tilde\varphi}{\sqrt{1 - v\cos^2\tilde\varphi}}\biggr). \tag{89}$$

Thus, $\Delta_{\varphi,\tilde\varphi}(a)$ in this case is power-dependent on the distance between the plates as a^{-4} with the dimensionless constant of proportionality, either positive or negative, depending on the value of the $\tilde\varphi$-parameter. In particular, we get

$$\Delta_{\varphi,0}(a) = \frac{\pi^2}{120}\frac{1}{a^4}, \quad \sqrt{|eB|}a \ll ma \ll 1 \tag{90}$$

and

$$\Delta_{\varphi,-\pi/2}(a) = -\frac{7}{8}\frac{\pi^2}{120}\frac{1}{a^4}, \quad \sqrt{|eB|}a \ll ma \ll 1. \tag{91}$$

In the limit of large distances between the plates, $ma \gg 1$, $\Delta_{\varphi,\tilde\varphi}(a)$ (86) takes form

$$\Delta_{\varphi,\tilde\varphi}(a) = \frac{2}{\pi^2}\int\limits_m^\infty {\rm d}\kappa\kappa(\kappa^2 - m^2)^{3/2}{\rm e}^{-2\kappa a}\int\limits_0^1 {\rm d}v\sqrt{1-v}\biggl\{a\frac{\kappa^2\cos 2\tilde\varphi - \tilde\mu_{v,\kappa^2}(\varphi,\tilde\varphi)}{\kappa^2 - 2\kappa m\cos\varphi\sin\tilde\varphi - \tilde\mu_{v,\kappa^2}(\varphi,\tilde\varphi)}$$

$$-\frac{(2\kappa\sin\tilde\varphi - m\cos\varphi)\tilde\mu_{v,\kappa^2}(\varphi,\tilde\varphi) - \kappa^2 m\cos\varphi\cos 2\tilde\varphi}{[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde\varphi - \tilde\mu_{v,\kappa^2}(\varphi,\tilde\varphi)]^2}\sin\tilde\varphi\biggr\}, \quad |eB| \ll m^2, \quad ma \gg 1. \tag{92}$$

Clearly, (92) is suppressed as $\exp(-2ma)$. In particular, we get

$$\Delta_{\varphi,0}(a) = \frac{1}{2\pi^{3/2}}\frac{m^{5/2}}{a^{3/2}}\mathrm{e}^{-2ma}\left[1+O\left(\frac{1}{ma}\right)\right], \quad |eB| \ll m^2, \quad ma \gg 1 \tag{93}$$

and

$$\Delta_{\varphi,-\pi/2}(a) = \left\{\begin{array}{ll} -\frac{3}{16\pi^{3/2}}\frac{m^{3/2}}{a^{5/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \varphi = 0 \\ \\ -\frac{\tan^2(\varphi/2)}{2\pi^{3/2}}\frac{m^{5/2}}{a^{3/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \varphi \neq 0 \end{array}\right\}, \quad |eB| \ll m^2, \quad ma \gg 1. \tag{94}$$

In the case of a strong magnetic field, $|B| \gg m^2|e|^{-1}$, one has

$$\Delta_{\varphi,\tilde{\varphi}}(a) = -\frac{|eB|}{\pi^2}\left[\int\limits_m^\infty \mathrm{d}\kappa\sqrt{\kappa^2-m^2}\Upsilon(\kappa)|_{n=0}\right.$$

$$\left. +2\sum_{n=1}^\infty \int\limits_{\sqrt{2n|eB|}}^\infty \mathrm{d}\kappa\sqrt{\kappa^2-2n|eB|}\Upsilon(\kappa)|_{m=0}\right], \quad |eB| \gg m^2. \tag{95}$$

In the limit of extremely small distances between the plates, $ma \ll \sqrt{|eB|}a \ll 1$, the analysis is similar to that of the limit of $\sqrt{|eB|}a \ll ma \ll 1$, yielding the same results as (88)-(91). Otherwise, in the limit of $\sqrt{|eB|}a \gg 1$, only the first term in square brackets on the right-hand side of (95) matters. In the limit of small distances between the plates this term becomes φ-independent, yielding

$$\Delta_{\varphi,\tilde{\varphi}}(a) = \frac{|eB|}{4a^2}\left[\frac{1}{6} - \frac{|\tilde{\varphi}|}{\pi}\left(1-\frac{|\tilde{\varphi}|}{\pi}\right)\right], \quad \sqrt{|eB|}a \gg 1, \quad ma \ll 1. \tag{96}$$

In particular, we get

$$\Delta_{\varphi,0}(a) = \frac{|eB|}{24a^2}, \quad \sqrt{|eB|}a \gg 1, \quad ma \ll 1, \tag{97}$$

$$\Delta_{\varphi,\pm\pi/4}(a) = -\frac{|eB|}{192a^2}, \quad \sqrt{|eB|}a \gg 1, \quad ma \ll 1 \tag{98}$$

and

$$\Delta_{\varphi,-\pi/2}(a) = -\frac{|eB|}{48a^2}, \quad \sqrt{|eB|}a \gg 1, \quad ma \ll 1. \tag{99}$$

In the limit of large distances between the plates, the first term in square brackets on the right-hand side of (95) yields

$$\Delta_{\varphi,\tilde{\varphi}}(a) = \frac{2|eB|}{\pi^2}\int\limits_m^\infty \mathrm{d}\kappa\kappa(\kappa^2-m^2)^{1/2}\mathrm{e}^{-2\kappa a}\left\{a\frac{\kappa^2\cos 2\tilde{\varphi} - m^2\sin(\varphi+\tilde{\varphi})\sin(\varphi-\tilde{\varphi})}{\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - m^2\sin(\varphi+\tilde{\varphi})\sin(\varphi-\tilde{\varphi})}\right.$$

$$\left. +\frac{\kappa^2 m\cos\varphi\cos 2\tilde{\varphi} - (2\kappa\sin\tilde{\varphi} - m\cos\varphi)m^2\sin(\varphi+\tilde{\varphi})\sin(\varphi-\tilde{\varphi})}{[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - m^2\sin(\varphi+\tilde{\varphi})\sin(\varphi-\tilde{\varphi})]^2}\sin\tilde{\varphi}\right\}, \quad \sqrt{|eB|}a \gg ma \gg 1, \tag{100}$$

which is obviously suppressed as $\exp(-2ma)$. In particular, we get

$$\Delta_{\varphi,0}(a) = \frac{|eB|}{2\pi^{3/2}}\frac{m^{3/2}}{a^{1/2}}\mathrm{e}^{-2ma}\left[1+O\left(\frac{1}{ma}\right)\right], \quad \sqrt{|eB|}a \gg ma \gg 1 \tag{101}$$

and

$$\Delta_{\varphi,-\pi/2}(a) = \left\{ \begin{array}{ll} -\frac{|eB|}{16\pi^{3/2}}\frac{m^{1/2}}{a^{3/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \varphi = 0 \\ \\ -\frac{|eB|\tan^2(\varphi/2)}{2\pi^{3/2}}\frac{m^{3/2}}{a^{1/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \varphi \neq 0 \end{array} \right\}, \quad \sqrt{|eB|}a \gg ma \gg 1. \tag{102}$$

It is appropriate in this section to consider also the limiting case of $m \to 0$. In view of the asymptotical behaviour of the boundary-independent piece of the Casimir pressure,

$$-\varepsilon_{\rm ren}^{\infty} = \frac{e^2B^2}{24\pi^2}\ln\frac{2|eB|}{m^2}, \quad m^2 \ll |eB| \tag{103}$$

and

$$-\varepsilon_{\rm ren}^{\infty} = \frac{1}{360\pi^2}\frac{e^4B^4}{m^4}, \quad m^2 \gg |eB|, \tag{104}$$

namely asymptotics (88) is relevant for this case, and the pressure from the vacuum of a confined massless spinor matter field is given by expression

$$F|_{m=0,\,B=0} = \frac{1}{8a^4}\left\{\frac{\pi^2}{30} - \int\limits_0^1 \mathrm{d}v\,\rho_{\tilde\varphi}(v)\left(1 - \frac{|\rho_{\tilde\varphi}(v)|}{\pi}\right)\right.$$

$$\left.\times\left[\frac{3}{2}\sqrt{1-v}\,\rho_{\tilde\varphi}(v)\left(1-\frac{|\rho_{\tilde\varphi}(v)|}{\pi}\right) + \frac{v\sin 2\tilde\varphi}{1 - v\cos^2\tilde\varphi}\left(\frac{1}{2} - \frac{|\rho_{\tilde\varphi}(v)|}{\pi}\right)\right]\right\}, \tag{105}$$

which is bounded from above and below by values

$$F|_{m=0,\,B=0,\,\tilde\varphi=0} = \frac{\pi^2}{240}\frac{1}{a^4} \tag{106}$$

and

$$F|_{m=0,\,B=0,\,\tilde\varphi=-\pi/2} = -\frac{7}{8}\frac{\pi^2}{240}\frac{1}{a^4}, \tag{107}$$

respectively; here, an additional factor of 1/2 has appeared due to diminishment of the number of degrees of freedom (a massless spinor can be either left or right).

6. Summary and discussion

We study an influence of a background (classical) magnetic field on the vacuum of a quantized charged spinor matter field which is confined to a bounded region of space; the sources of the magnetic field are outside of the bounded region, and the magnetic field strength lines are assumed to be orthogonal to a boundary. The confinement of the matter field (i.e. absence of the matter flux across the boundary) is ensured by boundary condition (32) which is compatible with the self-adjointness of the Dirac hamiltonan operator and which generalizes the well-known MIT bag boundary condition to the most extent. In the case which is relevant to the geometry of the Casimir effect (i.e. the spatial region bounded by two parallel planes separated by distance a) and the uniform magnetic field orthogonal to the planes, the most general extension of the MIT bag boundary condition is given by (55); the spectrum of the wave number vector along the magnetic field in this case depends on the number of the Landau level and on the sign of the one-particle energy, see (57). The Casimir pressure is shown to take the form of (71), where $\varepsilon_{\rm ren}^{\infty}$ is given by (45) and $\Upsilon(\kappa)$ is given by (72)-(76); the result for the case of the MIT bag boundary condition is obtained from (71) at $\varphi = 0$, $\tilde\varphi = -\pi/2$.

The Casimir effect is usually validated in experiments with nearly parallel plates separated by a distance of order $10^{-8} - 10^{-5}\,$m, see, e.g., [12]. The pressure from the vacuum of neutral massless spinor matter onto the bounding plates is given by a^{-4} times a constant of proportionality, either positive or negative, depending on a choice of boundary conditions, see (105)-(107). The situation with charged massive spinor matter is quite different. The Compton wavelength of the lightest charged particle, electron, is $m^{-1} \sim 10^{-12}\,$m, thus $ma \gg 1$ and, as has been shown in the preceding section, all the dependence of the Casimir pressure on the distance between the plates and a choice of boundary conditions at the plates is suppressed by factor $\exp(-2ma)$, see (92)-(94) and (100)-(102). Hence, the pressure from the electron-positron vacuum onto the plates separated by distance $a > 10^{-10}\,$m is well approximated by $F \approx -\varepsilon^{\infty}_{\rm ren}$, where $\varepsilon^{\infty}_{\rm ren}$ (45) is negative, i.e. the pressure is positive and the plates are repelled. Some possibilities to detect this new-type Casimir effect were pointed out in [10].

Acknowledgments
I would like to thank the Organizers of the XXIII International Conference on Integrable Systems and Quantum Symmetries for kind hospitality during this interesting and inspiring meeting. The work was supported by the National Academy of Sciences of Ukraine (project No.0112U000054), the Program of Fundamental Research of the Department of Physics and Astronomy of the National Academy of Sciences of Ukraine (project No.0112U000056) and the ICTP – SEENET-MTP grant PRJ-09 “Strings and Cosmology”.

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