Biquaternion Construction of SL(2,C) Yang-Mills Instantons

Jen-Chi Lee

Department of Electrophysics, National Chiao-Tung University, Hsinchu, Taiwan, R.O.C.

E-mail: `jcclee@cc.nctu.edu.tw`

Abstract. We use biquaternion to construct SL(2,C) ADHM Yang-Mills instantons. The solutions contain 16k-6 moduli parameters for the kth homotopy class, and include as a subset the SL(2,C) (M,N) instanton solutions constructed previously. In constrast to the SU(2) instantons, the SL(2,C) instantons inhereit jumping lines or singulariries which are not gauge artifacts and can not be gauged away.

1. Introduction

The classical exact solutions of Euclidean $SU(2)$ (anti)self-dual Yang-Mills (SDYM) equation were intensively studied by pure mathematicians and theoretical physicists in 1970s. The first BPST 1-instanton solution [1] with 5 moduli parameters was found in 1975. The CFTW k-instanton solutions [2] with $5k$ moduli parameters were soon constructed, and then the number of moduli parameters of the solutions for each homotopy class k was extended to $5k + 4$ (5,13 for k = 1,2) [3] based on the conformal symmetry of massless pure YM equation. The complete solutions with $8k - 3$ moduli parameters for each k-th homotopy class were finally worked out in 1978 by mathematicians ADHM [4] using theory in algebraic geometry. Through an one to one correspondence between anti-self-dual SU(2)-connections on S^4 and holomorphic vector bundles on CP^3, ADHM converted the highly nontrivial anti-SDYM equations into a much more simpler system of quadratic algebraic equations in quaternions. The explicit closed form of the complete solutions for $k = 2, 3$ had been worked out [5].

There are many important applications of instantons to algebraic geometry and quantum field theory. One important application of instantons in algebraic geometry was the classification of four-manifolds [6]. On the physics side, the non-perturbative instanton effect in QCD resolved the $U(1)_A$ problem [7]. Another important application of YM instantons in quantum field theory was the introduction of θ- vacua [8] in nonperturbative QCD, which created the strong CP problem.

In addition to $SU(2)$, the ADHM construction has been generalized to the cases of $SU(N)$ SDYM and many other SDYM theories with compact Lie groups [5, 9]. In this talk we are going to consider the classical solutions of non-compact $SL(2, C)$ SDYM system. YM theory based on $SL(2, C)$ was first discussed in 1970s [10, 11]. It was found that the complex $SU(2)$ YM field configurations can be interpreted as the real field configurations in $SL(2, C)$ YM theory. However, due to the non-compactness of $SL(2, C)$, the Cartan-Killing form or group metric of $SL(2, C)$ is not positive definite. Thus the action integral and the Hamiltonian of non-compact $SL(2, C)$ YM theory may not be positve. Nevertheless, there are still important motivations to study $SL(2,C)$ SDYM theory. For example, it was shown that the $4D$ $SL(2,C)$ SDYM equation can be dimensionally reduced to many important $1+1$ dimensional integrable systems [12], such as the KdV equation and the nonlinear Schrodinger equation.

2. SL(2,C) SDYM Equation

We first briefly review the $SL(2,C)$ YM theory. It was shown that [10] there are two linearly independent choices of $SL(2,C)$ group metric

$$g^a = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, g^b = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{2.1}$$

where I is the 3×3 unit matrix. In general, we can choose

$$g = \cos\theta g^a + \sin\theta g^b \tag{2.2}$$

where $\theta =$ real constant. Note that the metric is not positive definite due to the non-compactness of $SL(2,C)$. On the other hand, it was shown that $SL(2,C)$ group can be decomposed such that [13]

$$SL(2,C) = SU(2)\cdot P, P \in H \tag{2.3}$$

where $SU(2)$ is the maximal compact subgroup of $SL(2,C)$, $P \in H$ (not a group) and $H = \{P|P$ is Hermitain, positive definite, and $detP = 1\}$. The parameter space of H is a noncompact space R^3. The third homotopy group is thus [13]

$$\pi_3[SL(2,C)] = \pi_3[S^3 \times R^3] = \pi_3(S^3)\cdot\pi_3(R^3) = Z\cdot I = Z \tag{2.4}$$

where I is the identity group, and Z is the integer group.

On the other hand, Wu and Yang [10] have shown that a complex $SU(2)$ gauge field is related to a real $SL(2,C)$ gauge field. Starting from $SU(2)$ complex gauge field formalism, we can write down all the $SL(2,C)$ field equations. Let

$$G^a_\mu = A^a_\mu + iB^a_\mu \tag{2.5}$$

and, for convenience, we set the coupling constant $g = 1$. The complex field strength is defined as

$$F^a_{\mu\nu} \equiv H^a_{\mu\nu} + iM^a_{\mu\nu}, a, b, c = 1, 2, 3 \tag{2.6}$$

where

$$\begin{aligned} H^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}(A^b_\mu A^c_\nu - B^b_\mu B^c_\nu), \\ M^a_{\mu\nu} &= \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + \epsilon^{abc}(A^b_\mu B^c_\nu - A^b_\mu B^c_\nu), \end{aligned} \tag{2.7}$$

then $SL(2,C)$ Yang-Mills equation can be written as

$$\begin{aligned} \partial_\mu H^a_{\mu\nu} + \epsilon^{abc}(A^b_\mu H^c_{\mu\nu} - B^b_\mu M^c_{\mu\nu}) = 0, \\ \partial_\mu M^a_{\mu\nu} + \epsilon^{abc}(A^b_\mu M^c_{\mu\nu} - B^b_\mu H^c_{\mu\nu}) = 0. \end{aligned} \tag{2.8}$$

The $SL(2,C)$ SDYM equations are

$$\begin{aligned} H^a_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}H_{\alpha\beta}, \\ M^a_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}M_{\alpha\beta}. \end{aligned} \tag{2.9}$$

The Yang-Mills Equation above can be derived from the following Lagrangian

$$L_\theta = \frac{1}{4}[F^i_{\mu\nu}]^T g_{ij}[F^j_{\mu\nu}] = \cos\theta(\frac{1}{4}H^a_{\mu\nu}H^a_{\mu\nu} - \frac{1}{4}M^a_{\mu\nu}M^a_{\mu\nu}) + \sin\theta(\frac{1}{2}H^a_{\mu\nu}M^a_{\mu\nu}) \quad (2.10)$$

where $F^k_{\mu\nu} = H^k_{\mu\nu}$ and $F^{3+k}_{\mu\nu} = M^k_{\mu\nu}$ for $k = 1, 2, 3$. Note that L_θ is indefinite for any real value θ. We shall only consider the particular case for $\theta = 0$ in this talk, i.e.

$$L = \frac{1}{4}(H^a_{\mu\nu}H^a_{\mu\nu} - M^a_{\mu\nu}M^a_{\mu\nu}), \quad (2.11)$$

for the action density in discussing the homotopic classifications of our solutions.

3. Biquaternion construction of $SL(2, C)$ YM Instantons

Instead of quaternion in the $Sp(1)$ $(= SU(2))$ ADHM construction, we will use *biquaternion* to construct $SL(2, C)$ SDYM instantons. A quaternion x can be written as

$$x = x_\mu e_\mu, \;\; x_\mu \in R, \;\; e_0 = 1, e_1 = i, e_2 = j, e_3 = k \quad (3.12)$$

where e_1, e_2 and e_3 anticommute and obey

$$e_i \cdot e_j = -e_j \cdot e_i = \epsilon_{ijk}e_k; \;\; i, j, k = 1, 2, 3, \quad (3.13)$$

$$e_1^2 = -1, e_2^2 = -1, e_3^2 = -1. \quad (3.14)$$

A (ordinary) biquaternion (or complex-quaternion) z can be written as

$$z = z_\mu e_\mu, \;\; z_\mu \in C, \quad (3.15)$$

which will be used in this talk. Occasionally z can be written as

$$z = x + yi \quad (3.16)$$

where x and y are quaternions and $i = \sqrt{-1}$, not to be confused with e_1 in Eq.(3.12). For biquaternion, the biconjugation [14]

$$z^{\circledast} = z_\mu e^\dagger_\mu = z_0e_0 - z_1e_1 - z_2e_2 - z_3e_3 = x^\dagger + y^\dagger i, \quad (3.17)$$

will be heavily used in this talk. In contrast to the real number norm square of a quaternion, the norm square of a biquarternion used in this talk is defined to be

$$|z|^2_c = z^{\circledast}z = (z_0)^2 + (z_1)^2 + (z_2)^2 + (z_3)^2 \quad (3.18)$$

which is a *complex* number in general as a subscript c is used in the norm.

We are now ready to proceed the construction of $SL(2, C)$ instantons. We begin by introducing the $(k + 1) \times k$ biquarternion matrix $\Delta(x) = a + bx$

$$\Delta(x)_{ab} = a_{ab} + b_{ab}x, \;\; a_{ab} = a^\mu_{ab}e_\mu, b_{ab} = b^\mu_{ab}e_\mu \quad (3.19)$$

where a^μ_{ab} and b^μ_{ab} are complex numbers, and a_{ab} and b_{ab} are biquarternions. The biconjugation of the $\Delta(x)$ matrix is defined to be

$$\Delta(x)^{\circledast}_{ab} = \Delta(x)^\mu_{ba}e^\dagger_\mu = \Delta(x)^0_{ba}e_0 - \Delta(x)^1_{ba}e_1 - \Delta(x)^2_{ba}e_2 - \Delta(x)^3_{ba}e_3. \quad (3.20)$$

In contrast to the of $SU(2)$ instantons, the quadratic condition of $SL(2, C)$ instantons reads

$$\Delta(x)^{\circledast}\Delta(x) = f^{-1} = \text{symmetric, non-singular } k\times k \text{ matrix for } x \notin J, \tag{3.21}$$

from which we can deduce that $a^{\circledast}a, b^{\circledast}a, a^{\circledast}b$ and $b^{\circledast}b$ are all symmetric matrices. We stress here that it will turn out the choice of *biconjugation* operation is crucial for the follow-up discussion in this work. On the other hand, for $x \in J$, $\det \Delta(x)^{\circledast}\Delta(x) = 0$. The set J is called singular locus or "jumping lines" in the mathematical literatures and was discussed in [15]. In contrast to the $SL(2,C)$ instantons, there are no jumping lines for the case of $SU(2)$ instantons. In the $Sp(1)$ quaternion case, the symmetric condition on f^{-1} means f^{-1} is real. For the $SL(2,C)$ biquaternion case, however, it can be shown that symmetric condition on f^{-1} implies f^{-1} is *complex.*

To construct the self-dual gauge field, we introduce a $(k+1)\times 1$ dimensional biquaternion vector $v(x)$ satisfying the following two conditions

$$v^{\circledast}(x)\Delta(x) = 0, \tag{3.22}$$

$$v^{\circledast}(x)v(x) = 1. \tag{3.23}$$

Note that $v(x)$ is fixed up to a $SL(2,C)$ gauge transformation

$$v(x) \longrightarrow v(x)g(x), \quad g(x) \in \ 1\times 1 \text{ Biquaternion}. \tag{3.24}$$

Note also that in general a $SL(2,C)$ matrix can be written in terms of a 1×1 biquaternion as

$$g = \frac{q_\mu e_\mu}{\sqrt{q^{\circledast}q}} = \frac{q_\mu e_\mu}{|q|_c}. \tag{3.25}$$

The next step is to define the gauge field

$$G_\mu(x) = v^{\circledast}(x)\partial_\mu v(x), \tag{3.26}$$

which is a 1×1 biquaternion. Note that, unlike the case for $Sp(1)$, $G_\mu(x)$ needs not to be anti-Hermitian.

We can now define the $SL(2,C)$ field strength

$$F_{\mu\nu} = \partial_\mu G_\nu(x) + G_\mu(x)G_\nu(x) - [\mu \longleftrightarrow \nu]. \tag{3.27}$$

To show that $F_{\mu\nu}$ is self-dual, one first show that the operator

$$P = 1 - v(x)v^{\circledast}(x) \tag{3.28}$$

is a projection operator $P^2 = P$, and can be written in terms of Δ as

$$P = \Delta(x)f\Delta^{\circledast}(x). \tag{3.29}$$

The self-duality of $F_{\mu\nu}$ can now be proved as following

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu(v^{\circledast}(x)\partial_\nu v(x)) + v^{\circledast}(x)\partial_\mu v(x)v^{\circledast}(x)\partial_\nu v(x) - [\mu \longleftrightarrow \nu] \\ &= v^{\circledast}(x)b(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger)fb^{\circledast}v(x) \end{aligned} \tag{3.30}$$

where we have used Eqs.(3.19),(3.22) and (3.29). Finally the factor $(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger)$ above can be shown to be self-dual

$$\sigma_{\mu\nu} \equiv \frac{1}{4i}(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger) = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta}, \tag{3.31}$$

$$\bar{\sigma}_{\mu\nu} = \frac{1}{4i}(e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu) = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{\sigma}_{\alpha\beta}. \tag{3.32}$$

This proves the self-duality of $F_{\mu\nu}$. We thus have constructed many $SL(2,C)$ SDYM field configurations.

To count the number of moduli parameters for the $SL(2,C)$ k-instantons we have constructed , one uses transformations which preserve conditions Eq.(3.21), Eq.(3.22) and Eq.(3.23), and the definition of G_μ in Eq.(3.26) to bring b and a in Eq.(3.19) into a simple canonical form

$$b = \begin{bmatrix} 0_{1\times k} \\ I_{k\times k} \end{bmatrix}, \tag{3.33}$$

$$a = \begin{bmatrix} \lambda_{1\times k} \\ -y_{k\times k} \end{bmatrix} \tag{3.34}$$

where λ and y are biquaternion matrices with orders $1\times k$ and $k\times k$ respectively, and y is symmetric

$$y = y^T. \tag{3.35}$$

The constraints for the moduli parameters are

$$a_{ci}^{\circledast} a_{cj} = 0, i \neq j, \text{ and } y_{ij} = y_{ji}. \tag{3.36}$$

The total number of moduli parameters for k-instanton can be calculated through Eq.(3.36) to be

$$\#\text{ of moduli for } SL(2,C)\ k\text{-instantons} = 16k - 6, \tag{3.37}$$

which is twice of that of the case of $Sp(1)$. Roughly speaking, there are $8k$ parameters for instanton "biquaternion positions" and $8k$ parameters for instanton "sizes". Finally one has to subtract an overall $SL(2,C)$ gauge group degree of freesom 6. This picture will become more clear when we give examples of explicit constructions of $SL(2,C)$ instantons in the next section.

4. Examples of $SL(2,C)$ instantons and Jumping lines

In this section, we will explicitly construct examples of $SL(2,C)$ YM instantons to illustrate our prescription given in the last section. Example of $SL(2,C)$ instantons with jumping lines will also be given.

4.1. The $SL(2,C)$ (M,N) Instantons

In this first example, we will reproduce from the ADHM construction the $SL(2,C)$ (M,N) instanton solutions constructed in [13]. We choose the biquaternion λ_j in Eq.(3.34) to be $\lambda_j e_0$ with λ_j a *complex* number, and choose $y_{ij} = y_j\delta_{ij}$ to be a diagonal matrix with $y_j = y_{j\mu}e_\mu$ a quaternion. That is

$$\Delta(x) = \begin{bmatrix} \lambda_1 & \lambda_2 & ... & \lambda_k \\ x-y_1 & 0 & ... & 0 \\ 0 & x-y_2 & ... & 0 \\ . & ... & ... & ... \\ 0 & 0 & ... & x-y_k \end{bmatrix}, \tag{4.38}$$

which satisfies the constraint in Eq.(3.36). One can calculate the gauge potential as

$$
\begin{aligned}
G_\mu = v^{\circledast}\partial_\mu v &= \frac{1}{4}[e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu]\partial_\nu \ln(1+\frac{\lambda_1^2}{|x-y_1|^2}+...+\frac{\lambda_k^2}{|x-y_k|^2}) \\
&= \frac{1}{4}[e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu]\partial_\nu \ln(\phi)
\end{aligned}
\tag{4.39}
$$

where

$$
\phi = 1+\frac{\lambda_1^2}{|x-y_1|^2}+...+\frac{\lambda_k^2}{|x-y_k|^2}. \tag{4.40}
$$

For the case of $Sp(1)$, λ_j is a real number and $\lambda_j\lambda_j^\dagger = \lambda_j^2$ is a real number. So ϕ in Eq.(4.40) is a complex-valued function in general. If we choose $k=1$ and define $\lambda_1^2 = \frac{\alpha_1^2}{1+i}$, then

$$
\phi = 1+\frac{\frac{\alpha_1^2}{1+i}}{|x-y_1|^2}. \tag{4.41}
$$

The gauge potential is

$$
\begin{aligned}
G_\mu &= \frac{1}{4}[e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu]\partial_\nu \ln(1+\frac{\frac{\alpha_1^2}{1+i}}{|x-y_1|^2}) = \frac{1}{4}[e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu]\partial_\nu \ln(1+\frac{\alpha_1^2}{|x-y_1|^2}+i) \\
&= \frac{1}{2}[e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu]\frac{-\alpha_1^2(x-y_1)_\nu}{|x-y_1|^4+(|x-y_1|^2+\alpha_1^2)^2}[\frac{|x-y_1|^2+\alpha_1^2}{|x-y_1|^2}-i]
\end{aligned}
\tag{4.42}
$$

which reproduces the $SL(2,C)$ $(M,N)=(1,0)$ solution calculated in [13]. It is easy to generalize the above calculations to the general (M,N) cases, and it can be shown that the topological charge of these field configuations is $k=M+N$ [13].

4.2. $SL(2,C)$ CFTW k-instantons and jumping lines

For another subset of k-instanton field configurations, one chooses $\lambda_i = \lambda_i e_0$ (with λ_i a *complex* number) and y_i to be a *biquaternion* in Eq.(4.38). It is important to note that for these choices, the constraints in Eq.(3.36) are still satisfied *without* turning on the off-diagonal elements y_{ij} in Eq.(3.34). It can be shown that, for these field configurations, there are non-removable singularities which are zeros ($x \in J$) of

$$
\phi = 1+\frac{\lambda_1\lambda_1^{\circledast}}{|x-y_1|_c^2}+...+\frac{\lambda_k\lambda_k^{\circledast}}{|x-y_k|_c^2}, \tag{4.43}
$$

or

$$
\det \Delta(x)^{\circledast}\Delta(x) = |x-y_1|_c^2|x-y_2|_c^2\cdots|x-y_k|_c^2\phi = P_{2k}(x)+iP_{2k-1}(x)=0. \tag{4.44}
$$

For the k-instanton case, one encounters intersections of zeros of $P_{2k}(x)$ and $P_{2k-1}(x)$ polynomials with degrees $2k$ and $2k-1$ respectively

$$
P_{2k}(x)=0, \quad P_{2k-1}(x)=0. \tag{4.45}
$$

These new singularities can not be gauged away and do not show up in the field configurations of $SU(2)$ k-instantons. Mathematically, the existence of singular structures of the non-compact $SL(2,C)$ SDYM field configurations is consistent with the inclusion of "sheaves" by Frenkel-Jardim [16] recently, rather than just the restricted notion of "vector bundles", in the one to one correspondence between ASDYM and certain algebraic geometric objects.

5. Acknowledgments

This talk is based on a collaboration paper with S.H. Lai and I.H. Tsai. I thank the financial support of MoST, Taiwan.

[1] A. Belavin, A. Polyakov, A. Schwartz, Y. Tyupkin, "Pseudo-particle solutions of the Yang-Mills equations", Phys. Lett. B 59 (1975) 85.

[2] E.F. Corrigan, D.B. Fairlie, Phys. Lett. 67B (1977)69; G. 'tHooft, Phys. Rev. Lett., 37 (1976) 8; F. Wilczek, in "Quark Confinement and Field Theory", Ed. D.Stump and D. Weingarten, John Wiley and Sons, New York (1977).

[3] R. Jackiw, C. Rebbi, "Conformal properties of a Yang-Mills pseudoparticle", Phys. Rev. D 14 (1976) 517; R. Jackiw, C. Nohl and C. Rebbi, "Conformal properties of pseu-doparticle con gurations", Phys. Rev. D 15 (1977) 1642.

[4] M. Atiyah, V. Drinfeld, N. Hitchin, Yu. Manin, "Construction of instantons", Phys. Lett. A 65 (1978) 185.

[5] N. H. Christ, E. J. Weinberg and N. K. Stanton, "General Self-Dual Yang-Mills Solutions", Phys. Rev. D 18 (1978) 2013. V. Korepin and S. Shatashvili, "Rational parametrization of the three instanton solutions of the Yang-Mills equations", Math. USSR Izversiya 24 (1985) 307.

[6] S.K. Donaldson and P.B. Kronheimer, "The Geometry of Four Manifolds", Oxford University Press (1990).

[7] G. 't Hooft, "Computation of the quantum effects due to a four-dimensional pseudoparticle", Phys. Rev. D 14 (1976) 3432. G. 't Hooft, "Symmetry breaking through Bell-Jackiw anomalies", Phys. Rev. Lett. 37 (1976) 8.

[8] C. Callan Jr., R. Dashen, D. Gross, "The structure of the gauge theory vacuum", Phys. Lett. B 63 (1976) 334; "Toward a theory of the strong interactions", Phys. Rev. D 17 (1978) 2717. R. Jackiw, C. Rebbi, "Vacuum periodicity in a Yang-Mills quantum theory", Phys. Rev. Lett. 37 (1976) 172.

[9] R. Jackwi and C. Rebbi, Phys. Lett. 67B (1977) 189. C. W. Bernard, , N. H. Christ, A. H. Guth and E. J. Weinberg, Phys. Rev. D16 (1977) 2967.

[10] Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D12, 3843 (1975); Phys. Rev.D13, (1976) 3233.

[11] J. P. Hsu and E. Mac, J. Math. Phys. 18 (1977) 1377.

[12] L. J. Mason and G. A. J. Sparling, "Nonlinear Schrodinger and Korteweg-de Vries are reductions of self-dual Yang-Mills," Phys. Lett. A 137, 29–33 (1989).

[13] K. L. Chang and J. C. Lee, "On solutions of self-dual SL(2,C) gauge theory", Chinese Journal of Phys. Vol. 44, No.4 (1984) 59. J.C. Lee and K. L. Chang, "SL(2,C) Yang-Mills Instantons", Proc. Natl. Sci. Counc. ROC (A), Vol 9, No 4 (1985) 296.

[14] W. R. Hamilton, "Lectures on Quaternions", Macmillan & Co, Cornell University Library (1853).

[15] Sheng-Hong Lai, Jen-Chi Lee and I-Hsun Tsai, "Biquaternions and ADHM Construction of Non-Compact SL(2,C) Yang-Mills Instantons", Ann. Phys. 361,(2015) 14-32.

[16] I. Frenkel, M. Jardim, "Complex ADHM equations and sheaves on P^3", Journal of Algebra 319 (2008) 2913-2937. J. Madore, J.L. Richard and R. Stora, "An Introduction to the Twistor Programme", Phys. Rept. 49, No. 2 (1979) 113-130.