# 2d CFT/Gauge/Bethe correspondence and solvable quantum–mechanical systems

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**Abstract.** The holomorphic accessory parameters for the Heun (Fuchs with four singularities) and the Lamé equations are obtained by taking the classical limit of some null vector decoupling equations appearing in two-dimensional conformal field theory. In addition, a link between the aforementioned accessory parameters and the solution of some Bethe-like equations which occur in the supersymmetric gauge theories, is established.

## 1. Introduction

Let $C_{g,n}$ be the Riemann surface with genus $g$ and $n$ punctures. The basic objects of any two-dimensional conformal field theory (2dCFT) living on $C_g$ [1, 2] are the $n$-point correlation functions of primary physical vertex operators defined on $C_{g,n}$. Any correlation function can be factorized according to the pattern given by the pant decomposition of $C_{g,n}$ and written as a sum (or an integral for theories with a continuous spectrum) which includes the terms consisting of holomorphic and anti-holomorphic conformal blocks times the 3-point functions of the model for each pair of pants. The Virasoro conformal block $\mathcal{F}_{c,\Delta_p}[\Delta_i](\mathsf{Z})$ on $C_{g,n}$ depends on the cross ratios of the vertex operators locations denoted symbolically by $\mathsf{Z}$ and on the $3g-3+n$ intermediate conformal weights $\Delta_p$. Moreover, it depends on the $n$ external conformal weights $\Delta_i$ and on the central charge $c$.

Conformal blocks are fully determined by the underlying conformal symmetry. These functions possess an interesting, although not yet completely understood analytic structure. In general, they can be expressed only as a formal power series and no closed formula is known for its coefficients. Among the issues concerning conformal blocks which are still not fully understood there is the problem of their classical limit [3]. This is the limit in which all parameters of the conformal blocks tend to infinity in such a way that their ratios are fixed:

$$\Delta_i,\ \Delta_p,\ c\ \to\ \infty, \qquad \frac{\Delta_i}{c}\ =\ \frac{\Delta_p}{c}\ =\ \text{const.}\quad .$$

For the standard parametrization of the central charge $c = 1 + 6Q^2$, where $Q = b + \frac{1}{b}$ and for 'heavy' weights $(\Delta_p, \Delta_i) = \frac{1}{b^2}(\delta_p, \delta_i)$ with $\delta_p, \delta_i = \mathcal{O}(b^0)$ the classical limit corresponds to $b \to 0$. There exist many convincing arguments that in the classical limit the conformal blocks behave exponentially with respect to Z:

$$\mathcal{F} \stackrel{b\to 0}{\sim} \mathrm{e}^{\frac{1}{b^2}f}.$$

The function $f$ is known as the classical conformal block [3, 4].

In 2009 Alday, Gaiotto and Tachikawa (AGT) discovered an amazing relationship [5] which, in particular, identifies the Virasoro conformal blocks $\mathcal{F}_{c,\Delta_p}[\Delta_i](\mathsf{Z})$ on the Riemann surfaces $C_{g,n}$ with the instanton sectors $\mathcal{Z}_{\rm inst}$ of the Nekrasov partition functions [6, 7]:

$$\mathcal{Z}_{\rm Nekrasov}(\tilde{q}, \tilde{a}, \tilde{m}, \epsilon_1, \epsilon_2) = \mathcal{Z}_{\rm class}\mathcal{Z}_{\rm 1-loop}\mathcal{Z}_{\rm inst}$$

for certain class $T_{g,n}$ of ($\Omega$-deformed) four-dimensional supersymmetric $\mathcal{N} = 2$ SU(2) quiver gauge theories. The AGT correspondence works provided that certain relations between parameters of the conformal blocks and the Nekrasov functions are assumed. Let us remember that the Nekrasov partition function depends on the following set of parameters: $\tilde{q}$, $\tilde{a}$, $\tilde{m}$, $\epsilon_1$, $\epsilon_2$. The components of $\tilde{q} = \{\exp 2\pi\tau_1, \ldots, \exp 2\pi\tau_{3g-3+n}\}$ are the gluing parameters associated with the pant decomposition of $C_{g,n}$, where the $\tau_p = \theta_p/2\pi + 4\pi i/g_p^2$ are the complexified gauge couplings. The multiplet $\tilde{m} = \{m_1, \ldots, m_n\}$ contains the mass parameters. Moreover, $\tilde{a} = \{a_1, \ldots, a_{3g-3+n}\}$, where the $a$'s are the vacuum expectation values of the scalar fields in the vector multiplets. Finally, $\epsilon_1$, $\epsilon_2$ represent the complex $\Omega$-background parameters.

Soon after its discovery, the AGT conjecture has been extended to the SU(N)-gauge theories/conformal Toda correspondence [8].

On the other hand, at the same time Nekrasov and Shatashvili observed that in the limit where one of the $\Omega$-deformation parameters vanishes, $\epsilon_2 \to 0$ ($\epsilon_1 = \mathrm{const.}$), the $\mathcal{N} = 2$ SU(N) super Yang–Mills theories describe some N-particle quantum integrable systems (Gauge/Bethe correspondence) [9]. The AGT correspondence dictionary says that $b = \sqrt{\epsilon_2/\epsilon_1}$. Therefore, the Nekrasov–Shatashvili limit via the AGT hypothesis corresponds to the classical limit of the conformal blocks. Hence, one gets the triple correspondence relating two-dimensional conformal field theory to supersymmetric gauge theories and then to quantum integrable systems. The latter in the simplest case of SU(2) gauge group are just quantum–mechanical systems. This triple correspondence can be applied, in particular, to study the spectra of some Schrödinger operators [10, 11, 12] or to compute solutions of the monodromy problems for some ordinary differential equations [3, 13].

In the present note we give an overview of some part of our recent research related to the aforementioned dualities, cf. [14, 15, 16, 17, 18]. In particular, we show how it is possible to derive the so-called accessory parameters for the Fuchs (on $C_{0,4}$) and the Lamé equations by taking the classical limit of the so-called null vector decoupling equations, or equivalently, by solving certain Bethe-like equations.[1]

## 2. Classical conformal blocks vs. effective twisted superpotentials

To begin with, let us recall a definition and basic facts concerning the highest weight representation of the Virasoro algebra. Let $\mathcal{V}^n_{c,\Delta}$ be the free vector space generated by all vectors of the form:

$$|\,\nu^n_{\Delta,X}\,\rangle = L_{-X}|\,\nu_\Delta\,\rangle = L_{-n_k}\ldots L_{-n_2}L_{-n_1}|\,\nu_\Delta\,\rangle, \tag{1}$$

where $X = (n_k \geqslant \ldots \geqslant n_1 \geqslant 1)$ is an ordered set of positive integers of the length $|X| \equiv n_1 + \ldots + n_k = n$. In eq. above $L_n$'s are the Virasoro generators obeying

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{2}$$

[1] The same can be done for the Mathieu equation, cf. [11, 12] and a talk by A. R. Pietrykowski.

and $|\,\nu_\Delta\,\rangle$ is the highest weight state with the following property:

$$L_0|\,\nu_\Delta\,\rangle = \Delta|\,\nu_\Delta\,\rangle, \qquad L_n|\,\nu_\Delta\,\rangle = 0 \quad \forall\, n > 0. \tag{3}$$

The representation of the Virasoro algebra on the space:

$$\mathcal{V}_{c,\Delta} = \bigoplus_{n=0}^{\infty} \mathcal{V}^n_{c,\Delta}$$

defined by the relations (2) and (3) is called the Verma module with the central charge $c$ and the highest weight $\Delta$. The dimension of the subspace $\mathcal{V}^n_{c,\Delta} \subset \mathcal{V}_{c,\Delta}$ of all homogeneous elements of degree $n$ is given by the number of partitions $p(n)$ ($p(0) = 1$). The subspaces $\mathcal{V}^n_{c,\Delta}$ are eigenspaces of $L_0$ with the eigenvalues $\Delta + n$. On $\mathcal{V}^n_{c,\Delta}$ there exists the symmetric bilinear form $\langle\,\cdot\,|\,\cdot\,\rangle_{c,\Delta}$ uniquely defined by the relations:

$$\langle\,\nu_\Delta\,|\,\nu_\Delta\,\rangle_{c,\Delta} \;=\; 1, \qquad (L_n)^\dagger \;=\; L_{-n}.$$

The Gram matrix $G_{c,\Delta}$ of the form $\langle\,\cdot\,|\,\cdot\,\rangle_{c,\Delta}$ is block-diagonal in the basis $\{|\,\nu_{\Delta,X}\,\rangle\}$ with blocks:

$$\left[G^n_{c,\Delta}\right]_{XY} \;=\; \langle\,\nu^n_{\Delta,X}\,|\,\nu^n_{\Delta,Y}\,\rangle_{c,\Delta}. \tag{4}$$

The Verma module $\mathcal{V}_{c,\Delta}$ is irreducible if and only if the form $\langle\,\cdot\,|\,\cdot\,\rangle_{c,\Delta}$ is non-degenerate. The criterion for the irreducibility is vanishing of the determinant $\det G^n_{c,\Delta}$ of the Gram matrix, known as the Kac determinant which is given by the formula:

$$\det\, G^n_{c,\Delta} \;=\; C_n \prod_{\substack{r,s\in\mathbf{N},\\ s\leqslant r,\\ 1\leqslant rs\leqslant n}} \Phi_{rs}(c,\Delta)^{p(n-rs)}. \tag{5}$$

In the equation above $C_n$ is a constant and

$$\Phi_{rs}(c,\Delta) \;=\; \left(\Delta + \tfrac{r^2-1}{24}(c-13) + \tfrac{rs-1}{2}\right)\left(\Delta + \tfrac{s^2-1}{24}(c-13) + \tfrac{rs-1}{2}\right) + \tfrac{(r^2-s^2)^2}{16}.$$

The Kac determinant vanishes for

$$\Delta_{rs}(c) \;=\; \frac{Q^2}{4} - \frac{1}{4}\left(rb + \frac{s}{b}\right)^2, \tag{6}$$

where the central charge is given by $c = 1 + 6Q^2$ with $Q = b + \frac{1}{b}$. For the so-called degenerate conformal weights $\Delta_{rs}$ the representations $\mathcal{V}_{c,\Delta_{rs}}$ are reducible.

The non-zero element $|\,\chi_{rs}\,\rangle \in \mathcal{V}_{c,\Delta_{rs}}$ of degree $n = rs$ is called a null vector if

$$L_0\,|\,\chi_{rs}\,\rangle = (\Delta_{rs} + rs)\,|\,\chi_{rs}\,\rangle, \qquad L_k\,|\,\chi_{rs}\,\rangle = 0 \quad \forall\, k > 0.$$

Hence, $|\,\chi_{rs}\,\rangle$ is the highest weight state which generates its own Verma module $\mathcal{V}_{c,\Delta_{rs}+rs}$, which is a submodule of $\mathcal{V}_{c,\Delta_{rs}}$. One can prove that each submodule of the Verma module $\mathcal{V}_{c,\Delta_{rs}}$ is generated by a null vector. Then, the module $\mathcal{V}_{c,\Delta_{rs}}$ is irreducible if and only if it does not contain null vectors with positive degree.

Let $\mathcal{V}_\Delta$ be the Verma module with the highest weight state $|\,\nu_\Delta\,\rangle$. We define the chiral vertex operator (CVO) as the linear map

$$V^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0} : \mathcal{V}_{\Delta_2} \otimes \mathcal{V}_{\Delta_1} \to \mathcal{V}_{\Delta_3}$$

such that for all $|\,\xi_2\,\rangle \in \mathcal{V}_{\Delta_2}$ the operator

$$V(\xi_2|z) \equiv V^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\;z\;\;0}(|\,\xi_2\,\rangle \otimes\,\cdot\,) : \mathcal{V}_{\Delta_1} \to \mathcal{V}_{\Delta_3}$$

satisfies the following conditions:

$$\begin{aligned}
[L_n, V(\nu_2|z)] &= z^n\left(z\frac{\partial}{\partial z} + (n+1)\Delta_2\right) V(\nu_2|z)\,, \qquad n \in \mathbf{Z} && (7)\\
V(L_{-1}\xi_2|z) &= \frac{\partial}{\partial z} V(\xi_2|z)\,, && (8)\\
V(L_n\xi_2|z) &= \sum_{k=0}^{n+1} \binom{n+1}{k} (-z)^k\,[L_{n-k}, V(\xi_2|z)]\,, \qquad n > -1, && (9)\\
V(L_{-n}\xi_2|z) &= \sum_{k=0}^{\infty} \binom{n-2+k}{n-2} z^k\, L_{-n-k}\, V(\xi_2|z) \\
&+ (-1)^n \sum_{k=0}^{\infty} \binom{n-2+k}{n-2} z^{-n+1-k}\; V(\xi_2|z)\; L_{k-1}, \qquad n > 1 && (10)
\end{aligned}$$

and $\langle\,\nu_{\Delta_3}\,|V(\nu_{\Delta_2}|z)\,|\,\nu_{\Delta_1}\,\rangle = z^{\Delta_3-\Delta_2-\Delta_1}$. The commutation relation (7) defines the primary vertex operator corresponding to the highest weight state $|\,\nu_2\,\rangle \in \mathcal{V}_{\Delta_2}$. Eqs. (8)–(10) characterize the decendant CVO's.

The matrix elements of the primary chiral vertex operator $V_\Delta(z) \equiv V(\nu_\Delta|z)$ between the basis states (1) are building elements for the conformal blocks. Let us consider for example,

(i) the 4-point block on the sphere:

$$\begin{aligned}
\mathcal{F}_{c,\Delta}\!\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\!(x) &= x^{\Delta-\Delta_2-\Delta_1}\left(1 + \sum_{n=1}^{\infty} \mathcal{F}^n_{c,\Delta}\!\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] x^n\right), && (11)\\
\mathcal{F}^n_{c,\Delta}\!\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= \sum_{n=|I|=|J|} \langle\nu_{\Delta_4}, V_{\Delta_3}(1)\nu_{\Delta,I}\rangle \left[G_{c,\Delta}\right]^{IJ} \langle\nu_{\Delta,J}, V_{\Delta_2}(1)\nu_{\Delta_1}\rangle; && (12)
\end{aligned}$$

(ii) the 1-point block on the torus ($q = \mathrm{e}^{2\pi i\tau}$, where $\tau$ is the modular parameter of the torus):

$$\begin{aligned}
\mathcal{F}^{\tilde\Delta}_{c,\Delta}(q) &= q^{\Delta-\frac{c}{24}}\left(1 + \sum_{n=1}^{\infty} \mathcal{F}^{\tilde\Delta,n}_{c,\Delta} q^n\right), && (13)\\
\mathcal{F}^{\tilde\Delta,n}_{c,\Delta} &= \sum_{n=|I|=|J|} \left\langle\nu_{\Delta,I}, V_{\tilde\Delta}(1)\nu_{\Delta,J}\right\rangle \left[G_{c,\Delta}\right]^{IJ}. && (14)
\end{aligned}$$

The conformal blocks coefficients written explicitly in eqs. (12) and (14) are given in terms of the inverse

$$\left[G_{c,\Delta}\right]^{IJ} = \left(\left[G_{c,\Delta}\right]_{IJ}\right)^{-1}$$

of the Gramm matrix (4) and the matrix elements of $V_\Delta(z)$. To compute the latter it is enough to know the covariance properties (7) of the primary CVO w.r.t. the Virasoro algebra.

Due to the AGT correspondence the sphere and torus conformal blocks can be expressed through the SU(2) Nekrasov instanton partition functions respectively for

(i) the theory with four flavors $N_f = 4$:

$$x^{\Delta_1+\Delta_2-\Delta}\,\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) = \mathcal{Z}_{\rm inst}^{N_f=4,{\rm SU}(2)}(x,a,m_i,\epsilon_1,\epsilon_2) = (1-x)^{-\frac{(m_1+m_2)(m_3+m_4)}{2\epsilon_1\epsilon_2}}\,\mathcal{Z}_{\rm inst}^{N_f=4,{\rm U}(2)}(x,a,m_i,\epsilon_1,\epsilon_2), \tag{15}$$

where the following relations between the conformal block and the instanton function parameters are assumed:

$$c = 1+6\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2} \equiv 1+6Q^2, \qquad \Delta = \frac{(\epsilon_1+\epsilon_2)^2-4a^2}{4\epsilon_1\epsilon_2},$$

$$\Delta_1 = \frac{\frac{1}{4}(\epsilon_1+\epsilon_2)^2-\frac{1}{4}(m_1-m_2)^2}{\epsilon_1\epsilon_2}, \qquad \Delta_2 = \frac{\frac{1}{2}(m_1+m_2)(\epsilon_1+\epsilon_2-\frac{1}{2}(m_1+m_2))}{\epsilon_1\epsilon_2},$$

$$\Delta_3 = \frac{\frac{1}{2}(m_3+m_4)(\epsilon_1+\epsilon_2-\frac{1}{2}(m_3+m_4))}{\epsilon_1\epsilon_2}, \qquad \Delta_4 = \frac{\frac{1}{4}(\epsilon_1+\epsilon_2)^2-\frac{1}{4}(m_3-m_4)^2}{\epsilon_1\epsilon_2};$$

(ii) the theory with one massive hypermultiplet in the adjoint representation — the so-called $\mathcal{N}=2^*$ theory:

$$q^{\frac{c}{24}-\Delta}\mathcal{F}_{c,\Delta}^{\tilde\Delta}(q) = \mathcal{Z}_{\rm inst}^{\mathcal{N}=2^*,{\rm SU}(2)}(q,a,m,\epsilon_1,\epsilon_2) = \left[\prod_{n=1}^{\infty}(1-q^n)\right]^{1-2\tilde\Delta}\mathcal{Z}_{\rm inst}^{\mathcal{N}=2^*,{\rm U}(2)}(q,a,m;\epsilon_1,\epsilon_2), \tag{16}$$

where $\tilde\Delta = m(\epsilon_1+\epsilon_2-m)/(\epsilon_1\epsilon_2)$ and the relations among the central charge $c$, the intermediate weight $\Delta$, the $\Omega$-background parameters: $\epsilon_1, \epsilon_2$ and the v.e.v. $a$ are all the same as in the first example above.

As has been already mentioned, the conformal blocks exponentiate in the classical limit. In particular, in the case of our two canonical examples one can verify that

$$\mathcal{F}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) \overset{b\to 0}{\sim} \exp\left\{\frac{1}{b^2}f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)\right\}, \tag{17}$$

$$\mathcal{F}_{1+6Q^2,\Delta}^{\tilde\Delta}(q) \overset{b\to 0}{\sim} \exp\left\{\frac{1}{b^2}f_\delta^{\tilde\delta}(q)\right\}, \tag{18}$$

where $\Delta = \frac{1}{b^2}\delta$, $\Delta_i = \frac{1}{b^2}\delta_i$ and $\tilde\Delta = \frac{1}{b^2}\tilde\delta$. In eqs. above the so-called classical 4-point block on the sphere $f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)$ and the classical 1-point block on the torus $f_\delta^{\tilde\delta}(q)$ are available as the formal power series

$$f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) = (\delta-\delta_1-\delta_2)\log x + \sum_{n=1}^{\infty}\mathsf{f}_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]x^n, \tag{19}$$

$$f_\delta^{\tilde\delta}(q) = \left(\delta-\frac{1}{4}\right)\log q + \sum_{n=1}^{\infty}\mathsf{f}_\delta^{\tilde\delta,n}\,q^n \tag{20}$$

with the following coefficients:

$a)$ in the case of the classical spherical block

$$\begin{aligned}
\mathsf{f}^1_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right] &= \frac{(\delta+\delta_3-\delta_4)(\delta+\delta_2-\delta_1)}{2\delta},\\
\mathsf{f}^2_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right] &= \Big[16\delta^3(4\delta+3)\Big]^{-1}\Big[13\delta^5+\delta^4\left(18\delta_2-14\delta_1+18\delta_3-14\delta_4+9\right)\\
&+ \delta^3\Big(\delta_1^2+\delta_2^2-2\delta_1(\delta_2+6\delta_3-10\delta_4+6)\\
&+ 4\delta_2(5\delta_3-3\delta_4+3)+(\delta_3-\delta_4)(\delta_3-\delta_4+12)\Big)\\
&- 3\delta^2\Big(\delta_1^2(2\delta_3+2\delta_4-1)+2\delta_1(\delta_3^2+\delta_4^2+2\delta_3+\delta_2-2\delta_2\delta_3-2\delta_4(\delta_2+\delta_3+1))\\
&+ \delta_2^2(2\delta_3+2\delta_4-1)+2\delta_2(\delta_3-\delta_4-2)(\delta_3-\delta_4)-(\delta_3-\delta_4)^2\Big)\\
&+ 5\delta(\delta_1-\delta_2)^2(\delta_3-\delta_4)^2-3(\delta_1-\delta_2)^2(\delta_3-\delta_4)^2\Big],\ \ldots\ ;
\end{aligned}$$

$b)$ in the case of the classical torus block

$$\mathsf{f}^{\tilde\delta,1}_\delta=\frac{\tilde\delta^2}{2\delta},\qquad \mathsf{f}^{\tilde\delta,2}_\delta=\frac{\tilde\delta^2[24\delta^2\left(4\delta+1\right)+\tilde\delta^2\left(5\delta-3\right)-48\tilde\delta\delta^2]}{16\delta^3\left(4\delta+3\right)},\ \ldots .$$

The classical limits (17) and (18) are very nontrivial statements concerning the quantum conformal blocks. There is still no rigorous proof of these conjectures. However, there are many convincing arguments for the existence of the classical conformal blocks. In particular, remembering that $b=\sqrt{\epsilon_2/\epsilon_1}$, eqs. (17) and (18) are consistent with the AGT relations (15)-(16) and the Nekrasov–Shatashvili limit [9]:

$$\mathcal{Z}_{\rm inst}(\,\cdot\,,\epsilon_1,\epsilon_2)\ \stackrel{\epsilon_2\to0}{\sim}\ \exp\left\{\frac{1}{\epsilon_2}\,\mathcal{W}_{\rm inst}(\,\cdot\,,\epsilon_1)\right\}\tag{21}$$

of the Nekrasov instanton functions. Indeed, the following identities[2]

$$\begin{aligned}
f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) &= (\delta-\delta_1-\delta_2)\log x-\frac{(m_1+m_2)(m_3+m_4)}{2\epsilon_1^2}\,\log(1-x)\\
&+ \frac{1}{\epsilon_1}\,\mathcal{W}^{{\rm U}(2),N_f=4}_{\rm inst}(x,a,m_i,\epsilon_1), \qquad (22)\\
f^{\tilde\delta}_\delta(q) &= \left(\delta-\frac14\right)\log q-2\tilde\delta\log\left(\frac{\eta(q)}{q^{\frac{1}{24}}}\right)+\frac{1}{\epsilon_1}\,\mathcal{W}^{\mathcal{N}=2^*,{\rm U}(2)}_{\rm inst}(q,a,m,\epsilon_1) \qquad (23)
\end{aligned}$$

hold for the classical conformal weights $\delta$, $\delta_i$, $\tilde\delta$ defined as follows $\delta=\lim_{b\to0}b^2\Delta=\lim_{\epsilon_2\to0}\frac{\epsilon_2}{\epsilon_1}\Delta$. The instanton twisted superpotentials:

$$\mathcal{W}^{{\rm U}(2),N_f=4}_{\rm inst},\qquad\qquad \mathcal{W}^{\mathcal{N}=2^*,{\rm U}(2)}_{\rm inst}$$

in eqs. (22) and (23) are given in terms of the critical value of some functions $H^{{\rm U}(2)}_{\rm inst}(x_{ui})$. The latter can be derived from the Nekrasov instanton sums within the Nekrasov–Shatashvili limit. Indeed, the Nekrasov instanton partition functions can be approximated in the limit $\epsilon_2\to0$ by a kind of 'path integral' with the integrand:

$$\exp\left\{\frac{1}{\epsilon_2}\,H^{{\rm U}(2)}_{\rm inst}(x_{ui})\right\}.$$

[2] $\eta(q)$ is the Dedekind eta function.

Therefore, the saddle point method yields:

$$\mathcal{W}^{\mathrm{U}(2)}_{\mathrm{inst}} \equiv \lim_{\epsilon_2\to 0} \epsilon_2 \log \mathcal{Z}^{\mathrm{U}(2)}_{\mathrm{inst}} = H^{\mathrm{U}(2)}_{\mathrm{inst}}(x^{*}_{ui}),$$

where $x^{*}_{ui}$ denote the 'critical configurations' extremizing the 'actions' $H^{\mathrm{U}(2)}_{\mathrm{inst}}$. The extremality conditions (saddle point equations) for the 'actions' $H^{\mathrm{U}(2)}_{\mathrm{inst}}$ take the form of the Bethe-like equations. In particular, one gets

- in the case of the U(2) $N_f = 4$ theory:

$$-q\left[\prod_{v=1}^{2}\prod_{l=1}^{\infty}\frac{(x_{ui}-x_{vl}-\epsilon_1)(x_{ui}-x^0_{vl}+\epsilon_1)}{(x_{ui}-x_{vl}+\epsilon_1)(x_{ui}-x^0_{vl}-\epsilon_1)}\right]\left[\frac{\prod\limits_{\alpha=1}^{4}(x_{ui}+m_\alpha)}{\prod\limits_{v=1}^{2}(x_{ui}-a_v)(x_{ui}-a_v+\epsilon_1)}\right] = 1. \quad (24)$$

- in the case of the U(2) $\mathcal{N} = 2^*$ theory:

$$-q\Big(\prod_{v,j}\frac{(x_{ui}-x_{vj}-\epsilon_1)(x_{ui}-x^0_{vj}+\epsilon_1)}{(x_{ui}-x^0_{vj}-\epsilon_1)(x_{ui}-x_{vj}+\epsilon_1)}\frac{(x_{ui}-x_{vj}-m)(x_{ui}-x^0_{vj}+m)}{(x_{ui}-x^0_{vj}-m)(x_{ui}-x_{vj}+m)} \quad (25)$$
$$\times\frac{(x_{ui}-x_{vj}+m+\epsilon_1)(x_{ui}-x^0_{vj}-m-\epsilon_1)}{(x_{ui}-x^0_{vj}+m+\epsilon_1)(x_{ui}-x_{vj}-m-\epsilon_1)}\Big)\Big(\prod_{v}\frac{(x_{ui}-m-a_v)(x_{ui}+m+\epsilon_1-a_v)}{(x_{ui}-a_v)(x_{ui}+\epsilon_1-a_v)}\Big) = 1,$$

  where $u, v = 1, 2$; $i, j = 1, \ldots, \infty$.

The solutions $x^{*}_{ui} \equiv x_{ui}$ to the above equations can be found recursively in terms of certain formal expansions in the parameter $q$.

The problem of the computation of the Nekrasov–Shatashvili limit of the Nekrasov functions can be also well formulated and studied within the field theory framework, cf. eg. [17, 18].

## 3. Accessory parameters for the Heun and the Lamé equations

The Hilbert space of (the simplest) two-dimensional conformal field theories consists of the tensor products of the 'left' and 'right' irreducible Verma modules:

$$\mathcal{H} = \bigoplus_{\Delta,\bar{\Delta}\in\mathbf{S}} \mathcal{V}_\Delta \otimes \mathcal{V}_{\bar{\Delta}}. \quad (26)$$

The direct sum above is taken over the spectrum $\mathbf{S}$ of the theory, i.e. over the set of the highest weights of the model. In the case of 2d CFT's with the continuous spectrum the direct sum in eq. (26) becomes a direct integral. It is assumed that in the spectrum one can find a pair of weights equal to zero $(\Delta, \bar{\Delta}) = (0, 0) \in \mathbf{S}$. Hence, there exists in $\mathcal{H}$ the sector $\mathcal{V}_0 \otimes \mathcal{V}_0$ in which the Verma modules are generated from the highest weight state $|\,0\,\rangle \equiv |\,\nu_0\,\rangle$ with the highest weight $\Delta = 0$.

In $\mathcal{H}$ operate physical fields which are built out of the chiral vertex operators. In particular, the physical primary fields are made of the primary CVO's:

$$\mathsf{V}_{\Delta_2,\bar{\Delta}_2}(z,\bar{z}) = \sum_{\Delta_3,\bar{\Delta}_3,\Delta_1,\bar{\Delta}_1\in\mathbf{S}} C_{(\Delta_3,\bar{\Delta}_3),(\Delta_2,\bar{\Delta}_2),(\Delta_1,\bar{\Delta}_1)} V_{\Delta_2}(z)\otimes V_{\bar{\Delta}_2}(\bar{z}).$$

The symbols $C_{(\Delta_3,\bar{\Delta}_3),(\Delta_2,\bar{\Delta}_2),(\Delta_1,\bar{\Delta}_1)}$ denote the model dependent structure constants:

$$C_{(\Delta_3,\bar{\Delta}_3),(\Delta_2,\bar{\Delta}_2),(\Delta_1,\bar{\Delta}_1)} = \langle \nu_{\Delta_3}\otimes\nu_{\bar{\Delta}_3}|\mathsf{V}_{\Delta_2,\bar{\Delta}_2}(1,1)|\nu_{\Delta_1}\otimes\nu_{\bar{\Delta}_1}\rangle$$

which are nothing but the 3-point functions $\langle 0|\mathsf{V}_3(z_3,\bar{z}_3)\mathsf{V}_2(z_2,\bar{z}_2)\mathsf{V}_1(z_1,\bar{z}_1)|0\rangle$ evaluated at the special locations of the physical primary operators. Let us recall that according to the field–state correspondence the primary fields, acting on the 'vacuum' $|0\rangle \equiv |0\rangle\otimes|0\rangle$, generate the highest weight states in $\mathcal{H}$:

$$|\nu_\Delta\otimes\nu_{\bar{\Delta}}\rangle = \lim_{z,\bar{z}\to 0}\mathsf{V}_{\Delta,\bar{\Delta}}(z,\bar{z})|0\rangle, \qquad \langle\nu_\Delta\otimes\nu_{\bar{\Delta}}| = \lim_{z,\bar{z}\to\infty} z^{2\Delta}\bar{z}^{2\bar{\Delta}}\langle 0|\mathsf{V}_{\Delta,\bar{\Delta}}(z,\bar{z}).$$

Furthermore, each state in the Hilbert space corresponds to some local operator which generates this state from the vacuum. In particular, null states correspond to the so-called null fields. Since the physical space of states can not contain null vectors then the correlation functions (the vacuum expectation values) with the null fields must vanish. The last condition, using the conformal Ward identities [1, 2], can be converted to certain partial differential equations obeyed by the correlation functions with the degenerate primary operators. The latter are simply physical primary fields with the degenerate conformal weights (6).

Let us consider the projected 5-point function on the sphere in the diagonal theory ($\Delta_i = \bar{\Delta}_i$):

$$G_\Delta(z,x) \;\equiv\; \Big\langle \mathsf{V}_4(\infty,\infty)\mathsf{V}_3(1,1)\mathbf{1}_{\Delta,\Delta}\mathsf{V}_{-\frac{b}{2}}(z,\bar{z})\mathsf{V}_2(x,\bar{x})\mathsf{V}_1(0,0)\Big\rangle$$

where $\mathsf{V}_{\alpha=-\frac{b}{2}}$ is the degenerate field with the conformal weight

$$\Delta_{-\frac{b}{2}} \;=\; \Delta_{\alpha=-\frac{b}{2}} \;=\; \alpha(Q-\alpha) \;=\; -\frac{1}{2}-\frac{3}{4}\,b^2, \qquad Q \;=\; b+\frac{1}{b}$$

and $\mathsf{V}_i$'s are the four heavy primary operators ($\Delta_i = b^{-2}\,\delta_i$, $\delta_i = \mathcal{O}(1)$). The projected correlation functions are elements for building full physical correlators. Indeed, this can be done by summing or integrating such quantities over the spectrum of the theory. The function $G_\Delta(z,x)$ satisfies the following null vector decoupling (NVD) equation:

$$\left[\frac{\partial^2}{\partial z^2} - b^2\left(\frac{1}{z}-\frac{1}{1-z}\right)\frac{\partial}{\partial z}\right]G_\Delta(z,x) =$$
$$-b^2\left[\frac{\Delta_1}{z^2}+\frac{\Delta_2}{(z-x)^2}+\frac{\Delta_3}{(1-z)^2}+\frac{\Delta_1+\Delta_2+\Delta_3+\Delta_{-\frac{b}{2}}-\Delta_4}{z(1-z)}+\frac{x(1-x)}{z(z-x)(1-z)}\frac{\partial}{\partial x}\right]G_\Delta(z,x).$$

Our aim now is to consider the classical limit of the equation above. The key point here is an observation that in the limit $b\to 0$ only the operator with weight $\Delta_{-\frac{b}{2}}$ remains light ($\Delta_{-\frac{b}{2}} = \mathcal{O}(1)$) and its presence in the correlation function has no influence on the classical dynamics. Then, for $b\to 0$

$$G_\Delta(z,x) \;\sim\; \psi(z)\,\mathrm{e}^{-\frac{1}{b^2}\left(S^{\rm cl}(\delta_4,\delta_3,\delta)+S^{\rm cl}(\delta,\delta_2,\delta_1)-f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)-\bar{f}_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](\bar{x})\right)}. \tag{27}$$

Indeed, assuming that the light field does not contribute to the classical limit we are left with the projected 4-point function of the heavy operators:

$$\begin{aligned}
&\langle \mathsf{V}_4(\infty,\infty)\mathsf{V}_3(1,1)\mathbf{1}_{\Delta,\Delta}\mathsf{V}_2(x,\bar{x})\mathsf{V}_1(0,0)\rangle \;=\\
&\quad=\; C_{\Delta_4,\Delta_3,\Delta}\,C_{\Delta,\Delta_2,\Delta_1}\,\mathcal{F}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x)\bar{\mathcal{F}}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](\bar{x})\\
&\quad\overset{b\to 0}{\sim}\; \mathrm{e}^{-\frac{1}{b^2}\left(S^{\rm cl}(\delta_4,\delta_3,\delta)+S^{\rm cl}(\delta,\delta_2,\delta_1)-f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)-\bar{f}_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](\bar{x})\right)}.
\end{aligned}$$

The quantities $S^{\rm cl}(\delta_3,\delta_2,\delta_1)$ — the so-called classical 3-point actions — are the classical limits of the structure constants. The substitution of eq. (27) into the NVD equation leads, within the limit $b\to 0$, to the Fuchsian differential equation:

$$\partial_z^2\psi(z)+\left[\frac{\delta_1}{z^2}+\frac{\delta_2}{(z-x)^2}+\frac{\delta_3}{(1-z)^2}+\frac{\delta_1+\delta_2+\delta_3-\delta_4}{z(1-z)}+\frac{x(1-x)\mathcal{C}_2(x)}{z(z-x)(1-z)}\right]\psi(z)=0$$

with singularities located at $z_4=\infty, z_3=1, z_2=x, z_1=0$ and the accessory parameter $\mathcal{C}_2(x)$ determined by the classical 4-point block, cf. [13]:

$$\boxed{\mathcal{C}_2(x)=\frac{\partial}{\partial x}\,f_\delta\!\left[\begin{smallmatrix}\delta_3&\delta_2\\ \delta_4&\delta_1\end{smallmatrix}\right]\!(x)}\;.$$

On the other hand, it follows from eq. (22) that the parameter $\mathcal{C}_2(x)$ can be found once the solution of the saddle point equation (24) is known, cf. [15].

The above argumentation can be repeated in the case of the projected correlation functions on the torus:

$$\Big\langle\ \cdot\ \Big\rangle_{\tau,\Delta}\;=\;{\rm Tr}_{\mathcal{V}_\Delta\otimes\mathcal{V}_\Delta}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\ \cdot\ \right).$$

In particular, considering the NVD equation fulfilled by the projected torus 2-point function with the degenerate operator $\mathsf{V}_{\alpha=-\frac{b}{2}}$, it is possible to obtain in the classical limit $b\to 0$ the Weierstrassian-form Lamé equation:

$$\partial_z^2\psi(z)-\left[\,\kappa\,\wp(z)+\mathcal{B}\,\right]\psi(z)=0$$

with the accessory parameter/eigenvalue:[3]

$$\boxed{\frac{\mathcal{B}(\tau)}{4\pi^2}\;=\;q\frac{\partial}{\partial q}\,f_\delta^{\tilde\delta}(q)-\frac{\tilde\delta}{12}\,{\rm E}_2(\tau)}$$

determined by the classical 1-point block on the torus. Analogously, eq. (23) implies that the another way to compute the parameter $\mathcal{B}(\tau)$ is to solve the Bethe-like equation (25), cf. [16].

As a final remark in this section let us stress that the *holomorphic* accessory parameters $\mathcal{C}_2(x)$ and $\mathcal{B}(\tau)$ are closely related to that which emerge in the uniformization theory of the 4-punctured sphere and 1-punctured torus.[4] The accessory parameters for $C_{0,4}$ and $C_{1,1}$ can be found by taking the classical limit of the NVD equations obeyed by the Liouville 5-point function on the sphere and the 2-point function on the torus with $\mathsf{V}_{\alpha=-\frac{b}{2}}$. This means that one has to consider the projected correlation functions discussed above which are integrated over the Liouville field theory spectrum $\Delta=\Delta_\alpha$, $\alpha=\frac{Q}{2}+i\mathbf{R}^+$ with the structure constants $C_{\Delta_3,\Delta_2,\Delta_1}$ identified as the DOZZ Liouville 3-point functions [3, 19].

## 4. Concluding remarks

The main goal of the present article was to demonstrate that the calculation of the holomorphic accessory parameters for the Fuchs on $C_{0,4}$ and the Lamé equations can be reduced to the problem of computing corresponding classical conformal blocks or solving appropriate Bethe-like saddle point equations. The link between the solutions of certain Bethe-like equations and the classical conformal blocks yields a new method of calculating of the latter. This is of great importance, because the efficient methods of the computation of the classical blocks can be useful in studies in other areas of theoretical physics where classical conformal blocks have applications. Indeed, classical conformal blocks arise in a few other interesting contexts: the entanglement entropy in 2d CFT, 2d CFT/AdS$_3$ holography, matrix models, KdV equation and S-duality in 2d $\mathcal{N}=2$ SYM theories. We plan to examine these contexts soon.

[3] Here ${\rm E}_2(\tau)$ is the second Eisenstein series.

[4] Cf. discussion in ref. [13].

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