Twist deformations leading to κ-Poincaré Hopf algebra and their application to physics

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Abstract. We consider two twist operators that lead to kappa-Poincaré Hopf algebra, the first being an Abelian one and the second corresponding to a light-like kappa-deformation of Poincaré algebra. The adventage of the second one is that it is expressed solely in terms of Poincaré generators. In contrast to this, the Abelian twist goes out of the boundaries of Poincaré algebra and runs into envelope of the general linear algebra. Some of the physical applications of these two different twist operators are considered. In particular, we use the Abelian twist to construct the statistics flip operator compatible with the action of deformed symmetry group. Furthermore, we use the light-like twist operator to define a star product and subsequently to formulate a free scalar field theory compatible with kappa-Poincaré Hopf algebra and appropriate for considering the interacting ϕ^4 scalar field model on kappa-deformed space.

1. Introduction

It is known in the literature that the star product realization of the well known κ-deformed Minowski spacetime algebra [1, 2, 3, 4], $[x_0, x_j]_* = iax_j \equiv \frac{i}{\kappa}x_j$, for $j = 1, ..., n-1$ with remaining elements commuting [5], may arise from several different types of Drinfeld's twist operators [5]-[12], the Jordanian twist operator [13, 14, 15, 16] and the Abelian twist operator [17, 18, 19] being the most prominent ones, or at least being among those that have been mostly studied. That being said normaly assumes the existence of the algebra of functions $\mathcal{A}[[a]]$ where the role of the multiplication in the algebra is being played by the twisted star product $*$ given by

$$f * g = \mu \circ \mathcal{F}^{-1}(f \otimes g) = \bar{f}^\alpha(f)\bar{f}_\alpha(g), \tag{1}$$

for any two functions f and g in the algebra and where the twist operator is being symbolically written as $\mathcal{F} = f^\alpha \otimes f_\alpha$. [6] As was pointed out in [16], in the case that this twist is either Abelian or Jordanian, the star product (1) leads to the star commutation relation written above.

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[5] Here a is the formal parameter of deformation and $\kappa = \frac{1}{a}$ has the mass dimension. n is the number of generators of the spacetime algebra.

[6] $\mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha,$

On the other hand, when the focus is drawn toward κ-Poincaré Hopf algebra [1, 2, 3, 4], a different algebraic structure, although closely related to κ-deformed Minkowski spacetime algebra (see [4, 20] for the exact relation between the two), then the appearance of this structure from out of the original Poincaré Hopf algebra by means of the twist deformation method does not appear to be paved by as many proposals for the twists accomplishing this task, yet alone not in a format closed under the Poincaré algebra. Indeed, not only that there haven't been many proposals on offer, but also it was believed that the twist which would account for such a deformation does not even exist.

It is though true that some attempts have been made to get κ-Poincaré Hopf algebra from the Poincaré Hopf algebra, but neither of them finished with much of a success. The main problem being that the twist used either gives rise to a coalgebra (or at least the part of a coalgebra) that runs out of the Poincaré algebra and escapes into $igl(n)$ or the twist operator itself is not made of Poincaré generators only, but in building it up one needs to supplement the Poincaré generators by the elements of $igl(n)$.

In this paper we consider two types of Drinfeld's twists that can accomplish the aforementioned deformation that leads from the nondeformed Hopf algebra to the deformed one (κ-Poincaré Hopf algebra). The first one is the Abelian twist

$$\mathcal{F} = \exp\left\{i(\lambda x_k p_k \otimes A - (1-\lambda)A \otimes x_k p_k)\right\}, \tag{2}$$

where $0 \leq \lambda \leq 1$, is a real parameter and $A = ap_0$, and the second one is the so called light-like twist

$$\mathcal{F} = \exp\left\{ia^\alpha P^\beta \frac{\ln(1 + a \cdot P)}{a \cdot P} \otimes M_{\alpha\beta}\right\}, \tag{3}$$

where $a^2 = 0$ (therefrom the name). It was recently realized that the Abelian twist (2), despite the previous lack of success, may though reproduce the κ-Poincaré Hopf algebra. However, the price paid is that one needs to replace the Hopf algebra framework with a somewhat more general algebraic structure, like e.g. that of the Hopf algebroid [21, 22, 23]. Contrary to this, when the light-like twist (3) is concerned, everything is smooth there. This twist operator can reproduce κ-Poincaré Hopf algebra by staying only within the framework of the Hopf algebra. There is no need to go outside of this framework and to invoke certain more general algebraic structures. In addition to that, there are only Poincaré generators appearing in the twist (3).

In what follows we briefly describe how these twists came up onto the surface and discuss their physical applications. But before that, the notion of κ-Poincaré algebra in a given basis is introduced and the relation between particular basis of κ-Poincaré and κ-Minkowski spacetime realization, the operator ordering prescription and the star product is explained.

2. κ-Poincaré Hopf algebra

We take a brief look at κ-Poincaré algebra, which is one particular algebraic structure obtained by quantizing the usual Poincaré algebra and where the κ is a formal deformation parameter. Our interest shall be primarily focused on its coalgebraic sector, particularly on the coproducts for the Poincaré generators, that is the momentum generators $p_\mu = -i\partial_\mu$,

$$\begin{aligned}
\triangle_\varphi(p_0) &= p_0 \otimes \mathbf{1} + \mathbf{1} \otimes p_0, \\
\triangle_\varphi(p_i) &= \varphi(A \otimes \mathbf{1} + \mathbf{1} \otimes A)\left[\frac{p_i}{\varphi(A)} \otimes \mathbf{1} + e^A \otimes \frac{p_i}{\varphi(A)}\right]
\end{aligned} \tag{4}$$

and the Lorentz generators $M_{\mu\nu}$,

$$\begin{aligned}
\triangle_\varphi(M_{ij}) &= M_{ij} \otimes \mathbf{1} + \mathbf{1} \otimes M_{ij}, \\
\triangle_\varphi(M_{i0}) &= M_{i0} \otimes \mathbf{1} + e^A \otimes M_{i0} - ap_j\frac{1}{\varphi(A)} \otimes M_{ij},
\end{aligned} \tag{5}$$

where $A = -ia_0\partial_0 = ap_0$. Here we chose $a_\mu = (a, 0, ..., 0)$, $a \sim \frac{1}{\kappa}$.

It can be noted that the coalgebra is parametrized by the function $\varphi(A)$. It is a smooth function of A and each choice of this function corresponds to a certain basis of κ-Poincaré algebra. For example, if $\varphi(A) = 1$, then the coalgebra (4),(5) corresponds to the bicrossproduct basis of κ-Poincaré.

More direct interpretation of the function $\varphi(A)$ may be acquired if we turn to another interesting object, which is intimately related to κ-Poincaré algebra: κ-Minkowski spacetime,

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_j] = ia\hat{x}_j \equiv \frac{i}{\kappa}\hat{x}_j, \quad \text{for} \quad j = 1, ..., n-1. \tag{6}$$

In this context, each choice of $\varphi(A)$ corresponds to a particular differential representation of κ-Minkowski spacetime in terms of the generators x_μ, p_ν of the undeformed Heisenberg algebra

$$[x_\mu, x_\nu] = 0, \quad [p_\mu, p_\nu] = 0, \quad [p_\mu, x_\nu] = -i\eta_{\mu\nu}, \tag{7}$$

where $\eta_{\mu\nu} = diag(-1, 1, ..., 1)$. More explicitly, the realizations of κ-Minkowski spacetime that correspond to the coalgebra (4),(5) all belong to the family

$$\hat{x}_i = x_i\varphi(A), \tag{8}$$
$$\hat{x}_0 = x_0\psi(A) - ax_k p_k\gamma(A), \tag{9}$$

with two other functions, $\gamma(A)$ and $\psi(A)$, not being independent of $\varphi(A)$, but rather being related to it through the consistency relation $\frac{\varphi'}{\varphi}\psi = \gamma - 1$.

Furthermore, to each particular realization parametrized by the function $\varphi(A)$ there is a uniquely defined operator ordering prescription, so that for example one has

$$\begin{aligned}
: e^{ik_\mu\hat{x}^\mu} :_L &\equiv e^{-ik_0\hat{x}_0}e^{ik_i\hat{x}_i} = e^{-ik_0\hat{x}_0 + ik_i\hat{x}_i\varphi_S(ak_0)e^{ak_0}}, \\
: e^{ik_\mu\hat{x}^\mu} :_R &\equiv e^{ik_i\hat{x}_i}e^{-ik_0\hat{x}_0} = e^{-ik_0\hat{x}_0 + ik_i\hat{x}_i\varphi_S(ak_0)}, \\
: e^{ik_\mu\hat{x}^\mu} :_S &\equiv e^{ik_\mu\hat{x}^\mu},
\end{aligned} \tag{10}$$

for the left, right and totaly symmetric Weyl ordering, respectively. In the above expressions $\varphi_S(A) = \frac{A}{e^A - 1}$. Moreover, for the general ordering prescription labeled by $:\quad:_\varphi$ and corresponding to an arbitrary realization $\varphi(A)$, one can write

$$: e^{ik_\mu\hat{x}^\mu} :_\varphi = e^{-ik_0\hat{x}_0 + ik_i\hat{x}_i\frac{\varphi_S(ak_0)}{\varphi(ak_0)}}. \tag{11}$$

It is readily seen that from the later relation the left, right and Weyl totaly symmetric ordering can be recovered by taking the choices $\varphi = e^{-A}$, $\varphi = 1$, $\varphi = \varphi_S$, respectively. Of interest here will be the following family of operator orderings,

$$: e^{ik_\mu\hat{x}^\mu} :_\lambda = e^{-i\lambda k_0\hat{x}_0}e^{ik_i\hat{x}_i}e^{-i(1-\lambda)k_0\hat{x}_0}, \tag{12}$$

corresponding to the κ-Minkowski spacetime realization $\varphi(A) = e^{-\lambda A}$. It interpolates between the right, time-symmetric and left ordering, corresponding respectively to $\lambda = 0, \frac{1}{2}$ and 1.

The exponentials $: e^{ik_\mu\hat{x}^\mu} :_\varphi$ may be viewed as the plane waves on the κ-Minkowski spacetime. If further they are to be considered as the elements of the Borel group, then the notion of coproduct encodes the group multiplication rule

$$: e^{ik_\mu\hat{x}^\mu} :_\varphi : e^{iq_\mu\hat{x}^\mu} :_\varphi = : e^{i(k\oplus q)_\mu\hat{x}^\mu} :_\varphi = e^{i\mathcal{D}^\varphi(k,q)x}, \tag{13}$$

where the connection between the coproduct and the function $\mathcal{D}^\varphi(k, q)$ is unraveled by the relation $\triangle_\varphi(p_\mu) = \mathcal{D}_\mu^\varphi(p \otimes 1, 1 \otimes p)$.

Thus, to conclude, for each $\varphi(A)$ parametrizing the initial coalgebra, there is a unique operator ordering prescription, there is a unique realization of the accompanying structure (κ-Minkowski space) and there is a unique star product and a twist operator, as we shall see shortly.

A comment is needed regarding the algebra sector. In the algebra sector, the Lorentz algebra remains undeformed in this setting, while the commutator $[M_{\mu\nu}, p_\lambda]$ is deformed and depends on the realization $\varphi(A)$. However, it is also possible to formulate the κ-Poincaré algebra in the so called classical basis [20, 24, 25]. Classical here means that the algebraic sector is undeformed and the whole deformation is contained within the coalgebraic sector. This coalgebra is not contained within the coalgebra parametrization (4),(5) and needs to be treated separately. It looks as

$$\triangle(P_\mu) \;=\; P_\mu \otimes Z^{-1} + \mathbf{1} \otimes P_\mu - a_\mu(P_\lambda Z) \otimes P^\lambda + \frac{a_\mu}{2}\square\, Z \otimes aP, \tag{14}$$

$$\begin{aligned}\triangle(M_{\mu\nu}) \;=\;& M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &+\; a_\mu\left(-P^\lambda + \frac{a^\lambda}{2}\square\right) Z \otimes M_{\lambda\nu} + a_\nu\left(P^\lambda - \frac{a^\lambda}{2}\square\right) Z \otimes M_{\lambda\mu},\end{aligned} \tag{15}$$

where this time a_μ is the fourvector deformation parameter and

$$Z^{-1}(P) = aP + \sqrt{1 + a^2 P^2}, \qquad \square(P) = \frac{2}{a^2}(1 - \sqrt{1 + a^2 P^2}), \quad P_\mu = -i\partial_\mu. \tag{16}$$

Similarly as before, there exists a realization of the accompanying κ-Minkowski space,

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \tag{17}$$

that corresponds to the above coalgebra and it is given by

$$\hat{x}_\mu = x_\mu\left(aP + \sqrt{1 + a^2 P^2}\right) - (ax)P_\mu. \tag{18}$$

3. Twist operators and twisting

3.1. Abelian twist

Star product for the general parametrization $\varphi(A)$ can be derived [26] and the result is

$$(f \star_\varphi g)(x) = \lim_{\substack{u \to x \\ t \to x}} e^{x_j \partial_j^u\left(\frac{\varphi(A_u + A_t)}{\varphi(A_u)} - 1\right) + x_j \partial_j^t\left(\frac{\varphi(A_u + A_t)}{\varphi(A_t)}e^{A_u} - 1\right)} f(u)g(t), \tag{19}$$

where $A_u = -ia_0\partial_0^u$, $A_t = -ia_0\partial_0^t$. It corresponds to the coalgebra (4),(5) and also to the κ-Minkowski spacetime realization (9), both of them being parametrized with $\varphi(A)$. As a next step, we recall the general definition of the star product

$$f * g = \mu_*(f \otimes g) = \mu \circ \mathcal{F}^{-1}(f \otimes g) \tag{20}$$

and the result which says that for an arbitrary operator f commuting with $D \equiv x_i\partial_i$ (D is the dilatation operator) it holds [26]

$$: e^{Df} : \;=\; e^{D\ln(1+f)}. \tag{21}$$

Relying on these two, it is possible to extract the twist operator from the above star product,

$$\mathcal{F}_\varphi \;=\; \exp\left\{(D \otimes 1)\ln\frac{\varphi(A \otimes 1 + 1 \otimes A)}{\varphi(A \otimes 1)} + (1 \otimes D)(A \otimes 1 + \ln\frac{\varphi(A \otimes 1 + 1 \otimes A)}{\varphi(1 \otimes A)})\right\}. \tag{22}$$

For $\varphi = e^{-\lambda A}$ this reduces to the family of Abelian twists parametrized with the real number λ,

$$\mathcal{F} \;=\; \exp\left\{i(\lambda x_k p_k \otimes A - (1-\lambda)A \otimes x_k p_k)\right\}. \tag{23}$$

3.2. Light-like twist

There is a straightforward, but tedious and not always feasible procedure for constructing the twist operator from a given coalgebra. The method requires introducing the coordinate generators and considering their coproducts along with the coproducts for the symmetry generators (momenta). It also utilises the perturbative methods which in due course give a twist operator expressed in the form of an infinite series of terms, whose summuation is anything but trivial. However, the ultimate hope is that this series can be resummed and brought into a closed form. The method is the following [27]: Starting from the nontrivial coproducts $\triangle x$ and $\triangle p$, representing a given deformed coalgebra, one searches for the operator $\mathcal{F}$ that executes the following transformations:

$$\triangle(x_\mu) = \mathcal{F}\triangle_0(x_\mu)\mathcal{F}^{-1}, \quad \triangle(p_\mu) = \mathcal{F}\triangle_0(p_\mu)\mathcal{F}^{-1}. \tag{24}$$

Here

$$\triangle_0(x_\mu) = x_\mu \otimes \mathbf{1}, \quad \triangle_0(p_\mu) = p_\mu \otimes \mathbf{1} + \mathbf{1} \otimes p_\mu, \tag{25}$$

are the primitive coproducts for the coordinate and momentum generators. Next, one expresses $\triangle x$ and $\triangle p$ as the power series in the deformation parameter $a \sim \frac{1}{\kappa}$,

$$\triangle(x_\mu) = \sum_{k=0}^{\infty} \triangle_k(x_\mu), \quad \triangle(p_\mu) = \sum_{k=0}^{\infty} \triangle_k(p_\mu), \quad \triangle_k(x_\mu), \triangle_k(p_\mu) \sim a^k. \tag{26}$$

For the twist operator one may take the ansatz $\mathcal{F} = e^f$, where $f = \sum_{k=1}^{\infty} f_k$ and symbolically $f_k \propto a^k x p^{k+1}$ are the objects that need to be found. Then by using (24) and comparing both sides in these expressions, order by order in the deformation parameter, one gets the set of conditions

$$\Delta_1(x_\mu) = [f_1, \Delta_0(x_\mu)]$$

$$\Delta_2(x_\mu) = [f_2, \Delta_0(x_\mu)] + \frac{1}{2}[f_1, [f_1, \Delta_0(x_\mu)]]$$

$$\Delta_3(x_\mu) = [f_3, \Delta_0(x_\mu)] + \frac{1}{2}\left([f_1, [f_2, \Delta_0(x_\mu)]] + [f_2, [f_1, \Delta_0(x_\mu)]]\right) + \frac{1}{3!}[f_1, [f_1, [f_1, \Delta_0(x_\mu)]]]$$

...

$$\Delta_k(x_\mu) = [f_k, \Delta_0(x_\mu)] + ... + \frac{1}{k!}[f_1, [f_1, ...[f_1, \Delta_0(x_\mu)]]]$$

...

$$\tag{27}$$

and analogously for $\Delta(p_\mu)$. The interesting thing is that the above described procedure can be carried out in a full extent, when applied to the coalgebra (14),(15). Even more, in the special case when $a^2 = 0$, the infinite sum $f = \sum_{k=1}^{\infty} f_k$ can be evaluated and written in a compact form [28]. The result is the twist operator (3).

3.3. Twisting

Taking the Abelian twist (2) and applying it onto the primitive coproduct $\triangle_0(p) = p \otimes \mathbf{1} + \mathbf{1} \otimes p$ (by means of the twist deformation (24)), one recovers the momentum part (4) of the coalgebra (4),(5). However, if one repeats the story and applies the Abelian twist (2) onto the primitive coproduct of Lorentz generators, he does not get the part (5) of the κ-Poincaré algebra! Nevertheless, despite this inconvenience, the Abelian twist though may be used to reconstruct the full coalgebra (4),(5), as well as the appropriate antipodes, which for simplicity reasons were not written in this letter (see [29]). Though, to accomplish this, one needs to extend the

algebraic framework of the Hopf algebra and go beyond it. In this case an appeal to a broader algebraic framework, like for example that of the Hopf algebroid, appears to be necessary. It is also necessary to use certain tensor exchange identities [30] and when twisting, one does not twist the primitive coproduct itself, but the object that is compatible with the homomorphic property of the coproduct. The twist (2) satisfies the cocycle and counital conditions:

$$
\begin{aligned}
(\mathcal{F} \otimes \mathbf{1}) \cdot (\triangle \otimes id)\mathcal{F} &= (\mathbf{1} \otimes \mathcal{F})) \cdot (id \otimes \triangle)\mathcal{F}, \\
\mu \circ (\epsilon \otimes id)\mathcal{F} &= \mathbf{1} = \mu \circ (id \otimes \epsilon)\mathcal{F}.
\end{aligned}
\tag{28}
$$

The twist (3) for the light-like κ-deformation ($a^2 = 0$) also complies with the above conditions of cocyclicity and normalization. However, the adventage of this twist is that unlike the Abelian one, it is expressed in terms of Poincaré generators only (it doesn't contain the dilatation operator). Also, it is written in a covariant form. And finally, via twist deformation, it gives rise to the κ-Poincaré Hopf algebra written in a classical basis,

$$
\begin{aligned}
\Delta^{\mathcal{F}}(M_{\mu\nu}) &= \mathcal{F}\Delta_0(M_{\mu\nu})\mathcal{F}^{-1} = \Delta_0(M_{\mu\nu}) + (\delta_\mu^\alpha a_\nu - \delta_\nu^\alpha a_\mu)\left(P^\beta + \frac{1}{2}a^\beta P^2\right) Z \otimes M_{\alpha\beta}, \\
\Delta^{\mathcal{F}}(P_\mu) &= \mathcal{F}\Delta_0(P_\mu)\mathcal{F}^{-1} = \Delta_0(P_\mu) + \left[P_\mu a^\alpha - a_\mu\left(P^\alpha + \frac{1}{2}a^\alpha P^2\right) Z\right] \otimes P_\alpha, \\
S^{\mathcal{F}}(M_{\mu\nu}) &= \chi S(M_{\mu\nu})\chi^{-1} = -M_{\mu\nu} + (-a_\mu\delta_\nu^\beta + a_\nu\delta_\mu^\beta)\left(P^\alpha + \frac{1}{2}a^\alpha P^2\right) M_{\alpha\beta}, \\
S^{\mathcal{F}}(P_\mu) &= \chi S(P_\mu)\chi^{-1} = \left[-P_\mu - a_\mu\left(P_\alpha + \frac{1}{2}a_\alpha P^2\right) P^\alpha\right] Z, \\
\epsilon^{\mathcal{F}}(M_{\mu\nu}) &= 0, \quad \epsilon^{\mathcal{F}}(P_\mu) = 0,
\end{aligned}
\tag{29}
$$

where $Z = \frac{1}{1+aP}$ and $\chi^{-1} = \mu \circ \left[(S \otimes 1)\mathcal{F}^{-1}\right]$, with S and ϵ denoting the antiopode and counit, respectively and with the superscript $\mathcal{F}$ on this quantities refering to their deformed counterparts. Therefore, the twist (3) reproduces the coalgebra (14),(15) for the special case when $a^2 = 0$, that is it reproduces a deformed Poincaré Hopf algebra corresponding to a light-like deformation. The important point to be emphasized here is that to derive κ-Poincaré Hopf algebra by using the twist (3), one doesn't need to go beyond the Hopf algebra setting, since everything can be done within this framework.

4. Applications
4.1. Twisted statistics
We use this example to illustrate the application of the Abelian twist (2). Suppose first that we have a standard situation meaning that there are no effects induced by the noncommutativity ($a = 0$). Then assume that we have a certain symmetry algebra $\mathcal{G}$ (e.g. Poincaré) under which the system under consideration is invariant. Then the action of this symmetry algebra on the Hilbert space (here denoted by $\mathcal{A}$) of physical states is realized in some representation of $\mathcal{G}$. Therefore, if $|\phi\rangle$ is some state in the Hilbert space $\mathcal{A}$, and Λ is some element in the symmetry algebra, then the action of $\mathcal{G}$ on $\mathcal{A}$ is realized in some representation D of $\mathcal{G}$, $|\phi\rangle \longrightarrow D(\Lambda) |\phi\rangle$. If we further want to extend the action of this symmetry group from one-particle to two-particle (and generally many-particle states), we need to use the notion of the coproduct, so that the action of $\mathcal{G}$ on the two-particle states is performed as,

$$
|\phi\rangle \otimes |\psi\rangle \longrightarrow (D \otimes D)\Delta_0(\Lambda)|\phi\rangle \otimes |\psi\rangle,
\tag{30}
$$

where for $a = 0$ the coproduct Δ_0 is defined as $\Delta_0 : \Lambda \longrightarrow \Lambda \otimes \mathbf{1} + \mathbf{1} \otimes \Lambda$. Furthermore, this coproduct has to be compatible with the multiplication μ (usual pointwise multiplication) in

the algebra $\mathcal{A}$ of physical states. This is achieved by the requirement

$$\mu\left((D \otimes D)\Delta_0(\Lambda)|\phi\rangle \otimes |\psi\rangle\right) = D(\Lambda)\,\mu(|\phi\rangle \otimes |\psi\rangle). \tag{31}$$

When considering quantum mechanics and its premise of indistinguishable particles, the quantum statistics is usually implemented by restricting to subspaces which are composed of either totally symmetric or totally antysymmetric states. This is achieved by considering the so called statistics flip operator τ_0, which on an element $|\phi\rangle \otimes |\psi\rangle$ from $\mathcal{A} \otimes \mathcal{A}$ has the action

$$\tau_0(|\phi\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes |\phi\rangle, \tag{32}$$

and by using the projector $\frac{1}{2}(1 \pm \tau_0)$, with which help the symmetrization or antisymmetrization on a two particle Hilbert space is carried out. If we further want that these subspaces, built out by projectors, remain invariant under the action of the symmetry algebra (meaning that the process of (anty)symmetrization is invariant under the action of the symmetry algebra), we need to require that this τ_0 must commute with Δ_0 coproduct, through which the symmetry is implemented,

$$[\Delta_0(\Lambda), \tau_0] = 0. \tag{33}$$

Physically this means that the process of symmetrization or antisymmetrization is frame independent. The statistics thus remains invariant under the action of the symmetry algebra $\mathcal{G}$.

In the noncommutative case, when $a = \frac{1}{\kappa} \neq 0$, the deformed coproduct $\Delta_\varphi = \mathcal{F}_\varphi{}^{-1}\Delta_0\mathcal{F}_\varphi$, where $\mathcal{F}_\varphi$ is the twist element in (22), will no more be compatible with the multiplication μ. Therefore μ must be replaced with the new multiplication μ_*, so that the compatibility condition be satisfied,

$$\mu_*\left((D \otimes D)\Delta_\varphi(\Lambda)|\phi\rangle \otimes |\psi\rangle\right) = D(\Lambda)\,\mu_*(|\phi\rangle \otimes |\psi\rangle). \tag{34}$$

What about subspaces projected with $\frac{1}{2}(1 \pm \tau_0)$? The subspaces obtained in this way will no more be invariant under the action of the symmetry algebra. To correct this, one introduces a new statistics flip operator by means of the Abelian twist deformation, $\tau_\varphi = \mathcal{F}_\varphi^{-1}\tau_0\mathcal{F}_\varphi$. Unlike τ_0, this new τ_φ will now commute with Δ_φ and consequently may serve to project out subspaces irreducibly invariant under the twisted symmetry algebra. The role of the projector in this case is played by the operator $\frac{1}{2}(1 \pm \tau_\varphi)$. It gives rise to irreducible tensors of certain symmetry type.

Recall now the Abelian twist (22). One can use it to find the twisted statistics flip operator corresponding to the general class of realizations (8),(9) of κ-Minkowski space,

$$\tau_\varphi = e^{(D \otimes A - A \otimes D)}\tau_0, \tag{35}$$

and to find the corresponding universal R-matrix. The result for τ_φ may be used to derive the oscillator algebra of creation and annihilation operators for the scalar field on the noncommutative (NC) κ-deformed spacetime and to investigate the quantum particle statistics on it. This was illustrated in the example of the scalar field probing the background of the NC BTZ [31]. Deformed oscillator algebras of the similar or slightly different type on κ-deformed spaces are also considered in [32, 33, 34, 35, 36, 37, 38, 39]. Moreover, the twist (23) or its variant for $\lambda = \frac{1}{2}$ was used to construct $U(1)$ gauge theory in [40, 41]. Note that as A is directly proportional to the deformation parameter a, the twisted flip operator τ_φ goes over smoothly to the untwisted flip operator τ_0 as the deformation parameter $a \to 0$.

4.2. Scalar field propagation in the ϕ^4 κ-Minkowski model

We use this example to illustrate the application of the light-like twist (3). Here we consider the properties of the massive scalar field propagation within the ϕ^4 κ-Minkowski model based

on the light-like twist (3). Before writing the appropriate action, it is worthy to note [42] that the star product induced by the twist (3) has the property

$$\int d^n x \ \phi^\dagger \star_h \psi = \int d^n x \ \phi^* \cdot \psi \,, \tag{36}$$

that is, under the integration sign the star product between two functions may simply be dropped out and replaced by the pointwise multiplication. As a consequence, the free part (the first two terms) of the NC action

$$
\begin{aligned}
S_n[\phi] &= \int d^n x \ (\partial_\mu \phi)^\dagger \star_h (\partial^\mu \phi) + m^2 \int d^n x \ \phi^\dagger \star_h \phi \\
&+ \frac{\lambda}{4} \int d^n x \ \frac{1}{2} (\phi^\dagger \star_h \phi^\dagger \star_h \phi \star_h \phi + \phi^\dagger \star_h \phi \star_h \phi^\dagger \star_h \phi) \,,
\end{aligned}
\tag{37}
$$

which is built by using the same star product, will be completely reduced to a usual commutative free scalar field action. The only nontrivial contributions in the action (37) will come from the interaction terms. As for the interaction part of the action, a comment is in order. Actually, there are altogether six interaction terms that could in principle appear in the action (37), corresponding to six possible permutations between the fields $\phi^\dagger$ and ϕ. However, due to the properties of the star product, they reduce to only two mutually nonequivalent combinations and these two are that appearing in (37).

After expanding the star product, the efective action emerges,

$$
\begin{aligned}
S_n[\phi] &= \int d^n x \ \left[(\partial_\mu \phi^*)(\partial^\mu \phi) + m^2 \ \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 \right] \\
&+ i\frac{\lambda}{4} \int d^n x \left[a_\mu \ x^\mu \left(\phi^{*2} (\partial_\nu \phi) \partial^\nu \phi - \phi^2 (\partial_\nu \phi^*) \partial^\nu \phi^* \right) \right. \\
&+ \left. a_\nu \ x^\mu \left(\phi^2 (\partial_\mu \phi^*) \partial^\nu \phi^* - \phi^{*2} (\partial_\mu \phi) \partial^\nu \phi \right) + \frac{1}{2} a_\nu x^\mu \ \phi^* \phi \left((\partial_\mu \phi^*) \partial^\nu \phi - (\partial_\mu \phi) \partial^\nu \phi^* \right) \right] \,,
\end{aligned}
\tag{38}
$$

which can be treated within the standard framework of perturbative quantum field theory. This means that we may find the Feynman rules resulting from this effective action, by calculating 2-point and 4-point Green's functions. In a momentum space they look as

$$G \equiv G(k_1, k_2) = \frac{i}{k_1^2 + m^2} \delta^{(n)}(k_1 - k_2), \tag{39}$$

and for the vertex function in the momentum space,

$$\tilde{\Gamma}(k_1, k_2, k_3, k_4; a) = i \frac{\delta^4 S[\tilde{\phi}]}{\delta \tilde{\phi}(k_1) \delta \tilde{\phi}(k_2) \delta \tilde{\phi}^*(k_3) \delta \tilde{\phi}^*(k_4)} \,, \tag{40}$$

amounting to the following expression:

$$
\begin{aligned}
\tilde{\Gamma}(k_1, k_2, k_3, k_4; a) \;=\; & i(2\pi)^n \frac{\lambda}{2}\, a_\nu \bigg[\frac{a_\nu}{a^2} + \frac{1}{4}\Big(k_{4\mu}k_{3\nu} + k_{3\mu}k_{4\nu} - 2\delta_{\mu\nu}k_{4\rho}k_{3\rho} \\
& + \frac{1}{2}(k_{2\mu}k_{4\nu} - k_{4\mu}k_{2\nu} + k_{2\mu}k_{3\nu} - k_{3\mu}k_{2\nu}) \Big)\partial_\mu^{k_1} \\
& + \frac{1}{4}\Big(k_{4\mu}k_{3\nu} + k_{3\mu}k_{4\nu} - 2\delta_{\mu\nu}k_{4\rho}k_{3\rho} \\
& + \frac{1}{2}(k_{1\mu}k_{4\nu} - k_{4\mu}k_{1\nu} + k_{1\mu}k_{3\nu} - k_{3\mu}k_{1\nu}) \Big)\partial_\mu^{k_2} \\
& + \frac{1}{4}\Big(k_{1\mu}k_{2\nu} + k_{2\mu}k_{1\nu} - 2\delta_{\mu\nu}k_{1\rho}k_{2\rho} \Big)(\partial_\mu^{k_3} + \partial_\mu^{k_4}) \bigg] \delta^{(n)}(k_1 + k_2 - k_3 - k_4), \qquad (41)
\end{aligned}
$$

where we denote $\partial_\mu^k = \frac{\partial}{\partial k_\mu}$, and all four momenta k_i are flowing into the vertex. The coupling λ has to be dimensionally regularized (see [43] for details). Thus, while the free propagator remains unchanged, the vertex function gets heavily modified.

There is one more thing that one has to take into account when considering the scalar field theory with a deformed underlying symmetry. Namely, due to a deformation of the symmetry which in this case is not the standard Poincaré, but rather its quantum deformed counterpart disguised in the form of κ-Poincaré symmetry, the coalgebraic part of the symmetry will suffer from drastic changes. Since the coalgebra, particularly the coproduct for the momentum generator, is closely related to the momentum addition rule, the connection being disclosed by

$$
\Delta^{\mathcal{F}} p_\mu \equiv \mathcal{D}_\mu(p \otimes 1, 1 \otimes p), \qquad \mathcal{D}_\mu(p, k) = p \oplus k, \qquad (42)
$$

it is clear that the symmetry deformation will give rise to a reinterpretation of the energy-momentum conservation, with the actual form of the energy-momentum conservation being dictated by the form of coproduct for the momentum generators. We want to implement this conclusion into our formalism.

Before doing that, recall the coproduct from (29) which is relevant in this case,

$$
\Delta^{\mathcal{F}}(P_\mu) = \mathcal{F}\Delta_0(P_\mu)\mathcal{F}^{-1} = \Delta_0(P_\mu) + \left[P_\mu a^\alpha - a_\mu \left(P^\alpha + \frac{1}{2}a^\alpha P^2 \right) Z \right] \otimes P_\alpha,
$$

with $Z = \frac{1}{1+aP}$. The corresponding expansion up to the first order in a is

$$
\mathcal{D}_\mu(p, q) = p \oplus q = p_\mu(1 + aq) + q_\mu - a_\mu \frac{pq}{1 + ap} - \frac{1}{2}a_\mu(aq)\frac{p^2}{1 + ap}. \qquad (43)
$$

Going back to the vertex function (41), we consistently modify it in accordance with Eq.(43), so as to keep the track with the change in the coalgebra sector of the underlying symmetry. The modification includes the intervention in the argument of the δ-function in (41), resulting with

the Feynman rule which respects the κ-deformed momentum addition/subtraction rule:

$$
\begin{aligned}
\tilde{\Gamma}(k_1, k_2, k_3, k_4; a) &= i(2\pi)^n \frac{\lambda}{2} a_\nu \left[\frac{a_\nu}{a^2} + \frac{1}{4} \left(k_{4\mu} k_{3\nu} + k_{3\mu} k_{4\nu} - 2\delta_{\mu\nu} k_{4\rho} k_{3\rho} \right. \right. \\
&\quad + \frac{1}{2} (k_{2\mu} k_{4\nu} - k_{4\mu} k_{2\nu} + k_{2\mu} k_{3\nu} - k_{3\mu} k_{2\nu}) \Big) \partial_\mu^{k_1} \\
&\quad + \frac{1}{4} \Big(k_{4\mu} k_{3\nu} + k_{3\mu} k_{4\nu} - 2\delta_{\mu\nu} k_{4\rho} k_{3\rho} \\
&\quad + \frac{1}{2} (k_{1\mu} k_{4\nu} - k_{4\mu} k_{1\nu} + k_{1\mu} k_{3\nu} - k_{3\mu} k_{1\nu}) \Big) \partial_\mu^{k_2} \\
&\quad \left. + \frac{1}{4} \Big(k_{1\mu} k_{2\nu} + k_{2\mu} k_{1\nu} - 2\delta_{\mu\nu} k_{1\rho} k_{2\rho} \Big) (\partial_\mu^{k_3} + \partial_\mu^{k_4}) \right] \\
&\quad \times \left[\delta^{(n)}((k_1 \oplus k_2) \ominus (k_3 \oplus k_4)) + \delta^{(n)}((k_1 \oplus k_2) \ominus (k_4 \oplus k_3)) \right].
\end{aligned} \tag{44}
$$

For the more detailed explanation, see [43].

In order to investigate the properties of the massive scalar field propagation, it is most convenient to consider the connected 2-point Green's function $G_{(c,2)}^a$. The important piece of information for calculating the connected 2-point Green's function comes from the tad pole diagram contribution to the self-energy of the scalar particle, which is here calculated by using the Feynman rules (39),(44),

$$
\Pi_2^a = \int \frac{d^n \ell}{(2\pi)^n} \, \tilde{\Gamma}(k_1, \ell, \ell, k_4; a, \mu) \frac{i}{\ell^2 + m^2} . \tag{45}
$$

After regularizing (45) and isolating the divergences, one gets

$$
\Pi_2^a = \frac{\lambda m^2}{32\pi^2} \left[(1 + 3ak) \left(\frac{2}{\epsilon} + \psi(2) + \log \frac{4\pi\mu^2}{m^2} \right) - \frac{9}{4} ak \right] . \tag{46}
$$

Since Π_2^a is UV divergent, the Green's function $G_{(c,2)}^a$ will also diverge,

$$
G_{(c,2)}^a(k_1, k_4) \propto \left[\frac{i}{k_1^2 + m^2} + \frac{i}{k_1^2 + m^2} \Pi_2^a \frac{i}{k_1^2 + m^2} + \ldots \right],
$$

$$
G_{(c,2)}^a(k_1, k_4) \longrightarrow (2\pi)^n \delta^{(n)}(k_1 - k_4) \left[\frac{i}{k_1^2 + m^2 - \Pi_2^a} \right] . \tag{47}
$$

To remove the infinities from $G_{(c,2)}^a$, that is to renormalize it, one adds a mass counterterm to the Lagrangian. As a consequence, the resulting Green's function $\tilde{G}_{(c,2)}^a$, with the contribution from the mass counterterm included, is finite. More explicitly,

$$
\begin{aligned}
\tilde{G}_{(c,2)}^a(k_1, k_4) &= \left[G_{(c,2)}^a(k_1, k_4) + G_{(c,2)}^a(k_1, k_4)(-\delta m^2) G_{(c,2)}^a(k_1, k_4) + \ldots \right] \\
&\longrightarrow (2\pi)^n \delta^{(n)}(k_1 - k_4) \left[\frac{i}{k_1^2 + m^2 + \delta m^2 - \Pi_2^a} \right],
\end{aligned} \tag{48}
$$

where $\tilde{G}^a_{(c,2)}$ denotes the Green's function which includes the contribution from the mass counter term. It is easily seen that it can be rendered finite if

$$\delta m^2 = \frac{\lambda m^2}{32\pi^2}\left[(1+3ak)\frac{2}{\epsilon} + f\left(\frac{4-\epsilon}{2}, \frac{\mu^2}{m^2}, ak\right)\right].\tag{49}$$

Therefore, in the appropriate mass counterterm the UV divergent part is scaled with the particle statistics factor, which depends linearly on the noncommutativity parameter a and the ingoing momentum. Further discussion can be found in [43].

Acknowledgments

References

[1] Lukierski J, Nowicki A, Ruegg H and Tolstoy V N 1991 *Phys. Lett.* B **264** 331
[2] Lukierski J, Nowicki A and Ruegg H 1992 *Phys. Lett.* B **293**, 344
[3] Lukierski J and Ruegg H 1994 *Phys. Lett.* B **329** 189
[4] Majid S and Ruegg H 1994 *Phys. Lett.* B **334** 348
[5] Drinfeld V G 1983 *Soviet Math. Dokl.* **28** 667
[6] Drinfeld V G 1990 *Leningrad Math. J.* **1** 1419
[7] Chaichian M, Kulish P P, Nishijima K and Tureanu A 2004 *Phys. Lett.* B **604** 98
[8] Balachandran A P, Govindarajan T R, Mangano G, Pinzul A, Qureshi B A and Vaidya S 2007 *Phys. Rev.* D **75** 045009
[9] Aschieri P, Lizzi F and Vitale P 2008 *Phys. Rev.* D **77** 025037
[10] Aschieri P 2007 *Preprint* hep-th/0703013
[11] Aschieri P, Dimitrijevic M, Kulish P, Lizzi F and Wess J 2009 *Lecture Notes in Physics* **774** Springer , Chapter 7
[12] Aschieri P 2009 *Russ. J. Math. Phys.* **16** 371
[13] Kulish P P, Lyakhovsky V D and Mudrov A I 1999 *J. Math. Phys.* **40** 4569
[14] Gerstenhaber M, Giaquinto A and Schack S D 1992 *Quantum symmetry, Quantum Groups* (LNM 1510) (ed. P.P. Kulish), Springer Verlag, Berlin, pp. 9-46
[15] Tolstoy V 2004 *Proc. of International Workshop Supersymmetries and Quantum Symmetries (SQS03)* 242 (Russia, Dubna, eds: E. Ivanov and A. Pashnev, Publ. JINR, Dubna) *Preprint* math/0402433
[16] Borowiec A and Pachol A 2009, *Phys. Rev.* D **79** 045012
[17] Reshetikhin N Yu 1990 *Lett. Math. Phys.* **20** 331
[18] Bu Jong-Gepn, Kim Hyeong-Chan, Lee Youngone , Vac Chang Hyon and Yee Jae Hyung 2008 *Phys. Lett.* B **665** 95
[19] Govindarajan T R, Gupta K S, Harikumar E, Meljanac S and Meljanac D 2008 *Phys. Rev.* D **77** 105010
[20] Kowalski-Glikman J and Nowak S 2002 *Phys. Lett.* B **539** 126
[21] Lu J H 1996 *Int. J. Math.* B **7**
[22] Xu P 2001 *Commun. Math. Phys.* **216** 539-581
[23] Bhm G 2008 *Hopf algebroids* (Handbook of algebra) (*Preprint* 0805.3806[math.QA])
[24] Dimitrijevic M, Jonke L, Moller L, Tsouchnika E, Wess J and Wohlgenannt M 2003 *Eur. Phys. J. C* **31** 129
[25] Borowiec A and Pachol A 2010 **J. Phys.** A **43** 045203
[26] Meljanac S, Samsarov A, Stojic M and Gupta K S 2008 *Eur. Phys. J. C* **53** 295
[27] Juric T, Meljanac S and Strajn R 2014 *Int. J. Mod. Phys.* A **29** 5 1450022
[28] Juric T, Meljanac S, Pikutic D and Strajn R 2015 *J. High Energy Phys.* JHEP07 (2015) 055
[29] Jurić T, Meljanac S and Štrajn R 2013 *Phys. Lett.* A **377** 2472
[30] Meljanac S, Samsarov A and Strajn R 2012 *J. High Energy Phys.* JHEP08 (2012) 127
[31] Gupta K S, Meljanac S and Samsarov A 2012 *Phys. Rev.* D **85** 045029
[32] Arzano M and Marciano A 2007 *Phys. Rev.* D **76** 125005

[33] Daszkiewicz M, Lukierski J and Woronowicz M 2008 *Phys. Rev.* D **77** 105007
[34] Young C A S and Zegers R 2008 *Nucl. Phys.* B **797** 537
[35] Kim H C, Lee Y, Rim C and Yee J H 2009 *J. Math. Phys.* **50** 102304
[36] Govindarajan T R, Gupta K S, Harikumar E, Meljanac S and Meljanac D 2009 *Phys. Rev.* D **80** 025014
[37] Arzano M and Benedetti D 2009 *Int. J. Mod. Phys.* A **24** 4623
[38] Miao R X 2009 *J. Math. Phys.* **50** 063502
[39] Kim H C, Lee Y and Rim C 2010 *Mod. Phys. Lett.* A **25** 1491
[40] Dimitrijevic M and Jonke L 2011 *J. High Energy Phys.* JHEP12 (2011) 080
[41] Dimitrijevic M, Jonke L and Pachol A 2014 SIGMA **10** (2014) 063
[42] Meljanac S and Samsarov A 2011 *Int. J. Mod. Phys.* A **26** 1439
[43] Meljanac S, Samsarov A, Trampetic J and Wohlgenannt M 2011 *J. High Energy Phys.* JHEP12 (2011) 010