Solutions in the $\mathbb{C}P^N$ Skyrme type model

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Abstract. The extended Skyrme-Faddeev model possesses vortex solutions in a (3+1) dimensional Minkowski space-time with target space CP^N. The right choice of the potential leads to solutions with finite energy per unit of length. Such solutions have a form of waves propagating along vortices with the speed of light. Some possible applications are considered, such as the quantization scheme and the vortex-fermion system.

1. Introduction

The Skyrme type model is a large class of soliton models which is based on the non-linear sigma model in 2 or 3 spatial dimensions. The standard Skyrme model was originated by T.H.R. Skyrme [1, 2, 3] and it has soliton solutions which describe the hadrons after suitable quantization scheme [4]. The Skyrme-Faddeev model is a modification of the Skyrme model and supports the knotted solitons [5] as well as the vortices [6]. The significance of this model has increased noticeably when it has been conjectured that the model can be seen as a low-energy effective classical model of the underlying Yang-Mills theory [7, 8]. Remarkably, the exact soliton (vortex) solution of the model has been found within the integrable sector [9]. It has been shown that also in the case of the complex projective target space CP^N the extended Skyrme-Faddeev model (in which it has been imposed some special constraints for the parameters of the model) possesses an exact soliton solutions in the integrable sector [10]. It has not been clear, however, if the presence of the solutions in the considered model is related to the particular choice of the coupling constants like in the case of the exact solution or it is rather a general property of the model. We investigate the existence of the solutions of the model inside/outside the integrable sector, especially in absence of some particular relations between coupling constants. The key role is playing by the potential which usually work as a stabilizer for the solution. Many CP^N - type models are usually considered in context of instantons, i.e., the scale invariant solutions. The potential breaks such invariance and then the stable solutions appear. In the present paper we construct the potentials for combinations of the positive/negative winding numbers $\{n_i\}$.

There are several interesting topics related with the present solutions; the collective coordinate quantizations and the construction of the vortex-fermion systems. As was pointed out, the solitons describe the particle state after suitable quantization scheme. For the present model we employ the rotational zero-mode and discuss how the spectrum is estimated. The problem of the fermion can be more phenomenological. The existence of transverse zero energy modes of the Dirac equation in the presence of a string can make the string superconducting [11].

We propose two independent coupling scheme for the fermions and check how they work in two and three spatial dimensions, respectively.

The study of such models is promising and could be important for understanding some aspects of the strong coupling sector of the Yang-Mills theory.

2. The model

We start with a Skyrme type model

$$\mathcal{L} = -\frac{M^2}{2}\mathrm{Tr}(X^{-1}\partial_\mu X)^2 + \frac{1}{e^2}\mathrm{Tr}(X^{-1}\partial_\mu X X^{-1}\partial_\nu X)^2 \\ +\frac{\beta}{2}\left[\mathrm{Tr}(X^{-1}\partial_\mu X)^2\right]^2 + \gamma\left[\mathrm{Tr}(X^{-1}\partial_\mu X X^{-1}\partial_\nu X)\right]^2 - V \qquad (1)$$

where M is a coupling constant with dimension of mass whereas the coupling constants e^{-2}, β, γ are dimensionless. The variable $X(t,\vec{x})$ is an $SU(N+1)$- valued scalar. We shall discuss its explicit form below. All terms containing derivatives in the Lagrangian (1) possess the global symmetry $SU(N+1)_L \times SU(N+1)_R$ corresponding to transformations $X \mapsto g_L X g_R^\dagger$, where g_L, g_R are constant elements of $SU(N+1)$. Such a symmetry originates in the way how the variable X enters to the Lagrangian. In fact it is independent on particular form of X. One can make whole Lagrangian (1) invariant under this transformation for appropriate choice of the potential V. The first term is quadratic in X and corresponds with the Lagrangian of the CP^N model. The quartic term proportional to e^{-2} is the Skyrme term, and together with the other quartic terms they support the stability of the solution. Similarly, the form of V is (by construction) such that it is consistent with the stability requirement. Quite naturally X can be expressed as $X \sim \xi_L\xi_R^\dagger$, where the chiral transformation is given via $\xi_L \to \xi'_L = g_L\xi_L$, and $\xi_R \to \xi'_R = g_R\xi_R$.

In this paper we explore solutions with the CP^N target space. The standard Skyrme-Faddeev model on the CP^1 target space is parametrized by a real triplet unit vector $\vec{n}$, which has only two independent degrees of freedom. The target space (coset space) in the case of some higher dimensional $SU(N+1)$ Lie groups, i.e. $N \geqq 2$, can be chosen in several nonequivalent ways. Recently it has been proposed some formulation of the extended Skyrme-Faddeev model on the CP^N target space [10]. The coset space $CP^N = SU(N+1)/SU(N)\otimes U(1)$ is an example of a symmetric space and it can be naturally parametrized in terms of so called *principal variable*

$$X(g) = g\sigma(g)^{-1}, \qquad g \in SU(N+1) \qquad (2)$$

where σ being the order two automorphism under which the subgroup $SU(N)\otimes U(1)$ is invariant i.e. $\sigma(h) = h$ for $h \in SU(N)\otimes U(1)$. The coordinate $X(g)$ defined above satisfies $X(gh) = X(g)$. The Lagrangian has a global left symmetry such that $g \to \bar{g}g$ with $\bar{g}, g \in SU(N+1)$, which implies that $X \to \bar{g}X\sigma(\bar{g})^{-1}$ and then $X^{-1}\partial_\mu X \to \sigma(\bar{g})X^{-1}\partial_\mu X\sigma(\bar{g})^{-1}$. In addition it has a right local $SU(N)\times U(1)$ symmetry such that $g \to gk$ with $k \in SU(N)\times U(1)$ and $g \in SU(N+1)$, which implies that X is invariant.

There is an additional symmetry $X \to \mathcal{P}X$. The prefactor $\mathcal{P}$ is a constant, unitary matrix (in our considerations it is diagonal) **and seems to be essentially appeared by a choice of the gauge ξ_L, ξ_R**. The standard choice $\mathcal{P}_{\mathrm{I}} = (\mathbf{1}_{N+1\times N+1})$ coincides with definition used in [10]. On the other hand, if one choose

$$\mathcal{P}_{\mathrm{H}} = \begin{pmatrix} \mathbf{1}_{N\times N} & 0 \\ 0 & -1 \end{pmatrix}$$

the resulting matrix $X_{\rm H} \equiv \mathcal{P}_{\rm H} g\sigma(g)^{-1}$ becomes Hermitian. Notable thing is that both definitions of $\mathcal{P}$ have the same classical solutions because they are concerned with the symmetry of the model.

2.1. The parametrization

Let us shortly discuss the parametrization of the model. According to the previous paper [10] one can parametrize the model in terms of N complex fields u_i, where $i = 1, \dots, N$. Assuming $(N+1)$-dimensional defining representation where the $SU(N+1)$ valued element g is of the form

$$g \equiv \frac{1}{\vartheta}\left(\begin{array}{cc} \Delta & i\,u \\ i\,u^\dagger & 1 \end{array}\right) \qquad\qquad \vartheta \equiv \sqrt{1 + u^\dagger \cdot u} \tag{3}$$

and where Δ is the Hermitian $N \times N$-matrix

$$\Delta_{ij} = \vartheta\,\delta_{ij} - \frac{u_i\,u_j^*}{1+\vartheta} \quad \text{which satisfies} \quad \Delta\cdot u = u \quad u^\dagger \cdot \Delta = u^\dagger.$$

For g given by (3), one can has $\sigma(g) = g^{-1}$ [10], and so the principal variable $X_{\rm I}(g) := \mathcal{P}_{\rm I} g\sigma(g)^{-1}$ is given by

$$X_{\rm I}(g) = \mathcal{P}_{\rm I} g^2 = \left(\begin{array}{cc} \mathbf{1}_{N\times N} & 0 \\ 0 & -1 \end{array}\right) + \frac{2}{\vartheta^2}\left(\begin{array}{cc} -u\otimes u^\dagger & iu \\ iu^\dagger & 1 \end{array}\right)$$

Also for $X_{\rm H}(g) := \mathcal{P}_{\rm H} g\sigma(g)^{-1}$ we have

$$X_{\rm H}(g) = \mathcal{P}_{\rm H} g^2 = \left(\begin{array}{cc} \mathbf{1}_{N\times N} & 0 \\ 0 & 1 \end{array}\right) + \frac{2}{\vartheta^2}\left(\begin{array}{cc} -u\otimes u^\dagger & iu \\ -iu^\dagger & -1 \end{array}\right)$$

Apparently the $X_{\rm H}$ is Hermitian while the $X_{\rm I}$ is not.

The Lagrangian (1) reads

$$\mathcal{L} = -\frac{1}{2}\Big[M^2\eta_{\mu\nu} + C_{\mu\nu}\Big]\tau^{\nu\mu} - V \tag{4}$$

where the symbols $C_{\mu\nu}$ and $\tau_{\mu\nu}$ read

$$C_{\mu\nu} := M^2\eta_{\mu\nu} - \frac{4}{e^2}\Big[(\beta e^2 - 1)\tau_\rho^\rho\eta_{\mu\nu} + (\gamma e^2 - 1)\tau_{\mu\nu} + (\gamma e^2 + 2)\tau_{\nu\mu}\Big], \tag{5}$$

$$\tau_{\mu\nu} := -\frac{4}{\vartheta^4}\left[\vartheta^2\partial_\nu u^\dagger\cdot\partial_\mu u - (\partial_\nu u^\dagger\cdot u)(u^\dagger\cdot\partial_\mu u)\right]. \tag{6}$$

We shall discuss the specific form of the potential in the further part of the paper. It is enough to assume that $V = V(u^\dagger, u)$. The variation with respect to u_i^* leads to the equations

$$\begin{aligned}(1+u^\dagger\cdot u)\partial^\mu(C_{\mu\nu}\partial^\nu u_i) - C_{\mu\nu}&\left[(u^\dagger\cdot\partial^\mu u)\partial^\nu u_i + (u^\dagger\cdot\partial^\nu u)\partial^\mu u_i\right] \\ &+\frac{u_i}{4}(1+u^\dagger\cdot u)^2\left[\frac{\delta V}{\delta|u_i|^2} + \sum_{k=1}^{N}|u_k|^2\frac{\delta V}{\delta|u_k|^2}\right] = 0.\end{aligned} \tag{7}$$

In order to get the result (7) one needs to multiply the equation obtained directly from the variation with respect to u_i^* by the inverse matrix of Δ^2_{ki} i.e. $\Delta^{-2}_{ij} = \frac{1}{1+u^\dagger\cdot u}(\delta_{ij} + u_i u_j^\dagger)$. It leads to the term being a combination of partial derivatives of the potential. We introduce the dimensionless coordinates (t, ρ, φ, z) defined as

$$x^0 = r_0 t, \quad x^1 = r_0 \rho \cos\varphi, \quad x^2 = r_0 \rho \sin\varphi \quad x^3 = r_0 z \tag{8}$$

where the length scale r_0 is defined in terms of coupling constants M^2 and e^2 i.e.

$$r_0^2 = -\frac{4}{M^2 e^2}$$

and the light speed is $c = 1$ in the chosen unit system. The linear element ds^2 reads

$$ds^2 = r_0^2 (dt^2 - dz^2 - d\rho^2 - \rho^2 d\varphi^2).$$

The family of exact vortex solutions has been found for the model without potential $V = 0$ where in addition the coupling constants satisfy the condition $\beta e^2 + \gamma e^2 = 2$. The exact solutions have the form of vortices which depend on some specific combination of the coordinates i.e. one light-cone coordinate $x^3 + x^0$ and one complex coordinate $x^1 + ix^2$. The functions $u_i(x^3 + x^0, x^1 + ix^2)$ satisfy the zero curvature condition $\partial_\mu u_i \partial^\mu u_j = 0$ for all $i, j = 1, \ldots, N$ and therefore one can construct the infinite set of conserved currents.

The solutions can generally be written such as the form

$$u_j = f_j(\rho) e^{i(n_j \varphi + k_j \psi(w))} \tag{9}$$

where $\psi(w)$ is a real function of the light-cone coordinate and $f_i(\rho)$ is N-th element set of real functions. The constants n_i form the set of integer numbers and k_i are some real constants. The solutions in the integrable sector is thus holomorphic one

$$f_i(\rho) = c_i \rho^{n_i} \tag{10}$$

where c_i being complex constants.

We shall use the matrix notation for the convenience therefore we define two diagonal matrices

$$\lambda \equiv \mathrm{diag}(n_1, \ldots, n_N), \qquad \sigma \equiv \mathrm{diag}(k_1, \ldots, k_N). \tag{11}$$

In the matrix form the ansatz reads $u = f(\rho) \exp\left[i(\lambda\varphi + \sigma\psi(w))\right]$ where w is either $z + t$ or $z - t$. The components of $\tau_{\mu\nu}$ have the following form

$$\begin{aligned}
&\tau_{\rho\rho} \equiv \theta(\rho) = -\frac{4}{\vartheta^4}\left[\vartheta^2 f'^T.f' - (f'^T.f)(f^T.f')\right] \\
&\tau_{\varphi\varphi} \equiv \omega(\rho) = -\frac{4}{\vartheta^4}\left[\vartheta^2 f^T.\lambda^2.f - (f^T.\lambda.f)^2\right] \\
&\tau_{\varphi\rho} = -\tau_{\rho\varphi} \equiv i\zeta(\rho), \quad \zeta(\rho) = -\frac{4}{\vartheta^4}\left[\vartheta^2 f'^T.\lambda.f - (f^T.\lambda.f)(f'^T.f)\right] \\
&\tau_{t\rho} = -\tau_{\rho t} \equiv (i\partial_t\psi)\xi(\rho), \quad \tau_{z\rho} = -\tau_{\rho z} \equiv (i\partial_z\psi)\xi(\rho), \quad \xi(\rho) = -\frac{4}{\vartheta^4}\left[\vartheta^2 f'^T.\sigma.f - (f^T.\sigma.f)(f'^T.f)\right] \\
&\tau_{t\varphi} = \tau_{\varphi t} \equiv (\partial_t\psi)\eta(\rho), \quad \tau_{z\varphi} = \tau_{\varphi z} \equiv (\partial_z\psi)\eta(\rho), \quad \eta(\rho) = -\frac{4}{\vartheta^4}\left[\vartheta^2 f^T.\lambda.\sigma.f - (f^T.\lambda.f)(f^T.\sigma.f)\right] \\
&\tau_{tt} \equiv (\partial_t\psi)^2\chi(\rho), \quad \tau_{zz} \equiv (\partial_z\psi)^2\chi(\rho) \\
&\tau_{tz} = \tau_{zt} \equiv (\partial_t\psi)(\partial_z\psi)\chi(\rho), \quad \chi(\rho) = -\frac{4}{\vartheta^4}\left[\vartheta^2 f^T.\sigma^2.f - (f^T.\sigma.f)(f^T.\sigma.f)\right]
\end{aligned}$$

where the derivative with respect to ρ has been denoted by $\frac{d}{d\rho} ='$. The equations of motion written in dimensionless coordinates take the form

$$\begin{aligned}
&(1+f^T.f)\left[\frac{1}{\rho}\left(\rho\,\tilde{C}_{\rho\rho}f_k'\right)' + \frac{i}{\rho}\left(\frac{\tilde{C}_{\rho\varphi}}{\rho}\right)'(\lambda.f)_k - \frac{1}{\rho^4}\tilde{C}_{\varphi\varphi}(\lambda^2.f)_k\right] \\
&-2\left[\tilde{C}_{\rho\rho}(f^T.f')f_k' - \frac{1}{\rho^4}\tilde{C}_{\varphi\varphi}(f^T.\lambda.f)(\lambda.f)_k\right] \\
&+\frac{f_k}{4}(1+f^T.f)^2\left[\frac{\delta\tilde{V}}{\delta f_k^2} + \sum_{i=1}^{N} f_i^2\frac{\delta\tilde{V}}{\delta f_i^2}\right] = 0
\end{aligned} \tag{12}$$

for each $k = 1, \dots, N$, where we have introduced the symbols $\tilde{C}_{\mu\nu} := \frac{1}{r_0^2 M^2}C_{\mu\nu}$, and also $\tilde{V} := \frac{r_0^2}{M^2}V$. These of components $\tilde{C}_{\mu\nu}$ which appear in the equations of motion read

$$\begin{aligned}
\tilde{C}_{\rho\rho} &= -1 + (\beta e^2 - 1)\left(\theta + \frac{\omega}{\rho^2}\right) + (2\gamma e^2 + 1)\theta \\
\tilde{C}_{\varphi\varphi} &= -\rho^2 + \rho^2(\beta e^2 - 1)\left(\theta + \frac{\omega}{\rho^2}\right) + (2\gamma e^2 + 1)\omega \\
\tilde{C}_{\varphi\rho} &= -\tilde{C}_{\rho\varphi} = -3i\zeta\,.
\end{aligned} \tag{13}$$

2.2. The topological property
According to discussions in [12] and also in [10], we can define the topological charge in the present model. The field u_i provide a mapping from x^1x^2 plane into CP^N. However, for the finiteness of the energy, the field goes to a constant. Then the plane should be compactified into S^2 and the solutions define the mapping $S^2 \to CP^N$ which is classified into the homotopy classes of $\pi_2(CP^N)$. There exists a theorem describing in [12], $\pi_2(G/H) = \pi_1(H)_G$ where $\pi_1(H)_G$ is the subset of $\pi_1(H)$ formed by closed paths in H which can be contracted to a point in G. Thus, in the present case, the topological charges are given by

$$\pi_1(SU(N) \otimes U(1))_{SU(N+1)}. \tag{14}$$

The topological charges are equal to the number of poles of u_i, including those at infinity. And then, it can be obtained as

$$Q_{\rm top} = n_{\max} + |n_{\min}| \tag{15}$$

where the highest positive integer in the set $n_i, i = 1, 2, \cdots, N$ and $n_{\min}$ is the lowest negative integer in the same set.

3. The integrable sector and the potential

In [6] the authors argued that exact holomorphic solutions exist in the CP^1 model with special choice of potential V. We have tried to consider generalization to the CP^N case. Until now, however, we could solve the problem only for a quite limited case i.e., the first $K-$th integers n_i are equal and the remaining are all zero. The problem we shall solve is essentially an inverse problem. Therefore, first we assume holomorphic solutions and then explore appropriate potential which satisfies the equations of motion. We assume the solutions in the form

$$\begin{aligned}
u_i &= c_i\rho^n e^{in\varphi} \quad &&\text{for} \quad i = 1, \dots K \\
u_i &= c_i &&\text{for} \quad i = K+1, \dots N
\end{aligned} \tag{16}$$

where c_i are complex constants. The solutions satisfy the zero-curvature condition $\partial_\mu u_i \partial^\mu u_j = 0$. At least in this special case, the problem can be solved by the method of the separation of variables. For this reason we introduce the ansatz for potential of the form

$$V = \frac{\left(\sum_{k=1}^{K} |u_k|^2\right)^a Q(|u_{K+1}|^2, \ldots, |u_N|^2)}{(1 + \sum_{j=1}^{N} |u_j|^2)^4}$$

where a is a free constant. Plugging the solution (16) into the Eq.(7), one obtain the set of equations

$$\begin{cases} \frac{1}{\tilde{\mu}^2} \frac{R(\rho)}{\rho^4} - \left\{ a \left(\sum_{j=1}^{K} |u_j|^2\right)^{a-1} + (a - 4 + \sum_{k=K+1}^{N} \frac{|u_k|^2}{Q} \frac{\delta Q}{\delta |u_k|^2}) \left(\sum_{j=1}^{K} |u_j|^2\right)^a \right\} Q = 0 \\ \frac{\delta Q}{\delta |u_i|^2} + \sum_{k=K+1}^{N} |u_k|^2 \frac{\delta Q}{\delta |u_k|^2} + (a-4) Q = 0 \end{cases} \tag{17}$$

where the symbol $R(\rho)$ reads

$$R(\rho) = 32(\beta e^2 + \gamma e^2 - 2) n^4 \left(1 + \sum_{k=K+1}^{K} c_k^2\right)^2 \left[\alpha \sum_{k=1}^{K} |u_k|^2 - \epsilon \left(\sum_{k=1}^{N} |u_k|^2\right)^2 \right],$$

and the coefficients α and ϵ have the form

$$\alpha := 2\left(1 - \frac{1}{n}\right) \qquad \epsilon := \frac{2}{1 + \sum_{k=K+1}^{N} c_k^2} \left(1 + \frac{1}{n}\right). \tag{18}$$

The parameter a is fixed as $a = \alpha$ by comparing a power of ρ in the first set of Eq.(17). In order to satisfy Eq.(17), the function Q has to satisfy the condition

$$\left[\frac{1}{Q} \frac{\delta Q}{\delta |u_i|^2}\right]_{|u_k|^2 = c_k^2} = \epsilon \quad \text{for} \quad i = K+1, \cdots, N \tag{19}$$

and the value of $\tilde{\mu}^2$ is determined in terms of other constants :

$$\tilde{\mu}^2 = 32(\beta e^2 + \gamma e^2 - 2) n^4 \frac{\left(\sum_{i=1}^{K} c_i^2\right)^{2/n} \left(1 + \sum_{i=K+1}^{N} c_i^2\right)^2}{Q(c_{K+1}^2, \cdots, c_N^2)}. \tag{20}$$

Any function Q which satisfies the condition (19) provide a set of analytical solutions. The simplest example of the integrable potential is

$$V = \frac{\left(\sum_{i=1}^{K} |u_i|^2\right)^\alpha \left(1 + \sum_{i=K+1}^{N} |u_i|^2\right)^{4-\alpha}}{\left(1 + \sum_{i=1}^{N} |u_i|^2\right)^4} \tag{21}$$

with the coefficient

$$\tilde{\mu}^2 = 32(\beta e^2 + \gamma e^2 - 2) n^4 \left(\sum_{i=1}^{K} c_i^2\right)^{2/n} \left(1 + \sum_{i=K+1}^{N} c_i^2\right)^{-2/n}. \tag{22}$$

The question whether there exist exact solutions for arbitrary combination of winding numbers is still an open problem.

4. The typical classical solutions

Eq.(12) is highly nonlinear and therefore we solve it numerically. The standard successive overrelaxation method is used with a suitable rescaling of the coordinate $\rho := \sqrt{(1-y)/y}$ $(0 \leqq y \leqq 1)$ and the field $f_i := \sqrt{(1-g_i)/g_i}$ $(0 \leqq g_i \leqq 1)$.

4.1. The potential
The key point for the construction of the potential is careful analysis of the vacuum structure of the field. As a simple example, we start with the CP^1 ($O(3)$) case. The $O(3)$ σ model is usually defined as a vectorial triplet $\vec{n} = (n_1, n_2, n_3)$ with the constraint $\vec{n} \cdot \vec{n} = 1$. The well-known potential named "old-baby" type, i.e.potential with one vacuum, is of the form [13]

$$V(\vec{n}) = (1 - \vec{n}_\infty \cdot \vec{n}) \tag{23}$$

where $\vec{n}_\infty$ is a vacuum value of the field $\vec{n}$ at spatial infinity. For the choice $\vec{n}_\infty = (0,0,1)$ the potential becomes $V = (1-n_3)$, which is most well-known and is used.

Generally speaking, a potential can be deduced from the asymptotic structure of solutions of the model. Moreover, the potential has to have the form such that the model has solutions for all qualitatively different combinations of the integers (n_1, n_2). In the following subsections, we give an explicit form of the potentials for basic combinations of (n_1, n_2). The crucial point is that we explore only such potentials that numerical solutions and its analytical counterparts, both characterised by the same set of winding numbers (n_1, n_2), have the same asymptotics i.e. both behave as $\sim (\rho^{n_1}, \rho^{n_2})$ at spatial infinity.

4.1.1. The case: $n_1 > n_2 > 0$ An assumption that the solution and its holomorphic counterpart have the same asymptotic behavior at spatial infinity causes that an inverse of the principal variable X goes to ${X_\infty}^{-1} := \mathrm{diag}(-1,1,-1)$ for $\rho \to \infty$, which results in the following expression for the potential

$$V(u_i) = \mathrm{Tr}(1 - {X_\infty}^{-1}X) = 4\frac{1+|u_2|^2}{1+|u_1|^2+|u_2|^2}. \tag{24}$$

Note that for $\rho \to 0$ an inverse of the principal variable goes to $X_0^{-1} := \mathrm{diag}(1,1,1)$, then the expression $\mathrm{Tr}(1 - X_0^{-1}X)$ can be included as the "new-baby" potential which has two vacua [14]. Finally, the following expression can be considered as a general form of the potential

$$\begin{aligned} V &= [\mathrm{Tr}(1 - X_0^{-1}X)]^a[\mathrm{Tr}(1 - X_\infty^{-1}X)]^b \\ &= \frac{(|u_1|^2+|u_2|^2)^a(1+|u_2|^2)^b}{(1+|u_1|^2+|u_2|^2)^{a+b}} = \frac{(f_1^2+f_2^2)^a(1+f_2^2)^b}{(1+f_1^2+f_2^2)^{a+b}} \end{aligned} \tag{25}$$

where the integers a, b satisfy $a \geqq 0, b > 0$.

4.1.2. The case: $n_1 > 0 > n_2$ Assuming that for $n_2 < 0$ the field u_2 behaves at zero as its holomorphic counterpart i.e. $\sim \rho^{n_2}$ one gets that it tends to diverge as $\rho \to 0$. Then inverse of the principal variable X goes to $X_0^{-1} := \mathrm{diag}(1,-1,-1)$ for $\rho \to 0$. The general form of the potential takes the form

$$V = \frac{(1+f_1^2)^a(1+f_2)^b}{(1+f_1^2+f_2^2)^{a+b}}, \tag{26}$$

where the integers satisfy $a \geqq 0, b > 0$.

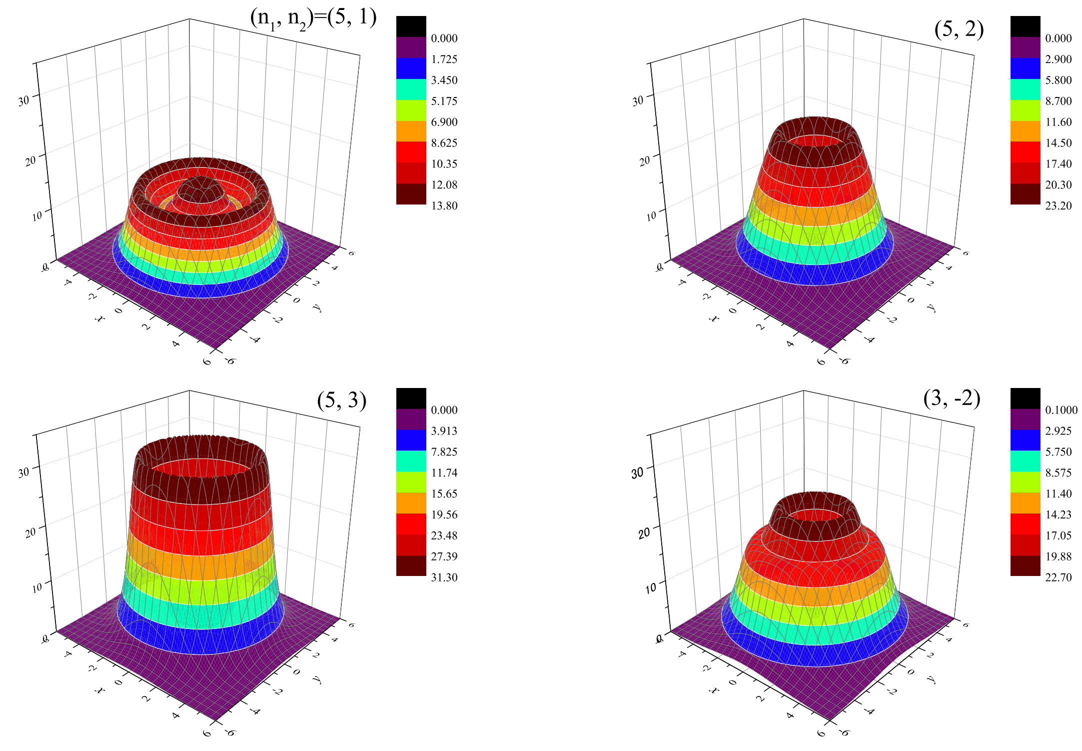


Figure 1. The plot of the CP^2 density of total energies $\mathcal{H}$ (in units of $4M^2$) for $Q_{\rm top} = 5$ for different pairs (n_1, n_2) and $(k_1, k_2) = (1.0, 1.0)$. The other parameters are $(\beta e^2, \gamma e^2, \tilde{\mu}^2) = (2.0, 2.0, 1.0)$.

4.1.3. The case: $n_1, n_2 < 0, \quad |n_1| > |n_2|$ The asymptotic values of inverse of the principal variable are given by constant matrices $X_\infty^{-1} = \mathrm{diag}(1,1,1)$ and $X_0^{-1} = \mathrm{diag}(-1,1,-1)$. Then the potential is

$$V \quad = \quad \frac{(1+f_2^2)^a(f_1^2+f_2^2)^b}{(1+f_1^2+f_2^2)^{a+b}} \tag{27}$$

where the integers satisfy $a \geqq 0, b > 0$.

In Fig.1 we plot some cases of the energy density per unit of length $\mathcal{H}$ for $Q_{\rm top} = 5$. For $N = 2$, it has two core structure where the constituent with the larger winding number n_i is located outward and the lower winding number is inward. It is thus easily see that if $n_1 \gg n_2$ the shape is a single torus, while $n_1 \sim n_2$ it looks like double torus.

In the following sections, we will give some possible applications of the vortex solutions.

5. The quantum state via a rigid body rotation

In this section, we explain a basic idea for quantization of vortices in the CP^2 model. A standard approach to the quantization of soliton solutions is based on the zero-mode quantization of the solution as a rigid body [4]. We start with the static case of (9), i.e., $u_j = f_j(\rho)e^{in_j\varphi} \quad j = 1, 2$. The constants n_j are some integers of both signs, however, we can study only the case $n_1 > n_2 > 0$ without loss of generality. We shall work with Hermitian variable $X_{\rm H}$ which is more convenient to study a problem of rotational zero-modes. In this section the subscript H is omitted for simplicity.

The Lagrangian (1) has a global $SU(3) \times SU(3)$ symmetry corresponding to the transformation $X \to \mathcal{A}X\mathcal{B}^\dagger$, where $\mathcal{A}, \mathcal{B} \in SU(3)$. Such symmetry is spontaneously broken to a residual one when Lagrangian is evaluated on solutions. There are two conditions for having residual symmetry. It must to preserve the topological charge and the asymptotic value of X i.e. X_∞. In terms of the Hermitian principal variable the topological charge is defined as

$$Q_{\text{top}} = \frac{1}{8\pi} \int d^2x \epsilon_{ij} \text{Tr}(X \partial_i X \partial_j X). \tag{28}$$

In order to preserve the topological charge, the transformation should be diagonal, i.e., $\mathcal{A} = \mathcal{B}$. The asymptotic form of the variable X reads $X_\infty = \text{diag}(-1, 1, 1)$ according to the leading behavior of u_j at spatial infinity. It follows that the unitary matrix $\mathcal{A}$ have to satisfy the condition

$$X_\infty \to \mathcal{A} X_\infty \mathcal{A}^\dagger = X_\infty. \tag{29}$$

If the transformation does not satisfy (29) then the resulting moments of inertia always diverge. The rotation matrices can be written in terms of the four Euler angles

$$\mathcal{A} = \mathcal{B} = e^{-i\lambda_- \alpha_1/2} e^{-i\lambda_7 \alpha_2/2} e^{-i\lambda_- \alpha_3/2} e^{-i\lambda_+ \alpha_4/2} \tag{30}$$

where the generators λ_- and λ_+ are defined as

$$\lambda_- := -(\lambda_3 - \sqrt{3}\lambda_8)/2, \quad \lambda_+ := -(\lambda_3 + \lambda_8/\sqrt{3}).$$

The generators $\{\lambda_6, \lambda_7, \lambda_-\}$ satisfies the $SU(2)$ algebra and the λ_+ commutes with them.

The standard procedure proceeds by promoting the parameter $\mathcal{A}$ to the status of dynamical variables $\mathcal{A}(t)$. Then the dynamical ansatz adopted in collective coordinate quantization is

$$X(\boldsymbol{r}; \mathcal{A}(t)) = \mathcal{A}(t) X(\boldsymbol{r}) \mathcal{A}^\dagger(t). \tag{31}$$

Plugging (31) into the Lagrangian (1) and assuming that $\mathcal{A}^\dagger \dot{\mathcal{A}} = -i\lambda_a \Omega^a/2$, where the symbol Ω^a represent the angular velocities, one gets the effective Lagrangian in the form

$$L_{\text{eff}} = -M_{\text{cl}} + \frac{1}{2} I_{ab} \Omega^a \Omega^b \quad a, b = 6, 7, -, + \tag{32}$$

where M_{cl} is the mass of static solution . By virtue of the axial symmetry of solution, we get the relation $I_{66} = I_{77}$ and the off-diagonal components vanish except $I_{+-} = I_{-+}$. Through the Legendre transformation, one can derive the quantum Hamiltonian which coincide with the Hamiltonian of a rotating symmetric top. The following set of eigenfunctions

$$\psi = D^l_{m,-m'}(\alpha_1, \alpha_2, \alpha_3) e^{iY\alpha_4} \tag{33}$$

diagolalize the Hamiltonian, where $D^l_{m,-m'}(\alpha_1, \alpha_2, \alpha_3)$ is the $SU(2)$ Wigner D-function. The energy spectrum is labelled by quantum numbers l, m and Y being eigenvalues of the Hamiltonian.

6. Fermions coupled with the principal variable

The fermion-vortex system was initially studied by [15]. It was shown that the normalizable zero-mode of the fermion appears due to the Index theorem. The Lagrangian of such a system has the form

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{1}{2} ig\phi \bar{\psi} \psi^c + \frac{1}{2} ig\phi \bar{\psi^c} \psi \tag{34}$$

where $\mu = 0, 1, 2, 3$ and a superscript c denotes charge conjugation. For the Dirac spinors and matrices, they keep four-components. However, due to the cylindrical symmetry of the vortex, there occurs an effective reduction to a two-space, one-time theory. The authors succeeded to find the analytical zero mode solutions of the spinors. A convenient redefinition of the Lagrangian in terms of Majorana spinors is presented in [16]. The Majorana fermions are defined as

$$\chi = \frac{1}{\sqrt{2}}[\psi_L + (\psi_L)^c], \quad \omega = \frac{1}{\sqrt{2}}[\psi_R + (\psi_R)^c] \tag{35}$$

where ψ_L, ψ_R are the left-, and right-handed spinors. One can see that the fermion Lagrangian can be rewritten in the form

$$\mathcal{L} = \bar{\chi} i\gamma^\mu \partial_\mu \chi - gf\phi\bar{\chi}\exp[i(n\theta + \frac{1}{2}\pi)\gamma_5]\chi + \bar{\omega} i\gamma^\mu \partial_\mu \omega - gf\phi\bar{\omega}\exp[-i(n\theta + \frac{1}{2}\pi), \gamma_5]\omega \tag{36}$$

where $n-$vortices have the form $\phi(\rho, \theta) = f(\rho)\exp(in\theta)$. If the fermion is a right-(or left-) handed field, χ(or ω) is identically zero. We can introduce the Lagrangian in a similar way for the CP^N vortices

$$\mathcal{L}_3 = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi} X^{\gamma_5} \psi \tag{37}$$

where $X^{\gamma_5} \equiv P_R X + P_L X^\dagger$, and $P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$. The corresponding Dirac equation has exactly $Q_{\rm top}$ normalizable modes.

The coupling of vortices to fermionic field can be studied in another approach following [17, 18, 19] where the fermionic sigma model is considered. The Lagrangian of such a model has the form

$$\mathcal{L} = \bar{\psi} i\gamma^\alpha \partial_\alpha \psi - m\bar{\psi}\vec{\tau} \cdot \vec{n}\psi \tag{38}$$

where $\alpha = 0, 1, 2$, $\vec{\tau}$ is a standard Pauli matrices and $\vec{n}$ is the real triplet scalar field. There are the normalizable modes on the plane and, what calls attention, a derivative expansion of the determinant corresponding to (38) indicates that the model has a proper topological charge of the bosonic part. Clearly, it is worth to study the fermion-CP^N vortex in this context. We point out that the field $\vec{\tau} \cdot \vec{n}$ is Hermitian, and therefore it is natural to employ the CP^N Hermitian field $X_{\rm H}$. The Lagrangian is thus

$$\mathcal{L}_2 = \bar{\psi} i\gamma^\alpha \partial_\alpha \psi - m\bar{\psi} X_{\rm H} \psi, \tag{39}$$

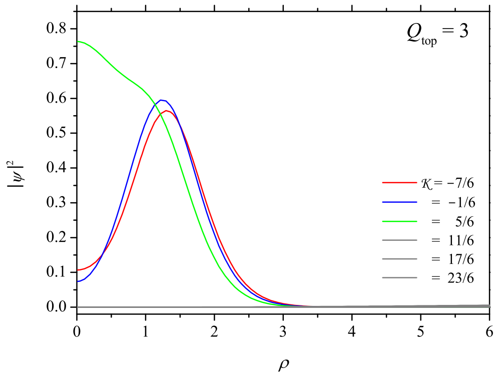
The corresponding Dirac equation can be solved numerically. The quantum number of the spinor consists of a funny combination of the angular momentum, the isospin and the Gell-Mann matrices

$$\mathcal{K} = l_3 + \frac{\tau_3}{2} - \frac{n_1}{2}\left(\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8\right) + \frac{n_2}{2}\left(\lambda_3 - \frac{1}{\sqrt{3}}\lambda_8\right) \tag{40}$$

where l_3 is a third component of the orbital angular momentum and the λ_i stand for the Gell-Mann matrices. All the spectra are thus labelled by $\mathcal{K}$. For instance they are of the form

$$\begin{aligned} &\mathcal{K} = -\frac{7}{6}, \ -\frac{1}{6}, \ \frac{5}{6}, \ \frac{11}{6}, \cdots \quad (n_1, n_2) = (3, 1), (2, -1), \cdots \\ &\mathcal{K} = -\frac{11}{6}, \ -\frac{5}{6}, \ \frac{1}{6}, \ \frac{7}{6}, \cdots \quad (n_1, n_2) = (4, 1), (3, -1), \cdots \end{aligned} \tag{41}$$

In Fig.2 we present the angular averaged densities $\sim \psi^\dagger \psi$ coupled with the vortex $(n_1, n_2) = (-3, -2), (3, -1)$. The number of states equals to the topological charge $Q_{\rm top} = 3, 4$.



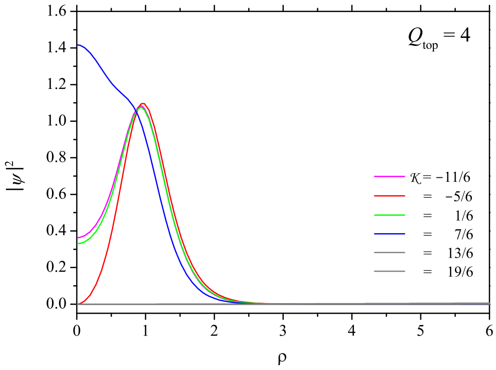


Figure 2. The angular averaged fermion densities coupled with the vortex with $(n_1, n_2) = (-3, -2)$ (left) and $(3, -1)$ (right).

7. Summary

We found solutions of the CP^N Skyrme-Faddeev type model. The model is written in terms of the principal variable which naturally describes the CP^N manifold. We observe that the model can be defined in several inequivalent ways which differ by the choice of X. For any such a choice the model has the same classical vortex solutions. However, the quantization and coupling to fermions fermions clearly depend on the choice of X (form of the prefactor $\mathcal{P}$). The existence of classical vortex solutions is very sensitive on the form of the potential. We proposed a systematic formulation of the potential and we manage to find solutions with many winding numbers. In the last sections, we gave some basic discussions of the collective quantization for the vortices and the coupling of fermions, i.e., fermion-CP^N vortex system.

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