

Exact scattering eigenstates in double quantum-dot systems with an interdot Coulomb interaction

Akinori Nishino¹, Naomichi Hatano² and Gonzalo Ordóñez³

¹Faculty of Engineering, Kanagawa University, 3-27-1, Rokkakubashi, Kanagawa-ku, Yokohama-shi, Kanagawa, 221-8686, Japan

²Institute of Industrial Science, The University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo, 153-8505, Japan

³Department of Physics and Astronomy, Butler University, 4600 Sunset Ave., Indianapolis, IN 46208, USA

E-mail: nishino@kanagawa-u.ac.jp

Abstract. We study a double quantum-dot system which consists of two leads of noninteracting electrons and two quantum dots with an interdot Coulomb interaction. We assume spinless electrons and consider arbitrary complex values of all lead-dot couplings and an interdot coupling. We construct exact many-electron scattering eigenstates whose incident states are free-electronic plane waves in the leads. Due to the interaction, some of the incident plane waves are scattered to two-body and three-body bound states. The binding strength of the many-body bound states is affected by the arrangement of the two quantum dots, by which we observe an interplay of the Coulomb interaction and quantum interference. We can understand the many-body bound states in terms of many-body resonances.

1. Introduction

Electronic transport in nanoscale systems is a rapidly developing research field of condensed matter physics. In particular, quantum-dot (QD) systems have been studied intensively in the last two decades. One of the significant experimental results is the measurement of a zero-bias peak of the electrical conductance across a QD at low temperatures [1–3], which is described by the Kondo effect in the Anderson model theoretically [4–6]. Recently, the subjects of interest have shifted to double quantum-dot (DQD) systems. The Kondo states realized in each QD are affected by the arrangement of the two QDs due to quantum interference [7–13] and, furthermore, the interdot Coulomb interaction yields the orbital Kondo effect [14–22]. For the interdot interaction larger than a critical value, correlation-induced resonances, which are different from the Kondo resonance, are observed in the conductance profile as an interplay of the Coulomb interaction and quantum interference [16, 23, 24].

Several methods have been proposed to compute the electrical conductance across nanoscale scatterers. The Landauer formula enables us to calculate it by using scattering eigenstates of open quantum systems [25, 26]. However, the application of the formula has been restricted to noninteracting cases. In previous papers [27–29], we obtained *exact* many-electron scattering eigenstates for the QD system described by the two-lead interacting resonant-level models (IRLM). By applying them in order to extend the Landauer formula, we analytically calculated the average electric current for the system under bias voltages in the first order of the interaction

parameters. The universal electric current that we obtained through a renormalized-group technique [30, 31] is consistent with those obtained by other approaches; specifically, we observed *negative differential conductance* for large bias voltages [30–35].

It is remarkable that, due to the interactions, some of the incident plane waves in our scattering eigenstates are scattered to *two-body bound states* that exponentially decay with the distance of the two electrons [27–29]. The two-body bound states promote the reflection of electrons at the QD and result in negative differential conductance in the universal electric current. We can understand the appearance of the two-body bound states in terms of one-body *resonant states* [36–41]. Due to the Coulomb interactions, some of the incident electrons are scattered to a resonant state and the corresponding anti-resonant state to form a two-body bound state. The resonant energy thereby determines the binding strength of the exponential decay of the two-body bound state. We remark that such many-body bound states in exact scattering eigenstates have been found in the Anderson model with spin degrees of freedom [42] and in optical systems of waveguide QED [43].

In this paper, we study many-electron scattering eigenstates for a DQD system with an interdot Coulomb interaction. We neglect spin degrees of freedom and assume a single energy level, called a gate energy, on each QD. We consider arbitrary complex values of all lead-dot couplings and dot-dot coupling [12, 16]. We construct exact many-electron scattering eigenstates with plane-wave incident states. Due to the Coulomb interaction, some of the incident plane waves are scattered to two-body and three-body bound states. We can understand the two-body bound states in terms of the one-body resonant states, as was done in the IRLM cases, while we can understand the three-body bound states in terms of the one-body and two-body resonant states. We also find that the many-body bound states are affected by the quantum interference brought by the arrangement of the two QDs. Indeed, the binding strength as a function of the detuning of the two gate energies behaves differently for parallel, serial and T-shaped DQD systems. Thus we observe an interplay of the Coulomb interactions and quantum interference in the many-body bound states.

The paper is organized as follows: In Section 2, the DQD system with an interdot Coulomb interaction is introduced as an open quantum system. In Section 3, the exact scattering eigenstates are presented for one, two and three electrons. The binding strength of the two-body bound states as a function of the detuning of the gate energies is investigated for parallel, serial and T-shaped DQD systems. In Section 4, the construction of the two- and three-electron scattering eigenstates is described in detail. Section 5 is devoted to a summary.

2. Double quantum-dot systems

We introduce a DQD system which consists of two QDs connected to two infinite leads of noninteracting spinless electrons. Figure 1 illustrates the arrangement of the two QDs and the two leads. The Hamiltonian of the DQD system is given by

$$\begin{aligned}
H = & \sum_{\ell=1,2} \int_{-\infty}^{\infty} dx c_{\ell}^{\dagger}(x) \frac{1}{i} \frac{d}{dx} c_{\ell}(x) + \sum_{\ell=1,2} \sum_{\alpha=1,2} (t_{\ell\alpha} c_{\ell}^{\dagger}(0) d_{\alpha} + t_{\ell\alpha}^{*} d_{\alpha}^{\dagger} c_{\ell}(0)) \\
& + t' d_1^{\dagger} d_2 + t'^{*} d_2^{\dagger} d_1 + \sum_{\alpha=1,2} \epsilon_{d\alpha} n_{d\alpha} + U n_{d1} n_{d2}.
\end{aligned} \tag{1}$$

Here, $c_{\ell}^{\dagger}(x)$ and $c_{\ell}(x)$ are the creation and the annihilation operators of an electron at a point x in the lead $\ell(= 1, 2)$ while d_{α}^{\dagger} and d_{α} are those on the QD $\alpha(= 1, 2)$. We have also defined the number operator $n_{d\alpha} = d_{\alpha}^{\dagger} d_{\alpha}$ of electrons on the QD α . We consider all the lead-dot couplings denoted by the parameters $t_{\ell\alpha}(\in \mathbb{C})$ and the dot-dot coupling denoted by $t'(\in \mathbb{C})$. We assume a single energy level $\epsilon_{d\alpha}(\in \mathbb{R})$ on each QD. The parameter $U(> 0)$ expresses the strength of the

interdot Coulomb repulsion. The first term of the Hamiltonian corresponds to the kinetic energy of electrons in the leads. Here we focus on the electrons with positive velocities and linearize the local dispersion relations of the leads in the vicinity of the Fermi energy under the assumption that other parameters are small compared with the Fermi energy.

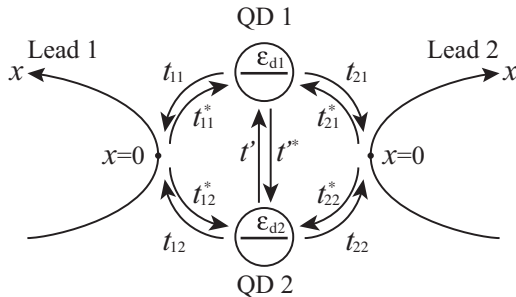


Figure 1. A schematic diagram of the DQD system.

The DQD system described by Eq. (1) includes various DQD systems such as a parallel DQD system ($t' = 0$), a serial DQD system ($t_{12} = t_{21} = 0$) and a T-shaped DQD system ($t_{12} = t_{22} = 0$) [12, 16]. In the previous work [44] studying a parallel DQD system, we employed an even-odd transformation that maps the two-lead system to decoupled two single-lead systems. The even-odd transformation, however, works only for the cases with $t_{11}t_{22} = t_{12}t_{21}$, which do not include the serial DQD system.

3. Exact scattering eigenstates

In the present section, we present exact solutions of one-, two- and three-electron scattering eigenstates. Details of the construction of two- and three-electron eigenstates will be given in the next section.

3.1. One-electron case

First, we construct the one-electron scattering eigenstates. The one-electron eigenstates are given in the form

$$|E\rangle = \left(\int dx \sum_m g_m(x) c_m^\dagger(x) + \sum_\alpha e_\alpha d_\alpha^\dagger \right) |0\rangle. \quad (2)$$

Here $|0\rangle$ is the vacuum state satisfying $c_m(x)|0\rangle = d_\alpha|0\rangle = 0$, ($m = 1, 2$, $\alpha = 1, 2$), and $g_m(x)$ and e_α are the eigenfunctions to be determined by the Schrödinger equations

$$\left(\frac{1}{i} \frac{d}{dx} - E \right) g_m(x) + \delta(x) \sum_\alpha t_{m\alpha} e_\alpha = 0, \quad (3a)$$

$$(\epsilon_{d\alpha} - E) e_\alpha + \sum_m t_{m\alpha}^* g_m(0) + t'_\alpha e_{\bar{\alpha}} = 0, \quad (3b)$$

where we put $t'_1 = t'$, $t'_2 = t'^*$ and $\bar{\alpha} = 3 - \alpha$.

We construct the one-electron scattering eigenfunctions $g_m(x)$ and e_α in the following way. We find that the eigenfunctions $g_m(x)$ are discontinuous at $x = 0$ due to the δ -function term in Eq. (3a). In fact, by integrating the Schrödinger equation (3a) around $x = 0$, we obtain the matching condition

$$g_m(0+) = g_m(0-) - i \sum_\alpha t_{m\alpha} e_\alpha. \quad (4)$$

Since the Schrödinger equations do not determine the value of the eigenfunctions at the discontinuous point, we assume

$$g_m(0) = \frac{1}{2}(g_m(0+) + g_m(0-)). \quad (5)$$

By applying Eqs. (4) and (5) to the Schrödinger equation (3b), we have

$$(\epsilon_{d\alpha} - i\Gamma_{\alpha\alpha} - E)e_\alpha + (t'_\alpha - i\Gamma_{\alpha\bar{\alpha}})e_{\bar{\alpha}} = -\sum_m t_{m\alpha}^* g_m(0-), \quad (6)$$

where $\Gamma_{\alpha\beta} = \sum_\ell t_{\ell\alpha}^* t_{\ell\beta}/2$ is the level-width matrix of the QDs. By solving Eq. (6) as coupled equations for e_1 and e_2 , we obtain

$$e_\alpha = \frac{\sum_m ((E - \epsilon_{d\bar{\alpha}} + i\Gamma_{\bar{\alpha}\bar{\alpha}})t_{m\alpha}^* - (i\Gamma_{\alpha\bar{\alpha}} - t'_\alpha)t_{m\bar{\alpha}}^*)g_m(0-)}{(E - \epsilon_{d1} + i\Gamma_{11})(E - \epsilon_{d2} + i\Gamma_{22}) - (i\Gamma_{12} - t'_1)(i\Gamma_{21} - t'_2)}. \quad (7)$$

Then, for given $g_m(0-)$, the values e_α and $g_l(0+)$ are determined by Eq. (7) and Eq. (4), respectively.

In order to study the electron transport by the Landauer formula [25,26], we need the scattering eigenstates whose incident states are plane waves in the leads. We note that the incident plane waves come from the part $x < 0$ of the leads since we focus on the electrons with positive velocities. Then we consider the eigenstate whose incident state is a plane wave with wave number k in the lead ℓ as follows:

$$g_{m,k}^{(\ell)}(x) = \frac{1}{\sqrt{2\pi}}\delta_{m\ell}e^{ikx}, \quad (x < 0), \quad (8)$$

which we call scattering boundary conditions. The reflected and transmitted waves are determined by the matching conditions (4) with Eq. (7). As a result, we obtain the exact one-electron scattering eigenfunctions satisfying the scattering boundary conditions (8) as

$$g_{m,k}^{(\ell)}(x) = \frac{1}{\sqrt{2\pi}}\left(\delta_{m\ell} - \sqrt{2\pi}i\sum_\alpha t_{m\alpha}e_{\alpha,k}^{(\ell)}\theta(x)\right)e^{ikx}, \quad (9a)$$

$$e_{\alpha,k}^{(\ell)} = \frac{1}{\sqrt{2\pi}}\frac{(k - \epsilon_{d\bar{\alpha}} + i\Gamma_{\bar{\alpha}\bar{\alpha}})t_{\ell\alpha}^* - (i\Gamma_{\alpha\bar{\alpha}} - t'_\alpha)t_{\ell\bar{\alpha}}^*}{(k - \epsilon_{d1} + i\Gamma_{11})(k - \epsilon_{d2} + i\Gamma_{22}) - (i\Gamma_{12} - t'_1)(i\Gamma_{21} - t'_2)}, \quad (9b)$$

where $\theta(x)$ is a step function: $\theta(x) = 1$ if $x > 0$, $\theta(0) = 1/2$ and $\theta(x) = 0$ if $x < 0$. The energy eigenvalue is given by $E = k$.

Let us rewrite the one-electron eigenfunction $e_{\alpha,k}^{(\ell)}$ in Eq. (9b) as follows:

$$e_{\alpha,k}^{(\ell)} = \frac{1}{\sqrt{2\pi}}\frac{(k - \epsilon_{d\bar{\alpha}} + i\Gamma_{\bar{\alpha}\bar{\alpha}})t_{\ell\alpha}^* - (i\Gamma_{\alpha\bar{\alpha}} - t'_\alpha)t_{\ell\bar{\alpha}}^*}{(k - \bar{\epsilon}_d + i\bar{\Gamma} - \frac{i}{2}\eta)(k - \bar{\epsilon}_d + i\bar{\Gamma} + \frac{i}{2}\eta)}, \quad (10)$$

where we define $\bar{\epsilon}_d = (\epsilon_{d1} + \epsilon_{d2})/2$, $\bar{\Gamma} = (\Gamma_{11} + \Gamma_{22})/2$, $\Delta\Gamma_\alpha = \Gamma_{\alpha\alpha} - \Gamma_{\bar{\alpha}\bar{\alpha}}$, $\Delta\epsilon_{d\alpha} = \epsilon_{d\alpha} - \epsilon_{d\bar{\alpha}}$ and

$$\eta = \sqrt{(i\Delta\epsilon_{d\alpha} + \Delta\Gamma_\alpha)^2 - 4(t' - i\Gamma_{12})(t'^* - i\Gamma_{21})}. \quad (11)$$

From this expression, we find resonant poles of the one-electron scattering at $k = \bar{\epsilon}_d - i\bar{\Gamma} \pm i\eta/2$ on the complex- k plane. ‘‘Eigenstates’’ with resonant energies are called *resonant states* [36–41]. It is known that the resonant states are characterized as the eigenstates with no incoming waves, which is called the Siegert boundary conditions [37]. Indeed, the resonant energy $k = \bar{\epsilon}_d - i\bar{\Gamma} \pm i\eta/2$ is determined by Eq. (6) with the Siegert boundary conditions $g_m(0-) = 0$. While the resonant states are unphysical in the framework of the standard quantum mechanics, we shall see in the two- and three-electron cases that such resonant states ‘‘emerge’’ as many-body bound states in the scattering eigenstates.

3.2. Two-electron case

Second, we consider the two-electron case, in which an effect of the interaction appears. The two-electron eigenstates are given in the form

$$|E\rangle = \left(\sum_{l,m} \int_{x_1 < x_2} dx_1 dx_2 g_{lm}(x_1, x_2) c_l^\dagger(x_1) c_m^\dagger(x_2) + \sum_{l,\alpha} \int dx_1 e_{l\alpha}(x_1) c_l^\dagger(x_1) d_\alpha^\dagger + f_{12} d_1^\dagger d_2^\dagger \right) |0\rangle. \quad (12)$$

Here we assume the anti-symmetry relations $g_{lm}(x_1, x_2) = -g_{ml}(x_2, x_1)$ and $f_{12} = -f_{21}$. The Schrödinger equations are given by

$$\left(\frac{1}{i} (\partial_1 + \partial_2) - E \right) g_{lm}(x_1, x_2) + \sum_{\alpha} (t_{m\alpha} e_{l\alpha}(x_1) \delta(x_2) - t_{l\alpha} \delta(x_1) e_{m\alpha}(x_2)) = 0, \quad (13a)$$

$$\left(\frac{1}{i} \frac{d}{dx_1} + \epsilon_{d\alpha} - E \right) e_{l\alpha}(x_1) + \sum_m t_{m\alpha}^* g_{lm}(x_1, 0) + t_{\bar{\alpha}}' e_{l\bar{\alpha}}(x_1) - t_{l\bar{\alpha}} \delta(x_1) f_{\alpha\bar{\alpha}} = 0, \quad (13b)$$

$$(\epsilon_{d1} + \epsilon_{d2} + U - E) f_{\alpha\bar{\alpha}} + \sum_l (t_{l\alpha}^* e_{l\bar{\alpha}}(0) - t_{l\bar{\alpha}}^* e_{l\alpha}(0)) = 0, \quad (13c)$$

where $\partial_i = \partial/\partial x_i$.

In a way similar to the one-electron case, we consider the two-electron scattering eigenstates satisfying the following scattering boundary conditions: the incident states are free-electronic plane waves with wave number k_1 in the lead ℓ_1 and with wave number k_2 in the lead ℓ_2 ,

$$g_{lm, k_1 k_2}^{(\ell_1 \ell_2)}(x_1, x_2) = \frac{1}{2\pi} \sum_P \text{sgn}(P) \delta_{l P_1} \delta_{m P_2} e^{i(k_{P_1} x_1 + k_{P_2} x_2)}, \quad (x_1, x_2 < 0), \quad (14)$$

where $P = (P_1, P_2)$ denotes a permutation of $(1, 2)$. We shall present how to construct the two-electron scattering eigenstates in the next section. As a result, we obtain the exact two-electron scattering eigenfunctions with energy eigenvalue $E = k_1 + k_2$ as

$$g_{m_1 m_2, k_1 k_2}^{(\ell_1 \ell_2)}(x_1, x_2) = \sum_P \text{sgn}(P) g_{m_1, k_{P_1}}^{(\ell_{P_1})}(x_1) g_{m_2, k_{P_2}}^{(\ell_{P_2})}(x_2) - \sum_{Q, \alpha, s} \text{sgn}(Q) i t_{m_{Q_1} \alpha} y_{m_{Q_2} \alpha, s} Z_{\alpha, k_1 k_2, s}^{(\ell_1 \ell_2)}(x_{Q_1 Q_2}) e^{iE x_{Q_2}} \theta(x_{Q_1}), \quad (15a)$$

$$e_{m_1 \alpha, k_1 k_2}^{(\ell_1 \ell_2)}(x_1) = \sum_P \text{sgn}(P) g_{m_1, k_{P_1}}^{(\ell_{P_1})}(x_1) e_{\alpha, k_{P_2}}^{(\ell_{P_2})} - \sum_s y_{m_1 \alpha, s} Z_{\alpha, k_1 k_2, s}^{(\ell_1 \ell_2)}(-x_1) e^{iE x_1}, \quad (15b)$$

$$f_{\alpha \bar{\alpha}, k_1 k_2}^{(\ell_1 \ell_2)} = \sum_P \text{sgn}(P) e_{\alpha, k_{P_1}}^{(\ell_{P_1})} e_{\bar{\alpha}, k_{P_2}}^{(\ell_{P_2})} + Z_{\alpha, k_1 k_2}^{(\ell_1 \ell_2)}. \quad (15c)$$

Here $x_{12} = x_1 - x_2$, $s = +$ or $-$, $Q = (Q_1, Q_2)$ is a permutation of $(1, 2)$ and

$$y_{m\alpha, s} = \frac{1}{2\eta_s} ((-\Delta\epsilon_{d\alpha} + i\Delta\Gamma_\alpha - i\eta_s) t_{m\bar{\alpha}} + 2(t_\alpha' - i\Gamma_{\alpha\bar{\alpha}}) t_{m\alpha}), \quad (16)$$

$$Z_{\alpha, k_1 k_2}^{(\ell_1 \ell_2)} = \frac{U}{k_1 + k_2 - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}} (e_{\alpha, k_1}^{(\ell_1)} e_{\bar{\alpha}, k_2}^{(\ell_2)} - e_{\alpha, k_2}^{(\ell_2)} e_{\bar{\alpha}, k_1}^{(\ell_1)}), \quad (17)$$

$$Z_{\alpha, k_1 k_2, s}^{(\ell_1 \ell_2)}(x) = Z_{\alpha, k_1 k_2}^{(\ell_1 \ell_2)} e^{i(\bar{\epsilon}_d - i\bar{\Gamma} + \frac{1}{2}\eta_s)x} \theta(-x) \quad (18)$$

with $\eta_\pm = \pm\eta$, where η is given in Eq. (11).

The first term of each eigenfunction in Eqs. (15) is the Slater determinant of the one-electron scattering eigenfunctions, which corresponds to the noninteracting case. The second term of the eigenfunction $g_{m_1 m_2, k_1 k_2}^{(\ell_1 \ell_2)}(x_1, x_2)$ corresponds to *two-body bound states* since the term exponentially decays as $e^{-\kappa_{\pm}|x_1 - x_2|}$ with the distance $|x_1 - x_2|$ between the two electrons. Here $\kappa_{\pm} = \bar{\Gamma} \mp \text{Re}(\eta)/2$ is the binding strength of the two-body bound states. We note that the complete eigenstate in Eq. (12) is stationary in time since the total energy eigenvalue E is real. Similarly, the second term of the eigenfunction $e_{m_1 \alpha, k_1 k_2}^{(\ell_1 \ell_2)}(x_1)$ corresponds to two-body bound states of two electrons one of which is on the QDs.

We observe an effect of quantum interference in the binding strength κ_{\pm} of the two-body bound states. Let us consider the following three cases: a parallel DQD system ($t_{11} = t_{12} = t_{21} = t_{22} = t$, $t' = 0$) [44], a serial DQD system ($t_{11} = t_{22} = \sqrt{2}t$, $t_{12} = t_{21} = 0$) and a T-shaped DQD system ($t_{11} = t_{21} = \sqrt{2}t$, $t_{12} = t_{22} = 0$). The binding strength κ_{\pm} is explicitly given by

$$\kappa_{\pm} = \begin{cases} \Gamma \mp \frac{1}{2} \text{Re}\left(\sqrt{-\Delta\epsilon_d^2 + 4\Gamma^2}\right) & \text{for the parallel DQD system,} \\ \Gamma \mp \frac{1}{2} \text{Re}\left(\sqrt{-\Delta\epsilon_d^2 - 4|t'|^2}\right) = \Gamma & \text{for the serial DQD system,} \\ \Gamma \mp \frac{1}{2} \text{Re}\left(\sqrt{(i\Delta\epsilon_d + 2\Gamma)^2 - 4|t'|^2}\right) & \text{for the T-shaped DQD system,} \end{cases} \quad (19)$$

where $\Delta\epsilon_d = \epsilon_{d1} - \epsilon_{d2}$ and $\Gamma = |t|^2$. The behavior of the binding strength κ_{\pm} as a function in the detuning $\Delta\epsilon_d$ in each case is illustrated in Fig. 2.

For the parallel DQD system, two binding strengths appear in the regime $0 < \Delta\epsilon_d < 2\Gamma$ but merge into $\kappa_{\pm} = \Gamma$ at $\Delta\epsilon_d = 2\Gamma$. For the T-shaped DQD system, two binding strengths appear for $\Delta\epsilon_d > 0$ and each monotonically increases/decreases with the detuning $\Delta\epsilon_d$. We can regard the existence of two different binding strengths as an effect of the quantum interference brought by the two paths connecting the left and the right leads. For the serial DQD system, on the other hand, only one binding strength appears, which takes a constant value $\kappa_{\pm} = \Gamma$. In this case, there is indeed only one path connecting the left and the right leads.

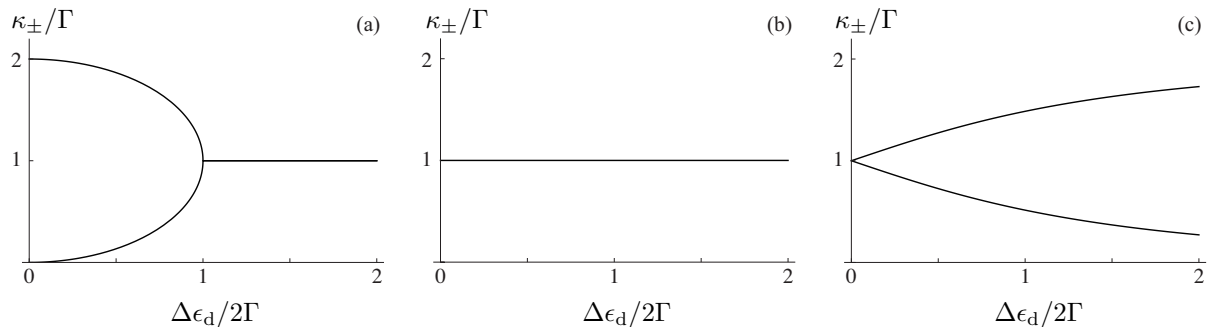


Figure 2. Binding strength κ_{\pm} of the two-body bound states as a function in the detuning $\Delta\epsilon_d$: (a) Parallel DQD system ($t_{11} = t_{12} = t_{21} = t_{22} = 1$, $t' = 0$); (b) Serial DQD system ($t_{11} = t_{22} = t' = \sqrt{2}$, $t_{12} = t_{21} = 0$); (c) T-shaped DQD system ($t_{11} = t_{21} = t' = \sqrt{2}$, $t_{12} = t_{22} = 0$).

We can understand the appearance of the two-body bound states as a consequence of the resonances of the one-body scattering. Let us inspect the set of wave numbers characterizing each term of the eigenfunctions in Eqs. (15). Due to the interaction, the wave-number set $\{k_1, k_2\}$ of the incident plane waves is not conserved and, as is expressed in the second terms

of Eq. (15a), is scattered to the set $\{\bar{\epsilon}_d - i\bar{\Gamma} + i\eta_s/2, E - \bar{\epsilon}_d + i\bar{\Gamma} - i\eta_s/2\}$, which is a pair of the resonant energy and the corresponding anti-resonant energy. Thus the unphysical resonant states “emerge” as physical two-body bound states. We note that such imaginary parts of the wave numbers are allowed since the total energy eigenvalue E is real.

We also find a resonant pole at $E = 2\bar{\epsilon}_d + U - 2i\bar{\Gamma}$ in the amplitude $Z_{\alpha, k_1 k_2}^{(\ell_1 \ell_2)}$ of the terms of the two-body bound states in Eqs. (15). We call it a *two-body resonance* since the resonant energy $E = 2\bar{\epsilon}_d + U - 2i\bar{\Gamma}$ cannot be expressed by the sum of the one-body resonant energies. The two-body resonant energy corresponds to the “eigenstate” with no incoming states (see Eq. (31) below), which is an extension of the Siegert boundary conditions [37]. The two-body resonance yields three-body bound states in the three-electron scattering eigenstates, which shall be described in the next subsection.

3.3. Three-electron case

Let us turn to the three-electron case. The three-electron eigenstates are given in the form

$$|E\rangle = \left(\sum_{l,m,n} \int_{x_1 < x_2 < x_3} dx_1 dx_2 dx_3 g_{lmn}(x_1, x_2, x_3) c_l^\dagger(x_1) c_m^\dagger(x_2) c_n^\dagger(x_3) \right. \\ \left. + \sum_{l,m,\alpha} \int_{x_1 < x_2} dx_1 dx_2 e_{lm\alpha}(x_1, x_2) c_l^\dagger(x_1) c_m^\dagger(x_2) d_\alpha^\dagger + \sum_m \int dx_1 f_{m12}(x_1) c_m^\dagger(x_1) d_1^\dagger d_2^\dagger \right) |0\rangle. \quad (20)$$

Here we assume the anti-symmetry

$$g_{lmn}(x_1, x_2, x_3) = -g_{mln}(x_2, x_1, x_3), \quad g_{lmn}(x_1, x_2, x_3) = -g_{lnm}(x_1, x_3, x_2), \quad (21a)$$

$$e_{lm\alpha}(x_1, x_2) = -e_{ml\alpha}(x_2, x_1), \quad f_{m12}(x_1) = -f_{m21}(x_1). \quad (21b)$$

The Schrödinger equations are given by

$$\left(\frac{1}{i} (\partial_1 + \partial_2 + \partial_3) - E \right) g_{lmn}(x_1, x_2, x_3) \\ + \sum_\alpha (t_{l\alpha} \delta(x_1) e_{mn\alpha}(x_2, x_3) - t_{m\alpha} \delta(x_2) e_{ln\alpha}(x_1, x_3) + t_{n\alpha} \delta(x_3) e_{lm\alpha}(x_1, x_2)) = 0, \quad (22a)$$

$$\left(\frac{1}{i} (\partial_1 + \partial_2) + \epsilon_{d\alpha} - E \right) e_{lm\alpha}(x_1, x_2) + \sum_n t_{n\alpha}^* g_{lmn}(x_1, x_2, 0) \\ + t'_\alpha e_{lm\bar{\alpha}}(x_1, x_2) + t_{l\bar{\alpha}} \delta(x_1) f_{m\alpha\bar{\alpha}}(x_2) - t_{m\bar{\alpha}} \delta(x_2) f_{l\alpha\bar{\alpha}}(x_1) = 0, \quad (22b)$$

$$\left(\frac{1}{i} \frac{d}{dx_1} + 2\bar{\epsilon}_d + U - E \right) f_{l\alpha\bar{\alpha}}(x_1) + \sum_m (t_{m\alpha}^* e_{lm\bar{\alpha}}(x_1, 0) - t_{m\bar{\alpha}}^* e_{lm\alpha}(x_1, 0)) = 0. \quad (22c)$$

We consider the three-electron scattering eigenstates satisfying the scattering boundary conditions:

$$g_{lmn, k_1 k_2 k_3}^{(\ell_1 \ell_2 \ell_3)}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_P \text{sgn}(P) \delta_{lP_1} \delta_{mP_2} \delta_{nP_3} e^{i \sum_i k_{P_i} x_i}, \quad (x_1, x_2, x_3 < 0), \quad (23)$$

where $P = (P_1, P_2, P_3)$ denotes a permutation of $(1, 2, 3)$. The exact three-electron scattering

eigenfunctions that shall be constructed in the next section are given by

$$\begin{aligned}
g_{m_1 m_2 m_3, k_1 k_2 k_3}^{(\ell_1 \ell_2 \ell_3)}(x_1, x_2, x_3) &= \sum_P \text{sgn}(P) g_{m_1, k_{P_1}}^{(\ell_{P_1})}(x_1) g_{m_2, k_{P_2}}^{(\ell_{P_2})}(x_2) g_{m_3, k_{P_3}}^{(\ell_{P_3})}(x_3) \\
&- \frac{1}{2} \sum_{\substack{P, Q \\ \alpha, s}} \text{sgn}(PQ) i t_{m_{Q_2} \alpha} y_{m_{Q_3} \alpha, s} g_{m_{Q_1}, k_{P_1}}^{(\ell_{P_1})}(x_{Q_1}) Z_{\alpha, k_{P_2} k_{P_3}, s}^{(\ell_{P_2} \ell_{P_3})}(x_{Q_2} x_{Q_3}) e^{i(k_{P_2} + k_{P_3}) x_{Q_3}} \theta(x_{Q_2}) \\
&+ \frac{U}{2\sqrt{2\pi}} \sum_{\substack{P, Q \\ \alpha, s, r}} \text{sgn}(PQ) i t_{m_{Q_1} \alpha} y_{m_{Q_2} \alpha, s} \frac{t_{\ell_{P_1} \alpha}^* y_{m_{Q_3} \bar{\alpha}, r} + t_{\ell_{P_1} \bar{\alpha}}^* y_{m_{Q_3} \alpha, r}}{(k_{P_1} - \bar{\epsilon}_d + i\Gamma - \frac{i}{2}\eta_{\bar{r}})(k_{P_1} - \bar{\epsilon}_d - U + i\Gamma - \frac{i}{2}\eta_{\bar{r}})} \\
&\times Z_{\alpha, k_{P_2} k_{P_3}, sr}^{(\ell_{P_2} \ell_{P_3})}(x_{Q_1} x_{Q_2}, x_{Q_2} x_{Q_3}) e^{i(k_{P_1} x_{Q_2} + (k_{P_2} + k_{P_3}) x_{Q_3})} \theta(x_{Q_1}) \\
&- \frac{U}{2\sqrt{2\pi}} \sum_{\substack{P, Q \\ \alpha, s, r}} \text{sgn}(PQ) i t_{m_{Q_1} \alpha} y_{m_{Q_2} \alpha, s} \frac{t_{\ell_{P_1} \alpha}^* y_{m_{Q_3} \bar{\alpha}, r} + t_{\ell_{P_1} \bar{\alpha}}^* y_{m_{Q_3} \alpha, r}}{(k_{P_1} - \bar{\epsilon}_d + i\Gamma - \frac{i}{2}\eta_{\bar{r}})(k_{P_1} - \bar{\epsilon}_d - U + i\Gamma - \frac{i}{2}\eta_{\bar{r}})} \\
&\times \tilde{Z}_{\alpha, k_{P_2} k_{P_3}, s}^{(\ell_{P_2} \ell_{P_3})}(x_{Q_1} x_{Q_2}, x_{Q_2} x_{Q_3}) e^{iE x_{Q_3}} \theta(x_{Q_1}), \tag{24a}
\end{aligned}$$

$$\begin{aligned}
e_{m_1 m_2 \alpha, k_1 k_2 k_3}^{(\ell_1 \ell_2 \ell_3)}(x_1, x_2) &= \sum_P \text{sgn}(P) g_{m_1, k_{P_1}}^{(\ell_{P_1})}(x_1) g_{m_2, k_{P_2}}^{(\ell_{P_2})}(x_2) e_{\alpha, k_{P_3}}^{(\ell_{P_3})} \\
&- \frac{1}{2} \sum_{P, Q, s} \text{sgn}(PQ) y_{m_{Q_2} \alpha, s} g_{m_{Q_1}, k_{P_1}}^{(\ell_{P_1})}(x_{Q_1}) Z_{\alpha, k_{P_2} k_{P_3}, s}^{(\ell_{P_2} \ell_{P_3})}(-x_{Q_2}) e^{i(k_{P_2} + k_{P_3}) x_{Q_2}} \\
&- \frac{1}{2} \sum_{\substack{P, Q \\ \beta, s}} \text{sgn}(PQ) i t_{m_{Q_1} \beta} y_{m_{Q_2} \beta, s} e_{\alpha, k_{P_1}}^{(\ell_{P_1})} Z_{\beta, k_{P_2} k_{P_3}, s}^{(\ell_{P_2} \ell_{P_3})}(x_{Q_1} x_{Q_2}) e^{i(k_{P_2} + k_{P_3}) x_{Q_2}} \theta(x_{Q_1}) \\
&- \frac{U}{2\sqrt{2\pi}} \sum_{P, Q, s, r} \text{sgn}(PQ) y_{m_{Q_1} \alpha, s} \frac{t_{\ell_{P_1} \alpha}^* y_{m_{Q_2} \bar{\alpha}, s} + t_{\ell_{P_1} \bar{\alpha}}^* y_{m_{Q_2} \alpha, s}}{(k_{P_1} - \bar{\epsilon}_d + i\bar{\Gamma} - \frac{i}{2}\eta_{\bar{r}})(k_{P_1} - \bar{\epsilon}_d - U + i\bar{\Gamma} - \frac{i}{2}\eta_{\bar{r}})} \\
&\times Z_{\alpha, k_{P_2} k_{P_3}, sr}^{(\ell_{P_2} \ell_{P_3})}(-x_{Q_1}, x_{Q_1} x_{Q_2}) e^{i(k_{P_1} x_{Q_1} + (k_{P_2} + k_{P_3}) x_{Q_2})} \\
&+ \frac{U}{2\sqrt{2\pi}} \sum_{P, Q, s, r} \text{sgn}(PQ) y_{m_{Q_1} \alpha, s} \frac{t_{\ell_{P_1} \alpha}^* y_{m_{Q_2} \bar{\alpha}, r} + t_{\ell_{P_1} \bar{\alpha}}^* y_{m_{Q_2} \alpha, r}}{(k_{P_1} - \bar{\epsilon}_d + i\bar{\Gamma} - \frac{i}{2}\eta_{\bar{r}})(k_{P_1} - \bar{\epsilon}_d - U + i\bar{\Gamma} - \frac{i}{2}\eta_{\bar{r}})} \\
&\times \tilde{Z}_{\alpha, k_{P_2} k_{P_3}, s}^{(\ell_{P_2} \ell_{P_3})}(-x_{Q_1}, x_{Q_1} x_{Q_2}) e^{iE x_{Q_2}}, \tag{24b}
\end{aligned}$$

$$\begin{aligned}
f_{m_1 \alpha \bar{\alpha}, k_1 k_2 k_3}^{(\ell_1 \ell_2 \ell_3)}(x_1) &= \sum_P \text{sgn}(P) g_{m_1, k_{P_1}}^{(\ell_{P_1})}(x_1) e_{\alpha, k_{P_2}}^{(\ell_{P_2})} e_{\bar{\alpha}, k_{P_3}}^{(\ell_{P_3})} + \frac{1}{2} \sum_P \text{sgn}(P) g_{m_1, k_{P_1}}^{(\ell_{P_1})}(x_1) Z_{\alpha, k_{P_2} k_{P_3}}^{(\ell_{P_2} \ell_{P_3})} \\
&- \frac{1}{2\sqrt{2\pi}} \sum_{P, s} \text{sgn}(P) \frac{t_{\ell_{P_1} \alpha}^* y_{m_1 \bar{\alpha}, s} + t_{\ell_{P_1} \bar{\alpha}}^* y_{m_1 \alpha, s}}{k_{P_1} - \bar{\epsilon}_d - U + i\bar{\Gamma} - \frac{i}{2}\eta_{\bar{s}}} Z_{\alpha, k_{P_2} k_{P_3}, s}^{(\ell_{P_2} \ell_{P_3})}(-x_1) e^{i(k_{P_2} + k_{P_3}) x_1} \\
&+ \frac{U}{2\sqrt{2\pi}} \sum_{P, s} \text{sgn}(P) \frac{t_{\ell_{P_1} \alpha}^* y_{m_1 \bar{\alpha}, s} + t_{\ell_{P_1} \bar{\alpha}}^* y_{m_1 \alpha, s}}{(k_{P_1} - \bar{\epsilon}_d + i\bar{\Gamma} - \frac{i}{2}\eta_{\bar{s}})(k_{P_1} - \bar{\epsilon}_d - U + i\bar{\Gamma} - \frac{i}{2}\eta_{\bar{s}})} \\
&\times Z_{\alpha, k_{P_2} k_{P_3}}^{(\ell_{P_2} \ell_{P_3})} e^{i(E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}) x_1} \theta(x_1), \tag{24c}
\end{aligned}$$

where

$$y_{m\alpha,s} = \frac{1}{2\eta_s} ((-\Delta\epsilon_{d\alpha} + i\Delta\Gamma_\alpha - i\eta_s)t_{m\bar{\alpha}} + 2(t'_\alpha - i\Gamma_{\alpha\bar{\alpha}})t_{m\alpha}), \quad (25a)$$

$$Z_{\alpha,k_1k_2}^{(\ell_1\ell_2)} = \frac{U}{k_1 + k_2 - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}} (e_{\alpha,k_1}^{(\ell_1)} e_{\bar{\alpha},k_2}^{(\ell_2)} - e_{\alpha,k_2}^{(\ell_2)} e_{\bar{\alpha},k_1}^{(\ell_1)}), \quad (25b)$$

$$Z_{\alpha,k_1k_2,s}^{(\ell_1\ell_2)}(x) = Z_{\alpha,k_1k_2}^{(\ell_1\ell_2)} e^{i(\bar{\epsilon}_d - i\bar{\Gamma} + \frac{i}{2}\eta_s)x} \theta(-x), \quad (25c)$$

$$Z_{\alpha,k_1k_2,sr}^{(\ell_1\ell_2)}(x,y) = Z_{\alpha,k_1k_2}^{(\ell_1\ell_2)} e^{i((\bar{\epsilon}_d - i\bar{\Gamma} + \frac{i}{2}\eta_s)x + (\bar{\epsilon}_d - i\bar{\Gamma} + \frac{i}{2}\eta_r)y)} \theta(-x)\theta(-y), \quad (25d)$$

$$\tilde{Z}_{\alpha,k_1k_2,s}^{(\ell_1\ell_2)}(x,y) = Z_{\alpha,k_1k_2}^{(\ell_1\ell_2)} e^{i((\bar{\epsilon}_d - i\bar{\Gamma} + \frac{i}{2}\eta_s)x + (2\bar{\epsilon}_d + U - 2i\bar{\Gamma})y)} \theta(-x)\theta(-y). \quad (25e)$$

The first term of each eigenfunction in Eqs. (24) corresponds to the noninteracting case. The second term of the eigenfunction $g_{m_1m_2m_3,k_1k_2k_3}^{(\ell_1\ell_2\ell_3)}(x_1, x_2, x_3)$ in Eq. (24a) is the two-body bound states which are the same as those in the two-electron case. We interpret both the third and fourth terms of the eigenfunction $g_{m_1m_2m_3,k_1k_2k_3}^{(\ell_1\ell_2\ell_3)}(x_1, x_2, x_3)$ as *three-body bound states* since, for the distance $|x_i - x_j|$ between any two of the three electrons, the third terms exponentially decay as $e^{-\kappa_\pm|x_{Q_1} - x_{Q_2}| - \kappa_\pm|x_{Q_2} - x_{Q_3}|}$ and the fourth terms also decay as $e^{-\kappa_\pm|x_{Q_1} - x_{Q_2}| - 2\bar{\Gamma}|x_{Q_2} - x_{Q_3}|}$. Similarly, three-body bound states of the three electrons one of which is on the QDs appear in the fourth and fifth terms of the eigenfunction $e_{m_1m_2\alpha,k_1k_2k_3}^{(\ell_1\ell_2\ell_3)}(x_1, x_2)$ in Eq. (24b).

In a way similar to the two-body bound states, we can also understand the three-body bound states in terms of resonances of the scattering. The wave-number set $\{k_1, k_2, k_3\}$ of the incident plane waves is scattered to the set $\{\bar{\epsilon}_d - i\bar{\Gamma} + i\eta_s/2, k_{P_1} - i(\eta_s - \eta_r)/2, k_{P_2} + k_{P_3} - \bar{\epsilon}_d + i\bar{\Gamma} - i\eta_r/2\}$ in the third term in Eq. (24a) and to the set $\{\bar{\epsilon}_d - i\bar{\Gamma} + i\eta_s/2, U + \bar{\epsilon}_d - i\bar{\Gamma} - i\eta_s/2, E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}\}$ in the fourth term in Eq. (24a). We interpret the third term in Eq. (24a) as “non-pure” three-body bound states since the resonant energies appearing in the former wave-number set are not the two-body resonant energy but the one-body (anti-)resonant energies. Here the one-body resonant energy $\bar{\epsilon}_d - i\bar{\Gamma} + i\eta_s/2$ does not pair with the one-body anti-resonant energy $k_{P_2} + k_{P_3} - \bar{\epsilon}_d + i\bar{\Gamma} - i\eta_r/2$ for $r \neq s$, which is a difference from the two-electron case. On the other hand, we interpret the fourth term in Eq. (24a) as “pure” three-body bound states since the two-body anti-resonant energy $E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}$ appears in the latter wave-number set. We remark that three-body resonances do not appear in the DQD system because only two electrons can interact at a time through the Coulomb interaction.

4. Construction of scattering eigenstates

4.1. Two-electron case

We demonstrate how to construct the two-electron scattering eigenstates in Eqs. (15). The eigenfunction $g_{lm}(x_1, x_2)$ is discontinuous at $x_1 = 0$ and $x_2 = 0$, and $e_{l\alpha}(x)$ is discontinuous at $x = 0$. In fact, by integrating the Schrödinger equations (13a) and (13b) around the discontinuous points, we obtain the matching conditions

$$g_{lm}(x, 0+) = g_{lm}(x, 0-) - i \sum_{\alpha} t_{m\alpha} e_{l\alpha}(x), \quad (26a)$$

$$e_{l\alpha}(0+) = e_{l\alpha}(0-) + it_{l\bar{\alpha}} f_{\alpha\bar{\alpha}} = 0. \quad (26b)$$

Since the Schrödinger equations (13) do not determine the values of the eigenfunctions at the discontinuous points, we assume

$$g_{lm}(x, 0) = \frac{1}{2} (g_{lm}(x, 0+) + g_{lm}(x, 0-)), \quad (27a)$$

$$e_{l\alpha}(0) = \frac{1}{2} (e_{l\alpha}(0+) + e_{l\alpha}(0-)). \quad (27b)$$

For $x \neq 0$, by using Eqs. (26a) and (27a), the coupled first-order differential equations (13b) for the eigenfunctions $e_{l\alpha}(x)$ and $e_{l\bar{\alpha}}(x)$ give the second-order differential equation

$$\partial_x^2 \tilde{e}_{l\alpha}(x) - i(\Delta\epsilon_{d\alpha} - i\Delta\Gamma_\alpha)\partial_x \tilde{e}_{l\alpha}(x) + (t_{\bar{\alpha}}^* - i\Gamma_{\alpha\bar{\alpha}})(t_{\alpha}^* - i\Gamma_{\bar{\alpha}\alpha})\tilde{e}_{l\alpha}(x) = G_{l\alpha}(x) \quad (28)$$

for $\tilde{e}_{l\alpha}(x) = e^{-i(E-\epsilon_{d\alpha}+i\Gamma_{\alpha\alpha})x} e_{l\alpha}(x)$. Here the inhomogeneous term is given by

$$G_{l\alpha}(x) = - \sum_m e^{-i(E-\epsilon_{d\alpha}+i\Gamma_{\alpha\alpha})x} ((E-\epsilon_{d\bar{\alpha}}+i\Gamma_{\bar{\alpha}\bar{\alpha}}+i\partial_x)t_{m\alpha}^* + (t_{\bar{\alpha}}^* - i\Gamma_{\alpha\bar{\alpha}})t_{m\bar{\alpha}}^*) g_{lm}(x, 0-). \quad (29)$$

General solutions for the function $e_{l\alpha}(x)$ are given by

$$e_{l\alpha}(x) = \sum_{s=\pm} C_{l\alpha,s} e^{i(E-\epsilon_{d\alpha}+i\Gamma_{\alpha\alpha}-i\lambda_{\alpha,s})x} + \sum_{s=\pm} \frac{1}{\lambda_{\alpha,s} - \lambda_{\alpha,\bar{s}}} e^{i(E-\epsilon_{d\alpha}+i\Gamma_{\alpha\alpha})x} \int_{x_0}^x dz e^{\lambda_{\alpha,s}(x-z)} G_{l\alpha}(z). \quad (30)$$

Here $C_{l\alpha,\pm}$ is an integration constant, $x_0 = -\infty$ if $x < 0$ and $x_0 = 0$ if $x > 0$, and $\lambda_{\alpha,\pm} = (i\Delta\epsilon_{d\alpha} + \Delta\Gamma_\alpha + \eta_\pm)/2$ are the roots of the characteristic equation associated with the second-order differential equation (28). By using Eqs. (26b) and (27b), we obtain

$$(E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma})f_{\alpha\bar{\alpha}} = \sum_l (t_{l\alpha}^* e_{l\bar{\alpha}}(0-) - t_{l\bar{\alpha}}^* e_{l\alpha}(0-)), \quad (31)$$

which gives

$$f_{\alpha\bar{\alpha}} = \frac{1}{E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}} \sum_l (t_{l\alpha}^* e_{l\bar{\alpha}}(0-) - t_{l\bar{\alpha}}^* e_{l\alpha}(0-)). \quad (32)$$

By using the matching conditions (26a) and (26b) and the solutions (30) and (32), we obtain the eigenfunctions satisfying arbitrary scattering boundary conditions. The construction scheme is indicated in Fig. 3.

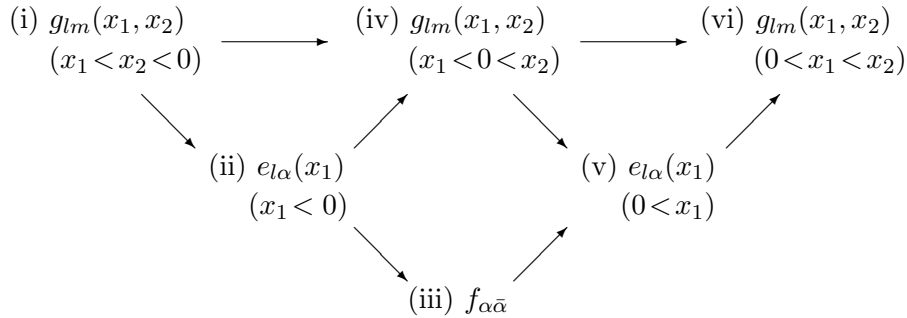


Figure 3. Construction scheme for the two-electron scattering eigenfunctions.

(i)→(ii): For $x < 0$, the scattering boundary conditions (14) read

$$g_{lm}(x, 0-) = \frac{1}{2\pi} \sum_P A_{lm,P} e^{ik_{P_1}x}, \quad A_{lm,P} = \delta_{l\ell_{P_1}} \delta_{m\ell_{P_2}} \text{sgn}(P). \quad (33)$$

Since the inhomogeneous term of the second-order differential equation (28) is given by

$$G_{l\alpha}(x) = -\frac{1}{2\pi} \sum_{P,m} ((k_{P_2} - \epsilon_{d\bar{\alpha}} + i\Gamma_{\bar{\alpha}\bar{\alpha}})t_{m\alpha}^* + (t_{\bar{\alpha}}^* - i\Gamma_{\alpha\bar{\alpha}})t_{m\bar{\alpha}}^*) A_{lm,P} e^{i(-k_{P_2} + \epsilon_{d\alpha} - i\Gamma_{\alpha\alpha})x}, \quad (34)$$

the general solution (30) is

$$e_{l\alpha}(x) = \sum_s C_{l\alpha,s} e^{i(E - \epsilon_{d\alpha} + i\Gamma_{\alpha\alpha} - i\lambda_{\alpha,s})x} + \frac{1}{\sqrt{2\pi}} \sum_{P,m} A_{lm,P} e_{\alpha,k_{P_2}}^{(m)} e^{ik_{P_1}x}. \quad (35)$$

To avoid the divergences in $x \rightarrow -\infty$, we take $C_{l\alpha,\pm} = 0$,

$$e_{l\alpha}(x) = \frac{1}{\sqrt{2\pi}} \sum_{P,m} A_{lm,P} e_{\alpha,k_{P_2}}^{(m)} e^{ik_{P_1}x}, \quad (x < 0). \quad (36)$$

(ii)→(iii): By using Eq. (32), we obtain

$$f_{\alpha\bar{\alpha}} = \frac{E - 2\bar{\epsilon}_d + 2i\bar{\Gamma}}{E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}} \sum_{P,l,m} A_{lm,P} e_{\alpha,k_{P_1}}^{(l)} e_{\bar{\alpha},k_{P_2}}^{(m)}. \quad (37)$$

(i), (ii)→(iv): For $x < 0$, we have

$$g_{lm}(x, 0+) = g_{lm}(x, 0-) - i \sum_{\alpha} t_{m\alpha} e_{l\alpha}(x) = \frac{1}{\sqrt{2\pi}} \sum_{P,n} A_{ln,P} e^{ik_{P_1}x} g_{m,k_{P_2}}^{(n)}(0+). \quad (38)$$

Then, for $x_1 < 0 < x_2$, we obtain

$$g_{lm}(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \sum_{P,n} A_{ln,P} e^{ik_{P_1}x_1} g_{m,k_{P_2}}^{(n)}(x_2). \quad (39)$$

(iv), (iii)→(v): For $x > 0$, we have

$$g_{lm}(0-, x) = \frac{1}{\sqrt{2\pi}} \sum_{P,n} A_{ln,P} g_{m,k_{P_2}}^{(n)}(x). \quad (40)$$

The general solution (30) is

$$e_{l\alpha}(x) = \sum_{P,m,n} A_{mn,P} g_{l,k_{P_1}}^{(m)}(x) e_{\alpha,k_{P_2}}^{(n)} + \sum_s \tilde{C}_{l\alpha,s} e^{i(E - \epsilon_{d\alpha} + i\Gamma_{\alpha\alpha} - i\lambda_{\alpha,s})x}. \quad (41)$$

We keep the integration constant $\tilde{C}_{l\alpha,\pm}$ since the second term is not divergent in $x \rightarrow \infty$. The integration constants $\tilde{C}_{l\bar{\alpha},\pm}$ satisfy the relation

$$\tilde{C}_{l\bar{\alpha},\pm} = -\frac{\lambda_{\alpha,\pm} \tilde{C}_{l\alpha,\pm}}{i(t_{\bar{\alpha}}^* - i\Gamma_{\alpha\bar{\alpha}})}. \quad (42)$$

Through the matching condition in Eq. (26b) for $e_{l\alpha}(x)$, we obtain

$$\sum_s \tilde{C}_{l\alpha,s} = \frac{it_{l\bar{\alpha}}U}{E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}} \sum_{P,m,n} A_{mn,P} e_{\alpha,k_{P_1}}^{(m)} e_{\bar{\alpha},k_{P_2}}^{(n)}. \quad (43)$$

By replacing α with $\bar{\alpha}$, we have

$$\sum_s \tilde{C}_{l\bar{\alpha},s} = -\sum_s \frac{\lambda_{\alpha,s} \tilde{C}_{l\alpha,s}}{i(t_{\bar{\alpha}}^* - i\Gamma_{\alpha\bar{\alpha}})} = \frac{it_{l\alpha}U}{E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}} \sum_{P,m,n} A_{mn,P} e_{\bar{\alpha},k_{P_1}}^{(m)} e_{\alpha,k_{P_2}}^{(n)}. \quad (44)$$

By solving the coupled equations (43) and (44) for $\tilde{C}_{l\alpha,+}$ and $\tilde{C}_{l\alpha,-}$, we obtain

$$\tilde{C}_{l\alpha,s} = \frac{1}{\eta_s} \frac{U}{E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}} (i\lambda_{\bar{\alpha},s} t_{l\bar{\alpha}} - (t'_\alpha - i\Gamma_{\alpha\bar{\alpha}}) t_{l\alpha}) \sum_{P,m,n} A_{mn,P} e_{\alpha,k_{P_1}}^{(m)} e_{\bar{\alpha},k_{P_2}}^{(n)}. \quad (45)$$

Hence, for $x > 0$, we have

$$\begin{aligned} e_{l\alpha}(x) &= \sum_{P,m,n} A_{mn,P} g_{l,k_{P_1}}^{(m)}(x) e_{\alpha,k_{P_2}}^{(n)} \\ &+ \frac{U}{E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}} \sum_s \frac{1}{\eta_s} (i\lambda_{\bar{\alpha},s} t_{l\bar{\alpha}} - (t'_\alpha - i\Gamma_{\alpha\bar{\alpha}}) t_{l\alpha}) \sum_{P,m,n} A_{mn,P} e_{\alpha,k_{P_1}}^{(m)} e_{\bar{\alpha},k_{P_2}}^{(n)} e^{i(E - \epsilon_{d\alpha} + i\Gamma_{\alpha\alpha} - i\lambda_{\alpha,s})x}. \end{aligned} \quad (46)$$

(iv), (v) \rightarrow (vi): For $x > 0$, we have

$$\begin{aligned} g_{lm}(x, 0+) &= g_{lm}(x, 0-) - i \sum_{\alpha} t_{m\alpha} e_{l\alpha}(x) \\ &= - \sum_{P,r,n} A_{rn,P} g_{m,k_{P_1}}^{(r)}(0+) g_{l,k_{P_2}}^{(n)}(x) - i \sum_{\alpha,s=\pm} t_{m\alpha} \tilde{C}_{l\alpha,s} e^{i(E - \epsilon_{d\alpha} + i\Gamma_{\alpha\alpha} - i\lambda_{\alpha,s})x}. \end{aligned} \quad (47)$$

Then, for $0 < x_1 < x_2$, we obtain

$$\begin{aligned} g_{lm}(x_1, x_2) &= \sum_{P,r,n} A_{rn,P} g_{l,k_{P_1}}^{(r)}(x_1) g_{m,k_{P_2}}^{(n)}(x_2) \\ &+ i \sum_{\alpha,s} t_{l\alpha} \tilde{C}_{m\alpha,s} e^{i((\epsilon_{d\alpha} - i\Gamma_{\alpha\alpha} + i\lambda_{\alpha,s})x_1 + (E - \epsilon_{d\alpha} + i\Gamma_{\alpha\alpha} - i\lambda_{\alpha,s})x_2)}. \end{aligned} \quad (48)$$

By inserting $A_{lm,P} = \delta_{l\ell_{P_1}} \delta_{m\ell_{P_2}} \text{sgn}(P)$ and $\tilde{C}_{m\alpha,s}$ in Eq. (45), we obtain

$$\begin{aligned} g_{lm}(x_1, x_2) &= \sum_P \text{sgn}(P) g_{l,k_{P_1}}^{(\ell_{P_1})}(x_1) g_{m,k_{P_2}}^{(\ell_{P_2})}(x_2) \\ &+ \frac{iU}{E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma}} \sum_P \text{sgn}(P) \sum_{\alpha,s} \frac{1}{\eta_s} t_{l\alpha} (i\lambda_{\bar{\alpha},s} t_{m\bar{\alpha}} - (t'_\alpha - i\Gamma_{\alpha\bar{\alpha}}) t_{m\alpha}) e_{\alpha,k_{P_1}}^{(\ell_{P_1})} e_{\bar{\alpha},k_{P_2}}^{(\ell_{P_2})} \\ &\times e^{i((\epsilon_{d\alpha} - i\Gamma_{\alpha\alpha} + i\lambda_{\alpha,s})x_1 + (E - \epsilon_{d\alpha} + i\Gamma_{\alpha\alpha} - i\lambda_{\alpha,s})x_2)}. \end{aligned} \quad (49)$$

Through the anti-symmetrization with respect to the variables x_1 and x_2 , we obtain the exact two-electron scattering eigenfunctions in Eqs. (15).

4.2. Three-electron case

We now describe an outline of the construction of the three-electron scattering eigenstates in Eqs. (24). It is clear from the Schrödinger equations (22) that the eigenfunction $g_{lmn}(x_1, x_2, x_3)$ is discontinuous at $x_i = 0$, ($i = 1, 2, 3$) and $e_{lm\alpha}(x_1, x_2)$ is also discontinuous at $x_i = 0$, ($i = 1, 2$). Then, in a way similar to the one- and two-electron cases, we assume

$$g_{lmn}(x_1, x_2, 0) = \frac{1}{2} (g_{lmn}(x_1, x_2, 0+) + g_{lmn}(x_1, x_2, 0-)), \quad (50a)$$

$$e_{lm\alpha}(x_1, 0) = \frac{1}{2} (e_{lm\alpha}(x_1, 0+) + e_{lm\alpha}(x_1, 0-)). \quad (50b)$$

We note that the function $f_{l\alpha\bar{\alpha}}(x_1)$ is a continuous function. The matching conditions for the functions $g_{lmn}(x_1, x_2, x_3)$ and $e_{lm\alpha}(x_1, x_2)$ are

$$g_{lmn}(x_1, x_2, 0+) - g_{lmn}(x_1, x_2, 0-) + i \sum_{\alpha} t_{n\alpha} e_{lm}(x_1, x_2) = 0, \quad (51a)$$

$$e_{lm\alpha}(x_1, 0+) - e_{lm\alpha}(x_1, 0-) - i t_{m\bar{\alpha}} f_{l\alpha\bar{\alpha}}(x_1) = 0. \quad (51b)$$

General solutions for the functions $e_{lm\alpha}(x_1, x_2)$ and $f_{l\alpha\bar{\alpha}}(x_1)$ are given by

$$e_{lm\alpha}(x_1, x_2) = \sum_{s=\pm} C_{lm\alpha,s}(x_{12}) e^{i(E - \epsilon_{d\alpha} + i\Gamma_{\alpha\alpha} - i\lambda_{\alpha,s})\bar{x}} + \sum_{s=\pm} \frac{1}{\lambda_{\alpha,s} - \lambda_{\alpha,\bar{s}}} e^{i(E - \epsilon_{d\alpha} + i\Gamma_{\alpha\alpha})\bar{x}} \int_{x_0}^{\bar{x}} e^{\lambda_{\alpha,s}(\bar{x}-z)} G_{lm\alpha}\left(z + \frac{x_{12}}{2}, z - \frac{x_{12}}{2}\right) dz, \quad (52)$$

$$f_{l\alpha\bar{\alpha}}(x_1) = C_{l\alpha\bar{\alpha}} e^{i(E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma})x_1} - i \sum_m \int_{x_0}^{x_1} dz e^{i(E - 2\bar{\epsilon}_d - U + 2i\bar{\Gamma})(x_1 - z)} (t_{m\alpha}^* e_{lm\bar{\alpha}}(z, 0-) - t_{m\bar{\alpha}}^* e_{lm\alpha}(z, 0-)). \quad (53)$$

Here we put $\bar{x} = (x_1 + x_2)/2$ and $x_{12} = x_1 - x_2$ and $x_0 = -\infty$ if $x < 0$ and $x_0 = 0$ if $x > 0$. The function $C_{l\alpha,\pm}(x_{12})$ is a function in the variable x_{12} and $C_{l\alpha\bar{\alpha}}$ is an integration constant. We also introduce

$$G_{lm\alpha}(z_1, z_2) = - \sum_n e^{i(-E + \epsilon_{d\alpha} - i\Gamma_{\alpha\alpha})\bar{z}} (t_{n\alpha}^* (E - \epsilon_{d\bar{\alpha}} + i\Gamma_{\bar{\alpha}\bar{\alpha}} + i\partial_{\bar{z}}) + t_{n\bar{\alpha}}^* (t'_{\alpha} - i\Gamma_{\alpha\bar{\alpha}})) g_{lmn}(z_1, z_2, 0-), \quad (54)$$

where $\bar{z} = (z_1 + z_2)/2$. In a way similar to the two-electron case, we construct the three-electron scattering eigenstates. The construction scheme is indicated in Fig. 4.

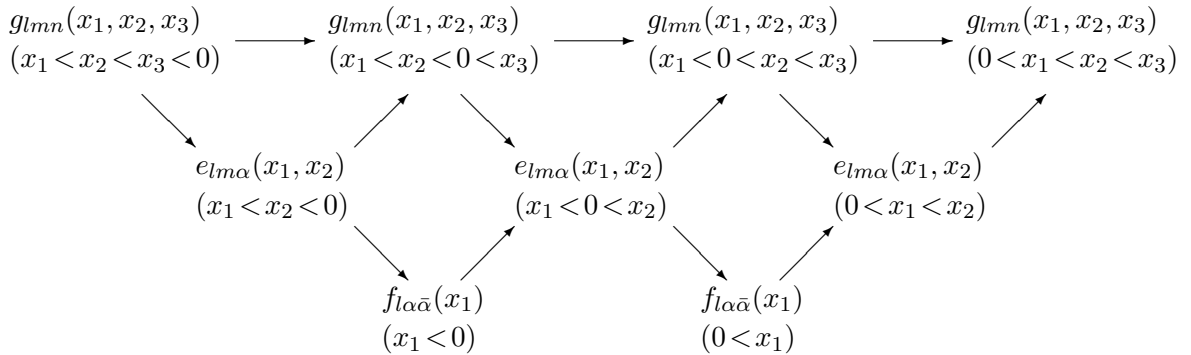


Figure 4. Construction scheme for three-electron scattering eigenfunctions.

5. Summary

We have constructed exact two- and three-electron scattering eigenstates for the DQD system with an interdot Coulomb interaction. The construction can be extended to the N -electron case straightforwardly. We have found that, due to the interdot Coulomb interaction, some of the incident plane waves are scattered to two- and three-body bound states. The binding strength

of the many-body bound states is affected by the arrangement of the two QDs, which is an effect of the quantum interference between different electron pathways. Thus we have observed an interplay of the interaction and quantum interference in the many-body bound states.

Let us summarize the relation of the many-body resonances and the many-body bound states in the DQD system. As is indicated in Table 1, we find the one- and two-body resonant poles in the scattering eigenstates, which produce the two- and three-body bound states. We also find that there is no three-body resonances in the three-electron scattering eigenstates, which implies that pure four-body bound states that are not a combination of the two- and three-body bound states do not appear in the scattering eigenstates for electrons more than three.

Table 1. Relation of many-body resonances and many-body bound states in the N -electron scattering eigenstates

N	Positions of resonant poles	Terms of many-body bound states
1	$k_1 = \bar{\epsilon}_d - i\bar{\Gamma} + \frac{i}{2}\eta_s$	
2	$k_i = \bar{\epsilon}_d - i\bar{\Gamma} + \frac{i}{2}\eta_s$ $k_1 + k_2 = 2\bar{\epsilon}_d + U - 2i\bar{\Gamma}$	$e^{(-\bar{\Gamma} + \frac{1}{2}\text{Re}(\eta_s)) x_1 - x_2 }$
3	$k_i = \bar{\epsilon}_d - i\bar{\Gamma} + \frac{i}{2}\eta_s$ $k_i = \bar{\epsilon}_d + U - i\bar{\Gamma} + \frac{i}{2}\eta_s$ $k_i + k_j = 2\bar{\epsilon}_d + U - 2i\bar{\Gamma}$	$e^{(-\bar{\Gamma} + \frac{1}{2}\text{Re}(\eta_s)) x_1 - x_2 }$ $e^{(-\bar{\Gamma} + \frac{1}{2}\text{Re}(\eta_s)) x_i - x_j + (-\bar{\Gamma} + \frac{1}{2}\text{Re}(\eta_r)) x_j - x_l }$ $e^{(-\bar{\Gamma} + \frac{1}{2}\text{Re}(\eta_s)) x_i - x_j - 2\bar{\Gamma} x_j - x_l }$

Acknowledgments

This work was supported by JSPS KAKENHI Grants Numbers 26400409, 15K05200 and 15K05207.

References

- [1] Cronenwett S M, Oosterkamp T H and Kouwenhoven L P 1998 *Science* **281** 540
- [2] Goldhaber-Gordon D, Shtrikman H, Mahalu D, Abusch-Magder D, Meirav U and Kastner M A 1998 *Nature (London)* **391** 156
- [3] W G van der Wiel, S De Franceschi, T Fujisawa, J M Elzerman, S Tarucha, and L P Kouwenhoven 2000 *Science* **289** 2105
- [4] Ng T K and Lee P A 1988 *Phys. Rev. Lett.* **61** 1768
- [5] Wingreen N S and Meir Y 1994 *Phys. Rev. B* **49** 11040
- [6] Costi T A, Hewson A C and Zlatić V 1994 *J. Phys. Condens. Matter* **6** 2519
- [7] Aono T, Eto M and Kawamura K 1998 *J. Phys. Soc. Jpn.* **67** 1860–1863
- [8] Georges A and Meir Y 1999 *Phys. Rev. Lett.* **82** 3508–3511
- [9] López R, Aguado R and Platero G 2002 *Phys. Rev. Lett.* **89** 136802
- [10] Kim T S and Hershfield S 2001 *Phys. Rev. B* **63** 245326
- [11] Zhang G M, Lü R, Liu Z R and Yu L 2005 *Phys. Rev. B* **72** 073308
- [12] Tanaka Y and Kawakami N 2005 *Phys. Rev. B* **72** 085304
- [13] Cornaglia P S and Grepel D R 2005 *Phys. Rev. B* **71** 075305
- [14] Boese D, Hofstetter W and Schoeller H 2002 *Phys. Rev. B* **66** 125315
- [15] Karrasch C, Enss T and Meden V 2006 *Phys. Rev. B* **73** 235337
- [16] Kashcheyevs V, Schiller A, Aharony A and Entin-Wohlman O 2007 *Phys. Rev. B* **75** 115313
- [17] Hübel A, Held K, Weis J and v Klitzing K 2008 *Phys. Rev. Lett.* **101** 186804
- [18] Okazaki Y, Sasaki S and Muraki K 2011 *Phys. Rev. B* **84** 161305
- [19] Büsser C A, Feiguin A E and Martins G B 2012 *Phys. Rev. B* **85** 241310(R)

- [20] Amasha S, Keller A J, Rau I G, Carmi A, Katine J A, Shtrikman H, Oreg Y and Goldhaber-Gordon D 2013 *Phys. Rev. Lett.* **110** 046406
- [21] Tosi L, Roura-Bas P and Aligia A A 2013 *Phys. Rev. B* **88** 235427
- [22] Nishikawa Y, Hewson A C, Crow D J G and Bauer J 2013 *Phys. Rev. B* **88** 245130
- [23] Meden V and Marquardt F 2006 *Phys. Rev. Lett.* **96** 146801
- [24] Nilsson H A, Karlström O, Larsson M, Caroff P, Pedersen J N, Samuelson L, Wacker A, Wernersson L E and Xu H Q 2010 *Phys. Rev. Lett.* **104** 186804
- [25] Imry Y 2002 *Introduction to Mesoscopic Physics* 2nd ed (Oxford Univ. Pr., Oxford)
- [26] Datta S 1995 *Electronic Transport in Mesoscopic Systems* (Cambridge University Press, Cambridge)
- [27] Nishino A, Imamura T and Hatano N 2009 *Phys. Rev. Lett.* **102** 146803
- [28] Nishino A, Imamura T and Hatano N 2011 *Phys. Rev. B* **83** 035306
- [29] Nishino A, Hatano N and Ordoez G 2015 *Phys. Rev. B* **91** 045140
- [30] Doyon B 2007 *Phys. Rev. Lett.* **99** 076806
- [31] Golub A 2007 *Phys. Rev. B* **76** 193307
- [32] Mehta P 2006 Ph.D. thesis Rutgers University
- [33] Boulat E, Saleur H and Schmitteckert P 2008 *Phys. Rev. Lett.* **101** 140601
- [34] C Karrasch, S Andergassen, M Pletyukhov, D Schuricht, L Borda, V Meden, and H Schoeller 2010 *Eur. Phys. Lett.* **90** 30003
- [35] S Andergassen, M Pletyukhov, D Schuricht, H Schoeller, and L Borda 2011 *Phys. Rev. B* **83** 205103
- [36] Gamow G 1928 *Z. Phys. A* **51** 204
- [37] Siegert A J F 1939 *Phys. Rev.* **56** 750
- [38] Peierls R E 1959 *Proc. Roy. Soc. London A* **253** 16
- [39] Brändas E E and Elandar N 1989 *Resonances* (Springer-Verlag, Berlin, Heidelberg)
- [40] Kukulín V I, Krasnopol'sky V M and Horáček J 1989 *Theory of Resonances: Principles and Applications* (Kluwer Academic Publishers, Dordrecht)
- [41] Hatano N, Sasada K, Nakamura H and Petrosky T 2008 *Prog. Theor. Phys.* **119** 187
- [42] Imamura T, Nishino A and Hatano N 2009 *Phys. Rev. B* **80** 245323
- [43] Zheng H, Gauthier D J and Baranger H U 2010 *Phys. Rev. A* **82** 063816
- [44] Nishino A, Imamura T and Hatano N 2012 *J. Phys.: Conf. Ser.* **343** 012087