

The classical Darboux III oscillator: factorization, Spectrum Generating Algebra and solution to the equations of motion

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Abstract. In a recent paper the so-called *Spectrum Generating Algebra* (SGA) technique has been applied to the N -dimensional Taub-NUT system, a maximally superintegrable Hamiltonian system which can be interpreted as a one-parameter deformation of the Kepler–Coulomb system. Such a Hamiltonian is associated to a specific Bertrand space of non-constant curvature. The SGA procedure unveils the symmetry algebra underlying the Hamiltonian system and, moreover, enables one to solve the equations of motion. Here we will follow the same path to tackle the Darboux III system, another maximally superintegrable system, which can indeed be viewed as a natural deformation of the isotropic harmonic oscillator where the flat Euclidean space is again replaced by another space of non-constant curvature.

1. Introduction

We consider the classical Hamiltonian in \mathbb{R}^N given by:

$$\mathcal{H}_\lambda(\mathbf{q}, \mathbf{p}) = \mathcal{T}_\lambda(\mathbf{q}, \mathbf{p}) + \mathcal{U}_\lambda(\mathbf{q}) = \frac{\mathbf{p}^2}{2m(1 + \lambda\mathbf{q}^2)} + \frac{m\omega^2\mathbf{q}^2}{2(1 + \lambda\mathbf{q}^2)}, \quad (1)$$

where ω and λ are real positive parameters, $\mathbf{q} = (q_1, \dots, q_N)$, $\mathbf{p} = (p_1, \dots, p_N) \in \mathbb{R}^N$ are conjugate coordinates and momenta, $\mathbf{q}^2 = \sum_{i=1}^N q_i^2$ and $\mathbf{p}^2 = \sum_{i=1}^N p_i^2$. The kinetic energy $\mathcal{T}_\lambda(\mathbf{q}, \mathbf{p})$ can be interpreted as the Lagrangian generating the geodesic motion of a particle of mass m on a conformally flat space known as the Darboux space of type III (D-III). As remarked in [1], such an N -dimensional (ND) curved space is the spherically symmetric generalization of the D-III surface [2, 3, 4], which was constructed in [5, 6]. Moreover, the central potential $\mathcal{U}_\lambda(\mathbf{q})$ was proven to be an ‘intrinsic oscillator’ potential on that space [6, 7]. We recall that this definition was first proposed in [6] and that a generalization of the Bertrand’s Theorem [8] to 3D conformally flat Riemannian spaces has been developed in [9, 10, 11]. In this respect, we shall say that the Hamiltonian (1) determines the D-III oscillator.

We also notice that in [6] the Hamiltonian \mathcal{H}_λ has been proven to be maximally superintegrable by taking advantage of its super-separability (like the usual harmonic oscillator,

the system turns out to be separable both in Cartesian and (hyper)spherical coordinates). This property has been further analysed in a number of subsequent papers (see, for instance, [11, 12, 13] and references therein), where its algebraic content was explained in terms of the *Demkov–Fradkin tensor* [14, 15] given by

$$\mathcal{I}_{ij} = p_i p_j + m^2 \left(\omega^2 - \frac{2\lambda}{m} \mathcal{H}_\lambda(\mathbf{q}, \mathbf{p}) \right) q_i q_j, \quad i, j = 1, \dots, N.$$

We recall here that the Hamiltonian \mathcal{H}_λ can be expressed in terms of hyperspherical coordinates r, θ_j and canonical momenta p_r, p_{θ_j} , ($j = 1, \dots, N-1$) defined by

$$q_j = r \cos \theta_j \prod_{k=1}^{j-1} \sin \theta_k \quad (1 \leq j \leq N-1), \quad q_N = r \prod_{k=1}^{N-1} \sin \theta_k,$$

such that

$$r = |\mathbf{q}|, \quad \mathbf{p}^2 = p_r^2 + \frac{\mathbf{L}^2}{r^2}, \quad \text{with} \quad \mathbf{L}^2 = \sum_{j=1}^{N-1} p_{\theta_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2 \theta_k},$$

where the radial coordinate r is canonically conjugated to the radial momentum p_r . Then, for a fixed value of $\mathbf{L}^2 = l^2$, the Hamiltonian (1) can be written as a one-degree of freedom radial system.

As a matter of fact the system associated with \mathcal{H}_λ can be considered as a genuine (maximally superintegrable) λ -deformation of the usual ND isotropic harmonic oscillator with frequency ω , since the limit $\lambda \rightarrow 0$ yields

$$\mathcal{H}_0(\mathbf{p}, \mathbf{q}) = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2 \mathbf{q}^2}{2}.$$

We remind also that both the classical and the quantum problems have been investigated (see [6, 12, 13]) in the special case characterized by positive values of the deformation parameter λ . Also, an in-depth discussion of the $\lambda < 0$ case for the classical system can be found in [13] where, however, the explicit formula of the trajectory has not been derived.

In the present work our scope is rather different from the one pursued in [6]. On one hand, we deal just with the classical system and restrict our considerations to the 3D case, with emphasis on the algebraic side of the problem. In this context we show that the so-called *Spectrum Generating Algebra* (SGA) technique [16, 17] provides us with all the necessary ingredients to achieve its solution, *at least in the $\lambda > 0$ case*, in full analogy with the classical Taub-NUT system [18]. In particular, we will derive the solution for both bounded and unbounded motions. On the other hand, through a direct approach, we will be able to handle the $\lambda < 0$ case as well, getting an explicit formula for $t = t(r)$ holding in the *punctured open ball*

$$0 < r < r_s \doteq \frac{1}{\sqrt{|\lambda|}},$$

and thus making some progress with respect to what had been already derived in the literature [6, 12, 13].

The paper is organized as follows:

- (i) In section 2 we perform the factorization of the classical Hamiltonian and derive the appropriate Poisson algebra for the ordinary isotropic harmonic oscillator, recovering the well-known solution to the equation of motion.

- (ii) In section 3, paraphrasing the procedure followed in [18] we solve the equations of motion in the $\lambda > 0$ case. As it happened for the Taub-NUT system, the solution is obtained in the form $t = t(r)$. As already noticed in [6], in contrast with the usual harmonic oscillator, for sufficiently high energy values there are unbounded orbits (the particle can escape to infinity) which in the quantum setting corresponds to the onset of a continuous spectrum. Coherently with the predictions of the Bertrand's Theorem, the bounded orbits will be closed and there will be stable circular trajectories.
- (iii) In section 4 the $\lambda < 0$ case will be tersely investigated. We will get behaviours that are similar to those exhibited by the Taub-NUT system for negative values of the deformation parameter η [18]. Indeed, in a sort of dual way with respect to the Taub-NUT case, the kinetic energy (and thus the metric) remains positive in the bounded region $0 < r < r_s$ while it takes negative values for $r > r_s$. So, in order to get an admissible metric for $r > r_s$, we should there resort to time-reversal symmetry to restore positivity. On the other hand, in that unbounded region the effective potential is a monotonic function of r having a constant limiting value at infinity, implying that no closed orbits can arise.
- (iv) The main results and some further perspectives are outlined in the concluding section.

2. The Euclidean case: isotropic harmonic oscillator

Before using the machinery of the SGA approach to tackle the $\lambda \neq 0$ case, we will firstly characterize the Euclidean case in order to check our future results in the (flat) limit $\lambda \rightarrow 0$ and to illustrate the usual SGA procedure. Therefore, let us consider the Hamiltonian

$$H_0(r, p) = T_0(r, p) + V_{0,\text{eff}}(r) = \frac{p^2}{2m} + \frac{l^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2, \quad (2)$$

where r is the radial coordinate and $p \equiv p_r$ is the radial momentum related to the particle of mass m , frequency ω and angular momentum l . The effective potential is so given by

$$V_{0,\text{eff}}(r) = \frac{l^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2, \quad (3)$$

Clearly, in order to have closed orbits, the values of the energy have to be confined in the region $V_{0,\text{eff}}(r_{\min}) < E < \infty$, where $V_{0,\text{eff}}(r_{\min}) = l\omega$ is the minimum of the effective potential, i.e. its value calculated at $r_{\min} = \sqrt{\frac{l}{m\omega}}$ (see lhs of figure 1).

Now, in order to apply the SGA technique, we proceed by factorizing the Hamiltonian (2) as follows

$$p^2 r^2 + m^2 \omega^2 r^4 - 2mr^2 H_0 = A_0^+ A_0^- + \gamma(H_0) = -l^2, \quad (4)$$

where A_0^\pm are unknown functions to be determined. In particular, taking into account (4), we obtain

$$A_0^\pm = \mp irp + m\omega r^2 - \frac{H_0}{\omega}, \quad (5)$$

and then

$$\gamma(H_0) = -\frac{H_0^2}{\omega^2}. \quad (6)$$

The Hamiltonian H_0 and A_0^\pm are functions of the canonical coordinates (r, p) and they have to close the following Poisson algebra [16, 17]:

$$\{H_0, A_0^\pm\} = \mp i\alpha(H_0)A_0^\pm, \quad \{A_0^+, A_0^-\} = i\beta(H_0).$$

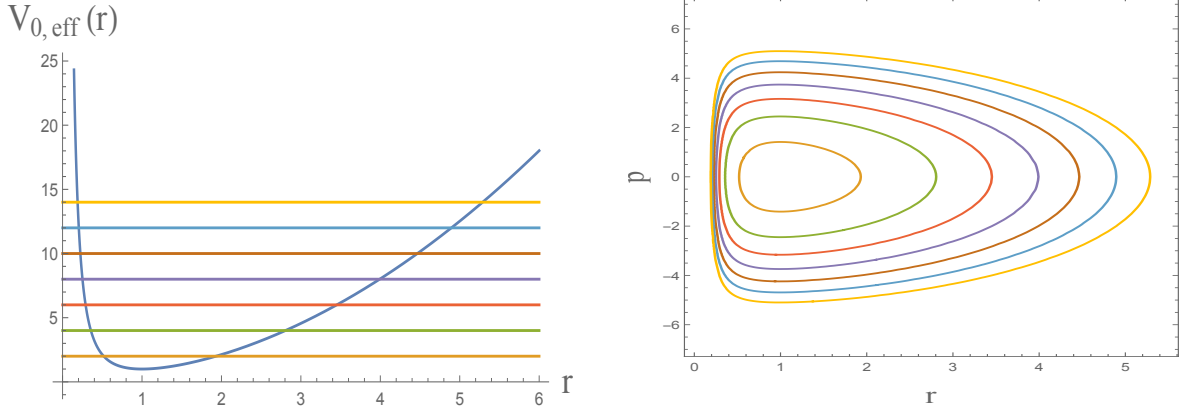


Figure 1. Effective potential $V_{0,\text{eff}}(r)$ (3) and phase plane (r, p) calculated for $m = \omega = l = 1$ and $E = 2, 4, 6, 8, 10, 12, 14$ (in appropriate units).

In particular, using the expressions (5), we obtain that

$$\alpha(H_0) = 2\omega, \quad \beta(H_0) = \frac{4H_0}{\omega}.$$

Summarizing, we have the following relations

$$\{H_0, A_0^\pm\} = \mp i2\omega A_0^\pm, \quad \{A_0^+, A_0^-\} = i\frac{4H_0}{\omega}. \quad (7)$$

At this point we can define the *time-dependent constants of motion*

$$Q_0^\pm = A_0^\pm e^{\mp i\alpha(H_0)t},$$

such that

$$\frac{dQ_0^\pm}{dt} = \{Q_0^\pm, H_0\} + \frac{\partial Q_0^\pm}{\partial t} = 0.$$

As we know, those dynamical variables can be written as $Q_0^\pm = q_0 e^{\pm i\theta_0}$ and allow us to determine the motion [16, 17, 18]. Indeed it holds:

$$\left(\mp irp + m\omega r^2 - \frac{H_0}{\omega} \right) e^{\mp 2i\omega t} = q_0 e^{\pm i\theta_0}.$$

By imposing (4) along (6), that is,

$$A_0^+ A_0^- - \frac{H_0^2}{\omega^2} = q_0^2 - \frac{H_0^2}{\omega^2} = -l^2,$$

we find that

$$\begin{cases} -irp + m\omega r^2 - \frac{H_0}{\omega} = q_0 e^{i(2\omega t + \theta_0)}, \\ irp + m\omega r^2 - \frac{H_0}{\omega} = q_0 e^{-i(2\omega t + \theta_0)}, \end{cases} \quad (8)$$

where

$$q_0 = \sqrt{-l^2 + \frac{H_0^2}{\omega^2}}.$$

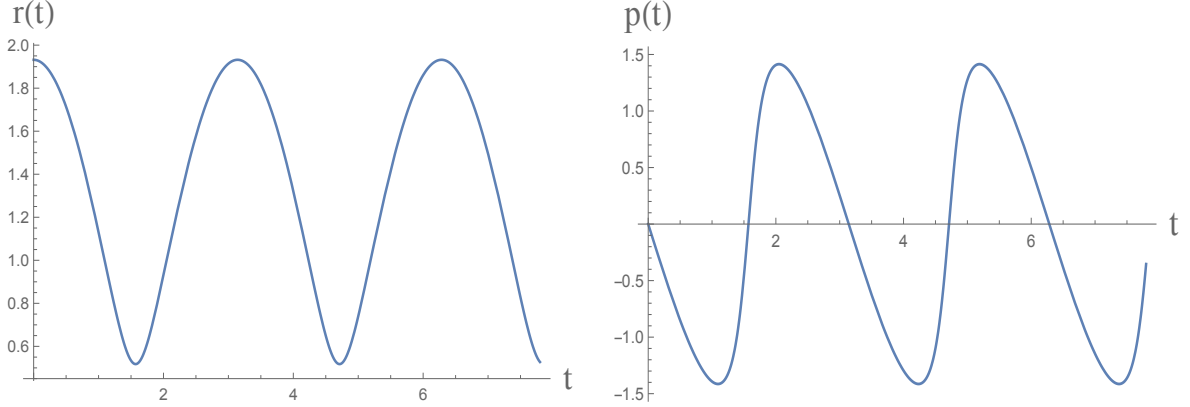


Figure 2. Trajectory $r(t)$ and momentum $p(t)$ calculated for $m = \omega = l = 1$ and $E = 2$.

Now, since the Hamiltonian does not depend explicitly on time, it is a constant of motion, i.e. the energy of the system $H_0 = E$. Therefore, summing and subtracting (8) we obtain (on the level surface $H_0 = E$)

$$\begin{cases} m\omega r^2 - \frac{E}{\omega} = q_0 \cos(2\omega t + \theta_0), \\ rp = -q_0 \sin(2\omega t + \theta_0). \end{cases} \quad (9)$$

Finally, using (9) we are able to write r and p as functions of t :

$$\begin{cases} r(t) = \sqrt{\frac{E}{m\omega^2} + \frac{q_0}{m\omega} \cos(2\omega t + \theta_0)}, \\ p(t) = -\frac{q_0 \sin(2\omega t + \theta_0)}{\sqrt{\frac{E}{m\omega^2} + \frac{q_0}{m\omega} \cos(2\omega t + \theta_0)}}. \end{cases} \quad (10)$$

In this way we have found $r(t)$ and $p(t)$ and then the motion has been fully determined by means of the SGA procedure. In the rhs of figure 1 the phase plane is depicted, while in figure 2 the curves (10) are represented.

In the following, we shall deal with the deformed case using the same procedure. Clearly, the results just obtained in this section have to be recovered in the Euclidean (i.e. $\lambda \rightarrow 0$) limit.

3. The deformed $\lambda > 0$ case: D-III oscillator

In this section we focus our attention on the D-III oscillator Hamiltonian (1) written as a 1D radial system by means of hyperspherical coordinates, namely

$$H(r, p) = T(r, p) + V_{\text{eff}}(r) = \frac{p^2}{2m(1 + \lambda r^2)} + \frac{l^2}{2mr^2(1 + \lambda r^2)} + \frac{m\omega^2 r^2}{2(1 + \lambda r^2)} = \mathcal{F}(r)H_0, \quad (11)$$

where m , ω and l are positive constants, λ is the deformation parameter, H_0 is the ‘undeformed’ isotropic oscillator Hamiltonian (2) and

$$\mathcal{F}(r) \doteq \frac{1}{1 + \lambda r^2}.$$

Multiplying both sides of (11) by $r^2(1 + \lambda r^2)$ we get:

$$r^2(1 + \lambda r^2)H = r^2 \left(\frac{p^2}{2m} + \frac{l^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 \right) = \frac{1}{2m}(r^2 p^2 + l^2 + m^2 \omega^2 r^4). \quad (12)$$

Now, as it has been done in the previous section for the undeformed case, at any r we can factorize (12) as:

$$r^2 p^2 + m^2 r^4 \left(\omega^2 - \frac{2\lambda}{m} H \right) - 2mr^2 H = A^+ A^- + \gamma(H) = -l^2, \quad (13)$$

where A^+ , A^- are unknown functions of r , p . We make the following *ansatz* for A^+ , A^- :

$$A^\pm = \left(\mp irp + mr^2 \sqrt{\omega^2 - \frac{2\lambda}{m} H} - \frac{H}{\sqrt{\omega^2 - \frac{2\lambda}{m} H}} \right) e^{\pm f(r,p)}.$$

The ‘arbitrary function’ $f(r, p)$ will be determined by requiring the closure of the Poisson algebra generated by H and A^\pm :

$$\begin{aligned} \{H, A^\pm\} &= \mp i\alpha(H)A^\pm, \\ \{A^+, A^-\} &= i\beta(H), \end{aligned}$$

where the functions α , β have to be determined. Inserting A^\pm in (13) we get

$$\gamma(H) = -\frac{H^2}{\omega^2 - \frac{2\lambda}{m} H},$$

and requiring that A^\pm obey the proper Poisson brackets we arrive at

$$f(r, p) = -\frac{i\lambda r p \sqrt{\omega^2 - \frac{2\lambda}{m} H}}{m(\omega^2 - \frac{\lambda}{m} H)}, \quad \alpha(H) = \frac{2(\omega^2 - \frac{2\lambda}{m} H)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m} H}, \quad \beta(H) = \frac{4H}{\sqrt{\omega^2 - \frac{2\lambda}{m} H}}.$$

Hence we find that

$$\begin{aligned} A^\pm &= \left(\mp irp + mr^2 \sqrt{\omega^2 - \frac{2\lambda}{m} H} - \frac{H}{\sqrt{\omega^2 - \frac{2\lambda}{m} H}} \right) \exp \left\{ \mp \frac{i\lambda r p \sqrt{\omega^2 - \frac{2\lambda}{m} H}}{m(\omega^2 - \frac{\lambda}{m} H)} \right\}, \\ \{H, A^\pm\} &= \mp i \frac{2(\omega^2 - \frac{2\lambda}{m} H)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m} H} A^\pm, \quad \{A^+, A^-\} = i \frac{4H}{\sqrt{\omega^2 - \frac{2\lambda}{m} H}}. \end{aligned}$$

We notice that in the limit $\lambda \rightarrow 0$ one gets back the undeformed Poisson algebra (7) as expected. We also point out that the requirement $\omega^2 - \frac{2\lambda}{m} H > 0$ implies the upper bound $E < \frac{m\omega^2}{2\lambda}$. Moreover, for bounded motion, the energy has to be greater than the minimum of the effective potential $V_{\text{eff}}(r)$. The latter turns out to be

$$V_{\text{eff}}(r_{\min}) = \frac{l^2}{m} \left(\sqrt{\lambda^2 + \frac{m^2 \omega^2}{l^2}} - \lambda \right), \quad r_{\min}^2 = \frac{l^2}{m^2 \omega^2} \left(\lambda + \sqrt{\lambda^2 + \frac{m^2 \omega^2}{l^2}} \right).$$

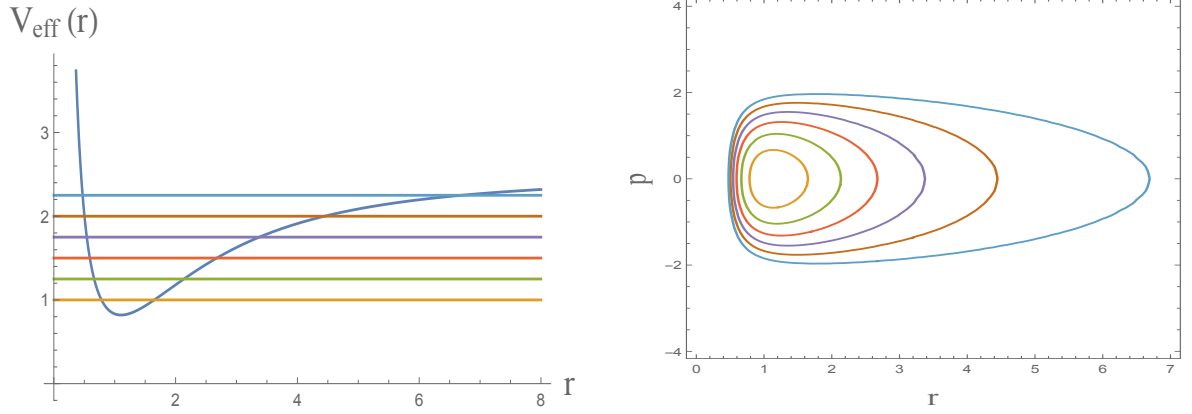


Figure 3. Effective potential $V_{\text{eff}}(r)$ (11) and phase plane (r, p) calculated for $m = \omega = l = 1$ and $E = 1.00, 1.25, 1.50, 1.75, 2.00, 2.25$. The deformation parameter is fixed at the value $\lambda = 0.20$.

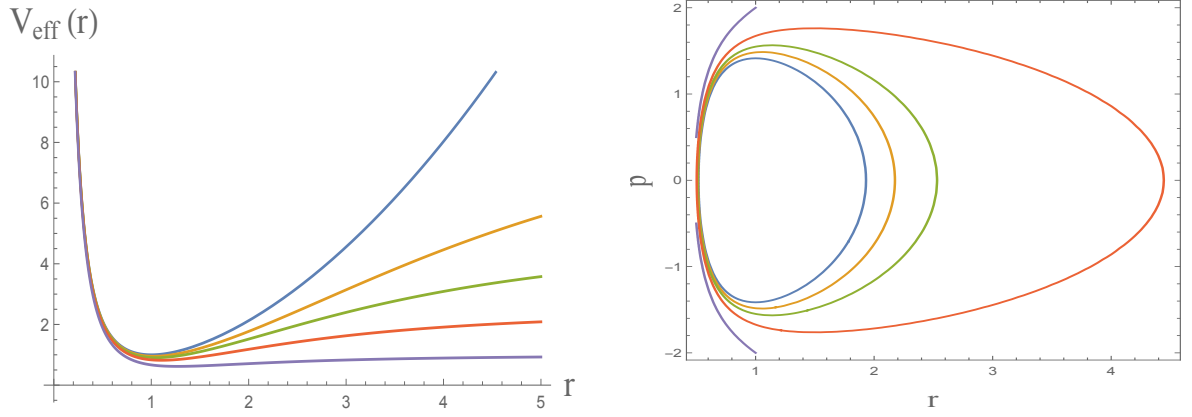


Figure 4. Effective potential $V_{\text{eff}}(r)$ (11) and phase plane (r, p) calculated for $m = \omega = l = 1$ and $E = 2$. The deformation parameter take the values $\lambda = 0, 0.05, 0.10, 0.20, 0.50$.

As a consequence the energy belongs to the interval (see lhs of figures 3 and 4)

$$V_{\text{eff}}(r_{\min}) < E < \frac{m\omega^2}{2\lambda}.$$

Next, as usual, in order to find the corresponding trajectory we define the *time-dependent constants of the motion*

$$Q^{\pm} = A^{\pm} e^{\mp i\alpha(H)t},$$

from which we can write (on-shell):

$$\left(\mp irp + mr^2 \sqrt{\omega^2 - \frac{2\lambda}{m}E} - \frac{E}{\sqrt{\omega^2 - \frac{2\lambda}{m}E}} \right) \exp \left\{ \mp i \left(\frac{\lambda rp \sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} + \frac{2(\omega^2 - \frac{2\lambda}{m}E)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} t \right) \right\} = q e^{\pm i\theta},$$

or else

$$\begin{cases} -irp + mr^2\sqrt{\omega^2 - \frac{2\lambda}{m}E} - \frac{E}{\sqrt{\omega^2 - \frac{2\lambda}{m}E}} = q \exp \left\{ i \left(\frac{\lambda rp\sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} + \frac{2(\omega^2 - \frac{2\lambda}{m}E)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} t + \theta \right) \right\}, \\ +irp + mr^2\sqrt{\omega^2 - \frac{2\lambda}{m}E} - \frac{E}{\sqrt{\omega^2 - \frac{2\lambda}{m}E}} = q \exp \left\{ -i \left(\frac{\lambda rp\sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} + \frac{2(\omega^2 - \frac{2\lambda}{m}E)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} t + \theta \right) \right\}, \end{cases} \quad (14)$$

where now

$$q = \sqrt{-l^2 + \frac{E^2}{\omega^2 - \frac{2\lambda}{m}E}}.$$

Once again, summing and subtracting (14) we obtain:

$$\begin{cases} mr^2\sqrt{\omega^2 - \frac{2\lambda}{m}E} - \frac{E}{\sqrt{\omega^2 - \frac{2\lambda}{m}E}} = q \cos \left(\frac{\lambda rp\sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} + \frac{2(\omega^2 - \frac{2\lambda}{m}E)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} t + \theta \right), \\ rp = -q \sin \left(\frac{\lambda rp\sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} + \frac{2(\omega^2 - \frac{2\lambda}{m}E)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} t + \theta \right). \end{cases}$$

Taking the sum of the square of these two equations we recover (13) restricted to the level surface $H = E$. Finally, thanks to the above relations, we are able to express t as a function of r :

$$t(r) = \frac{1}{\Omega_{[\lambda]}(E)} \left[\arccos \left(\frac{mr^2(\omega^2 - \frac{2\lambda}{m}E) - E}{q\sqrt{\omega^2 - \frac{2\lambda}{m}E}} \right) - \frac{\lambda\sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} \sqrt{2mEr^2 - l^2 - m^2r^4(\omega^2 - \frac{2\lambda}{m}E)} - \theta \right], \quad (15)$$

where

$$\Omega_{[\lambda]}(E) = \frac{2(\omega^2 - \frac{2\lambda}{m}E)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} \equiv \alpha(E)$$

is the angular frequency of the motion. In the rhs of figures 3 and 4 some representations of the resulting phase space can be found.

At this point, defining the quantities

$$a_{[\lambda]}^2 \doteq \frac{E}{m(\omega^2 - \frac{2\lambda}{m}E)}, \quad \epsilon_{[\lambda]} \doteq \sqrt{1 - \frac{(\omega^2 - \frac{2\lambda}{m}E)l^2}{E^2}},$$

$a_{[\lambda]}^2$ being the square of the major semi-axes of the ellipse and $\epsilon_{[\lambda]}$ a parameter directly related to its eccentricity, equation (15) can be cast into the following form:

$$\Omega_{[\lambda]}(E)t(r) + \theta = \arccos \left[-\frac{1}{\epsilon_{[\lambda]}} \left(1 - \left(\frac{r}{a_{[\lambda]}} \right)^2 \right) \right] - \frac{\lambda a_{[\lambda]}^2}{1 + \lambda a_{[\lambda]}^2} \sqrt{\epsilon_{[\lambda]}^2 - \left[1 - \left(\frac{r}{a_{[\lambda]}} \right)^2 \right]^2}, \quad (16)$$

whose structure is very similar to the one obtained for the classical Taub-NUT system [18]. In the limit $\lambda \rightarrow 0$ the equation (16) can be easily inverted to obtain the trajectory $r(t)$ given in (10), showing that it just represents the λ -deformation of the solution of the Euclidean isotropic harmonic oscillator. Then, we can conclude that the motion has been solved by means of the SGA method.

Now we turn to investigate the unbounded motion for positive values of the deformation parameter (see also [6]). The latter will arise for $E > \frac{m\omega^2}{2\lambda}$. In this case

$$\omega^2 - \frac{2\lambda}{m}E = -\left(\frac{2\lambda}{m}E - \omega^2\right) < 0,$$

therefore all the previous functions have to be adapted to this situation. In particular, we can write:

$$\tilde{A}^\pm = i \left(\mp rp + mr^2 \sqrt{\frac{2\lambda}{m}H - \omega^2} + \frac{H}{\sqrt{\frac{2\lambda}{m}H - \omega^2}} \right) \exp \left\{ \pm \frac{\lambda rp \sqrt{\frac{2\lambda}{m}H - \omega^2}}{m(\omega^2 - \frac{\lambda}{m}H)} \right\},$$

$$\tilde{\alpha}(H) = -i \frac{2\left(\frac{2\lambda}{m}H - \omega^2\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}H}, \quad \tilde{\beta}(H) = -i \frac{4H}{\sqrt{\frac{2\lambda}{m}H - \omega^2}}.$$

All the functions are purely imaginary and the time-dependent constants of motion (also purely imaginary) take the form (see [17]):

$$\tilde{Q}^\pm = \tilde{A}^\pm e^{\mp i \tilde{\alpha}(H)t} = i \tilde{q} e^{\mp \theta}, \quad \tilde{q} \doteq \sqrt{l^2 + \frac{E^2}{\frac{2\lambda}{m}E - \omega^2}}.$$

This leads to the two following equations (again on-shell):

$$\begin{cases} -rp + mr^2 \sqrt{\frac{2\lambda}{m}E - \omega^2} + \frac{E}{\sqrt{\frac{2\lambda}{m}E - \omega^2}} = \tilde{q} \exp \left\{ -\left(\frac{\lambda rp \sqrt{\frac{2\lambda}{m}E - \omega^2}}{m(\omega^2 - \frac{\lambda}{m}E)} - \frac{2\left(\frac{2\lambda}{m}E - \omega^2\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} t + \theta \right) \right\}, \\ +rp + mr^2 \sqrt{\frac{2\lambda}{m}E - \omega^2} + \frac{E}{\sqrt{\frac{2\lambda}{m}E - \omega^2}} = \tilde{q} \exp \left\{ +\left(\frac{\lambda rp \sqrt{\frac{2\lambda}{m}E - \omega^2}}{m(\omega^2 - \frac{\lambda}{m}E)} - \frac{2\left(\frac{2\lambda}{m}E - \omega^2\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} t + \theta \right) \right\}, \end{cases}$$

implying that

$$\begin{cases} mr^2 \sqrt{\frac{2\lambda}{m}E - \omega^2} + \frac{E}{\sqrt{\frac{2\lambda}{m}E - \omega^2}} = \tilde{q} \cosh \left(\frac{\lambda rp \sqrt{\frac{2\lambda}{m}E - \omega^2}}{m(\omega^2 - \frac{\lambda}{m}E)} - \frac{2\left(\frac{2\lambda}{m}E - \omega^2\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} t + \theta \right), \\ rp = \tilde{q} \sinh \left(\frac{\lambda rp \sqrt{\frac{2\lambda}{m}E - \omega^2}}{m(\omega^2 - \frac{\lambda}{m}E)} - \frac{2\left(\frac{2\lambda}{m}E - \omega^2\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} t + \theta \right). \end{cases}$$

We notice that the equation (13) (for $H = E$) is now recovered by taking the difference of the square of these two equations. Also in this case, due to the above relations, we are able to obtain

t as a function of r , namely:

$$t(r) = \frac{1}{\zeta_{[\lambda]}(E)} \left[\operatorname{arccosh} \left(\frac{mr^2(\frac{2\lambda}{m}E - \omega^2) + E}{\tilde{q}\sqrt{\frac{2\lambda}{m}E - \omega^2}} \right) + \frac{\lambda\sqrt{\frac{2\lambda}{m}E - \omega^2}}{m(\frac{\lambda}{m}E - \omega^2)} \sqrt{2mEr^2 - l^2 + m^2r^4(\frac{2\lambda}{m}E - \omega^2)} - \theta \right], \quad (17)$$

where we defined the function

$$\zeta_{[\lambda]}(E) \doteq \frac{2(\frac{2\lambda}{m}E - \omega^2)^{\frac{3}{2}}}{\frac{\lambda}{m}E - \omega^2}.$$

Similarly to the case of bounded motion, by defining the quantities

$$\tilde{a}_{[\lambda]}^2 \doteq \frac{E}{m(\frac{2\lambda E}{m} - \omega^2)}, \quad \tilde{e}_{[\lambda]} \doteq \sqrt{1 + \frac{(\frac{2\lambda E}{m} - \omega^2)l^2}{E^2}},$$

the expression (17) can be written more elegantly as:

$$\zeta_{[\lambda]}(E)t(r) + \theta = \operatorname{arccosh} \left[\frac{1}{\tilde{e}_{[\lambda]}} \left(1 + \left(\frac{r}{\tilde{a}_{[\lambda]}} \right)^2 \right) \right] + \frac{\lambda\tilde{a}_{[\lambda]}^2}{1 - \lambda\tilde{a}_{[\lambda]}^2} \sqrt{\left[1 + \left(\frac{r}{\tilde{a}_{[\lambda]}} \right)^2 \right]^2 - \tilde{e}_{[\lambda]}^2}. \quad (18)$$

Equation (18) might be compared with the formula (15) given in [6], having in mind the relation existing between the Hamiltonian H defined in the present paper and the one defined in the aforementioned reference, here denoted as \bar{H} , calculated for $\kappa = \frac{1}{\lambda}$ and $b_j = 0$ in [6]. Such relation is given by $\bar{H} = \lambda H - \frac{m\omega^2}{2}$. In [6], the equation of motion has been solved assuming that $\bar{E} := 2\bar{H} = 2\lambda E - m\omega^2 > 0$.

4. The deformed $\lambda < 0$ case

In this case the deformation parameter is negative and the Hamiltonian can be written as follows

$$H(r, p) = T(r, p) + V_{\text{eff}}(r) = \frac{p^2}{2m(1 - |\lambda|r^2)} + \frac{l^2}{2mr^2(1 - |\lambda|r^2)} + \frac{m\omega^2 r^2}{2(1 - |\lambda|r^2)}. \quad (19)$$

It is clear that it presents a singularity at the point $r_s = \frac{1}{\sqrt{|\lambda|}}$.

As a matter of fact, owing to the above singularity, the domain of definition of the effective potential splits in two subdomains. The first one, corresponding to the punctured open ball $0 < r < r_s$, is characterized by a positive kinetic energy term in the Hamiltonian and the effective potential exhibits a typical confining shape. Viceversa, the second one, $r_s < r < \infty$, is characterized by a negative kinetic energy and the effective potential has no critical points (see figure 5 and lhs of figure 6).

The interesting region is the open set $0 < r < r_s$. In fact, in this case there will be only closed orbits for energy values $V_{\text{eff}}(r_{\min}) < E < \infty$, where

$$V_{\text{eff}}(r_{\min}) = \frac{l^2}{m} \left(\sqrt{\lambda^2 + \frac{m^2\omega^2}{l^2}} + |\lambda| \right), \quad r_{\min}^2 = \frac{l^2}{m^2\omega^2} \left(-|\lambda| + \sqrt{\lambda^2 + \frac{m^2\omega^2}{l^2}} \right).$$

In that bounded region the solution can be immediately obtained by means of the SGA technique by letting in (16) $\lambda \rightarrow -|\lambda|$. Clearly, since the term $\omega^2 - \frac{2\lambda}{m}E = \omega^2 + \frac{2|\lambda|}{m}E$ is always positive, no

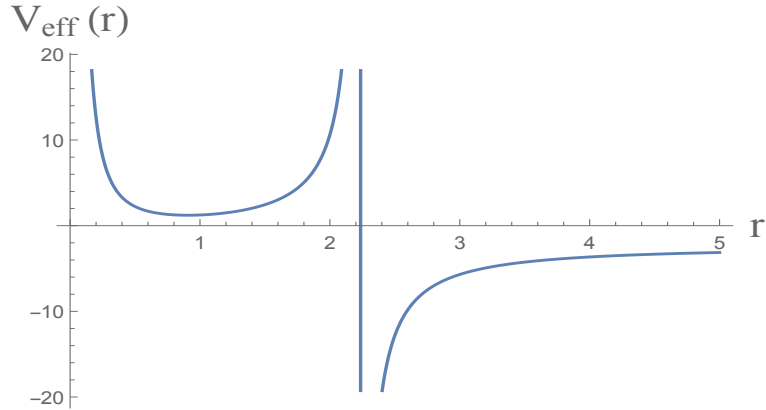


Figure 5. Effective potential $V_{\text{eff}}(r)$ (19) calculated for $m = \omega = l = 1$ and $\lambda = -0.20$. The singularity is located at the point $r_s = \sqrt{5} \simeq 2.24$.

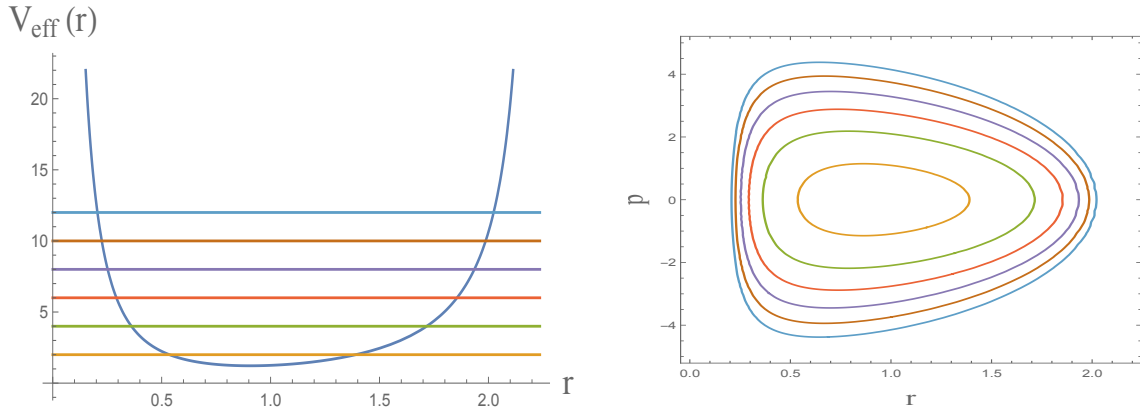


Figure 6. Effective potential $V_{\text{eff}}(r)$ (19) and phase plane (r, p) calculated for $m = \omega = l = 1$ and $E = 2, 4, 6, 8, 10, 12$ in the region $0 < r < r_s$. The deformation parameter is fixed at the value $\lambda = -0.2$.

further restrictions on the energy will apply, i.e. the energy has no upper bound, as expected. For this case the phase plane is depicted in the rhs of figure 6.

Viceversa, in order to obtain the solution in the unbounded region $r > r_s$, as we already noticed in the case of the Taub-NUT system [18], we have to factorise the new Hamiltonian $\tilde{H} = -H$, the latter being defined with an overall change of sign. This allows us to restore a positive kinetic energy and then to obtain a physically meaningful Hamiltonian (see also [13]). However, this case is not so interesting since no closed orbits are allowed.

5. Conclusions

To summarize the key points of our paper, we can say that we have successfully applied the SGA approach to the classical D-III oscillator, identifying the structure of its Poisson algebra, and solving the equations of motion by finding an explicit expression in the form $t = t(r)$. In particular, after having calculated the solution to the equation of motion for the usual isotropic harmonic oscillator (which in our framework just represents the Euclidean limit of the D-III system), we have found the trajectory $t = t(r)$ of the D-III oscillator, for positive values of the deformation parameter λ , both in the case of bounded and unbounded motion. Furthermore,

the case of λ negative has also been analyzed. For the latter situation, as we have already seen in the classical Taub-NUT system [18], the most interesting features of the dynamics are somehow hidden ‘beyond the singularity’, in the open set $0 < r < r_s$, where for suitable initial conditions the particle remains trapped during its whole life. As a matter of fact, in that region the effective potential shows a typical confining shape and the kinetic energy is always positive definite. We were then enabled to recover an explicit formula for the trajectory, restricted to that domain, simply by taking $t = t(r)$ for $\lambda = -|\lambda|$.

In the quantum case we expect similar features, however *in that punctured open ball* we might miss the algebraic nature of the spectrum and, accordingly, an analytic expression for the wave functions might not be available.

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