Minimal surfaces in q-deformed $\mathbf{AdS}_5\times\mathbf{S}^5$

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Abstract. We study minimal surfaces in q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$. For this purpose, it is convenient to introduce a coordinate system which describes the spacetime only inside the singularity surface and treat the singularity surface as the holographic screen. In particular, we consider minimal surfaces whose boundary shapes are 1) a straight line and 2) a circle. In the $q\to 1$ limit, the solutions correspond to a 1/2 BPS straight line Wilson loop and a 1/2 BPS circular one, respectively. A remarkable point is that the classical Euclidean actions have no linear divergence unlike the original ones. This finiteness indicates that the q-deformation may be regarded as a UV regularization.

1. Introduction

The AdS/CFT correspondence is a realization of the equivalence between string theories and gauge theories. The most well-studied example is a duality between type IIB string theory on the $\mathrm{AdS}_5\times\mathrm{S}^5$ background and $\mathcal{N}=4$ super Yang-Mills theory at large N limit [1]. A great discovery in the recent is an integrable structure behind the AdS/CFT [2]. On the string-theory side, the Green-Schwarz string action on $\mathrm{AdS}_5\times\mathrm{S}^5$ is constructed based on a supersymmetric coset [3] and its classical integrability has been shown in [4]. Although the essential mechanism of the duality has not been fully understood yet, the integrability has played a crucial role in checking conjectured relations in the AdS/CFT.

To reveal a deeper structure behind gauge/gravity dualities beyond the conformal invariance, it would be worth considering integrable deformations of the AdS/CFT. On the string-theory side, a good way is to employ the Yang-Baxter sigma model approach [5]. This is a systematic way to consider integrable deformations of 2D non-linear sigma models. In this approach, an integrable deformation is specified by picking up a skew-symmetric classical r-matrix satisfying the modified classical Yang-Baxter equation (mCYBE). The deformed action is classically integrable in the sense that a Lax pair exists.

The original argument was restricted to principal chiral models, then it was generalized to the symmetric coset case [6][1]. Based on this generalization, a q-deformed action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring has been constructed in [11] by picking up the Drinfeld-Jimbo type r-matrix [12] and the resulting action exhibits a quantum group symmetry. The metric and NS-NS two-form of the deformed background have been found in [13]. In particular, a singularity surface exists in the deformed AdS part. For this deformed string theory, many works have been done. Some special limits of deformed $\mathrm{AdS}_n\times\mathrm{S}^n$ were studied in [14]. A mirror description were proposed in [15,16].

[1] For earlier developments on q-deformations of $su(2)$ and $sl(2)$, see [7–10].

The fast-moving string limits were considered in [17]. Giant magnon solutions have been argued in [15,18]. The deformed Nuemann models were derived in [19]. A possible holographic setup was proposed in [20] and minimal surfaces have been studied in [20,21]. Two-parameter deformations have been studied in [14, 22]. For some arguments towards the complete supergravity solution, see [23–25]. More recently, another integrable deformation called the λ-deformation has been proposed in [26, 27]. This deformation is closely linked to the Yang-Baxter deformation by a Poisson-Lie duality [26, 28–30].

A generalization of the Yang-Baxter sigma model to the (non-modified) classical Yang-Baxter equation (CYBE) has been considered in [31] . In this reformulation, the Lax pair and the kappa transformation should be reconstructed and this generalization is not so trivial. An advantage in comparison to the mCYBE case is that one may also consider partial deformations of $\mathrm{AdS}_5\times\mathrm{S}^5$. So far, in a series of works [32–38] , many kinds of classical r-matrices have been identified with the well-known type IIB supergravity solutions including the γ-deformations of S^5 [39], gravity duals for noncommutative gauge theories [40] and Schrödinger spacetimes [41], in addition to new backgrounds [32]. This identification may be referred to as the gravity/CYBE correspondence (For a short summary, see [42]). The solutions of the CYBE may be regarded as a moduli space of the $\mathrm{AdS}_5\times\mathrm{S}^5$ string background. In the recent, this identification has been generalized to integrable deformations of 4D Minkowski spacetime in [43]. Then Lax pairs for string theories on Yang-Baxter deformed backgrounds have been derived explicitly in [44]. Another remarkable feature of the CYBE case is that non-integrable backgrounds can be described. The well-known example is $\mathrm{AdS}_5 \times T^{1,1}$ [45] and the non-integrability of this background has been shown by the existence of chaotic string solutions in [46, 47]. It has been found that Yang-Baxter deformations can also reproduce this background in [48]. This result indicates that the gravity/CYBE correspondence is not restricted to a class of integrable backgrounds but also applicable to a wider class of gravity solutions.

In this article, we focus on the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring [11]. An interesting issue is to consider a holographic relation in the q-deformed geometry. A proposal is that the singularity surface in the deformed AdS may be regarded as the holographic screen [20]. For this purpose, it is useful to introduce a coordinate system which describes the spacetime enclosed by the singularity surface [20]. With this coordinate system, minimal surfaces whose boundary shapes are a straight line and a circle have been considered in [20, 21]. The solutions are reduced to the well-known results [49–52] in the $q \to 1$ limit. A remarkable feature is that the classical Euclidean actions have no linear divergence, in comparison to the original ones. This result indicates that the q-deformation may be regarded as a UV regularization.

This article is organized as follows. Section 2 gives a brief review of a coordinate system for the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ which describes the spacetime only inside the singularity surface and the associated Poincaré coordinates. In section 3, we consider two types of minimal surfaces. The former is a static solution whose boundary is a straight line. The classical action does not have the standard linear divergence but a logarithmic divergence, unlike the usual AdS/CFT case. The solution corresponds to a 1/2 BPS straight line Wilson loop [49,50] in the undeformed limit $q \to 1$. The latter is a circular solution which is constructed by supposing whose boundary shape is a circle. A remarkable point is that in the circular case, the resulting classical action is finite even though there is no UV cut-off. The solution corresponds to a 1/2 BPS circular Wilson loop [51, 52] in the undeformed limit $q \to 1$. Section 4 is devoted to conclusion and discussion.

2. A q-deformed $\mathbf{AdS}_5\times\mathbf{S}^5$ background

We consider the bosonic part of the classical action of a q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring [11]. In the following, we focus on minimal surfaces ending at the singularity surface in the deformed

AdS. For this purpose, it is helpful to employ a coordinate system which describes the spacetime only inside the singularity surface [20]. In this coordinate system, the singularity surface is located at the boundary.

2.1. The bosonic part of the q-deformed action

Let us first introduce the metric part and the WZ term of the bosonic action with the coordinate system [20] , then the associated Poincaré coordinates [21].

The bosonic action (in the conformal gauge) is composed of the metric part and the Wess-Zumino (WZ) term which describes the coupling of string to an NS-NS two-form.

With the coordinate system proposed in [20] , the metric for the deformed AdS and sphere parts are given by, respectively,

$$\begin{aligned} ds^2_{\mathrm{AdS}_5} &= R^2\sqrt{1+C^2}\Big[-\cosh^2\chi\, dt^2 + \frac{d\chi^2}{1+C^2\cosh^2\chi} \\ &\quad + \frac{(1+C^2\cosh^2\chi)\sinh^2\chi}{(1+C^2\cosh^2\chi)^2+C^2\sinh^4\chi\sin^2\zeta}\,(d\zeta^2+\cos^2\zeta\, d\psi_1^2) + \frac{\sinh^2\chi\sin^2\zeta\, d\psi_2^2}{1+C^2\cosh^2\chi}\Big]\,, \end{aligned} \tag{1}$$

$$\begin{aligned} ds^2_{\mathrm{S}^5} &= R^2\sqrt{1+C^2}\,\Big[\cos^2\gamma\, d\phi^2 + \frac{d\gamma^2}{1+C^2\cos^2\gamma} \\ &\quad + \frac{(1+C^2\cos^2\gamma)\sin^2\gamma}{(1+C^2\cos^2\gamma)^2+C^2\sin^4\gamma\sin^2\xi}\,(d\xi^2+\cos^2\xi\, d\phi_1^2) + \frac{\sin^2\gamma\sin^2\xi\, d\phi_2^2}{1+C^2\cos^2\gamma}\Big]\,. \end{aligned} \tag{2}$$

Here the coordinates $(t\,,\psi_1\,,\psi_2\,,\zeta\,,\chi)$ describe the deformed AdS_5 , while the the coordinates $(\phi\,,\phi_1\,,\phi_2\,,\xi\,,\gamma)$ parameterize the deformed S^5 . The deformation is characterized by a real parameter $C\in[0,\infty)$. When $C=0$, the geometry is reduced to the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ with the curvature radius R .

It should be mentioned that a curvature singularity exists at $\chi=\infty$ in this coordinate system as well as the original one [13]. In [20], it has been shown that it takes infinite affine time for a massless particle to reach the singularity surface, while it does not take infinite time to reach the singularity in the coordinate time. This is the same feature as in the usual AdS space with the global coordinates. Hence it seems likely to treat the singularity surface as the boundary in the holographic setup for the q-deformed geometry.

The WZ term for the AdS part and the sphere part are given by, respectively,

$$\mathcal{L}^{\mathrm{WZ}}_{\mathrm{AdS}} = \frac{\sqrt{\lambda}}{4\pi}\,C\sqrt{1+C^2}\,\epsilon^{\mu\nu}\frac{r^4\sin 2\zeta}{[z^2+C^2(z^2+r^2)]^2+C^2r^4\sin^2\zeta}\,\partial_\mu\varphi\,\partial_\nu\zeta\,, \tag{3}$$

$$\mathcal{L}^{\mathrm{WZ}}_{\mathrm{S}} = -\frac{\sqrt{\lambda}}{4\pi}\,C\sqrt{1+C^2}\,\epsilon^{\mu\nu}\frac{\sin^4\gamma\sin 2\xi}{(1+C^2\cos^2\gamma)^2+C^2\sin^4\gamma\sin^2\xi}\,\partial_\mu\phi_1\partial_\nu\xi\,. \tag{4}$$

Here the totally anti-symmetric tensor $\epsilon^{\mu\nu}$ is normalized as $\epsilon^{01}=+1$ and the coupling λ is defined as

$$\sqrt{\lambda}\equiv\frac{R^2}{\alpha'}\,. \tag{5}$$

Each component of the WZ term is proportional to C , and hence it vanishes when $C=0$.

2.2. Poincaré coordinates

To consider minimal surfaces in the deformed AdS (1) , it is helpful to introduce the associated Poincaré coordinates.

Let us first perform a coordinate transformation,

$$\cosh\chi \equiv \frac{1}{\cos\theta}\,, \tag{6}$$

and then the Wick rotation $t \to i\tau$ to move to the Euclidean signature. Performing the following coordinate transformation,

$$z \equiv \mathrm{e}^{\tau}\cos\theta\,, \qquad r \equiv \mathrm{e}^{\tau}\sin\theta\,, \tag{7}$$

the Poincaré analogue of the deformed Euclidean AdS_5 is obtained as

$$\begin{aligned} ds^2_{\mathrm{AdS}_5} &= R^2\sqrt{1+C^2}\Big[\frac{dz^2+dr^2}{z^2+C^2(z^2+r^2)} + \frac{C^2(z\,dz+r\,dr)^2}{z^2\big(z^2+C^2(z^2+r^2)\big)} \\ &\quad + \frac{\big(z^2+C^2(z^2+r^2)\big)r^2}{\big(z^2+C^2(z^2+r^2)\big)^2+C^2r^4\sin^2\zeta}\,(d\zeta^2+\cos^2\zeta\,d\psi_1^2) + \frac{r^2\sin^2\zeta\,d\psi_2^2}{z^2+C^2(z^2+r^2)}\Big]\,. \end{aligned} \tag{8}$$

When $C=0$, (8) is reduced to the Euclidean AdS_5 metric with the Poincaré coordinates.

After the Wick rotation has been performed, the space-like path is the only sensible measure rather than time-like and null ones. As argued in [21], the space-like proper distance to the singularity surface is finite when $C\neq 0$, unlike the undeformed case with $C=0$. This property might be crucial in the next section.

3. Minimal surfaces

In this section, we consider minimal surfaces in the deformed AdS. The gauge-theory dual has not been unveiled yet, but at least when $C=0$, these solutions should correspond to Wilson loops in the $\mathcal{N}$=4 super Yang-Mills theory. The minimal surfaces may be a good clue to seek for the dual gauge theory.

In the following, we consider minimal surfaces in the deformed AdS_2 subspace for two cases, 1) a straight line solution 2) a circular solution. In the undeformed limit, the solutions are reduced to minimal surfaces dual for a 1/2 BPS straight Wilson loop [49, 50] and a 1/2 BPS circular one [51, 52], respectively.

3.1. A straight line solution

Let us first study a straight string solution. Due to the deformation, it seems difficult to construct a straight line solution with the Poincaré coordinates (8), while it is possible to construct a static configuration of the string world-sheet in the global Lorentzian deformed AdS_5 (1). Then the solution ends with two lines on the boundary.

Let us consider a static configuration of the string world-sheet with the ansatz :

$$\begin{aligned} &t=\kappa\tau\,, \qquad \chi=\chi(\sigma)\,, \qquad \zeta=\psi_1=\psi_2=0\,, \\ &\phi=\phi_1=\phi_2=\gamma=\xi=0\,. \end{aligned} \tag{9}$$

Then the full metric is reduced to that of a deformed AdS_2 subspace,

$$ds^2_{\mathrm{AdS}_2} = R^2\sqrt{1+C^2}\,\Big[-\cosh^2\chi dt^2 + \frac{d\chi^2}{1+C^2\cosh^2\chi}\Big]\,. \tag{10}$$

Note that the WZ terms vanish under this ansatz (9). Then the Lagrangian is given by

$$\mathcal{L}_{\mathrm{AdS}_2} = -\frac{\sqrt{\lambda}}{4\pi}\sqrt{1+C^2}\left[\kappa^2\cosh^2\chi + \frac{\chi'^2}{1+C^2\cosh^2\chi}\right]. \tag{11}$$

The Virasoro constraint is

$$0 = -\kappa^2\cosh^2\chi + \frac{\chi'^2}{1+C^2\cosh^2\chi}\,, \tag{12}$$

and which gives the following equation,

$$\chi' = -\kappa\cosh\chi\sqrt{1+C^2\cosh^2\chi}\,. \tag{13}$$

By integrating (13) with the boundary condition $\chi(\sigma=0)=\infty$,

$$-\int_\infty^\chi \frac{d\tilde{\chi}}{\cosh\tilde{\chi}\sqrt{1+C^2\cosh^2\tilde{\chi}}} = \kappa\int_0^\sigma d\tilde{\sigma}\,, \tag{14}$$

the following expression is obtained,

$$\kappa\sigma = \arctan\Big[\frac{1}{C}\Big] - \arctan\Big[\frac{\sinh\chi}{\sqrt{1+C^2\cosh^2\chi}}\Big]. \tag{15}$$

By introducing σ_0 as $\sinh\chi(\sigma_0)=0$, $\kappa\sigma_0$ is related to C through,

$$\kappa\sigma_0 = \arctan\Big[\frac{1}{C}\Big]. \tag{16}$$

Then the solution (15) can be expressed as

$$\cosh^2\chi = \frac{1+\tan^2\big[\arctan[\frac{1}{C}]-\kappa\sigma\big]}{1-C^2\tan^2\big[\arctan[\frac{1}{C}]-\kappa\sigma\big]}\,. \tag{17}$$

Note that (17) describes a solution which stretches from the boundary to the center of the deformed AdS. The other direction of the AdS part can be described by the negative σ region. In total, the solution stretches from one boundary to the other boundary through the center.

Next, let us convert the solution (17) in the global coordinates into the one in the Poincaré coordinates. Note that it is divided into two sections according to the positive (or negative) σ region. For the positive σ region, the solution is given by

$$\begin{aligned} z^{(+)}(\tau,\sigma) &= \mathrm{e}^{\kappa\tau}\sqrt{\cos^2(\arctan[1/C]-\kappa\sigma) - C^2\sin^2(\arctan[1/C]-\kappa\sigma)}\,, \\ r^{(+)}(\tau,\sigma) &= \sqrt{1+C^2}\,\mathrm{e}^{\kappa\tau}\sin(\arctan[1/C]-\kappa\sigma)\,, \end{aligned} \tag{18}$$

and, for the negative σ region, the one is

$$\begin{aligned} z^{(-)}(\tau,\sigma) &= \mathrm{e}^{\kappa\tau}\sqrt{\cos^2(\arctan[1/C]+\kappa\sigma) - C^2\sin^2(\arctan[1/C]+\kappa\sigma)}\,, \\ r^{(-)}(\tau,\sigma) &= -\sqrt{1+C^2}\,\mathrm{e}^{\kappa\tau}\sin(\arctan[1/C]+\kappa\sigma)\,. \end{aligned} \tag{19}$$

Here τ is taken as $-\infty<\tau<\infty$ and the $(\pm)$-signatures correspond to the solution in the positive σ region $\big(0\le\sigma\le\sigma_0\big)$ and that of the negative σ region $\big(-\sigma_0\le\sigma<0\big)$, respectively.

The positive σ region is mapped to $\theta \equiv \arctan(r/z)$,

$$\theta(\sigma = 0) = \frac{\pi}{2} \quad \text{(boundary)} \quad \longrightarrow \quad \theta(\sigma = \sigma_0) = 0 \quad \text{(origin)}\,, \tag{20}$$

while the negative σ region covers,

$$\theta(\sigma = -\sigma_0) = 0 \quad \text{(origin)} \quad \longrightarrow \quad \theta(\sigma = 0) = -\frac{\pi}{2} \quad \text{(boundary)}\,. \tag{21}$$

Note that the contribution coming from the negative σ region is the same value as that from the positive σ region.

Then the classical Euclidean action is evaluated as

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi}\sqrt{1+C^2}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \int_{\epsilon}^{\sigma_0} d\sigma\, 2\kappa^2 \Big(\frac{1+\tan^2\big[\arctan[\frac{1}{C}]-\kappa\sigma\big]}{1-C^2\tan^2\big[\arctan[\frac{1}{C}]-\kappa\sigma\big]} \Big) \\ &= \frac{T\sqrt{\lambda}}{\pi}\,\frac{\sqrt{1+C^2}}{C}\,\kappa\,\mathrm{arctanh}\Big[C\cot\big(\kappa\epsilon+\arctan[C]\big)\Big]\,, \end{aligned} \tag{22}$$

where T is the interval of the world-sheet τ and taken to be large. When $C \neq 0$, the above expression can be expanded in terms of ϵ as

$$S = \frac{T\sqrt{\lambda}}{2\pi}\,\frac{\sqrt{1+C^2}}{C}\,\kappa\,\log\left[\frac{2C}{(1+C^2)\,\kappa\epsilon}\right] + \mathcal{O}(\epsilon)\,. \tag{23}$$

Note that the divergence becomes logarithmic, unlike the usual AdS/CFT case.

The Legendre transformation It would be worth mentioning about an additional contribution coming from the boundary [52]. In the undeformed case, it is well recognized that the classical action has a linear divergence and it can be removed by considering a Legendre transformation. The origin of this additional contribution is the surface term which appears in taking a variation of the classical action to obtain the equations motion. This is just because the minimal surface has the boundary. Thus it is important to discuss this contribution in the deformed case as well.

By taking account of a Legendre transformation, the total action is written as

$$\tilde{S} \;=\; S + S_L\,, \qquad S_L = 2\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \int_{\epsilon}^{\sigma_0} d\sigma\, \partial_\sigma\Big(z\,\frac{\partial L}{\partial(\partial_\sigma z)}\Big)\,. \tag{24}$$

The term S_L is evaluated as

$$\begin{aligned} S_L &= 2\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \Big(z\,\frac{\partial L}{\partial(\partial_\sigma z)}\Big|_{\sigma_0} - z\,\frac{\partial L}{\partial(\partial_\sigma z)}\Big|_{\epsilon}\Big) \\ &= -\frac{T\sqrt{\lambda}}{\pi}\,\sqrt{1+C^2}\,\kappa\,\cot\big(\kappa\epsilon+\arctan[C]\big)\,. \end{aligned}$$

In the non-zero C case, S_L can be expanded in terms of ϵ as

$$S_L \;=\; -\frac{T\sqrt{\lambda}}{\pi}\,\frac{\sqrt{1+C^2}}{C}\,\kappa + \mathcal{O}(\epsilon)\,. \tag{25}$$

Thus, even though the boundary contribution (25) has been taken into account, the logarithmic divergent term in (23) cannot be canceled out. There might be a proper method to regularize (23), or it may be divergent essentially.

It is worth noting the undeformed limit of the classical action (23). By taking the $C \to 0$ limit (ϵ : fixed), the well-known result in the undeformed case is reproduced,

$$S = \frac{T\sqrt{\lambda}}{\pi}\,\kappa\cot[\kappa\epsilon] = \frac{T\sqrt{\lambda}}{\pi}\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\,. \tag{26}$$

In the undeformed limit $C \to 0$ with ϵ fixed, S_L (25) results in

$$S_L = -\frac{T\sqrt{\lambda}}{\pi}\,\kappa\cot[\kappa\epsilon]\,, \tag{27}$$

and it cancels out the divergent term in (26) as usual.

3.2. A circular solution

Next, we shall consider minimal surfaces which ends up with a circle at the boundary of the q-deformed AdS_5 with the Poincaré coordinates (8) .

Let us consider an ansatz with the conformal gauge :

$$z = \sqrt{a^2 - r^2}\,, \qquad r = r(\sigma)\,, \qquad \psi_1 = \psi_1(\tau)\,, \qquad \psi_2 = \zeta = 0\,, \tag{28}$$

with $0 \le \tau < 2\pi$, $0 \le \sigma < \infty$. Here a is the radius of the circle at the boundary. Note that (28) is a consistent ansatz and the WZ term vanishes under (28) . Then the geometry with the metric (8) is reduced to the following deformed AdS_2 ,

$$ds^2_{\mathrm{AdS}_2} \;=\; \frac{R^2\sqrt{1+C^2}\,r^2}{(1+C^2)a^2 - r^2}\Big[\frac{a^2\,dr^2}{r^2(a^2-r^2)} + d\psi_1^2\Big]\,. \tag{29}$$

This metric leads to a solution of the string equation of motion :

$$z = a\tanh\sigma\,, \qquad r = \frac{a}{\cosh\sigma}\,, \qquad \psi_1 = \tau\,. \tag{30}$$

The next task is to evaluate the classical Euclidean action. By following the undeformed case, let us formally introduce a cut-off ϵ for the coordinate z . It may be regarded as a cut-off σ_0 for the string world-sheet coordinate σ through the classical solution (30) for z ,

$$\epsilon = a\tanh\sigma_0\,. \tag{31}$$

Then the classical action is evaluated as

$$\begin{aligned} S \;&=\; \frac{\sqrt{\lambda}}{4\pi}\sqrt{1+C^2}\int_0^{2\pi} d\tau \int_{\sigma_0}^{\infty} d\sigma \Big[\frac{2}{\sinh^2\sigma + C^2\cosh^2\sigma}\Big] \\ &=\; \sqrt{\lambda}\frac{\sqrt{1+C^2}}{C}\Big(\mathrm{arccot}[C] - \mathrm{arccot}\Big[\frac{Ca}{\epsilon}\Big]\Big)\,. \end{aligned} \tag{32}$$

Note that the second term in (32) can be expanded in terms of ϵ as

$$-\sqrt{\lambda}\frac{\sqrt{1+C^2}}{C}\,\mathrm{arccot}\Big[\frac{Ca}{\epsilon}\Big] = -\sqrt{\lambda}\frac{\sqrt{1+C^2}}{C}\,\Big(\frac{\epsilon}{Ca} + \mathcal{O}(\epsilon^2)\Big)\,, \tag{33}$$

where C has been fixed in this expansion. It is easy to see that the cut-off can be removed for non-vanishing C. By taking $\epsilon \to 0$, the classical action (32) becomes,

$$S = \sqrt{\lambda}\frac{\sqrt{1+C^2}}{C}\,\mathrm{arccot}[C]\,. \tag{34}$$

It should be mentioned that the action (34) is finite even if there is no UV cut-off for the string world-sheet (equivalently for the radial direction of the deformed AdS). The result would come from the finiteness of the space-like proper distance to the singularity surface. Then the deformation parameter C works as a UV regularization and one does not need to introduce ϵ any more in evaluating the classical action. This point becomes clear by taking the $C \to 0$ limit of (34). By expanding (34) in terms of C, we obtain the following expression:

$$S = -\sqrt{\lambda} + \sqrt{\lambda}\,\frac{\pi}{2C} + \mathcal{O}(C)\,. \tag{35}$$

However one needs to include ϵ so as to reproduce the regularized result in the undeformed limit [51, 52] as shown below. For this purpose, it is helpful to consider the undeformed limit of the classical action (32) by taking $C \to 0$ while keeping ϵ finite. This corresponds to keeping the boundary of the solution away from the singularity surface. Then the result in the undeformed case [51, 52] is reproduced as

$$S = -\sqrt{\lambda} + \sqrt{\lambda}\,\frac{a}{\epsilon}\,. \tag{36}$$

Note that the linear divergence is also reproduced in the undeformed limit and it can be canceled by taking account of a Legendre transformation as usual.

The Legendre transformation It is of importance to consider the boundary term via a Legendre transformation [52]. In the present case, the total derivative term is given by

$$\tilde{S} \;=\; S + S_L\,, \qquad S_L = \int_0^{2\pi} d\tau \int_{\sigma_0}^{\infty} d\sigma\, \partial_\sigma\Big(z\,\frac{\partial L}{\partial(\partial_\sigma z)}\Big)\,. \tag{37}$$

The term S_L is a total derivative and evaluated as

$$\begin{aligned} S_L &= -\int_0^{2\pi} d\tau\, z\,\frac{\partial L}{\partial(\partial_\sigma z)}\bigg|_{\sigma=\sigma_0} = -\sqrt{\lambda}\,\frac{\sqrt{1+C^2}\,\tanh\sigma_0}{\sinh^2\sigma_0 + C^2\cosh^2\sigma_0} \\ &= -\sqrt{\lambda}\,\sqrt{1+C^2}\,\frac{\epsilon\,(a^2-\epsilon^2)}{a(\epsilon^2+C^2a^2)}\,. \end{aligned} \tag{38}$$

A remarkable point is that (38) vanishes in the limit $\epsilon \to 0$ when $C \neq 0$, hence it does not contribute to the final expression of the action.

The next is to consider the undeformed limit of S_L so as to cancel the divergent term in the undeformed limit of S (32). The term S_L in (38) vanishes when $C \neq 0$, hence this is the first place one need to introduce ϵ so as to keep the boundary away from the singularity surface. By taking the $C \to 0$ limit with ϵ fixed, the resulting expression of S_L is given by

$$S_L = -\sqrt{\lambda}\,\frac{a}{\epsilon} + \mathcal{O}(\epsilon)\,, \tag{39}$$

and it cancels the divergent term in (36) as usual.

It would be worth mentioning about the cut-off ϵ and the deformation parameter C. When $C \neq 0$, there is no strict need to introduce ϵ in evaluating the classical action and the Legendre term vanishes. However, if the limit $\epsilon \to 0$ is taken first, or ϵ is not turned on, the finite result cannot be reproduced correctly in the $C \to 0$ limit. In this scene, there is a subtlety of the order the two limits: 1) $\epsilon \to 0$ and 2) $C \to 0$. At least so far, a possible resolution is to take the limit $C \to 0$ first while keeping ϵ finite.

4. Conclusion and discussion

In this article, we have discussed a q-deformation of the $AdS_5 \times S^5$ superstring. It has been conjectured in [20] that the singularity surface may be treated as a holographic screen in the deformed theory. To look for some support for this conjecture, we have further considered minimal surfaces by employing the coordinate system which is enclosed by the singularity surface. A remarkable feature is that the classical Euclidean actions have no linear divergence unlike the original ones. This finiteness comes from the fact that the q-deformation may be regarded as a UV regularization. In the undeformed limit, the linear divergent term is also reproduced. This result may indicate that our conjecture would make sense. To obtain more support, it would be nice to consider other minimal surfaces like cusped solutions by employing the Poincaré coordinates and calculate the corresponding quark-antiquark potantial by following [53] .

There are a lot of issues to be studied. The most interesting issue is to find out the gauge-theory side dual to the q-deformation of the $AdS_5 \times S^5$ superstring. To tackle this issue, it would be useful to look for some clues of the corresponding gauge theory by employing the Poincaré coordinates.

We believe that our results on minimal surfaces can play an important role in unveiling a possible gauge-theory dual.

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