Link between the relativistic canonical quantum mechanics of arbitrary spin and the corresponding field theory

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Abstract. The new relativistic equations of motion for the particles with arbitrary spin and nonzero mass have been introduced. The axiomatic level description of the relativistic canonical quantum mechanics of the arbitrary mass and spin has been given. The 64-dimensional $\mathrm{Cl}^{\mathrm{R}}(0,6)$ algebra in terms of Dirac gamma matrices has been suggested. The link between the relativistic canonical quantum mechanics of the arbitrary spin and the covariant local field theory has been found. Different methods of the Dirac equation derivation have been reviewed. The manifestly covariant field equations for an arbitrary spin that follow from the quantum mechanical equations have been considered. The covariant local field theory equations for spin s = (1,1) particle-antiparticle doublet, spin s = (1,0,1,0) particle antiparticle multiplet, spin s = (3/2,3/2) particle-antiparticle doublet, spin s = (2,2) particle-antiparticle doublet, spin s = (2,0,2,0) particle-antiparticle multiplet and spin s = (2,1,2,1) particle-antiparticle multiplet have been introduced. The Maxwell-like equations for the boson with spin s = 1 and nonzero mass have been introduced as well.

1. Introduction

Recently in [1, 2] the interesting results in the area of relativistic quantum mechanics and quantum field theory have been presented. This article contains brief exposition of the results [1, 2], which were reported on the conference. The general forms of quantum-mechanical and covariant equations for arbitrary spin are presented. The corresponding relativistic quantum mechanics of arbitrary spin is given as the brief version of the system of axioms. The partial cases of the spin s=(0,0) and spin s=(3/2,3/2) particle-antiparticle doublets are considered in explicit forms. The brief review of the different investigations in the area of relativistic canonical quantum mechanics (RCQM) is given and the brief analysis of the existing approaches to the field theory of arbitrary spin is initiated.

Note that in the Dirac model [3, 4] the quantum-mechanical interpretation is not evident. It has been demonstrated in [1, 2, 5, 6] that the quantum-mechanical interpretation is much more clear in the Foldy–Wouthuysen (FW) model [5, 6]. Nevertheless, the complete quantum-mechanical picture is possible only in the framework of RCQM. This assertion is one of the main conclusions proved in [1, 2].

The relativistic quantum mechanics under consideration is called *canonical* due to three main reasons. (i) The model under consideration has direct link with nonrelativistic quantum mechanics based on nonrelativistic Schrödinger equation. The principles of heredity and correspondence with other models of physical reality leads directly to nonrelativistic Schrödinger quantum mechanics. (ii) The FW model is already called by many authors as the canonical representation of the Dirac equation or a canonical field model, see, e. g., the paper [6]. And the difference between the field model given by FW and the RCQM is minimal – in corresponding equations it is only the presence and absence of beta matrix. (iii) The list of relativistic quantum-mechanical models is long. The Dirac model and the FW model are called by the "'old"' physicists as the relativistic quantum mechanics as well (one of my tasks in this paper is to show in visual and demonstrative way that these models have only weak quantum-mechanical interpretation). Further, the fractional relativistic quantum mechanics and the proper-time relativistic quantum mechanics can be listed (recall matrix formulation by W. Heisenberg, Feynman's sum over path's quantum theory, many-worlds interpretation by H. Everett), etc. Therefore, in order to avoid a confusion the model under consideration must have its proper name. Due to the reasons (i)–(iii) the best name for it is RCQM.

The general and fundamental goals here are as follows: (i) visual and demonstrative generalization of existing RCQM for the case of arbitrary spin, (ii) more complete formulation of this model on axiomatic level (on the test example of spin s=(1/2,1/2) particle-antiparticle doublet), (iii) vertical and horizontal links between the three different models of physical reality: relativistic quantum mechanics of arbitrary spin in canonical form, canonical (FW type) field theory of any spin, locally covariant (Dirac and Maxwell type) field theory of arbitrary spin.

2. Concepts, definitions and notations

The concepts, definitions and notations here are the same as in [1, 2]. For example, in the Minkowski space-time

$$\mathrm{M}(1,3) = \{x \equiv (x^{\mu}) = (x^0 = t,\ \overrightarrow{x} \equiv (x^j))\}; \quad \mu = \overline{0,3},\ j = 1,2,3, \tag{1}$$

x^{μ} denotes the Cartesian (covariant) coordinates of the points of the physical space-time in the arbitrary-fixed inertial reference frame (IRF). We use the system of units with $\hbar = c = 1$. The metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = g^{\mu}_{\nu},\ (g^{\mu}_{\nu}) = \mathrm{diag}\,(1,-1,-1,-1)\,; \quad x_{\mu} = g_{\mu\nu}x^{\mu}, \tag{2}$$

and summation over the twice repeated indices is implied.

Note that the square-root operator equation $i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x)$, which is the main equation of RCQM, has been rejected by Dirac in his consideration in [4] (chapter 11, section 67), see [2] for the details. Nevertheless, today, contrary to the year 1928, the definition of the pseudo-differential (non-local) operator

$$\widehat{\omega} \equiv \sqrt{\widehat{\overrightarrow{p}}^{2} + m^2} = \sqrt{-\Delta + m^2} \geq m > 0, \quad \widehat{\overrightarrow{p}} \equiv (\widehat{p}^j) = -i\nabla, \quad \nabla \equiv (\partial_{\ell}), \tag{3}$$

is well known. The action of the operator (3) in the coordinate representation (see, e. g. [7]) is given by

$$\widehat{\omega} f(t,\overrightarrow{x}) = \int d^3y K(\overrightarrow{x} - \overrightarrow{y}) f(t,\overrightarrow{y}), \quad f \in \mathrm{H}^{3,\mathrm{N}}, \tag{4}$$

where the function $K(\overrightarrow{x} - \overrightarrow{y})$ has the form $K(\overrightarrow{x} - \overrightarrow{y}) = -\frac{2m^2 K_2(m|\overrightarrow{x} - \overrightarrow{y}|)}{(2\pi)^2|\overrightarrow{x} - \overrightarrow{y}|^2}$ and $K_{\nu}(z)$ is the modified Bessel function (Macdonald function), $|\overrightarrow{a}|$ designates the norm of the vector $\overrightarrow{a}$, $\mathrm{H}^{3,\mathrm{N}}$ is the Hilbert space of N-component functions.

Further, the following integral form

$$(\widehat{\omega} f)(t,\overrightarrow{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{i\overrightarrow{k}\overrightarrow{x}}\widetilde{\omega}\widetilde{f}(t,\overrightarrow{k}), \quad \widetilde{\omega}\equiv\sqrt{\overrightarrow{k}^2+m^2}, \quad \widetilde{f}\in\widetilde{\mathrm{H}}^{3,\mathrm{N}}, \tag{5}$$

of the operator $\widehat{\omega}$ is used often, see, e. g., [6, 8], where f and $\widetilde{f}$ are linked by the 3-dimensional Fourier transformations

$$f(t,\overrightarrow{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{i\overrightarrow{k}\overrightarrow{x}}\widetilde{f}(t,\overrightarrow{k}) \Leftrightarrow \widetilde{f}(t,\overrightarrow{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{-i\overrightarrow{k}\overrightarrow{x}}\widetilde{f}(t,\overrightarrow{x}), \tag{6}$$

(in (6) $\overrightarrow{k}$ belongs to the spectrum $\mathrm{R}^3_{\overrightarrow{k}}$ of the operator $\widehat{\overrightarrow{p}}$, and the parameter $t\in(-\infty,\infty)\subset \mathrm{M}(1,3)$).

Note that the space of states $\mathrm{H}^{3,\mathrm{N}}$ is invariant with respect to the Fourier transformation (6). Therefore, both $\overrightarrow{x}$-realization $\mathrm{H}^{3,\mathrm{N}}$ and $\overrightarrow{k}$-realization $\widetilde{\mathrm{H}}^{3,\mathrm{N}}$ of the space of states are suitable for the purposes of our consideration. In the $\overrightarrow{k}$-realization the Schrödinger–Foldy equation has the algebraic-differential form

$$i\partial_t\widetilde{f}(t,\overrightarrow{k}) = \sqrt{\overrightarrow{k}^2+m^2}\widetilde{f}(t,\overrightarrow{k}); \quad \overrightarrow{k}\in\mathrm{R}^3_{\overrightarrow{k}}, \quad \widetilde{f}\in\widetilde{\mathrm{H}}^{3,\mathrm{N}}. \tag{7}$$

Below in the places, where misunderstanding is impossible, the symbol "tilde" is omitted.

Thus, today on the basis of above given definitions the difficulties, which stopped Dirac in 1928, can be overcome.

The name of the person, whose contribution in the theoretical model based on the square-root operator equation was decisive, is Leslie Lawrance Foldy (1919–2001). His interesting biography is presented in [9]. In our investigations we always marked the role of L. Foldy. Taking into account the L. Foldy's contribution in the construction of RCQM and his proof of the principle of correspondence between RCQM and non-relativistic quantum mechanics, we propose [10, 11] and [1, 2] *to call the N-component square-root operator equation* $i\partial_t f(x) = \sqrt{m^2-\Delta}f(x)$ *as the Schrödinger–Foldy equation.* Note here that this equation, which is a direct sum of one component spinless Salpeter equations [12], has been introduced in the formula (21) of [6]. Furthermore, note here that the nonlocal Poincaré group representation generators are known from the formulae (B-25)–(B-28) of the L. Foldy's paper [6].

Contrary to the times of papers [3, 5, 6, 12], the RCQM today is enough approbated and generally accepted theory. The spinless Salpeter equation has been introduced in [12]. The allusion on the RCQM and the first steps are given in [6], where the Salpeter equation for the 2s+1-component wave function was considered and the cases of s=1/2, s=1 were presented as an examples. In [13] Foldy continued his investigations [6] by the consideration of the relativistic particle systems with interaction. The interaction was introduced by the specific group-theoretical method.

After that in the RCQM were developed both the construction of mathematical foundations and the solution of concrete quantum-mechanical problems for different potentials. The brief review of 24 articles devoted to contemporary RCQM has been given in [2]. In the papers [10, 11], where we started our investigations in RCQM, this relativistic model for the test case of the spin s=(1/2,1/2) particle-antiparticle doublet is formulated. In [10], this model is considered as the system of the axioms on the level of the von Neumann monograph [14], where the mathematically well-defined consideration of the nonrelativistic quantum mechanics was given. Furthermore, in [10, 11] the operator link between the spin s=(1/2,1/2) particle-antiparticle doublet RCQM and the Dirac theory is given and Foldy's synthesis of *covariant particle equations* is extended to the start from the RCQM of the spin s=(1/2,1/2) particle-antiparticle doublet. In [1] the same procedure is fulfilled for the spin s=(1,1), s=(1,0,1,0), s=(3/2,3/2), s=(2,2), s=(2,0,2,0) and spin s=(2,1,2,1) RCQM. The corresponding equations, which follow from the RCQM for the covariant local field theory, are introduced.

Taking into account the 24 RCQM results reviewed in [2] the following conclusion is given. Here and in [1, 2] I am not going to formulate a new relativistic quantum mechanics! The foundations of RCQM based on the spinless Salpeter equation are already formulated in [6, 7] and in Refs. [9–35] given in [2].

3. Brief analysis of the covariant equations for an arbitrary spin

One of the goals of [1] is the link between the RCQM of an arbitrary spin and the different approaches to the covariant local field theory of an arbitrary spin. Surely, at least the brief analysis of the existing covariant equations for an arbitrary spin should be presented.

Note that in [1] and here only the first-order particle and the field equations (together with their canonical nonlocal pseudo-differential representations) are considered. The second order equations (like the Klein–Gordon–Fock equation) are not the subject of this investigation.

Different approaches to the description of the field theory of an arbitrary spin can be found in [6, 15–24]. Here and in [1, 2] only the approach started in [6] is the basis for further application. Other results given in [15–24] are not used here.

Note only some general deficiencies of the known equations for arbitrary spin. The consideration of the partial cases, when the substitution of the fixed value of spin is fulfilled, is not successful in all cases. For example, for the spin $s > 1$ existing equations have the redundant components and should be complemented by some additional conditions. Indeed, the known equations [25, 26] for the spin s=3/2 (and their confirmation in [27]) should be essentially complemented by the additional conditions. The main difficulty in the models of an arbitrary spin is the interaction between the fields of higher-spin. Even the quantization of higher-spin fields generated the questions. These and other deficiencies of the known equations for higher-spin are considered in Refs. [50–61] given in [2] (a brief review of deficiencies see in [28]).

Equations suggested in [1] and here are free of these deficiencies. The start of such consideration is taken from [6], where the main foundations of the RCQM are formulated. In the text of [1, 2] and here the results of [6] are generalized and extended. The operator link between the results of [5] and [6] (between the canonical FW type field theory and the RCQM) is suggested. Note that the cases s=3/2 and s=2 are not presented in [6], especially in explicit demonstrative forms. The results of [1, 2] are closest to the given in [29, 30]. The difference is explained in the section 5 below after the presentation of the results [1, 2].

Even this brief analysis makes us sure in the prospects of the investigations started in [1]. The successful description of the arbitrary spin field models is not the solved problem today.

4. Axioms of the relativistic canonical quantum mechanics of an arbitrary spin

The RCQM of the arbitrary spin given in sections 2 and 18 of [1] can be formulated at the level of von Neumann's consideration [14]. The difference with [14] is only in relativistic invariance and in the consideration of multicomponent and multidimensional objects.

The partial case of axiomatic formulation is already given in section 7 of [1] at the example of spin s=1/2 particle-antiparticle doublet. The RCQM of the arbitrary spin particle-antiparticle doublet (or particle singlet) can be formulated similarly as the corresponding generalization of this partial case.

Below the brief presentation of the *list of the axioms* is given. Note that some particular content of these axioms is already given in section 2 of [1], where the RCQM of the arbitrary spin particle singlet has been formulated.

4.1. On the space of states

The space of states of isolated arbitrary spin particle singlet in an arbitrarily-fixed inertial frame of reference (IFR) in its $\overrightarrow{x}$-realization is the Hilbert space

$$\mathrm{H}^{3,\mathrm{N}} = \mathrm{L}_2(\mathrm{R}^3) \otimes \mathrm{C}^{\otimes \mathrm{N}} = \{f = (f^{\mathrm{N}}) : \mathrm{R}^3 \rightarrow \mathrm{C}^{\otimes \mathrm{N}}; \quad \int d^3x |f(t, \overrightarrow{x})|^2 < \infty\}, \quad \mathrm{N} = 2s+1, \quad (8)$$

of complex-valued N-component square-integrable functions of $x \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ (similarly, in momentum, $\overrightarrow{p}$-realization). In (8) d^3x is the Lebesgue measure in the space $\mathrm{R}^3 \subset \mathrm{M}(1,3)$ of the eigenvalues of the position operator $\overrightarrow{x}$ of the Cartesian coordinate of the particle in an arbitrary-fixed IFR. Further, $\overrightarrow{x}$ and $\overrightarrow{p}$ are the operators of canonically conjugated dynamical variables of the spin s=(1/2,1/2) particle-antiparticle doublet, and the vectors f, $\tilde{f}$ in $\overrightarrow{x}$- and $\overrightarrow{p}$-realizations are linked by the 3-dimensional Fourier transformation (the variable t is the parameter of time-evolution).

The mathematical correctness of the consideration demands the application of the rigged Hilbert space

$$\mathrm{S}^{3,\mathrm{N}} \equiv \mathrm{S}(\mathrm{R}^3) \times \mathrm{C}^{\mathrm{N}} \subset \mathrm{H}^{3,\mathrm{N}} \subset \mathrm{S}^{3,\mathrm{N}*}. \quad (9)$$

where the Schwartz test function space $\mathrm{S}^{3,\mathrm{N}}$ is the core (i. e., it is dense both in $\mathrm{H}^{3,\mathrm{N}}$ and in the space $\mathrm{S}^{3,\mathrm{N}*}$ of the N-component Schwartz generalized functions). The space $\mathrm{S}^{3,\mathrm{N}*}$ is conjugated to that of the Schwartz test functions $\mathrm{S}^{3,\mathrm{N}}$ by the corresponding topology (see, e. g. [31]).

Strictly speaking, the mathematical correctness of consideration demands to make the calculations in the space $\mathrm{S}^{3,\mathrm{N}*}$ of generalized functions, i. e. with the application of cumbersome functional analysis (see, e. g. [32]). Nevertheless, one can take into account the properties of the Schwartz test function space $\mathrm{S}^{3,\mathrm{N}}$ in the triple (9). The space $\mathrm{S}^{3,\mathrm{N}}$ is dense both in quantum-mechanical space $\mathrm{H}^{3,\mathrm{N}}$ and in the space of generalized functions $\mathrm{S}^{3,\mathrm{N}*}$. Therefore, any physical state $f \in \mathrm{H}^{3,\mathrm{N}}$ can be approximated with an arbitrary precision by the corresponding elements of the Cauchy sequence in $\mathrm{S}^{3,\mathrm{N}}$, which converges to the given $f \in \mathrm{H}^{3,\mathrm{N}}$. Further, taking into account the requirement to measure the arbitrary value of the quantum-mechanical model with non-absolute precision, it means that all concrete calculations can be fulfilled within the Schwartz test function space $\mathrm{S}^{3,\mathrm{N}}$. Thus, such consideration allows us to perform, without any loss of generality, all necessary calculations in the space $\mathrm{S}^{3,\mathrm{N}}$ at the level of correct differential and integral calculus. More detailed consideration see in [1, 2].

Note finally that in the case of arbitrary spin particle-antiparticle doublet the dimension of spaces (8), (9) is M=2N=2(2s+1).

4.2. On the time evolution of the state vectors

The time dependence of the state vectors $f \in \mathrm{H}^{3,\mathrm{N}}$ (time t is the parameter of evolution) is given either in the integral form by the unitary operator

$$u\,(t_0, t) = \exp\left[-i\widehat{\omega}(t - t_0)\right]; \quad \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad (10)$$

(below $t_0 = t$ is put), or in the differential form by the Schrödinger–Foldy equation of motion

$$(i\partial_0 - \widehat{\omega})f(x) = 0. \quad (11)$$

with the wave function

$$f \equiv \mathrm{column}(f^1, f^2, ..., f^{\mathrm{N}}), \quad \mathrm{N} = 2s+1, \quad f \in \mathrm{H}^{3,\mathrm{N}}. \quad (12)$$

Note that here the operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$ is the relativistic analog of the energy operator (Hamiltonian) of nonrelativistic quantum mechanics. The Minkowski space-time M(1,3) is pseudo Euclidean with metric $g = \mathrm{diag}(+1, -1, -1, -1)$. The step from the particle singlet of arbitrary spin to the corresponding particle-antiparticle doublet is evident.

Thus, for the arbitrary spin particle-antiparticle doublet the system of two N-component equations $(i\partial_0 - \widehat{\omega})f(x) = 0$ and $(i\partial_0 - \widehat{\omega})f(x) = 0$ is used. Therefore, the corresponding Schrödinger–Foldy equation is given by (11), where the 2N-component wave function is the direct sum of the particle and antiparticle wave functions, respectively. Due to the historical tradition of the physicists the antiparticle wave function is put in the down part of the 2N-column.

The general solution of the Schrödinger–Foldy equation of motion (11) (in the case of particle-antiparticle arbitrary spin doublet) has the form

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} a^{2\mathrm{N}} \left(\overrightarrow{k}\right) \mathrm{d}_{2\mathrm{N}}, \quad kx \equiv \omega t - \overrightarrow{k}\overrightarrow{x}, \quad \omega \equiv \sqrt{\overrightarrow{k}^2 + m^2}, \tag{13}$$

where the orts of the N-dimensional Cartesian basis can be found in [1] in formulae (10).

The action of the pseudo-differential (non-local) operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$ is explained in (4), (5).

4.3. On the fundamental dynamical variables

The dynamical variable $\overrightarrow{x} \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ (as well as the variable $\overrightarrow{k} \in \mathrm{R}^3_{\overrightarrow{k}}$) represents the external degrees of freedom of the arbitrary spin particle-antiparticle doublet. The spin $\overrightarrow{s}$ of the particle-antiparticle doublet is the first in the list of the carriers of the internal degrees of freedom. Taking into account the Pauli principle and the fact that experimentally an antiparticle is observed as the mirror reflection of a particle, the operators of the charge sign and the spin of the arbitrary particle-antiparticle doublet are taken in the form

$$g \equiv -\Gamma^0_{2\mathrm{N}} \equiv -\sigma^3_{2\mathrm{N}} = \begin{vmatrix} -\mathrm{I_N} & 0 \\ 0 & \mathrm{I_N} \end{vmatrix}, \quad \overrightarrow{s}_{2\mathrm{N}} = \begin{vmatrix} \overrightarrow{s}_{\mathrm{N}} & 0 \\ 0 & -\hat{C}\overrightarrow{s}_{\mathrm{N}}\hat{C} \end{vmatrix}, \quad \mathrm{N} = 2s + 1, \tag{14}$$

where $\Gamma^0_{2\mathrm{N}}$ is the $2\mathrm{N} \times 2\mathrm{N}$ Dirac Γ^0 matrix, $\sigma^3_{2\mathrm{N}}$ is the $2\mathrm{N} \times 2\mathrm{N}$ Pauli σ^3 matrix, $\hat{C}$ is the operator of complex conjugation in the form of $\mathrm{N} \times \mathrm{N}$ diagonal matrix, the operator of involution in $\mathrm{H}^{3,2\mathrm{N}}$, and $\mathrm{I_N}$ is $\mathrm{N} \times \mathrm{N}$ unit matrix. Thus, the spin is given by the generators of SU(2) algebra! The spin matrices $\overrightarrow{s}_{2\mathrm{N}}$ (14) satisfy the commutation relations

$$\left[s^j_{2\mathrm{N}}, s^\ell_{2\mathrm{N}}\right] = i\varepsilon^{j\ell n} s^n_{2\mathrm{N}}, \quad \varepsilon^{123} = +1, \tag{15}$$

of the algebra of SU(2) group, where $\varepsilon^{j\ell n}$ is the Levi-Civita tensor and $s^j = \varepsilon^{j\ell n} s_{\ell n}$ are the Hermitian $2\mathrm{N} \times 2\mathrm{N}$ matrices (14) – the generators of a 2N-dimensional reducible representation of the spin group SU(2) (universal covering of the SO(3)⊂SO(1,3) group).

4.4. On the relativistic invariance of the theory

The relativistic invariance of the model under consideration (the relativistic invariance of the Schrödinger–Foldy equation (11)) requires, as a first step, consideration of its invariance with respect to the proper ortochronous Lorentz $\mathrm{L}^\uparrow_+ = \mathrm{SO(1,3)} = \{\Lambda = (\Lambda^\mu_\nu)\}$ and Poincaré $\mathrm{P}^\uparrow_+ = \mathrm{T(4)}\times)\mathrm{L}^\uparrow_+ \supset \mathrm{L}^\uparrow_+$ groups. This invariance in an arbitrary relativistic model is the implementation of the Einstein's relativity principle in the special relativity form. Note that the mathematical correctness requires the invariance mentioned above to be considered as the invariance with respect to the *universal coverings* $\mathcal{L} = \mathrm{SL(2,C)}$ and $\mathcal{P} \supset \mathcal{L}$ of the groups $\mathrm{L}^\uparrow_+$ and $\mathrm{P}^\uparrow_+$, respectively.

For the group $\mathcal{P}$ we choose real parameters $a = (a^{\mu}) \in$M(1,3) and $\varpi \equiv (\varpi^{\mu\nu} = -\varpi^{\nu\mu})$ with well-known physical meaning. For the standard $\mathcal{P}$ generators $(p_{\mu}, j_{\mu\nu})$ we use commutation relations in the manifestly covariant form

$$[p_{\mu}, p_{\nu}] = 0, \; [p_{\mu}, j_{\rho\sigma}] = ig_{\mu\rho}p_{\sigma} - ig_{\mu\sigma}p_{\rho}, \tag{16}$$

$$[j_{\mu\nu}, j_{\rho\sigma}] = -i\left(g_{\mu\rho}j_{\nu\sigma} + g_{\rho\nu}j_{\sigma\mu} + g_{\nu\sigma}j_{\mu\rho} + g_{\sigma\mu}j_{\rho\nu}\right).$$

The following assertion should be noted. Not a matter of fact that *non-covariant* objects such as the Lebesgue measure d^3x and *non-covariant (non-Lie)* generators of algebras are explored, the model of RCQM of arbitrary spin is a relativistic invariant in the following sense. The Schrödinger–Foldy equation (11) and the set of its solution $\{f\}$ (13) are invariant with respect to the irreducible unitary representation of the group $\mathcal{P}$, the N$\times$N matrix-differential generators of which are given by the following nonlocal operators

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_{\ell} = i\partial_{\ell}, \tag{17}$$

$$\widehat{j}_{\ell n} = x_{\ell}\widehat{p}_n - x_n\widehat{p}_{\ell} + s_{ln} \equiv \widehat{m}_{\ell n} + s_{\ell n},$$

$$\widehat{j}_{0\ell} = -\widehat{j}_{\ell 0} = t\widehat{p}_{\ell} - \frac{1}{2}\left\{x_{\ell}, \widehat{\omega}\right\} - \left(\frac{s_{\ell n}\widehat{p}_n}{\widehat{\omega} + m} \equiv \breve{s}_{\ell}\right), \tag{18}$$

where the orbital parts of the generators are not changed under the transition from one spin to another. Under such transitions only the spin parts (14), (15) of the expressions (17), (18) are changed. Indeed, the direct calculations visualize that generators (17), (18) commute with the operator of equation (11) and satisfy the commutation relations (16) of the Lie algebra of the Poincaré group $\mathcal{P}$. In formulae (17), (18), the SU(2)-spin generators $s^{\ell n}$ have particular specific forms for each representation of the SU(2) group (see the list of examples in [1]).

Note that the generators (17), (18) are known from the formulae (B-25) (B-28) of the paper [6].

Note also that together with the generators (17), (18) another set of 10 operators commutes with the operator of equation (11), satisfies the commutation relations (16) of the Lie algebra of Poincaré group $\mathcal{P}$, and, therefore, can be chosen as the Poincaré symmetry of the model under consideration. This second set is given by the generators $\widehat{p}^0$, $\widehat{p}^{\ell}$ from (17) together with the orbital parts of the generators $\widehat{j}^{\ell n}$, $\widehat{j}^{0\ell}$ from (18).

Thus, the irreducible unitary representation of the Poincaré group $\mathcal{P}$ in the space (9), with respect to which the Schrödinger-Foldy equation (11) and the set of its solution $\{f\}$ (13) are invariant, is given by a series converges in this space

$$(a, \varpi) \rightarrow U(a, \varpi) = \exp(-ia^0\widehat{p}_0 - i\overrightarrow{a}\overrightarrow{\widehat{p}} - \frac{i}{2}\varpi^{\mu\nu}\widehat{j}_{\mu\nu}) \tag{19}$$

where the generators $(\widehat{p}^{\mu}, \widehat{j}^{\mu\nu})$ are given in (17), (18) with the arbitrary values of the SU(2) spins $\overrightarrow{s} = (s^{\ell n})$ (14), (15).

The validity of this assertion is verified by the following three steps. (i) The calculation that the $\mathcal{P}$-generators (17), (18) commute with the operator $i\partial_0 - \widehat{\omega}$ of the Schrödinger–Foldy equation (11). (ii) The verification that the $\mathcal{P}$-generators (17), (18) satisfy the commutation relations (16) of the Lie algebra of the Poincaré group $\mathcal{P}$. (iii) The proof that generators (17), (18) realize the spin s(s+1) representation of this group. Therefore, the Bargman–Wigner classification on the basis of the corresponding Casimir operators calculation should be given. These three steps can be made by direct and non-cumbersome calculations.

The expression (19) is well known, but rather formal. In fact the transition from a Lie algebra to a finite group transformations is a rather non-trivial action. The mathematical justification of (19) can be fulfilled in the framework of Schwartz test function space and will be given in next special publication.

The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^{\mu}\widehat{p}_{\mu} = m^2 \mathrm{I}_{\mathrm{N}}, \tag{20}$$

$$W = w^{\mu}w_{\mu} = m^2 \overrightarrow{s}^2 = \mathrm{s}(\mathrm{s}+1)m^2 \mathrm{I}_{\mathrm{N}}, \tag{21}$$

where I_{N} is the $\mathrm{N} \times \mathrm{N}$ unit matrix and s =1/2, 1, 3/2, 2, ...

Note that together with the generators (17), (18) another set of 10 operators commutes with the operator of equation (11), satisfies the commutation relations (16) of the Lie algebra of Poincaré group $\mathcal{P}$, and, therefore, can be chosen as the Poincaré symmetry of the model under consideration. This second set is given by the generators $\widehat{p}^0$, $\widehat{p}^{\ell}$ from (17) together with the orbital parts of the generators $\widehat{j}^{\ell n}$, $\widehat{j}^{0\ell}$ from (18). Thus, this second set of Poincaré generators is given by

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_{\ell} = i\partial_{\ell}, \quad \widehat{m}_{\ell n} = x_{\ell}\widehat{p}_n - x_n\widehat{p}_{\ell}, \quad \widehat{m}_{0\ell} = -\widehat{m}_{\ell 0} = t\widehat{p}_{\ell} - \frac{1}{2}\left\{x_{\ell}, \widehat{\omega}\right\}. \tag{22}$$

Note that in the case s=0 only generators (22) form the Poincaré symmetry.

Note that the modern definition of $\mathcal{P}$ invariance (or $\mathcal{P}$ symmetry) of the equation of motion (11) in $\mathrm{H}^{3,\mathrm{N}}$ is given by the following assertion, see, e. g. [33]. *The set* $\mathrm{F} \equiv \{f\}$ *of all possible solutions of the equation (11) is invariant with respect to the* $\mathcal{P}^{\mathrm{f}}$*-representation of the group* $\mathcal{P}$*, if for arbitrary solution* f *and arbitrarily-fixed parameters* (a, ϖ) *the assertion*

$$(a, \varpi) \rightarrow U(a, \varpi)\{f\} = \{f\} \equiv \mathrm{F} \tag{23}$$

is valid. In [2] this axiom is considered together with very useful axiom *on the dynamic and kinematic aspects of the relativistic invariance.*

4.5. On the Clifford–Dirac algebra

This axiom is additional and is not necessary. Nevertheless, such axiom is very useful for the dimensions, where the Γ matrices exist. Application of the Clifford–Dirac algebra is the useful method of calculations in RCQM. Three different definitions of the Clifford algebra and their equivalence are considered in [34]. In different approaches to the relativistic quantum mechanics the matrix representation of the Clifford algebra in terms of the Dirac gamma matrices is used. This representation is called the Clifford–Dirac algebra.

For our purposes the anticommutation relations of the Clifford–Dirac algebra generators are taken in the general form

$$\Gamma^{\bar{\mu}}_{2\mathrm{N}}\Gamma^{\bar{\nu}}_{2\mathrm{N}} + \Gamma^{\bar{\nu}}_{2\mathrm{N}}\Gamma^{\bar{\mu}}_{2\mathrm{N}} = 2g^{\bar{\mu}\bar{\nu}}; \quad \bar{\mu}, \bar{\nu} = \overline{0,4}, \quad (g^{\bar{\mu}\bar{\nu}}) = (+----), \tag{24}$$

where $\Gamma^{\bar{\mu}}_{2\mathrm{N}}$ are the $2\mathrm{N} \times 2\mathrm{N}$ Dirac $\Gamma^{\bar{\mu}}$ matrices ($2\mathrm{N} \times 2\mathrm{N}$ generalization of the Dirac 4×4 γ matrices), $\Gamma^4_{2\mathrm{N}} \equiv \Gamma^0_{2\mathrm{N}}\Gamma^1_{2\mathrm{N}}\Gamma^2_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}}$. Here and in our publications (see, e. g. the last years articles [35–39]) we use the $\gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$ matrix instead of the γ^5 matrix of other authors. Our γ^4 is equal to $i\gamma^5_{\mathrm{standard}}$. Notation γ^5 is used in [35–39] for a completely different matrix $\gamma^5 \equiv \gamma^1\gamma^3\hat{C}$. As well as the element $\Gamma^4_{2\mathrm{N}} \equiv \Gamma^0_{2\mathrm{N}}\Gamma^1_{2\mathrm{N}}\Gamma^2_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}}$ of (24) is dependent the algebra basis is formed by 4=1+3 independent elements. Therefore, such Clifford algebra over the field of complex numbers is denoted $\mathrm{Cl}^{\mathrm{C}}(1{,}3)$ and the dimension of the algebra is $2^4 = 16$.

The best consideration of this axiom is given in [2], where the complete analysis is presented. Moreover, in [2] new Clifford–Dirac algebra over the field of real numbers is introduced. Two important representations of this algebra are defined as $\mathrm{Cl}^{\mathrm{R}}(4{,}2)$, $\mathrm{Cl}^{\mathrm{R}}(0{,}6)$ and the dimension of this algebra is $2^6 = 64$.

4.6. Briefly on other axioms

Other axioms of arbitrary spin RCQM are given in [2]. The list of these axioms is as follows. *On the external and internal degrees of freedom, on the algebra of observables, on the main and additional conservation laws, on the stationary complete sets of operators, on the solutions of the Schrödinger-Foldy equation, on the mean value of the operators of observables, on the principles of heredity and the correspondence, on the second quantization (external axiom), on the physical interpretation.* All axioms of this section eventually need to be reconciled with three levels of description used in this paper: RCQM, canonical FW and Dirac models. Nevertheless, this interesting problem cannot be considered in few pages. The readers of this paper can compare the axioms of RCQM given above with the main principles of the Dirac model given in B. Thaller's monograph [40] on the high mathematical level.

5. General description of the arbitrary spin field theory

The step by step consideration of the different partial examples in [1] (sections 21–27) enabled us to rewrite them in the general form, which is valid for arbitrary spin. Therefore, the generalization of the consideration given in [1] leads to the general formalism of the arbitrary spin fields.

The formalism presented below in this section is valid for an arbitrary particle-antiparticle multiplet in general and for the particle-antiparticle doublet in particular.

5.1. The canonical (FW type) model of the arbitrary spin particle-antiparticle field

The operator, which transform the RCQM of the arbitrary spin particle-antiparticle multiplet into the corresponding canonical particle-antiparticle field, is given by

$$v_{2\mathrm{N}} = \left| \begin{array}{cc} \mathrm{I_N} & 0 \\ 0 & \hat{C}\mathrm{I_N} \end{array} \right|, \quad v_{2\mathrm{N}}^{-1} = v_{2\mathrm{N}}^{\dagger} = v_{2\mathrm{N}}, \quad v_{2\mathrm{N}}v_{2\mathrm{N}} = \mathrm{I}_{2\mathrm{N}}, \quad \mathrm{N} = 2s+1, \tag{25}$$

where $\hat{C}\mathrm{I_N}$ is the $\mathrm{N}\times\mathrm{N}$ operator of complex conjugation. Indeed, the operator (25) translates any operator from canonical field FW representation into the RCQM representation and vice versa:

$$v_{2\mathrm{N}}\hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}v_{2\mathrm{N}} = \hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}, \quad v_{2\mathrm{N}}\hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}v_{2\mathrm{N}} = \hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}. \tag{26}$$

Here $\hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the RCQM of the 2N-component particle-antiparticle doublet in the anti-Hermitian form, e. g., the operator $(\partial_0 + i\widehat{\omega})$ of equation of motion (11), the operator of spin $\overrightarrow{s}_{2\mathrm{N}}$ (14) taken in anti-Hermitian form, etc., $\hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the canonical field theory of the 2N-component particle-antiparticle doublet in the anti-Hermitian form. Thus, the only warning is that operators here must be taken in anti-Hermitian form, see section 9 in [1] for the details and see [41, 42] for the mathematical correctness of anti-Hermitian operators application.

Further, the operator (25) translates

$$\phi = v_{2\mathrm{N}}f, \quad f = v_{2\mathrm{N}}\phi, \tag{27}$$

the solution (13) of the Schrödinger–Foldy equation (11) into the solution

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k \left[e^{-ikx}a^{\mathrm{N}}(\overrightarrow{k})\mathrm{d_N} + e^{ikx}a^{*\check{\mathrm{N}}}(\overrightarrow{k})\mathrm{d}_{\check{\mathrm{N}}}\right], \tag{28}$$

$\mathrm{N} = 1, 2, ..., \mathrm{N}, \quad \check{\mathrm{N}} = \mathrm{N}+1, \mathrm{N}+2, ..., 2\mathrm{N}$, of the FW equation

$$(i\partial_0 - \Gamma_{2\mathrm{N}}^0\widehat{\omega})\phi(x) = 0, \quad \Gamma_{2\mathrm{N}}^0 \equiv \sigma_{2\mathrm{N}}^3 = \left| \begin{array}{cc} \mathrm{I_N} & 0 \\ 0 & -\mathrm{I_N} \end{array} \right|, \tag{29}$$

$\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$, $\quad \mathrm{N} = 2s+1$, and *vice versa*. Thus, the transformation (25), (26) translates the matrices $\Gamma^0_{2\mathrm{N}}$ and

$$\Gamma^j_{2\mathrm{N}} = \begin{vmatrix} 0 & \Sigma^j_{\mathrm{N}} \\ -\Sigma^j_{\mathrm{N}} & 0 \end{vmatrix}, \quad j = 1,2,3, \tag{30}$$

into the RCQM representation and *vice versa*

$$\bar{\Gamma}^{\bar{\mu}}_{2\mathrm{N}} = v_{2\mathrm{N}}\Gamma^{\bar{\mu}}_{2\mathrm{N}}v_{2\mathrm{N}}, \quad \Gamma^{\bar{\mu}}_{2\mathrm{N}} = v_{2\mathrm{N}}\bar{\Gamma}^{\bar{\mu}}_{2\mathrm{N}}v_{2\mathrm{N}}. \tag{31}$$

In (30) Σ^j_{N} are the $\mathrm{N} \times \mathrm{N}$ Pauli matrices. Matrices $\Gamma^\mu_{2\mathrm{N}}$ (29), (30) together with matrix $\Gamma^4_{2\mathrm{N}} \equiv \Gamma^0_{2\mathrm{N}}\Gamma^1_{2\mathrm{N}}\Gamma^2_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}}$ satisfy the anticommutation relations (24) of the Clifford–Dirac algebra.

The formulas mentioned below are found from the corresponding formulas of RCQM with the help of the operator (25) on the basis of its properties (26), (27). Thus, for the general form of arbitrary spin canonical particle-antiparticle field the equation of motion of the FW type is given by (29). The general solution has the form (28), where $a^{\mathrm{N}}(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the particle and $a^{\check{\mathrm{N}}}(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the antiparticle, $\{\mathrm{d}\}$ is 2N-component Cartesian basis.

It is evident from (28) that the model under consideration is not quantum mechanics. Indeed, contrary to (13) the solution (28) contains positive and negative frequency terms and, as a consequence, equation (29) is dealing with positive and negative energies (contrary to equation (11)).

The spin operator, which follows from (14), has the form

$$\overrightarrow{s}_{2\mathrm{N}} = \begin{vmatrix} \overrightarrow{s}_{\mathrm{N}} & 0 \\ 0 & \overrightarrow{s}_{\mathrm{N}} \end{vmatrix}, \quad \mathrm{N} = 2s+1, \tag{32}$$

where $\overrightarrow{s}_{\mathrm{N}}$ are $\mathrm{N} \times \mathrm{N}$ generators of arbitrary spin irreducible representations of SU(2) algebra, which satisfy the commutation relations $\left[s^j_{\mathrm{N}}, s^\ell_{\mathrm{N}}\right] = i\varepsilon^{j\ell n}s^n_{\mathrm{N}}, \quad \varepsilon^{123} = +1$.

The generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (29) and the set $\{\phi\}$ of its solutions (28) are invariant, are given by

$$\widehat{p}^0 = \Gamma^0_{2\mathrm{N}}\widehat{\omega} \equiv \Gamma^0_{2\mathrm{N}}\sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell, \quad \widehat{j}^{\ell n} = x^\ell\widehat{p}^n - x^n\widehat{p}^\ell + s^{\ell n}_{2\mathrm{N}} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{2\mathrm{N}}, \tag{33}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^\ell - \frac{1}{2}\Gamma^0_{2\mathrm{N}}\left\{x^\ell, \widehat{\omega}\right\} + \Gamma^0_{2\mathrm{N}}\frac{(\overrightarrow{s}_{2\mathrm{N}} \times \overrightarrow{p})^\ell}{\widehat{\omega} + m}, \tag{34}$$

where arbitrary spin SU(2) generators $\overrightarrow{s}_{2\mathrm{N}} = (s^{\ell n}_{2\mathrm{N}})$ have the form (32), $\Gamma^0_{2\mathrm{N}}$ is given in (29).

Note that together with the generators (33), (34) another set of 10 operators commutes with the operator of equation (29), satisfies the commutation relations (16) of the Lie algebra of Poincaré group $\mathcal{P}$, and, therefore, can be chosen as the Poincaré symmetry of the model under consideration. This second set is given by the generators $\widehat{p}^0$, $\widehat{p}^\ell$ from (33) together with the orbital parts of the generators $\widehat{j}^{\ell n}$, $\widehat{j}^{0\ell}$ from (33), (34), respectively. In another way this set follows from the set (22) after the transformation (25), (26).

The calculation of the Casimir operators $p^2 = \widehat{p}^\mu\widehat{p}_\mu$, $W = w^\mu w_\mu$ (w^μ is the Pauli–Lubanski pseudovector) for the fixed value of spin completes the brief description of the model.

5.2. The locally covariant model of the arbitrary spin particle-antiparticle field

The operator, which transform the canonical (FW type) model of the arbitrary spin particle-antiparticle field into the corresponding locally covariant particle-antiparticle field, is the generalized FW operator and is given by

$$V^{\mp} = \frac{\mp\overrightarrow{\Gamma}_{2\mathrm{N}} \cdot \overrightarrow{p} + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}}, \quad V^{-} = (V^{+})^{\dagger}, \quad V^{-}V^{+} = V^{+}V^{-} = \mathrm{I}_{2\mathrm{N}}, \quad \mathrm{N} = 2s + 1, \tag{35}$$

where $\Gamma^{j}_{2\mathrm{N}}$ are known from (30) and Σ^{j}_{N} are the $\mathrm{N} \times \mathrm{N}$ Pauli matrices.

Note that in formulas (35) and in all formulas before the end of the subsection the values of N are only even. Therefore, the canonical field equation (29) describes the larger number of multiplets then the generalized Dirac equation (36) given below.

The formulas (36)–(41) below are found from the corresponding formulas (28), (29), (32)–(34) of canonical field model on the basis of the operator (35).

For the general form of arbitrary spin locally covariant particle-antiparticle field the Dirac-like equation of motion follows from the equation (29) after the transformation (35) and is given by

$$\left[i\partial_0 - \Gamma^0_{2\mathrm{N}}(\overrightarrow{\Gamma}_{2\mathrm{N}} \cdot \overrightarrow{p} + m)\right]\psi(x) = 0. \tag{36}$$

The general solution has the form

$$\psi(x) = V^{-}\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\left[e^{-ikx}a^{\mathrm{N}}(\overrightarrow{k})\mathrm{v}^{-}_{\mathrm{N}}(\overrightarrow{k}) + e^{ikx}a^{*\breve{\mathrm{N}}}(\overrightarrow{k})\mathrm{v}^{+}_{\breve{\mathrm{N}}}(\overrightarrow{k})\right], \tag{37}$$

where amplitudes and notation $\breve{\mathrm{N}}$ are the same as in (28); $\left\{\mathrm{v}^{-}_{\mathrm{N}}(\overrightarrow{k}), \mathrm{v}^{+}_{\breve{\mathrm{N}}}(\overrightarrow{k})\right\}$ are 2N-component Dirac basis spinors with properties of orthonormalisation and completeness similar to 4-component Dirac spinors from [43].

The spin operator is given by

$$\overrightarrow{s}_{\mathrm{D}} = V^{-}\overrightarrow{s}_{2\mathrm{N}}V^{+}, \tag{38}$$

where operator $\overrightarrow{s}_{2\mathrm{N}}$ is known from (32). The explicit forms of few partial cases of spin operators (38) are given in formulae (259)–(261), (284)–(286), (359) of [1] for the particle-antiparticle multiplets s=(1,0,1,0), s=(3/2,3/2), s=(2,1,2,1), respectively.

The generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant field equation (36) and the set $\{\psi\}$ of its solutions (37) are invariant, have the form

$$\widehat{p}^0 = \Gamma^0_{2\mathrm{N}}(\overrightarrow{\Gamma}_{2\mathrm{N}} \cdot \overrightarrow{p} + m), \quad \widehat{p}^{\ell} = -i\partial_{\ell}, \quad \widehat{j}^{\ell n} = x^{\ell}_{\mathrm{D}}\widehat{p}^{n} - x^{n}_{\mathrm{D}}\widehat{p}^{\ell} + s^{\ell n}_{\mathrm{D}} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{\mathrm{D}}, \tag{39}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^{\ell} - \frac{1}{2}\left\{x^{\ell}_{\mathrm{D}}, \widehat{p}^0\right\} + \frac{\widehat{p}^0(\overrightarrow{s}_{\mathrm{D}} \times \overrightarrow{p})^{\ell}}{\widehat{\omega}(\widehat{\omega} + m)}, \tag{40}$$

where the spin matrices $\overrightarrow{s}_{\mathrm{D}} = (s^{\ell n}_{\mathrm{D}})$ are given in (38) and the operator $\overrightarrow{x}_{\mathrm{D}}$ has the form

$$\overrightarrow{x}_{\mathrm{D}} = \overrightarrow{x} + \frac{i\overrightarrow{\Gamma}_{2\mathrm{N}}}{2\widehat{\omega}} - \frac{\overrightarrow{s}^{\Gamma}_{2\mathrm{N}} \times \overrightarrow{p}}{\widehat{\omega}(\widehat{\omega} + m)} - \frac{i\overrightarrow{p}(\overrightarrow{\Gamma}_{2\mathrm{N}} \cdot \overrightarrow{p})}{2\widehat{\omega}^2(\widehat{\omega} + m)}, \tag{41}$$

where specific spin matrices $\overrightarrow{s}^{\Gamma}_{2\mathrm{N}}$ are given by

$$\overrightarrow{s}^{\Gamma}_{2\mathrm{N}} \equiv \overrightarrow{s}_{\mathrm{FW}} = \left(s^1_{2\mathrm{N}}, s^2_{2\mathrm{N}}, s^3_{2\mathrm{N}}\right) = \frac{i}{2}(\Gamma^2_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}},\, \Gamma^3_{2\mathrm{N}}\Gamma^1_{2\mathrm{N}},\, \Gamma^1_{2\mathrm{N}}\Gamma^2_{2\mathrm{N}}). \tag{42}$$

Note that for corresponding partial cases of $\overrightarrow{\mathcal{x}}_{\rm D}$ in [1] ((267) for s=(1,0,1,0), once more (267) but for s=(3/2,3/2), (333) for s=(2,0,2,0), (363) for s=(2,1,2,1)) the corresponding $\overrightarrow{s}^{\Gamma}_{\rm 2N}$ are given by

$$\mathrm{s} = (1,0,1,0) \,\text{and}\, \mathrm{s} = (3/2,3/2): \quad \overrightarrow{s}^{\Gamma}_{8} \equiv \left(s^1_8, s^2_8, s^3_8\right) = \frac{i}{2}(\Gamma^2_8\Gamma^3_8,\, \Gamma^3_8\Gamma^1_8,\, \Gamma^1_8\Gamma^2_8), \tag{43}$$

for $\overrightarrow{\mathcal{x}}_{\rm D}$ [1] (267), where the Γ^j_8 matrices are given in (253),

$$\mathrm{s} = (2,0,2,0): \quad \overrightarrow{s}^{\Gamma}_{12} \equiv \left(s^1_{12}, s^2_{12}, s^3_{12}\right) = \frac{i}{2}(\Gamma^2_{12}\Gamma^3_{12},\, \Gamma^3_{12}\Gamma^1_{12},\, \Gamma^1_{12}\Gamma^2_{12}), \tag{44}$$

for $\overrightarrow{\mathcal{x}}_{\rm D}$ [1] (333), where the Γ^j_{12} matrices are given in (324),

$$\mathrm{s} = (2,1,2,1): \quad \overrightarrow{s}^{\Gamma}_{16} \equiv \left(s^1_{16}, s^2_{16}, s^3_{16}\right) = \frac{i}{2}(\Gamma^2_{16}\Gamma^3_{16},\, \Gamma^3_{16}\Gamma^1_{16},\, \Gamma^1_{16}\Gamma^2_{16}), \tag{45}$$

for $\overrightarrow{\mathcal{x}}_{\rm D}$ [1] (363), where the Γ^j_{16} matrices are given in (353).

It is easy to verify that the generators (39), (40) for any N commute with the operator of equation (36), and satisfy the commutation relations (16) of the Lie algebra of the Poincaré group. The last step in the brief description of the model is the calculation of the Casimir operators $p^2 = \widehat{p}^\mu\widehat{p}_\mu$, $W = w^\mu w_\mu$ (w^μ is the Pauli–Lubanski pseudovector) for the fixed value of spin.

5.3. The example of spin s=(0,0) particle-antiparticle doublet

The completeness of simplest spin multiplets and doublets consideration of [1] is achieved by the supplementation of this example. The formalism follows from the general formalism of arbitrary spin after the substitution s=0.

The Schrödinger–Foldy equation of RCQM is given by (11) for N=1, i. e. it is 2-component equation. The solution is given by (13) for N=1. The Poincaré group $\mathcal{P}$ generators, with respect to which the equation (11) for s=(0,0) is invariant, are given by (17), (18) taken in the form of 2×2 matrices with spin terms equal to zero, i. e. the corresponding generators are given by 2×2 matrices (22).

The corresponding FW type equation of canonical field theory is given by

$$(i\partial_0 - \sigma^3\widehat{\omega})\phi(x) = 0, \quad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad \widehat{\omega} \equiv \sqrt{-\Delta + m^2}. \tag{46}$$

The general solution is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx}a^1(\overrightarrow{k})\mathrm{d}_1 + e^{ikx}a^{*2}(\overrightarrow{k})\mathrm{d}_2\right]. \tag{47}$$

The Poincaré group $\mathcal{P}$ generators, with respect to which the equation (45) and the set $\{\phi\}$ of its solutions (47) are invariant, have the form

$$\widehat{p}^0 = \sigma^3\widehat{\omega} \equiv \sigma^3\sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell, \quad \widehat{j}^{\ell n} = x^\ell\widehat{p}^n - x^n\widehat{p}^\ell, \tag{48}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^\ell - \frac{1}{2}\sigma^3\left\{x^\ell, \widehat{\omega}\right\}. \tag{49}$$

Generators (48), (49) are the partial 2×2 matrix form of operators (33), (34) taken with the spin terms equal to zero.

5.4. The example of covariant field equation for spin s=(3/2,3/2) particle-antiparticle doublet Consider the nontrivial partial example of covariant field equation for arbitrary spin. Such example is given by covariant field equation for spin s=(3/2,3/2) particle-antiparticle doublet. This case presents the demonstrative example how new equations can be derived by the developed in [1] and here methods.

Now contrary to [1] equation for spin s=3/2 is found as the simple partial case of general equation (36):

$$\left[i\partial_0 - \Gamma_8^0(\overrightarrow{\Gamma}_8 \cdot \overrightarrow{p} + m)\right] \psi(x) = 0. \tag{50}$$

Here the Γ_8^μ matrices are given by

$$\Gamma_8^0 = \begin{vmatrix} \mathrm{I}_4 & 0 \\ 0 & -\mathrm{I}_4 \end{vmatrix}, \quad \Gamma_8^j = \begin{vmatrix} 0 & \Sigma^j \\ -\Sigma^j & 0 \end{vmatrix}, \tag{51}$$

where Σ^j are the 4×4 Pauli matrices

$$\Sigma^j = \begin{vmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{vmatrix}, \tag{52}$$

and σ^j are the standard 2×2 Pauli matrices. The matrices Σ^j satisfy the similar commutation relations as the standard 2×2 Pauli matrices and have other similar properties. The matrices Γ_8^μ (51) satisfy the anticommutation relations of the Clifford–Dirac algebra in the form (24) with N=4.

Note that equation (50) is not the ordinary direct sum of the two Dirac equations. Therefore, it is not the complex Dirac–Kahler equation [44]. Moreover, it is not the standard 16-component Dirac–Kahler equation [45]. Furthermore, for the same reason it is not the spin 3/2 equation from [29, 30].

The solution of equation (50) is derived as a partial case from the solution (37) of the general equation (36) and is given by

$$\psi(x) = V_8^- \phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} b^{\mathrm{A}}(\overrightarrow{k}) \mathrm{v}_{\mathrm{A}}^-(\overrightarrow{k}) + e^{ikx} b^{*\mathrm{B}}(\overrightarrow{k}) \mathrm{v}_{\mathrm{B}}^+(\overrightarrow{k})\right], \tag{53}$$

where $\mathrm{A} = \overline{1,4}$, $\mathrm{B} = \overline{5,8}$ and the 8-component spinors $(\mathrm{v}_{\mathrm{A}}^-(\overrightarrow{k}), \mathrm{v}_{\mathrm{B}}^+(\overrightarrow{k}))$ are given by (257) in [1].

The spinors $(\mathrm{v}_{\mathrm{A}}^-(\overrightarrow{k}), \mathrm{v}_{\mathrm{B}}^+(\overrightarrow{k}))$ satisfy the relations of the orthonormalization and completeness similar to the corresponding relations for the standard 4-component Dirac spinors, see, e. g., [43].

In the covariant local field theory, the operators of the SU(2) spin, which satisfy the corresponding commutation relations $\left[s_{8\mathrm{D}}^j, s_{8\mathrm{D}}^\ell\right] = i\varepsilon^{j\ell n} s_{8\mathrm{D}}^n$ and commute with the operator $\left[i\partial_0 - \Gamma_8^0(\overrightarrow{\Gamma}_8 \cdot \overrightarrow{p} + m)\right]$ of equation (50), are derived from the pure matrix operators (279) of [1] with the help of transition operator V_8^-: $\overrightarrow{s}_{8\mathrm{D}} = V_8^- \overrightarrow{s}_8 V_8^+$. The explicit form of the transition operator $V_8^\mp$ is given in (249)–(251) of [1]. The explicit form of these s=(3/2,3/2) SU(2) generators was given already by formulae (284)–(287) in [1].

The equations on eigenvectors and eigenvalues of the operator $s_{8\mathrm{D}}^3$ (286) in [1] follow from the equations (280) of [1] and the transformation V_8^-. In addition to it, the action of the operator $s_{8\mathrm{D}}^3$ (286) in [1] on the spinors $(\mathrm{v}_{\mathrm{A}}^-(\overrightarrow{k}), \mathrm{v}_{\mathrm{B}}^+(\overrightarrow{k}))$ (257) in [1] also leads to the result $s_{8\mathrm{D}}^3 \mathrm{v}_1^-(\overrightarrow{k}) = \frac{3}{2}\mathrm{v}_1^-(\overrightarrow{k})$, $s_{8\mathrm{D}}^3 \mathrm{v}_2^-(\overrightarrow{k}) = \frac{1}{2}\mathrm{v}_2^-(\overrightarrow{k})$, $s_{8\mathrm{D}}^3 \mathrm{v}_3^-(\overrightarrow{k}) = -\frac{1}{2}\mathrm{v}_3^-(\overrightarrow{k})$, $s_{8\mathrm{D}}^3 \mathrm{v}_4^-(\overrightarrow{k}) = -\frac{3}{2}\mathrm{v}_4^-(\overrightarrow{k})$,

$$s_{8\mathrm{D}}^3 \mathrm{v}_5^+(\overrightarrow{k}) = \frac{3}{2}\mathrm{v}_5^+(\overrightarrow{k}),\ s_{8\mathrm{D}}^3 \mathrm{v}_6^+(\overrightarrow{k}) = \frac{1}{2}\mathrm{v}_6^+(\overrightarrow{k}),\ s_{8\mathrm{D}}^3 \mathrm{v}_7^+(\overrightarrow{k}) = -\frac{1}{2}\mathrm{v}_7^+(\overrightarrow{k}),\ s_{8\mathrm{D}}^3 \mathrm{v}_8^+(\overrightarrow{k}) = -\frac{3}{2}\mathrm{v}_8^+(\overrightarrow{k}). \tag{54}$$

In order to verify equations (54) the identity $(\tilde{\omega}+m)^2+(\overrightarrow{k})^2=2\tilde{\omega}(\tilde{\omega}+m)$ is used. In the case $\mathrm{v}_{\mathrm{B}}^{+}(\overrightarrow{k})$ in the expression $s_{8\mathrm{D}}^{3}(\overrightarrow{k})$ (286) of [1] the substitution $\overrightarrow{k}\rightarrow-\overrightarrow{k}$ is made.

The equations (54) determine the interpretation of the amplitudes in solution (53). Nevertheless, the direct quantum-mechanical interpretation of the amplitudes should be made in the framework of the RCQM and is already given in [1] (section 14 in paragraph after equations (183)).

The explicit form of the $\mathcal{P}$-generators of the fermionic representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant equation (50) and the set $\{\psi\}$ of its solutions (53) are invariant, is derived as a partial case from the generators (39), (40). The corresponding generators are given by

$$\widehat{p}^{0}=\Gamma_{8}^{0}(\overrightarrow{\Gamma}_{8}\cdot\overrightarrow{p}+m),\quad \widehat{p}^{\ell}=-i\partial_{\ell},\quad \widehat{j}^{\ell n}=x_{\mathrm{D}}^{\ell}\widehat{p}^{n}-x_{\mathrm{D}}^{n}\widehat{p}^{\ell}+s_{8\mathrm{D}}^{\ell n}\equiv\widehat{m}^{\ell n}+s_{8\mathrm{D}}^{\ell n}, \tag{55}$$

$$\widehat{j}^{0\ell}=-\widehat{j}^{\ell 0}=x^{0}\widehat{p}^{\ell}-\frac{1}{2}\left\{x_{\mathrm{D}}^{\ell},\widehat{p}^{0}\right\}+\frac{\widehat{p}^{0}(\overrightarrow{s}_{8\mathrm{D}}\times\overrightarrow{p})^{\ell}}{\widehat{\omega}(\widehat{\omega}+m)}, \tag{56}$$

where the spin matrices $\overrightarrow{s}_{8\mathrm{D}}=(s_{8\mathrm{D}}^{\ell n})$ are given by (284)–(286) in [1] and the operator $\overrightarrow{x}_{\mathrm{D}}$ has the form

$$\overrightarrow{x}_{\mathrm{D}}=\overrightarrow{x}+\frac{i\overrightarrow{\Gamma}_{8}}{2\widehat{\omega}}-\frac{\overrightarrow{s}_{8}^{\Gamma}\times\overrightarrow{p}}{\widehat{\omega}(\widehat{\omega}+m)}-\frac{i\overrightarrow{p}(\overrightarrow{\Gamma}_{8}\cdot\overrightarrow{p})}{2\widehat{\omega}^{2}(\widehat{\omega}+m)}, \tag{57}$$

with specific spin matrices $\overrightarrow{s}_{8}^{\Gamma}$ given in (43).

It is easy to verify that the generators (55), (56) with SU(2) spin (284)–(286) from [1] commute with the operator $\left[i\partial_{0}-\Gamma_{8}^{0}(\overrightarrow{\Gamma}_{8}\cdot\overrightarrow{p}+m)\right]$ of equation (50), satisfy the commutation relations (16) of the Lie algebra of the Poincaré group and the corresponding Casimir operators are given by $p^{2}=\widehat{p}^{\mu}\widehat{p}_{\mu}=m^{2}\mathrm{I}_{8}$, $W=w^{\mu}w_{\mu}=m^{2}\overrightarrow{s}_{8\mathrm{D}}^{2}=\frac{3}{2}\left(\frac{3}{2}+1\right)m^{2}\mathrm{I}_{8}$.

The conclusion that equation (50) describes the local field of fermionic particle-antiparticle doublet of the spin s=(3/2,3/2) and mass $m>0$ (and its solution (53) is the local fermionic field of the above mentioned spin and nonzero mass) follows from the analysis of equations (50) and the above given calculation of the Casimir operators p^{2}, $W=w^{\mu}w_{\mu}$.

6. Brief conclusions

Hence, the equation (50) describes the spin s=(3/2,3/2) particle-antiparticle doublet on the same level, on which the standard 4-component Dirac equation describes the spin s=(1/2,1/2) particle-antiparticle doublet. Moreover, the external argument in the validity of such interpretation is the link with the corresponding RCQM of spin s=(3/2,3/2) particle-antiparticle doublet, where the quantum-mechanical interpretation is direct and evident. Therefore, the fermionic spin s=(3/2,3/2) properties of equation (50) are proved.

Contrary to the bosonic spin s=(1,0,1,0) properties of the equation (50) found in [1] (section 22), the fermionic spin s=(1/2,1/2,1/2,1/2) properties of this equation are evident. The fact that equation (50) describes the multiplet of two fermions with the spin s=1/2 and two antifermions with that spin can be proved much more easier then the above given consideration. The proof is similar to that given in the standard 4-component Dirac model. The detailed consideration can be found in sections 7, 9, 10 of [1]. Therefore, equation (50) has more extended property of the Fermi–Bose duality then the standard Dirac equation [35–39]. This equation has the property of the Fermi–Bose triality. The property of the Fermi–Bose triality of the manifestly covariant equation (50) means that this equation describes on equal level (i) the spin s=(1/2,1/2,1/2,1/2) multiplet of two spin s=(1/2,1/2) fermions and two spin s=(1/2,1/2) antifermions, (ii) the spin s=(1,0,1,0) multiplet of the vector and scalar bosons together with their antiparticles, (iii) the spin s=(3/2,3/2) particle-antiparticle doublet.

It is evident that equation (50) is new in comparison with the Pauli–Fierz [25], Rarita–Schwinger [26] and confirmed by Davydov [27] equations for the spin s=3/2 particle. Contrary to 16-component equations from [25–27] equation (50) is 8-component and does not need any additional condition. Formally equation (50) looks like to have some similar features with the Bargman–Wigner equation [29] for arbitrary spin, when the spin value is taken 3/2. The transformation $V_8^{\mp} = \frac{\mp\overrightarrow{\Gamma}_8\cdot\overrightarrow{p}+\widehat{\omega}+m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}$ looks like the transformation of Pursey [30] in the case of s=3/2. Nevertheless, the difference is clear. Equation (50) is not the ordinary direct sum of the two Dirac equations. Furthermore, the given here model is derived from the first principles of RCQM (not from the FW type representation of the canonical field theory). Our consideration is original and new. The link with corresponding RCQM, the proof of the symmetry properties and relativistic invariance, the well defined spin operator (284)–(286) in [1], the features of the Fermi–Bose duality (triality) of the equation (50), the interaction with electromagnetic field and many other characteristics are suggested firstly.

Interaction, quantization and Lagrange approach in the above given spin s=(3/2,3/2) model are completely similar to the Dirac 4-component theory and standard quantum electrodynamics. For example, the Lagrange function of the system of interacting 8-component spinor and electromagnetic fields (in the terms of 4-vector potential $A^{\mu}(x)$) is given by

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{i}{2}\left(\overline{\psi}(x)\Gamma_8^{\mu}\frac{\partial\psi(x)}{\partial x^{\mu}} - \frac{\partial\overline{\psi}(x)}{\partial x^{\mu}}\Gamma_8^{\mu}\psi(x)\right)-m\overline{\psi}(x)\psi(x)+q\overline{\psi}(x)\Gamma_8^{\mu}\psi(x)A_{\mu}(x), \quad (58)$$

where $\overline{\psi}(x)$ is the independent Lagrange variable and $\overline{\psi} = \psi^{\dagger}\Gamma_8^0$ in the space of solutions $\{\psi\}$. In Lagrangian (101) $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the electromagnetic field tensor in the terms of potentials, which play the role of variational variables in this Lagrange approach. Thus, the difficulties mentioned in [28] are absent here.

Therefore, the covariant local quantum field theory model for the interacting particles with spin s=3/2 and photons can be constructed in complete analogy to the construction of the modern quantum electrodynamics. This model can be useful for the investigations of processes with interacting hyperons and photons.

Other results mentioned in abstract are proved similarly. The consideration of these results is presented in [1] and [2].

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