Symmetry operators of the two-component Gross–Pitaevskii equation with a Manakov-type nonlocal nonlinearity

A V Shapovalov[1,2], A Yu Trifonov[1,2], and A L Lisok[2]

[1] Tomsk State University, Lenin ave., 36, Tomsk, Russia, 634050

[2] Tomsk Polytechnic University, Lenin ave., 30, Tomsk, Russia, 634034

Abstract.
We consider an integro-differential 2-component multidimensional Gross–Pitaevskii equation with a Manakov-type cubic nonlocal nonlinearity. In the framework of the WKB–Maslov semiclassical formalism, we obtain a semiclassically reduced 2-component nonlocal Gross–Pitaevskii equation determining the leading term of the semiclassical asymptotic solution. For the reduced Gross–Pitaevskii equation we construct symmetry operators which transform arbitrary solution of the equation into another solution. Constructing the symmetry operator is based on the Cauchy problem solution technique and uses an intertwining operator which connects two solutions of the reduced Gross–Pitaevskii equation. General structure of the symmetry operator is illustrated with a 1D case for which a family of symmetry operators is found explicitly and a set of exact solutions is generated.

1. Introduction

Symmetry properties of an equation are related to analytical methods of exact solution finding to the equation under consideration. Symmetries are usually can be found explicitly for rather narrow sets of equations. A way for extending classes of equations with symmetries is to use methods of symmetry analysis along with approximate asymptotic methods. For instance, in [1–3] a theory of approximate symmetries was developed for partial differential equations (PDEs) in the framework of the perturbation theory where solutions of an equation and its symmetries were presented as regular asymptotic power series in a formal small parameter.

Another way to relate ideas of symmetry and approximate methods was explored in [4, 5] where WKB–Maslov semiclassical approximation method [6, 7] as well as symmetry properties of nonlocal and nonlinear equations with partial derivatives were considered. Note that in contrast to the perturbation method, the semiclassical asymptotics depends singularly on a small asymptotic parameter.

To specify the problem statement, consider a nonlinear operator $\hat{F}$ acting in a functional space $\mathcal{L}$, $\hat{F}:\mathcal{L}\to\mathcal{L}$. By definition, an operator $\hat{A}:\mathcal{L}\to\mathcal{L}$ is a symmetry operator for $\hat{F}$ if

$$\hat{F}(u)=0\to\hat{F}(\hat{A}(u))=0, \tag{1}$$

or

$$\hat{F}(\hat{A}(u))=\hat{\Phi}(u,\hat{F}(u)),\quad \hat{\Phi}(u,0)=0,\quad \forall u\in\mathcal{L}, \tag{2}$$

where $\hat{\Phi}$ is an operator Lagrangian multiplier.

Symmetry operators can be used directly, for example, for generation of exact solutions from a known particular solution of the equation. Important application of symmetry operators is related to the Lie groups. Let $\hat{A}(\alpha)$ be a Lie group of symmetry operators with a generator $\hat{X}$ and a group parameter α, $\hat{X} = \frac{d\hat{A}(\alpha)}{d\alpha}|_{\alpha=0}$. From (2) one can obtain

$$\hat{F}'(u)\hat{X}(u) = \hat{R}(u)\hat{F}(u), \quad \forall u \in \mathcal{L}. \tag{3}$$

Here, $\hat{R}(u)$ for a given u is a linear operator acting on $\hat{F}(u)$, $\hat{R}$ plays the role of an operator Lagrangian multiplier, $\hat{F}'$ is the derivative of $\hat{F}$.

Unlike (2), equation (3) is the linear operator equation determining an operator $\hat{X}$. For partial differential operators $\hat{F}$ and $\hat{X}$ the determining equation (3) can be solved effectively for many important equations of mathematical physics. $\hat{X}$ is called a symmetry for $\hat{F}$ and $\hat{X}$ is the main object of study in the group analysis of PDEs [8–10].

Finding the symmetry operators $\hat{A}$ for an operator $\hat{F}$ of general form is ill-posed problem on account of complexity of the determining equation (2). Therefore, symmetry operators for nonlinear equations are explored a little. Nevertheless, the problem of symmetry operators finding can be not unreasonable for nonlinear equations of a special form.

In the present work we consider a 2-component generalized multidimensional Gross–Pitaevskii equation (GPE) with a Manakov-type nonlocal nonlinearity. The GPE is the basic model equation in the theory of the Bose–Einstein condensates (see, e.g., the review paper [11]). Nonlocal generalizations of the GPE are introduced to avoid the problem of the wave function collapse and for some other problems [12, 13].

Our results use as a main tool the WKB–Maslov formalism of semiclassical asymptotics (e.g., [6]). In terms of this formalism the leading term of the semiclassical asymptotic solution is determined by a reduced 2-component nonlocal GPE (R2GPE).

Here, we construct a class of symmetry operators for the R2GPE using the method developed in [5] for solving the Cauchy problem for the 1-component nonlocal GPE.

This article is structured as follows. In Section 2, following [5], we briefly describe the method of the Cauchy problem solution for the 2-component multidimensional GPE with a Manakov-type nonlocal nonlinearity in the semiclassical approximation. In the framework of the semiclassical approximation, we obtain the reduced 2-component nonlocal GPE which determines the leading term of the semiclassical asymptotics. In Section 3, based on the results of Section 2, we formulate a general structure describing a class of symmetry operators for the R2GPE. The core of this structure is an intertwining operator that connects two linear operators associated with the R2GPE in the framework of the semiclassical formalism. Section 4 contains an illustrative example where a set of symmetry operators is obtained in explicit form for a particular case of 1D R2GPE. A family of exact solutions is generated with the symmetry operators obtained. Concluding remarks are in Section 5.

2. Solution of the Cauchy problem

Consider the generalized 2-component Gross–Pitaevskii equation with the nonlocal Manakov-type interaction term:

$$\hat{F}(\Psi)(\vec{x},t) = \{-i\hbar\partial_t\mathbb{I} + \hat{H}_0(t)\mathbb{I} + \hbar(\vec{\sigma},\vec{H}(t)) + \varkappa\mathbb{I}\hat{V}(\Psi)(t)\}\Psi(\vec{x},t) = 0, \tag{4}$$

$$\hat{V}(\Psi)(t) = V(\Psi)(\hat{z},t) = \int\limits_{\mathbb{R}^n} d\vec{y}\,\Psi^+(\vec{y},t)V(\hat{z},\hat{w},t)\Psi(\vec{y},t), \tag{5}$$

where $\partial_t = \partial/\partial t$, $\Psi(\vec{x},t) = (\Psi_1(\vec{x},t), \Psi_2(\vec{x},t))^{\intercal}$, $\Psi^+(\vec{x},t) = (\Psi_1^*(\vec{x},t), \Psi_2^*(\vec{x},t))$. Here $\Psi_1(\vec{x},t)$ and $\Psi_2(\vec{x},t)$ are smooth complex scalar functions which belong to a complex Schwartz space

$\mathbb{S}$ in the space variable $\vec{x} \in \mathbb{R}^n$ at each time t, Ψ_1^* and Ψ_2^* denote the complex conjugate to Ψ_1 and Ψ_2, respectively, $\Psi^\intercal$ denotes the transpose of Ψ; $\mathbb{I} = \mathbb{I}_{2\times 2}$ is an 2×2 identity matrix; $\vec{H}(t) = (H_1(t), H_2(t), H_3(t))^\intercal$ is a vector function where $H_k(t)$, $k = 1, 2, 3$, are smooth real functions of t, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^\intercal$ is the column vector where σ_k are the Pauli matrices; $(\vec{\sigma}, \vec{H}(t)) = \sigma_1 H_1(t) + \sigma_2 H_2(t) + \sigma_3 H_3(t)$. We call $\hat{V}(\Psi)(t)$ in (5) the nonlocal Manakov-type nonlinear term. The linear operators $\hat{H}_0(t) = H_0(\hat{z}, t)$ and $V(\hat{z}, \hat{w}, t)$ in (4) and (5) are Hermitian Weyl-ordered functions [14] of time t and of noncommuting operators

$$\hat{z} = (\hat{\vec{p}}, \vec{x}) = (-i\hbar\partial/\partial\vec{x}, \vec{x}), \quad \hat{w} = (-i\hbar\partial/\partial\vec{y}, \vec{y}), \quad \vec{x}, \vec{y} \in \mathbb{R}^n, \tag{6}$$

with commutators $[\hat{z}_k, \hat{z}_j]_- = [\hat{w}_k, \hat{w}_j]_- = i\hbar J_{kj}$, $[\hat{z}_k, \hat{w}_j]_- = 0$, $k, j = \overline{1, 2n}$, where $[\hat{A}, \hat{B}]_- = \hat{A}\hat{B} - \hat{B}\hat{A}$, $J = \|J_{kj}\|_{2n\times 2n}$ is a unit symplectic matrix: $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}_{2n\times 2n}$, and $I = I_{n\times n}$ is an $n \times n$ identity matrix. The space $\mathbb{S}$ is used to provide for existence of the moments of $\Psi(\vec{x}, t)$ and convergence of the integral in (5). In what follows, we use the norm $\|\Psi\|$, $\Psi \in \mathbb{S}(\mathbb{R}^n_x, \mathbb{C}^2)$, of the space $L_2(\mathbb{R}^n_x, \mathbb{C}^2)$, i.e., $\|\Psi\| = \sqrt{(\Psi, \Psi)}$, $(\Phi, \Psi) = \int\limits_{\mathbb{R}^n} d\vec{x}\, \Phi^+(\vec{x})\Psi(\vec{x})$ denotes the Hermitian inner product of the functions $\Phi, \Psi \in \mathbb{S}(\mathbb{R}^n_x, \mathbb{C}^2)$.

From equation (4) it follows that the squared norm of a solution $\Psi(\vec{x}, t)$ is conserved, $\|\Psi(t)\|^2 = \|\Psi(0)\|^2 = \text{const}$.

Equation (4) with variable coefficients of general form cannot be integrated exactly by known methods. Therefore, analytical solutions to this equation we will construct approximately using the method of semiclassical asymptotics as $\hbar \to 0$. The semiclassical solutions of the 2-component nonlocal GPE (4) can be constructed using an appropriate complex WKB–Maslov ansatz [6, 7].

Following [5], we introduce a class $\mathcal{K}_\hbar^t$ of two-component functions

$$\mathcal{K}_\hbar^t = \left\{\Phi : \Phi(\vec{x}, t, \hbar) = \sum_{k=0}^{\infty} \hbar^{k/2} u^{(k)}(t) \widetilde{\Phi}^{(k)}(\vec{x}, t, \hbar)\Big]\right\}, \tag{7}$$

where $u^{(k)}(t)$ is a two-component vector-function smoothly dependent on t; $\tilde{\Phi}^{(k)}(\vec{x}, t, \hbar) \in \mathcal{C}_\hbar^t$, $k = \overline{1, \infty}$. Here

$$\mathcal{C}_\hbar^t = \left\{\Phi : \Phi(\vec{x}, t) = \varphi\Big(\frac{\Delta\vec{x}}{\sqrt{\hbar}}, t\Big)\mathcal{R}\big(S(t, \hbar), \vec{P}(t, \hbar), \Delta\vec{x}\big)\right\}, \quad \mathcal{R}\big(S, \vec{\xi}, \vec{\zeta}\big) = \exp\Big[\frac{i}{\hbar}(S + \langle\vec{\xi}, \vec{\zeta}\rangle)\Big],$$

where $\vec{\xi}, \vec{\zeta} \in \mathbb{R}^n$, the angle brackets $\langle ., . \rangle$ denote the Euclidean inner product of vectors: $\langle\vec{\xi}, \vec{\zeta}\rangle = \sum\limits_{j=1}^{n} \xi_j \zeta_j$; the function $\varphi(\vec{\xi}, t, \hbar)$ belongs to the Schwarz space $\mathbb{S}$ in the variable $\vec{\xi}$ and smoothly depends on t. Here $\Delta\vec{x} = \vec{x} - \vec{X}(t, \hbar)$, and the real function $S(t, \hbar)$ and the $2n$-dimensional vector function $Z(t, \hbar) = (\vec{P}(t, \hbar), \vec{X}(t, \hbar))$, which characterize the class $\mathcal{C}_\hbar^t$, regularly depend on $\sqrt{\hbar}$ in the neighborhood of $\hbar = 0$ and *are to be determined*. Note that $\mathcal{C}_\hbar^t \subset \mathbb{S}$.

In accordance with [5], we call the functions of the class $\mathcal{K}_\hbar^t$ *two-component trajectory-concentrated functions*. The functions belonging to the class $\mathcal{C}_\hbar^t$ at any fixed time $t \in \mathbb{R}^1$ are *concentrated*, as $\hbar \to 0$, in the neighborhood of a point lying on the phase curve $z = Z(t, 0)$, $t \in \mathbb{R}^1$ [15]. This property motivates the term "trajectory-concentrated functions".

In the next section 3 we construct symmetry operators using the complex WKB–Maslov asymptotic solutions of the Cauchy problem to equation (4) with the initial condition:

$$\Psi(\vec{x}, t)\big|_{t=s} = \psi(\vec{x}) = u_0 \cdot \phi(\vec{x}), \quad \phi(\vec{x}) \in \mathcal{C}_\hbar^0, \tag{8}$$

where $u_0 = (u_0^1, u_0^2)^\intercal$ is a constant 2-component vector and $\phi(\vec{x})$ is a scalar complex function,

$$\mathcal{C}_\hbar^0 = \Big\{\phi : \phi(\vec{x}, \hbar) = \varphi\Big(\frac{\Delta\vec{x}}{\sqrt{\hbar}}\Big)\mathcal{R}\big(S_0(\hbar), \vec{P}_0(\hbar), \Delta\vec{x}_0\big)\Big\}, \quad \Delta\vec{x}_0 = \vec{x} - \vec{X}_0(\hbar).$$

Let us use $\widehat{O}(\hbar^\nu)$ to designate an operator $\hat{F}$ such that for any function $\Phi \in \mathcal{K}_\hbar^t$ the following asymptotic estimate is valid:

$$\frac{\|\hat{F}\Phi\|}{\|\Phi\|} = O(\hbar^\nu), \qquad \hbar \to 0. \tag{9}$$

In accordance with (9), it may be shown (see [15,16]) that

$$\Delta\hat{z} = \hat{z} - Z(t, \hbar) = \widehat{O}(\hbar^{1/2}), \quad \hbar \to 0. \tag{10}$$

Let us expand the operators $\hat{H}_0(t) = H_0(\hat{z}, t)$ and $V(\hat{z}, \hat{w}, t)$ in (4) and (5) as Taylor series in the operators $\Delta\hat{z} = \hat{z} - Z(t, \hbar)$ and $\Delta\hat{w} = \hat{w} - Z(t, \hbar)$, respectively, and restrict ourselves to quadratic terms. Then, in view of (10), the solution of the Cauchy problem (4) and (8) asymptotic in a formal small parameter $\hbar$ ($\hbar \to 0$) can be constructed accurate to $O(\hbar^{3/2})$ (see [5,16]). The leading-order term of the asymptotics can be found by reducing the generalized GPE (4) to an equation with a quadratic nonlocal operator.

The higher-order corrections to the leading-order term can be found using perturbation theory [16]. Thus the study of GPEs with a quadratic nonlocal operator is crucial for the construction of semiclassical asymptotics for the GPE of the form (4) in the class of trajectory concentrated functions (7). Without loss of generality, we consider a reduced 2-component nonlocal GPE of the form

$$\Big\{-i\hbar\partial_t\mathbb{I} + H_{\rm qu}(\hat{z}, t)\mathbb{I} + \hbar(\vec{\sigma}, \vec{H}(t)) + \varkappa\mathbb{I}\int_{\mathbb{R}^n} d\vec{y}\, \Psi^+(\vec{y}, t)V_{\rm qu}(\hat{z}, \hat{w}, t)\Psi(\vec{y}, t)\Big\}\Psi(\vec{x}, t) = 0, \tag{11}$$

where the linear operators $H_{\rm qu}(\hat{z}, t)$ and $V_{\rm qu}(\hat{z}, \hat{w}, t)$ are Hermitian and quadratic in $\hat{z}$, $\hat{w}$, respectively:

$$H_{\rm qu}(\hat{z}, t) = \frac{1}{2}\langle\hat{z}, \mathcal{H}_{zz}(t)\hat{z}\rangle + \langle\mathcal{H}_z(t), \hat{z}\rangle, \tag{12}$$

$$V_{\rm qu}(\hat{z}, \hat{w}, t) = \frac{1}{2}\langle\hat{z}, W_{zz}(t)\hat{z}\rangle + \langle\hat{z}, W_{zw}(t)\hat{w}\rangle + \frac{1}{2}\langle\hat{w}, W_{ww}(t)\hat{w}\rangle. \tag{13}$$

Here, $\mathcal{H}_{zz}(t)$, $W_{zz}(t)$, $W_{zw}(t)$, and $W_{ww}(t)$ are $2n \times 2n$ matrices; $\mathcal{H}_z(t)$ is a $2n$ vector; $\langle z, w\rangle = \sum\limits_{j=1}^{2n} z_j w_j$; $z, w \in \mathbb{R}^{2n}$.

We call equation (11) with the linear operators $H_{\rm qu}$ and $V_{\rm qu}$ given by (12) and (13), respectively, *a reduced 2-component nonlocal Gross–Pitaevskii equation* (R2GPE).

The R2GPE can be integrated explicitly similar to [5] and its symmetries are of interest in the symmetry analysis of integro-differential equations (IDEs).

In study of symmetries of nonlocal equations the problem arises how to choose a structure of symmetries for non-differential equations. The R2GPE provides an example of IDE, when symmetry operators can be found explicitly.

Let us consider briefly a method for solving the Cauchy problem (8) for the R2GPE (11), following the scheme described elsewhere [5,16].

We denote the Weyl-ordered symbol of an operator $\hat{A}(t) = A(\hat{z}, t)$ by $A(z, t)$ and define the expectation value for $\hat{A}(t)$ over the state $\Psi(\vec{x}, t)$ as

$$A_\Psi(t) = \frac{1}{\|\Psi\|^2}(\Psi, \hat{A}(t)\Psi) = \frac{1}{\|\Psi\|^2}\int\limits_{\mathbb{R}^n} d\vec{x}\, \Psi^+(\vec{x}, t)\hat{A}(t)\Psi(\vec{x}, t).$$

As $\|\Psi\|^2$ does not depend on time, we have from (11),(12), and (13)

$$\dot{A}_\Psi(t) = \frac{1}{\|\Psi\|^2}\int\limits_{\mathbb{R}^n} d\vec{x}\, \Psi^+(\vec{x}, t)\Big\{\frac{\partial \hat{A}(t)}{\partial t} + \frac{i}{\hbar}[\mathbb{I}H_{\text{qu}}(\hat{z}, t), \hat{A}(t)]_- + i[(\vec{\sigma}, \vec{H}(t)), \hat{A}(t)]_- + \\ + \frac{i\tilde{\varkappa}}{\hbar}\int\limits_{\mathbb{R}^n} d\vec{y}\, \Psi^+(\vec{y}, t)[\mathbb{I}V_{\text{qu}}(\hat{z}, \hat{w}, t), \hat{A}(t)]_- \Psi(\vec{y}, t)\Big\}\Psi(\vec{x}, t), \qquad (14)$$

where $\dot{A}_\Psi(t) = dA_\Psi(t)/dt$ and $\tilde{\varkappa} = \varkappa\|\Psi\|^2 = \varkappa\|\psi\|^2$.

Let $z_\Psi(t) = \big(z_{\Psi\, l}(t)\big)$, $\vec{\eta}_\Psi(t)$ and $\Delta^{(2)}_\Psi(t) = \big(\Delta^{(2)}_{\Psi\, kl}(t)\big)$ denote the expectation values over $\Psi(\vec{x}, t)$ for the operators

$$\hat{z}_l\mathbb{I}, \quad \vec{\sigma}, \quad \hat{\Delta}^{(2)}_{kl}\mathbb{I} = \frac{1}{2}\Big(\mathbb{I}\Delta\hat{z}_k\Delta\hat{z}_l + \mathbb{I}\Delta\hat{z}_l\Delta\hat{z}_k,\Big), \quad k, l = \overline{1, 2n},$$

respectively. Here, $\Delta\hat{z}_l = \hat{z}_l - (z_\Psi)_l(t)$, $\hat{z}_l$ are given in (6), and we set $Z(t, \hbar) = (z_\Psi)(t)$. We call $z_\Psi(t)$ the first moments and $\Delta^{(2)}_\Psi(t)$ do the second centered moments of $\Psi(\vec{x}, t)$.

From (11), (12), (13), and (14) we immediately obtain a dynamical system in matrix notation:

$$\begin{cases} \dot{z}_\Psi = J\{\mathcal{H}_z(t) + [\mathcal{H}_{zz}(t) + \tilde{\varkappa}(W_{zz}(t) + W_{zw}(t))]z_\Psi\}, \\ \dot{\Delta}^{(2)}_\Psi = J[\mathcal{H}_{zz}(t) + \tilde{\varkappa}W_{zz}(t)]\Delta^{(2)}_\Psi - \Delta^{(2)}_\Psi[\mathcal{H}_{zz}(t) + \tilde{\varkappa}W_{zz}(t)]J, \\ \dot{\vec{\eta}}_\Psi(t) = 2\vec{H}(t) \times \vec{\eta}_\Psi(t). \end{cases} \qquad (15)$$

Following [5], we call (15) *the Hamilton–Ehrenfest system* (HES) of the second order for the R2GPE (11).

The HES is of the second order, as equations (15) contain the first and second moments. For brevity, we use a shorthand notation for the total set of the first and second moments of $\Psi(\vec{x}, t)$:

$$\mathfrak{g}_\Psi(t) = \big(z_\Psi(t), \vec{\eta}_\Psi(t), \Delta^{(2)}_\Psi(t)\big). \qquad (16)$$

Then the R2GPE (11) can be written equivalently as

$$\hat{L}(t, \mathfrak{g}_\Psi(t))\Psi(\vec{x}, t) = \Big\{ - i\hbar\partial_t\mathbb{I} + \hat{H}_q(t, \mathfrak{g}_\Psi(t))\mathbb{I} + \hbar(\vec{\sigma}, \vec{H}(t))\Big\}\Psi(\vec{x}, t) = 0, \qquad (17)$$

$$\hat{H}_q(t, \mathfrak{g}_\Psi(t)) = \frac{1}{2}\langle\hat{z}, \mathcal{H}_{zz}(t)\hat{z}\rangle + \langle\mathcal{H}_z(t), \hat{z}\rangle + \frac{\tilde{\varkappa}}{2}\langle\hat{z}, W_{zz}(t)\hat{z}\rangle + \\ + \frac{\tilde{\varkappa}}{2}\langle z_\Psi(t), W_{ww}(t)z_\Psi(t)\rangle + \tilde{\varkappa}\langle\hat{z}, W_{zw}(t)z_\Psi(t)\rangle + \frac{\tilde{\varkappa}}{2}\,\mathrm{Sp}\Big[W_{ww}(t)\Delta^{(2)}_\Psi(t)\Big]. \qquad (18)$$

The Cauchy problem (17) and (8) for the R2GPE gives the Cauchy problem for the HES (15) which can be written in a concise form as

$$\dot{\mathfrak{g}}_\Psi(t) = \Gamma(t, \mathfrak{g}_\Psi(t)), \qquad (19)$$

$$\mathfrak{g}_\Psi(t)\Big|_{t=s} = \mathfrak{g}_\psi, \tag{20}$$

where $\Gamma\big(t, \mathfrak{g}_\Psi(t)\big)$ designates the rhs of (15).

The system (17), (19) allows us to reduce the Cauchy problem for the R2GPE (17) to the Cauchy problem for a linear PDE because the Cauchy problem (20) for HES (19) can be solved independently of equation (17).

Let

$$\mathfrak{g}(t, \mathbf{C}) = \big(Z(t, \mathbf{C}), \vec{\eta}(t, \mathbf{C}), \Delta^{(2)}(t, \mathbf{C})\big) \tag{21}$$

be the general solution of the HES (19) and $\mathbf{C} = (C_1, C_2, \dots, C_N)$ denote the set of integration constants.

Consider a linear PDE with coefficients depending on the parameters $\mathbf{C}$:

$$\hat{L}(t, \mathbf{C})\Phi(\vec{x}, t, \mathbf{C}) = \big\{ -i\hbar\partial_t \mathbb{I} + \hat{H}_q(t, \mathbf{C})\mathbb{I} + \hbar(\vec{\sigma}, \vec{H}(t))\big\}\Phi(\vec{x}, t, \mathbf{C}) = 0, \tag{22}$$

where

$$\begin{aligned} \hat{H}_q(t, \mathbf{C}) = \frac{1}{2}\langle \hat{z}, \mathcal{H}_{zz}(t)\hat{z}\rangle + \langle \mathcal{H}_z(t), \hat{z}\rangle + \frac{\tilde{\varkappa}}{2}\langle \hat{z}, W_{zz}(t)\hat{z}\rangle + \tilde{\varkappa}\langle \hat{z}, W_{zw}(t)Z(t, \mathbf{C})\rangle + \\ + \frac{\tilde{\varkappa}}{2}\langle Z(t, \mathbf{C}), W_{ww}(t)Z(t, \mathbf{C})\rangle + \frac{\tilde{\varkappa}}{2}\,\mathrm{Sp}\Big[W_{ww}(t)\Delta^{(2)}(t, \mathbf{C})\Big]. \end{aligned} \tag{23}$$

The operator $\hat{H}_q(t, \mathbf{C})$ (23) of (22) is obtained from (18) where the general solution $\mathfrak{g}(t, \mathbf{C})$ of the HES (19) stands for the moments $\mathfrak{g}_\Psi(t)$. We call (22) *the associated linear equation* (ALE) for the R2GPE (17).

A solution of the ALE (22) can be found in the form

$$\Phi(\vec{x}, t, \mathbf{C}) = \phi(\vec{x}, t, \mathbf{C}) \cdot v(t, \mathbf{C}), \tag{24}$$

where a scalar complex function $\phi(\vec{x}, t, \mathbf{C})$ and a 2-component complex function $v(t, \mathbf{C})$ satisfy the equations

$$\big\{ -i\hbar\partial_t + \hat{H}_q(t, \mathbf{C})\big\}\phi(\vec{x}, t, \mathbf{C}) = 0, \tag{25}$$

$$\big\{ -i\mathbb{I}\frac{d}{dt} + (\vec{\sigma}, \vec{H}(t))\big\}v(t, \mathbf{C}) = 0. \tag{26}$$

Let $\Phi(\vec{x}, t, \mathbf{C}[\psi])$ denote the solution of the Cauchy problem for the ALE (22) with the initial condition (8):

$$\Phi(\vec{x}, t, \mathbf{C}[\psi])\Big|_{t=s} = \psi(\vec{x}) = u_0\phi(\vec{x}). \tag{27}$$

Then for (24), (25), and (26) we have, respectively,

$$\phi(\vec{x}, t, \mathbf{C}[\psi])\Big|_{t=s} = \phi(\vec{x}), \qquad v(t, \mathbf{C}[\psi])\Big|_{t=s} = u_0. \tag{28}$$

Here, the integration constants $\mathbf{C}$ in (25) and (26) have been replaced by the functionals $\mathbf{C} = \mathbf{C}[\psi]$ determined from the algebraic conditions

$$\mathfrak{g}(t, \mathbf{C})\Big|_{t=s} = \mathfrak{g}_\psi. \tag{29}$$

Then the solution of the Cauchy problem (8) and (17), (18) for the R2GPE (see [5, 16] for details) is

$$\Psi(\vec{x}, t) = \Phi(\vec{x}, t, \mathbf{C}[\psi]) = \varphi(\vec{x}, t) \cdot u(t), \tag{30}$$

where $\varphi(\vec{x},t) = \phi(\vec{x},t,\mathbf{C}[\psi])$ is the scalar complex function and $u(t) = v(t,\mathbf{C}[\psi])$.

Define $\mathbf{C}[\Psi](t)$ by the algebraic condition

$$\mathfrak{g}(t,\mathbf{C}[\Psi](t)) = \mathfrak{g}_\Psi(t). \tag{31}$$

From the uniqueness of the solution of the Cauchy problem for the HES (20) it follows that

$$\mathfrak{g}(t,\mathbf{C}[\Psi](t)) = \mathfrak{g}(t,\mathbf{C}[\psi])$$

and, hence,

$$\mathbf{C}[\Psi](t) = \mathbf{C}[\psi],$$

i.e., the functionals $\mathbf{C}[\Psi](t)$ are the integrals of (4).

Also, we have

$$\mathfrak{g}(t,\mathbf{C}[\psi]) = \mathfrak{g}_\psi(t), \tag{32}$$

where $\mathfrak{g}_\psi(t)$ is the solution of the HES (19) with the initial condition (29).

Let us now turn to the construction of symmetry operators for the R2GPE (11). To do this, we use an operator intertwining a pair of ALEs of the form (22).

3. Symmetry operators

In this section, we apply the approach developed in [5] for the 1-component nonlocal GPE to find symmetry operators for the reduced 2-component nonlocal Gross–Pitaevskii equation (11) or, equivalently, (17).

According to definition (1), the nonlinear symmetry operator $\hat{A}(t)$ maps any solution $\Psi(\vec{x},t)$ of equation (17) into its another solution:

$$\Psi_A(\vec{x},t) = (\hat{A}(t)\Psi)(\vec{x},t). \tag{33}$$

The main idea of the symmetry operator construction is as follows. Let us take an operator $\hat{a}$ acting on the initial function $\psi(\vec{x})$ of the Cauchy problem (8) and set

$$\psi_a(\vec{x}) = \hat{a}\psi(\vec{x}) = \Psi_A(t)\big|_{t=s}, \quad \hat{A}(t)\big|_{t=s} = \hat{a}. \tag{34}$$

Define the functions $\Psi(\vec{x},t)$ and $\Psi_A(\vec{x},t)$ of the form (30) in (33) as two solutions of the Cauchy problem for the R2GPE (11) with two initial functions $\psi(\vec{x})$ and $\psi_a(\vec{x})$, respectively. Then the desired symmetry operator $\hat{A}(t)$ should relate $\Psi(\vec{x},t)$ and $\Psi_A(\vec{x},t)$.

Introduce the notation $\mathfrak{g}_{\psi_a}$ for the first and second moments of $\psi_a(\vec{x})$ similar to (16). From the solution of the Cauchy problem for the HES (20) with the initial condition $\mathfrak{g}_\Psi(t)\Big|_{t=s} = \mathfrak{g}_{\psi_a}$, by analogy with (32), we have

$$\mathfrak{g}(t,\mathbf{C}[\psi_a]) = \mathfrak{g}_{\psi_a}(t). \tag{35}$$

According to (30), the solutions $\Psi(\vec{x},t)$ and $\Psi_A(\vec{x},t)$ of the R2GPE (17) are found as

$$\Psi(\vec{x},t) = \Phi(\vec{x},t,\mathbf{C})\big|_{\mathbf{C}=\mathbf{C}[\psi]} \tag{36}$$

and

$$\Psi_A(\vec{x},t) = \Phi(\vec{x},t,\mathbf{C}')\big|_{\mathbf{C}'=\mathbf{C}'[\psi_a]}, \tag{37}$$

where $\Phi(\vec{x},t,\mathbf{C})$ and $\Phi(\vec{x},t,\mathbf{C}')$ are the solutions of two ALEs of the form (22) with two different sets of integration constants $\mathbf{C}$ and $\mathbf{C}'$, respectively, and the corresponding linear operators $\hat{L}(t,\mathbf{C})$ and $\hat{L}(t,\mathbf{C}')$.

The essential point in the construction of the symmetry operator $\hat{A}(t)$ in (33) is a linear operator $\hat{M}(t, s, \mathbf{C}', \mathbf{C})$ intertwining the operators $\hat{L}(t, \mathbf{C}')$ and $\hat{L}(t, \mathbf{C})$:

$$\hat{L}(t, \mathbf{C}')\hat{M}(t, s, \mathbf{C}', \mathbf{C}) = \hat{R}(t, s, \mathbf{C}', \mathbf{C})\hat{L}(t, \mathbf{C}). \tag{38}$$

Here, the linear operator $\hat{R}(t, s, \mathbf{C}', \mathbf{C})$ is a Lagrangian multiplier, and the initial condition is $\hat{M}(t, s, \mathbf{C}', \mathbf{C})|_{t=s} = \hat{a}$.

From (38) we have that $\Phi(\vec{x}, t, \mathbf{C}') = \hat{M}(t, s, \mathbf{C}', \mathbf{C})\Phi(\vec{x}, t, \mathbf{C})$ for two arbitrary sets of constants $\mathbf{C}'$ and $\mathbf{C}$, and this is especially true for $\Phi(\vec{x}, t, \mathbf{C}'[\psi_a])$ and $\Phi(\vec{x}, t, \mathbf{C}[\psi])$ with the constants $\mathbf{C}'[\psi_a]$ and $\mathbf{C}[\psi]$.

The intertwining operator $\hat{M}(t, s, \mathbf{C}', \mathbf{C})$ relates the functions $\Phi(\vec{x}, t, \mathbf{C}'[\psi_a])$ and $\Phi(\vec{x}, t, \mathbf{C}[\psi])$ that in turn relates $\Psi_A(\vec{x}, t)$ and $\Psi(\vec{x}, t)$ according to (37) and (36).

To find the operator $\hat{M}(t, s, \mathbf{C}', \mathbf{C})$, we consider a linear intertwining operator $\widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})$ for $\hat{L}(t, \mathbf{C}')$ and $\hat{L}(t, \mathbf{C})$ satisfying the conditions

$$\hat{L}(t, \mathbf{C}')\widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C}) = \widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})\hat{L}(t, \mathbf{C}), \tag{39}$$

$$\widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})\Big|_{t=s} = \hat{\mathbb{I}}, \tag{40}$$

where $\hat{\mathbb{I}}$ is an identity operator. We call $\widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})$ *the fundamental intertwining operator* for $\hat{L}(t, \mathbf{C}')$ and $\hat{L}(t, \mathbf{C})$.

With the use of $\widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})$, the operator $\hat{M}(t, s, \mathbf{C}', \mathbf{C})$ entering into (38) can be presented as

$$\hat{M}(t, s, \mathbf{C}', \mathbf{C}) = \widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})\widehat{B}(t, \mathbf{C}).$$

Here, $\widehat{B}(t, \mathbf{C}) \in \mathcal{B}$ is the linear symmetry operator of ALE (22) satisfying the conditions

$$[\hat{L}(t, \mathbf{C}), \widehat{B}(t, \mathbf{C})]_- = 0, \quad \widehat{B}(t, \mathbf{C})|_{t=s} = \hat{a}, \tag{41}$$

and $\mathcal{B}$ is the family of linear symmetry operators of the ALE (41).

Hence, given the operator $\widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})$ of (39) and the family $\mathcal{B}$ of linear symmetry operators of the ALE (41), we can construct the family of nonlinear symmetry operators for the GPE (4) as follows:

Let $\widehat{\mathcal{D}}(t, s, \mathbf{C}', \mathbf{C})$ be the fundamental intertwining operator (39) for $\hat{L}(t, \mathbf{C}')$ and $\hat{L}(t, \mathbf{C})$ and let $\widehat{B}(t, \mathbf{C})$ be the linear symmetry operator of the ALE (22) satisfying the conditions (41), then

$$(\hat{A}(t)\Psi)(\vec{x}, t) = \widehat{\mathcal{D}}(t, s, \mathbf{C}'[\hat{a}\psi], \mathbf{C}[\Psi](t))\widehat{B}(t, \mathbf{C}[\Psi](t))\Psi(\vec{x}, t) \tag{42}$$

defines the family of nonlinear symmetry operators for the GPE (11).

Here, $\mathbf{C}'[\hat{a}\psi]$ and $\mathbf{C}[\Psi](= \mathbf{C}[\psi])$ can be found from (32) and (31), respectively, and $\widehat{B} \in \mathcal{B}$.

Note that the symmetry operator $\hat{A}(t)$ from (42) is nonlinear, as the operators $\widehat{\mathcal{D}}$ and $\widehat{B}$ depend on the parameters $\mathbf{C}$ being functionals of the function Ψ.

4. Symmetry operators in the 1D case

As an illustration of the general formula (42), we will explicitly construct here the symmetry operators for the 1D R2GPE (11) of the form

$$\hat{F}(\Psi)(x, t) = \{-i\hbar\partial_t\mathbb{I} + H_{\text{qu}}(\hat{p}, x)\mathbb{I} + \hbar(\vec{\sigma}, \vec{\mathcal{B}}) + \varkappa\mathbb{I}\hat{V}_{\text{qu}}(\Psi)\}\Psi(x, t) = 0, \tag{43}$$

where $H_{\text{qu}}(\hat{p}, x) = \dfrac{1}{2}\left(\mu\hat{p}^2 + \rho(x\hat{p} + \hat{p}x) + \omega x^2\right)$, $\hat{V}_{\text{qu}}(\Psi) = \dfrac{1}{2}\int\limits_{-\infty}^{+\infty} dy\ \left(ax^2 + 2bxy + cy^2\right)|\Psi(y)|^2$, $\hat{p} = -i\hbar\partial/\partial x$; a, b, and c are the real parameters of the nonlocal operator $\hat{V}_{\text{qu}}(\Psi)$; μ, ω, and ρ

are the parameters of the linear operator $H_{\rm qu}(\hat{p},x)$; $x,y\in\mathbb{R}^1$, $\vec{\mathcal{B}}$ is a real constant vector. The initial condition (8) is written as

$$\Psi(x,t)\big|_{t=0}=\psi(x)=\phi(x)\cdot u_0, \tag{44}$$

where $\phi(x)$ is a scalar function and $u_0=(u_0^1,u_0^2)^\intercal$ is a constant 2-component vector.

The Hamilton–Ehrenfest system (19) for the first-order moments becomes

$$\begin{cases}\dot{p}=-\rho p-\omega_0 x,\\ \dot{x}=\mu p+\rho x,\end{cases} \tag{45}$$

for the vector $\vec{\eta}(t)=\langle\Psi|\vec{\sigma}|\Psi\rangle$ we similarly obtain

$$\dot{\vec{\eta}}(t)=2\vec{\mathcal{B}}\times\vec{\eta}(t),\quad \vec{\eta}^2(t)=1, \tag{46}$$

and for the second-order moments with $\Delta_{21}^{(2)}=\Delta_{12}^{(2)}$ we have

$$\begin{cases}\dot{\Delta}_{11}^{(2)}=-2\rho\Delta_{11}^{(2)}-2\tilde{\omega}\Delta_{21}^{(2)},\\ \dot{\Delta}_{21}^{(2)}=\mu\Delta_{11}^{(2)}-\tilde{\omega}\Delta_{22}^{(2)},\\ \dot{\Delta}_{22}^{(2)}=2\mu\Delta_{21}^{(2)}+2\rho\Delta_{22}^{(2)},\end{cases} \tag{47}$$

where $\omega_0=\omega+\tilde{\varkappa}(a+b)$, $\tilde{\omega}=\omega+\tilde{\varkappa}a$. We assume that $\bar{\Omega}^2=\omega_0\mu-\rho^2>0$, $\Omega^2=\tilde{\omega}\mu-\rho^2>0$ and introduce the notations

$$\bar{\Omega}=\sqrt{\omega_0\mu-\rho^2},\qquad \Omega=\sqrt{\tilde{\omega}\mu-\rho^2}. \tag{48}$$

Then the general solution of system (45) is

$$\begin{aligned}&X(t,\mathbf{C})=C_1\sin\bar{\Omega}t+C_2\cos\bar{\Omega}t,\\ &P(t,\mathbf{C})=\frac{1}{\mu}(\bar{\Omega}C_1-\rho C_2)\cos\bar{\Omega}t-\frac{1}{\mu}(\bar{\Omega}C_2+\rho C_1)\sin\bar{\Omega}t,\end{aligned} \tag{49}$$

and for system (47) we have

$$\begin{aligned}\Delta_{22}^{(2)}(t,\mathbf{C})&=C_3\sin 2\Omega t+C_4\cos 2\Omega t+C_5,\\ \Delta_{21}^{(2)}(t,\mathbf{C})&=\frac{1}{\mu}(\Omega C_3-\rho C_4)\cos 2\Omega t-\frac{1}{\mu}(\Omega C_4+\rho C_3)\sin 2\Omega t-\frac{\rho}{\mu}C_5,\\ \Delta_{11}^{(2)}(t,\mathbf{C})&=\frac{1}{\mu^2}\big((\rho^2-\Omega^2)C_3+2\rho\Omega C_4\big)\sin 2\Omega t+\\ &\quad+\frac{1}{\mu^2}\big((\rho^2-\Omega^2)C_4-2\rho\Omega C_3\big)\cos 2\Omega t+\frac{\tilde{\omega}}{\mu}C_5.\end{aligned} \tag{50}$$

Note that all solutions of system (49) and (47) are localized. Similarly, solution of system (46) subject to the normalization condition $\vec{\eta}^2(t)=1$ reads

$$\vec{\eta}(t,\mathbf{C})=\vec{n}\cos C_6+\vec{e}_\theta\sin C_6\cos(2\mathcal{B}t+C_7)+\vec{e}_\varphi\sin C_6\sin(2\mathcal{B}t+C_7). \tag{51}$$

Here, we use the notations: $\mathbf{C}=(C_1,\dots,C_7)$ are arbitrary integration constants, $\mathcal{B}=|\vec{\mathcal{B}}|$, and

$$\begin{aligned}&\vec{n}=\frac{\vec{\mathcal{B}}}{\mathcal{B}}=(\sin\theta\cos\varphi,\,\sin\theta\sin\varphi,\,\cos\theta),\\ &\vec{e}_\theta=(\cos\theta\cos\varphi,\,\cos\theta\sin\varphi,\,-\sin\theta),\\ &\vec{e}_\varphi=(\sin\varphi,\,-\cos\varphi,\,0),\\ &\vec{n}\times\vec{e}_\theta=-\vec{e}_\varphi,\quad \vec{n}\times\vec{e}_\varphi=\vec{e}_\theta,\quad \vec{e}_\varphi\times\vec{e}_\theta=\vec{n}.\end{aligned} \tag{52}$$

The 1D associated linear equation (22) for the 1D R2GPE (43) takes the form

$$\hat{L}(t,\mathbf{C})\Phi(\vec{x},t,\mathbf{C}) = \big\{-i\hbar\partial_t\mathbb{I} + \hat{H}_q(t,\mathbf{C})\mathbb{I} + \hbar(\vec{\sigma},\vec{\mathcal{B}})\big\}\Phi(\vec{x},t,\mathbf{C}) = 0, \tag{53}$$
$$\hat{H}_q(t,\mathbf{C}) = \frac{\mu\hat{p}^2}{2} + \frac{\tilde{\omega}x^2}{2} + \frac{\rho(x\hat{p}+\hat{p}x)}{2} + \tilde{\varkappa}bxX(t,\mathbf{C}) + \tilde{\varkappa}\frac{c}{2}\big[X^2(t,\mathbf{C}) + \Delta_{22}^{(2)}(t,\mathbf{C})\big].$$

Denote by $\widehat{\mathcal{D}}(t,\mathbf{C}',\mathbf{C})$ the fundamental intertwining operator in (39), (40) as $s=0$. To find the operator $\widehat{\mathcal{D}}(t,\mathbf{C}',\mathbf{C})$, we introduce a function $\Upsilon(\vec{x},t,\mathbf{C})$ by the conditions

$$\Phi(x,t,\mathbf{C}) = \hat{\mathcal{K}}(x,t,\mathbf{C})\Upsilon(\vec{x},t,\mathbf{C}), \tag{54}$$
$$\hat{\mathcal{K}}(x,t,\mathbf{C}) = \hat{K}(x,t,\mathbf{C})\mathcal{G}(t,\mathbf{C}),$$
$$\hat{K}(x,t,\mathbf{C}) = \exp[-X(t,\mathbf{C})\partial_x]\exp\Big\{\frac{i}{\hbar}[S(t,\mathbf{C}) + P(t,\mathbf{C})x]\Big\}, \tag{55}$$

where

$$S(t,\mathbf{C}) = \int_0^t \Big\{P(t,\mathbf{C})\dot{X}(t) - H_{\varkappa}(t,\mathbf{C})\Big\}dt, \tag{56}$$
$$H_{\varkappa}(t,\mathbf{C}) = \frac{\mu}{2}P^2(t,\mathbf{C}) + \frac{1}{2}X^2(t,\mathbf{C})\big[\omega_0 + \tilde{\varkappa}(b+c)\big] + \rho P(t,\mathbf{C})X(t,\mathbf{C}) + \tilde{\varkappa}\frac{c}{2}\Delta_{22}^{(2)}(t,\mathbf{C}),$$

and $\mathcal{G}(t)$ is the Cauchy matrix of the system

$$\Big\{-i\mathbb{I}\frac{d}{dt} + (\vec{\sigma},\vec{\mathcal{B}})\Big\}\mathcal{G} = 0, \quad \mathcal{G}\big|_{t=0} = \mathbb{I}. \tag{57}$$

From (57) we have

$$\mathcal{G}(t) = \exp[-i(\vec{\sigma},\vec{\mathcal{B}})t] = \mathbb{I}\cos\big(\mathcal{B}t\big) - \frac{i}{\mathcal{B}}(\vec{\sigma},\vec{\mathcal{B}})\sin\big(\mathcal{B}t\big), \tag{58}$$

where $\mathcal{B} = |\vec{\mathcal{B}}|$. In the issue relation (58) can be written as

$$\mathcal{G}(t) = \frac{1}{\mathcal{B}}\begin{pmatrix} \mathcal{B}\cos\mathcal{B}t - i\mathcal{B}_3\sin\mathcal{B}t & -(i\mathcal{B}_1+\mathcal{B}_2)\sin\mathcal{B}t \\ -(i\mathcal{B}_1-\mathcal{B}_2)\sin\mathcal{B}t & \mathcal{B}\cos\mathcal{B}t + i\mathcal{B}_3\sin\mathcal{B}t \end{pmatrix}. \tag{59}$$

For $\Upsilon(x,t,\mathbf{C})$ we have from (22) and (54)

$$\hat{L}_0(x,t)\mathbb{I}\Upsilon = 0, \quad \Upsilon(\vec{x},t,\mathbf{C}) = \phi(x,t,\mathbf{C})u_0, \tag{60}$$
$$\hat{L}_0(x,t)\mathbb{I} = \hat{\mathcal{K}}^{-1}(x,t,\mathbf{C})\hat{L}(x,t,\mathbf{C})\hat{\mathcal{K}}(x,t,\mathbf{C}) =$$
$$= \big(-i\hbar\partial_t + \frac{\mu\hat{p}^2}{2} + \frac{\tilde{\omega}x^2}{2} + \frac{\rho(x\hat{p}+\hat{p}x)}{2}\big)\mathbb{I}, \tag{61}$$
$$\hat{L}_0(x,t)\phi(x,t,\mathbf{C}) = 0. \tag{62}$$

For the associated linear equation (62) we can construct the following set of symmetry operators linear in x and $\hat{p}$:

$$\hat{a}(t,\mathbf{C}) = \frac{1}{\sqrt{2\hbar}}\big[C(t)\hat{p} - B(t)x\big], \tag{63}$$
$$\hat{a}^+(t,\mathbf{C}) = \frac{1}{\sqrt{2\hbar}}\big[C^*(t)\hat{p} - B^*(t)x\big]. \tag{64}$$

Here the functions $B(t)$ and $C(t)$ are solutions of the linear Hamiltonian system

$$\begin{cases}\dot{B} = -\rho B - \tilde{\omega} C, \\ \dot{C} = \mu B + \rho C.\end{cases} \tag{65}$$

The Cauchy matrix $\mathcal{X}(t)$ for system (65) can easily be found as

$$\mathcal{X}(t) = \begin{pmatrix} \cos\Omega t - \dfrac{\rho}{\Omega}\sin\Omega t & -\dfrac{\tilde{\omega}}{\Omega}\sin\Omega t \\ \dfrac{\mu}{\Omega}\sin\Omega t & \cos\Omega t + \dfrac{\rho}{\Omega}\sin\Omega t \end{pmatrix}, \quad \mathcal{X}(t)\Big|_{t=0} = \mathbb{I}_{2\times 2}. \tag{66}$$

The set of solutions normalized by the condition [6]

$$B(t)C^*(t) - C(t)B^*(t) = 2i \tag{67}$$

can be written as

$$B(t) = e^{i\Omega t}\frac{(-\rho + i\Omega)}{\sqrt{\Omega\mu}}, \qquad C(t) = e^{i\Omega t}\sqrt{\frac{\mu}{\Omega}}.$$

Equation (67) results in the following commutation relations for the symmetry operators (63) and (64):

$$\big[\hat{a}(t,\mathbf{C}), \hat{a}^+(t,\mathbf{C})\big]_- = 1.$$

Then the symmetry operator $\hat{A}(t)$ in (42) for equation (43) with the initial condition (44) can be presented as

$$(\hat{A}(t)\Psi)(x,t) = \widehat{\mathcal{D}}(t, \mathbf{C}[\hat{a}\psi], \mathbf{C}[\Psi](t))\widehat{B}(t, \mathbf{C}[\Psi](t))\Psi(x,t). \tag{68}$$

Here, $\widehat{B}(t,\mathbf{C})$ is the symmetry operator of the associated linear equation (53), $\widehat{B}(t,\mathbf{C})|_{t=0} = \hat{a}$, and the fundamental intertwining operator $\widehat{\mathcal{D}}(t,\mathbf{C}',\mathbf{C})$ reads

$$\begin{aligned}\widehat{\mathcal{D}}(t,\mathbf{C}',\mathbf{C}) = \exp\Big\{i\frac{C_2' - C_2}{2\hbar\mu}\Big(\bar{\Omega}(C_1' - C_1) - \rho(C_2' - C_2)\Big)\Big\} \times \\ \times\hat{\mathcal{K}}^{-1}(x,t,\mathbf{C}')\exp\Big\{\frac{i}{\hbar}\hat{b}(t,\mathbf{C}',\mathbf{C})\Big\}\hat{\mathcal{K}}(x,t,\mathbf{C}),\end{aligned} \tag{69}$$

where

$$\hat{b}(t,\mathbf{C}',\mathbf{C}) = b_x(t,\mathbf{C}',\mathbf{C})\hat{p} - b_p(t,\mathbf{C}',\mathbf{C})x = \langle b(t,\mathbf{C}',\mathbf{C}), J\hat{z}\rangle,$$

and

$$b(t,\mathbf{C}',\mathbf{C}) = \begin{pmatrix} b_p(t,\mathbf{C}',\mathbf{C}) \\ b_x(t,\mathbf{C}',\mathbf{C})\end{pmatrix} = \frac{1}{\mu}\mathcal{X}(t)\begin{pmatrix}\bar{\Omega}(C_1' - C_1) - \rho(C_2' - C_2) \\ \mu(C_2' - C_2)\end{pmatrix}.$$

The matrix $\mathcal{X}(t)$ is given by (66).

The symmetry operator $\hat{A}(t)$ in (68) for the 1D R2GPE (43) has the structure of a linear pseudodifferential operator whose parameters are functionals of the function $\Psi(x,t)$ on which the operator acts. Therefore, the operator $\hat{A}(t)$ is nonlinear.

Note that for some values of the parameters the pseudodifferential operator can become a differential one.

The symmetry operator $\hat{A}(t)$ depends on the symmetry operator $\widehat{B}(t,\mathbf{C})$ of the associated linear equation (53). Consider

$$\widehat{B}(t,\mathbf{C}) = \widehat{B}_\nu(t,\mathbf{C}) = \frac{1}{\sqrt{\nu!}}\Big[\hat{a}^+(t,\mathbf{C})\Big]^\nu\mathbb{I}, \quad \nu\in\mathbb{Z}_+, \tag{70}$$

where the operators $\hat{a}^+(t, \mathbf{C})$ are defined in (64).

Substituting (70) and (69) in (68), we obtain the symmetry operator for the 1D R2GPE (43), which we denote by $\hat{A}_\nu(t)$.

Following calculations given in section 4 of Ref. [5], we can generate a countable family of exact solutions for the 1D R2GPE (43) with the use of the family of the symmetry operators $\hat{A}_\nu(t)$, $\nu \in \mathbb{Z}_+$ and a starting solution $\Psi_0(x,t)$ of equation (43). Since the construction of these solutions for the 1D R2GPE (43) is similar to the same problem for the reduced one-component GPE [5], we present here only the solutions themselves without detail calculations.

We find the starting solution $\Psi_0(x,t)$ using a stationary solution of the of the Hamilton–Ehrenfest system (45), (47) that can be obtained from the general solution (49), (50) if we take integration constants as $\mathbf{C} = \mathbf{C}^0 = (C_1^0, \ldots, C_7^0)$, where we set $C_1^0 = C_2^0 = C_3^0 = C_4^0 = C_7^0 = 0$, $\sin C_6^0 = 0$, and C_5^0 is an arbitrary real constant. The stationary solution is

$$\begin{aligned} &X(t,\mathbf{C}) = P(t,\mathbf{C}) = 0,\\ &\Delta_{22}^{(2)}(t,\mathbf{C}) = C_5^0, \quad \Delta_{21}^{(2)}(t,\mathbf{C}) = -\frac{\rho}{\mu}C_5^0, \quad \Delta_{11}^{(2)}(t,\mathbf{C}) = \frac{\tilde{\omega}}{\mu}C_5^0,\\ &\vec{\eta}(t,\mathbf{C}) = \zeta\vec{n}, \quad \zeta = \pm 1. \end{aligned} \tag{71}$$

The function

$$\Phi_0(x,t,\mathbf{C}^0) = \Big(\frac{\Omega}{\pi\hbar\,\mu}\Big)^{1/4} \exp\Big\{-\frac{i}{2\hbar}\frac{\rho}{\mu}x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\Big\} \exp\Big\{-\frac{i}{2}\Omega t - \frac{i}{2\hbar}\tilde{\varkappa} c C_5^0 t\Big\} v_\zeta(t) \tag{72}$$

is a particular solution of the corresponding associated linear equation (53). Here, $v_\zeta(t) = \exp\big(-i\zeta\mathcal{B}t\big) f_\zeta$, and f_ζ is given by the system

$$(\vec{\sigma}, \vec{\mathcal{B}}) f_\zeta = \zeta\mathcal{B} f_\zeta \tag{73}$$

and has the form

$$f_\zeta = \frac{e^{i\alpha}}{\sqrt{2}} \begin{pmatrix} \zeta\sqrt{1+\zeta\,\cos\theta}\,\exp(-i\varphi/2) \\ \sqrt{1-\zeta\,\cos\theta}\,\exp(i\varphi/2) \end{pmatrix}, \ \alpha = \text{const}, \ \zeta = \pm 1. \tag{74}$$

Taking $C_5^0 = (\hbar\mu/2\Omega)$ in (72), we find the desired starting solution $\Psi_0(x,t)$ of the 1D R2GPE (43):

$$\begin{aligned} \Psi_0(x,t) = \Phi_0(x,t,\mathbf{C}^0)\Big|_{C_5^0 = (\hbar\mu/2\Omega)} &= \Big(\frac{\Omega}{\pi\hbar\,\mu}\Big)^{1/4} \exp\Big\{-\frac{i}{2\hbar}\frac{\rho}{\mu}x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\Big\}\times\\ &\times \exp\Big\{-\frac{i}{2}\Omega t - \frac{i\mu}{4\Omega}\tilde{\varkappa} c t\Big\} v_\zeta(t). \end{aligned} \tag{75}$$

Then the symmetry operator $\hat{A}_\nu(t)$ determined by (68) transforms the solution $\Psi_0(x,t)$ of (75) into a solution $\Psi_\nu(x,t)$ of the 1D R2GPE (43) according to the following relation:

$$\begin{aligned} \Psi_\nu(x,t) = (\hat{A}_\nu(t)\Psi_0)(x,t) &= \Big(\frac{\Omega}{\pi\hbar\,\mu}\Big)^{1/4}\Big(\frac{1}{\sqrt{2}}\Big)^\nu \exp\Big\{-\frac{i}{2\hbar}\frac{\rho}{\mu}x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\Big\}\times\\ &\times H_\nu\Big(\sqrt{\frac{\Omega}{\hbar\mu}}x\Big) \exp\Big\{-i\Big(\nu+\frac{1}{2}\Big)\Big(\frac{\tilde{\varkappa} c\mu}{2\Omega} + \Omega\Big)t\Big\} v_\zeta(t), \end{aligned} \tag{76}$$

where $H_\nu(\zeta)$ are the Hermite polynomials [17] The functions (76) constitute a countable set of particular solutions to equation (43) which are generated from $\Psi_0(x,t)$ by the nonlinear symmetry operator $\hat{A}_\nu(t)$.

5. Concluding remarks

In this paper, which is a sequel to [4] and [5], we continued our study of symmetry properties of nonlinear integro-differential equations using WKB-Maslov semiclassical approximation.

In [5] we considered the 1-component multidimensional Gross–Pitaevskii equation with nonlocal nonlinear term and developed an approach to construct a class of symmetry operators for the semiclassically reduced Gross–Pitaevskii equation. The approach is based on our method proposed to construct the semiclassical asymptotic solution of the Cauchy problem to the nonlocal GPE.

In this work we apply the approach of Ref. [5] to the 2-component Gross–Pitaevskii equation with the nonlocal Manakov-type nonlinear interaction term (4), (5). In the semiclassical approximation the leading term of the asymptotic expansion is determined by the R2GPE (11) (or (17)) which contains nonlocal terms as a finite number of moments of the unknown function $\Psi(\vec{x}, t)$. We assign the R2GPE to the class of nearly linear equations according to the definition of Ref. [4] similar to the semiclassically reduced GPE considered in [5].

The general expression for the symmetry operator (42) is similar to the one obtained in [5] for the scalar (1-component) nonlocal GPE. To explore how the symmetry operators for a system (multi-component equation) transform as compared to a scalar equation, we have considered here the simplest case of the 2-component nonlocal GPE when the vector $\vec{H}(t)$ in (4) depends on t and does not depend on $\vec{x}$. For this case solution (8) can be presented as the product of a 2-component vector $u(t)$ and a scalar function $\varphi(\vec{x}, t)$ that leads to slight modification of the symmetry operator. Finding such symmetry operators in explicit form and generating exact solutions are illustrated with an example of 1D R2GPE considered in section 4.

Acknowledgments

The work was supported in part by Tomsk State University International Competitiveness Improvement Programm and by program "Nauka" No 1.676.2014/K.

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