On complex Riemannian foliations

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Abstract. In this paper we make some general considerations about the geometry of complex Riemannian foliations, we introduce a leafwise characteristic connection and we write Einstein equations with respect to it. Next, using an one-to-one correspondence between leafwise holomorphic Riemannian metrics and leafwise anti-Kählerian metrics, we focus on the Einstein condition for a leafwise holomorphic Riemannian metric and the associated real leafwise anti-Kählerian metric on a manifold endowed with a complex foliation.

1. Introduction

The importance of (holomorphic) complex Riemannian metrics in mathematical physics is without question (see for instance [3, 16, 20]). We recall that a (holomorphic) complex Riemannian manifold is a complex manifold M, together with a (holomorphic) complex tensor field G that is a complex scalar product (i.e., nondegenerate, symmetric, $\mathbb{C}$-bilinear form) on each holomorphic tangent space of M. The (holomorphic) complex Riemannian geometry possesses an underlying real geometry consisting of a pseudo-Riemannian metric of neutral signature and there is a strong relation between holomorphic Riemannian metrics and anti-Kählerian metrics (also known as Kähler-Norden metrics [10, 18, 19, 21]). In [2], it is proved that an anti-Kählerian metric must be the real part of a holomorphic metric. There is studied the Einstein condition for anti-Kählerian metrics and some generalized Einstein conditions on holomorphic Riemannian manifolds are investigated in [19]. Also, we notice that in [6, 7, 8], a study of connections and curvatures on (holomorphic) complex Riemannian manifolds is given, and the anti-Hermitian metrics which are locally conformal with anti-Kählerian metrics are investigated in [5].

In this paper we work in the category of smooth manifolds endowed with complex foliations, and we check that more of the techniques used in the study of (holomorphic) complex Riemannian geometry can be adjusted to the foliated picture. Thus, we generalize the main notions from the geometry of complex Riemannian manifods to that of complex Riemannian foliations. Incidentally the classical case is contained in the foliation formalism by taking the foliation which consists of the leaf M only.

The structure of the paper is as follows. In the second section, we brief recall the definition of a complex foliation which is used for instance in [4, 13, 14] , we define the leafwsise complex Riemannian metrics and we present some examples. Also, following [2, 7, 11] we generalize some notions from complex Riemannian manifolds to that of complex Riemannian foliations (not necessarily leafwise holomorphic). More exactly we consider the associated leafwise characteristic connection, we study its properties and we write Einstein equations with respect to it. In the last section we study the Einstein condition for leafwise holomorphic Riemannian metrics.

The main methods used here are similar and closely related to those used in the study of complex Riemannian manifolds [7, 8, 11] and anti-Kählerian manifolds [3, 2, 19, 21]. For this reason the most proofs are omitted here.

2. Complex Riemannian foliations

Let M be a smooth $(2m+n)$-dimensional manifold endoweed with a smooth regular foliation $\mathcal{F}$ of codimension n. Then the dimension of the foliation $\mathcal{F}$ is $2m$. We denote by $T\mathcal{F}$ the tangent bundle along the leaves and by $T^*\mathcal{F}$ its dual.

An *almost complex structure* along the leaves of $\mathcal{F}$ is defined as a smooth real vector bundle automorphism $J_{\mathcal{F}}$ of $T\mathcal{F}$ satisfying $J_{\mathcal{F}}^2 = -\mathrm{Id}_{\mathcal{F}}$. Given such a leafwise almost complex structure, we obtain the decomposition $T_{\mathbb{C}}\mathcal{F} = T\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}\mathcal{F} \oplus T^{0,1}\mathcal{F}$, where

$$T^{1,0}\mathcal{F} = \{X - iJ_{\mathcal{F}}X \mid X \in \Gamma(T\mathcal{F})\} \text{ and } T^{1,0}\mathcal{F} = \{X + iJ_{\mathcal{F}}X \mid X \in \Gamma(T\mathcal{F})\}.$$

Moreover, if the Nijenhuis tensor along the leaves

$$N_{J_{\mathcal{F}}} := [J_{\mathcal{F}}X, J_{\mathcal{F}}Y] - J_{\mathcal{F}}[J_{\mathcal{F}}X, Y] - J_{\mathcal{F}}[X, J_{\mathcal{F}}Y] - [X, Y], \text{ for } X, Y \in \Gamma(T\mathcal{F}) \tag{2.1}$$

vanishes, then $J_{\mathcal{F}}$ is called a *complex structure* on $\mathcal{F}$ and $(\mathcal{F}, J_{\mathcal{F}})$ is called a complex foliation on M. In this case, the complex foliation $\mathcal{F}$ can be defined by an open cover $\{U_i\}$, $i \in I$, of M and diffeomorphisms $\phi_i : \Omega_i \times \mathcal{O}_i \to U_i$ (where Ω_i is an open polydisc in $\mathbb{C}^m$ and $\mathcal{O}_i$ is an open ball in $\mathbb{R}^n$) such that, for every pair $(i,j) \in I \times I$ with $U_i \cap U_j \neq \phi$, the coordinate change $\phi_{ij} = \phi_j^{-1} \circ \phi_i : \phi_i^{-1}(U_i \cap U_j) \to \phi_j^{-1}(U_i \cap U_j)$ is of the form $(z', x') = (\phi_{ij}^1(z,x), \phi_{ij}^2(x))$ with $\phi_{ij}^1(z,x)$ holomorphic in z for every x fixed. A such adapted atlas will be called *leafwise complex.*

If we set $z^a = u^a + iu^{m+a}$, $a = 1, \ldots, m$, then the complex structure along the leaves $J_{\mathcal{F}} : T\mathcal{F} \to T\mathcal{F}$ is given by

$$J_{\mathcal{F}}\left(\frac{\partial}{\partial u^a}\right) = \frac{\partial}{\partial u^{m+a}}\,,\; J_{\mathcal{F}}\left(\frac{\partial}{\partial u^{m+a}}\right) = -\frac{\partial}{\partial u^a}\,,\; a = 1, \ldots, m$$

and its complex extension to $T_{\mathbb{C}}\mathcal{F}$ is given by

$$J_{\mathcal{F}}\left(\frac{\partial}{\partial z^a}\right) = i\frac{\partial}{\partial z^a}\,,\; J_{\mathcal{F}}\left(\frac{\partial}{\partial \overline{z}^a}\right) = -i\frac{\partial}{\partial \overline{z}^a},\; a = 1, \ldots, m.$$

Remark 2.1. The above notion of complex foliations is also related to laminations. For instance, a lamination by Riemann surfaces is a topological space locally homeomorphic to a model space of type $D \times T$, where D is the open unit disc in $\mathbb{C}$ and T is a topological space. The transition functions between two such charts is assumed to be of the form $D \times T \to D \times T'$ and defined by $(z,x) \mapsto (f(z,x), g(x))$, where $f(z,x)$ is holomorphic in z and continuous in x, and $g(x)$ is a continuous function of x (see Section 2 in [9]).

We have the following simple examples of complex foliations.

Example 2.1. Any complex manifold M of $\dim_{\mathbb{C}} M = m$ is a complex foliation of complex dimension m and codimension 0.

Example 2.2. Let M be an open set of $\mathbb{C}^m \times N$, where N is an n-dimensional smooth manifold. For every $t \in N$, the set $M^t = \{z \in \mathbb{C}^m \mid (z,t) \in M\}$ is an open set of $\mathbb{C}^m$ called the *section* of M along t. Then, sections of M are leaves of a complex foliation $\mathcal{F}$ of dimension m called the complex *canonical* foliation of M.

Example 2.3. Let F be a complex manifold of $\dim_{\mathbb{C}} F = m$ and N an n-dimensional smooth manifold. Every locally trivial fibration $F \hookrightarrow M \to N$ whose cocycle takes values in the automorphism group $\mathrm{Aut}(F)$ of (the complex manifold) F is a complex foliation, the fibers being the leaves. If the fibration is trivial, that is $M = F \times N$, we say that $\mathcal{F}$ is a *complex product foliation*.

For more examples of complex foliations, see for instance [13, 14] and references therein.

Definition 2.1. A *leafwise complex Riemannian metric* on $(M, \mathcal{F}, J_{\mathcal{F}})$ is a covariant symmetric 2-tensor field $G : \Gamma(T_{\mathbb{C}}\mathcal{F}) \times \Gamma(T_{\mathbb{C}}\mathcal{F}) \to \mathbb{C}$, which is non-degenerate at each point (z, x) of $(M, \mathcal{F})$ and satisfies

$$G(\overline{Z}_1, \overline{Z}_2) = \overline{G(Z_1, Z_2)} \text{ for every } Z_1, Z_2 \in \Gamma(T_{\mathbb{C}}\mathcal{F}), \tag{2.2}$$

$$G(Z_1, Z_2) = 0 \text{ for every } Z_1 \in \Gamma(T^{1,0}\mathcal{F}) \text{ and } Z_2 \in \Gamma(T^{0,1}\mathcal{F}). \tag{2.3}$$

It is easy to see that the relation (2.3) is equivalent to

$$G(J_{\mathcal{F}}Z_1, J_{\mathcal{F}}Z_2) = -G(Z_1, Z_2) \text{ for every } Z_1, Z_2 \in \Gamma(T_{\mathbb{C}}\mathcal{F}), \tag{2.4}$$

where, we have denoted again by $J_{\mathcal{F}}$ the $\mathbb{C}$-linear extension of $J_{\mathcal{F}}$ to $T_{\mathbb{C}}\mathcal{F}$. Thus, a leafwise complex Riemannian metric on $(M, \mathcal{F}, J_{\mathcal{F}})$ is completely determined by its values on $\Gamma(T^{1,0}\mathcal{F})$.

Definition 2.2. The pair $(M, \mathcal{F}, J_{\mathcal{F}}, G)$ consisting by a smooth $(2m+n)$-dimensional manifold M endowed with a complex foliation $(\mathcal{F}, J_{\mathcal{F}})$ and with a leafwise complex Riemannian metric G on $(M, \mathcal{F}, J_{\mathcal{F}})$, will be called a *complex Riemannian foliation*.

If $(z^1, \ldots, z^m, x^1, \ldots, x^n)$ is an adapted local coordinate system (leafwise complex) on $(M, \mathcal{F}, J_{\mathcal{F}})$, such that $\Gamma(T_{\mathbb{C}}\mathcal{F}) = span\{\partial/\partial z^a, \partial/\partial z^{\overline{a}}\}$, we put

$$G_{AB}(z, x) = G\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right), \; A, B \in \{1, \ldots, m, \overline{1}, \ldots, \overline{m}\}. \tag{2.5}$$

Then, for a leafwise complex Riemannian metric G, the defining conditions (2.2) and (2.3) can be expressed locally in the form

$$G_{\overline{A}\,\overline{B}} = \overline{G_{AB}} \text{ and } G_{a\overline{b}} = G_{\overline{a}b} = 0. \tag{2.6}$$

Definition 2.3. A leafwise complex Riemannian metric G on $(M, \mathcal{F}, J_{\mathcal{F}})$ is called *leafwise holomorphic Riemannian metric* if the local components $G_{ab}(z, x)$ are leafwise holomorphic functions, that is

$$\frac{\partial G_{ab}}{\partial z^{\overline{c}}} = 0\,, \text{ for every } c \in \{1 \ldots, m\}. \tag{2.7}$$

As in the case of complex Riemannian manifolds (see [7]) for a given leafwise complex Riemannian metric G on $(M, \mathcal{F}, J_{\mathcal{F}})$, we define the leafwise tensor field $\widetilde{G}$ on $(M, \mathcal{F}, J_{\mathcal{F}})$ by setting

$$\widetilde{G}(Z_1, Z_2) = (G \circ J_{\mathcal{F}})(Z_1, Z_2) := G(J_{\mathcal{F}}Z_1, Z_2) \text{ for every } Z_1, Z_2 \in \Gamma(T_{\mathbb{C}}\mathcal{F}). \tag{2.8}$$

This metric is called *leafwise twin metric*, and locally, it satisfies

$$\widetilde{G}_{ab} = iG_{ab} \text{ and } \widetilde{G}_{\overline{a}\,\overline{b}} = -iG_{\overline{a}\,\overline{b}}. \tag{2.9}$$

Also, we notice that given a leafwise complex Riemannian metric G on $(M, \mathcal{F}, J_{\mathcal{F}})$ it induces a leafwise real Riemannian metric g on the underlying real foliated manifold $(M, \mathcal{F}, J_{\mathcal{F}})$ by setting

$$g(X, Y) = 2Re\, G(\widehat{X}, \widehat{Y})\,,\; X, Y \in \Gamma(T\mathcal{F}), \tag{2.10}$$

where $\widehat{X} = (1/2)(X - iJ_{\mathcal{F}}X), \widehat{Y} = (1/2)(Y - iJ_{\mathcal{F}}Y) \in \Gamma(T^{1,0}\mathcal{F})$, and this real metric satisfies

$$g(J_{\mathcal{F}}X, J_{\mathcal{F}}Y) = -g(X, Y) \text{ for every } X, Y \in \Gamma(T\mathcal{F}), \tag{2.11}$$

or, equivalently

$$g(J_{\mathcal{F}}X, Y) = g(X, J_{\mathcal{F}}Y) \text{ for every } X, Y \in \Gamma(T\mathcal{F}). \tag{2.12}$$

Such a leafwise real metric will be called a *leafwise anti-Hermitian metric*, or *leafwise Norden metric* and $(M, \mathcal{F}, J_{\mathcal{F}}, g)$ will be called the *realization* of $(M, \mathcal{F}, J_{\mathcal{F}}, G)$.

Conversely, every leafwise anti-Hermitian metric on the underlying real foliated manifold $(M, \mathcal{F}, J_{\mathcal{F}})$ induces a leafwise complex Riemannian metric on the $(M, \mathcal{F}, J_{\mathcal{F}})$ by setting

$$G(\widehat{X}, \widehat{Y}) = \frac{1}{2}\left(g(X, Y) - ig(X, J_{\mathcal{F}}Y)\right), \tag{2.13}$$

where $X, Y \in \Gamma(T\mathcal{F})$ and $\widehat{X} = (1/2)(X - iJ_{\mathcal{F}}X), \widehat{Y} = (1/2)(Y - iJ_{\mathcal{F}}Y) \in \Gamma(T^{1,0}\mathcal{F})$ as above, and next we extend G to have the conditions (2.2) and (2.3), which is possible because of (2.11).

We recall (see [17]) that a leafwise (or tangential) connection ∇ on $(M, \mathcal{F})$ can be seen as a linear map $\nabla : \Gamma(T_{\mathbb{C}}\mathcal{F}) \times \Gamma(T_{\mathbb{C}}\mathcal{F}) \to \Gamma(T_{\mathbb{C}}\mathcal{F})$ with $\nabla_{fX}Y = f\nabla_X Y$ and $\nabla_X(gY) = X(g) \cdot Y + g\nabla_X Y$ for every $f, g \in C^\infty(M) \otimes_{\mathbb{R}} \mathbb{C}$ and $X, Y \in \Gamma(T_{\mathbb{C}}\mathcal{F})$. Given any leafwise connection D on $(M, \mathcal{F}, J_{\mathcal{F}})$, with respect to an adapted coordinate system (z, x) (leafwise complex), we put

$$D_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} = L_{AB}^C(z, x) \frac{\partial}{\partial z^C}.$$

We notice that the leafwise covariant differentiation, which is defined for leafwise real vector fields in $\Gamma(T\mathcal{F})$, can be extended by complex linearity on leafwise complex vector fields from $\Gamma(T_{\mathbb{C}}\mathcal{F})$. Then $L_{\overline{A}\,\overline{B}}^{\overline{C}} = \overline{L_{AB}^C}$, where $\overline{\overline{A}} = A$.

Definition 2.4. A leafwise (real) connection D on $(M, \mathcal{F}, J_{\mathcal{F}})$ is called *leafwise almost complex* if $DJ_{\mathcal{F}} = 0$.

By direct calculus, we easy obtain

Proposition 2.1. *A leafwise connection D on $(M, \mathcal{F}, J_{\mathcal{F}})$ is leafwise almost complex if and only if $L_{ab}^{\bar{c}} = L_{a\bar{b}}^{c} = 0$*

Let us consider the leafwise (or tangential) Levi-Civita connection of a tangential Riemannian metric (see Proposition 5.18 in [17]). We have

Definition 2.5. A leafwise anti-Hermitian metric g on $(M, \mathcal{F}, J_{\mathcal{F}})$ is called *leafwise anti-Kählerian metric* if the leafwise Levi-Civita connection of g is leafwise almost complex.

In the following we present some examples of leafwise anti-(Kählerian) Hermitian metrics on manifolds endowed with complex foliations.

Example 2.4. Let (M, J, g) be a locally conformal anti-Kähler manifold, with a parallel Lee form and a non light-like Lee vector field (see [5]). Then, its vertical foliation (defined by Lee and anti-Lee vector fields) carries a complex structure with respect to which the induced leafwise metric is anti-Hermitian (see Theorem 5 in [5]).

Example 2.5. It is well known [3] that every parallelisable complex manifold G (including complex Lie groups) can be endowed with anti-Kählerian metrics. Thus, it is easy to see that every product complex foliation defined by trivial fibration $M = G \times N \to N$ where G is a parallelisable complex manifold and N is a paracompact smooth manifold, can be endowed with leafwise anti-Kählerian metrics.

Example 2.6. We consider an m-dimensional smooth manifold M and let $\pi : TM \to M$ its tangent bundle with the total space $\mathcal{T}M$, called the *tangent manifold* of M. Let (x^a, y^a), $a = 1, \dots, m$ the local coordinates on the manifold $\mathcal{T}M$, where (x^a) are the local coordinates on M and (y^a) are the vector coordinates with respect to the basis $\{\frac{\partial}{\partial x^a}\}$. The fibers of TM define the *vertical foliation* $\mathcal{V}$, and we shall denote by $V = T\mathcal{V}$ the tangent bundle along the vertical leaves, which is the vertical bundle. Now, we consider the total space $\mathcal{T}V$ of the vertical bundle $V \to TM$, which is a $3m$-dimensional manifold, called the *vertical tangent manifold* of M. The iterated tangent manifold $\mathcal{T}(\mathcal{T}M)$ has local coordinates $(x^a, y^a, \xi^a, \eta^a)$, where ξ, η are vector coordinates with respect to the natural basis $\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^a}\}$, and the vertical tangent manifold $\mathcal{T}V$ may be seen as the submanifold of $\mathcal{T}(\mathcal{T}M)$ defined by $\xi^a = 0$. For the projection $p : \mathcal{T}V \to M$ given by $p(x, y, \eta) = x$ its kernel is the Whitney sum $W = V_1 \oplus V_2$, where $V_2 \cong V_1 = V$. The vertical bundle $W = span\{\frac{\partial}{\partial y^a}, \frac{\partial}{\partial \eta^a}\}$ is called the *double vertical bundle*, which defines the *double vertical foliation* $\mathcal{W}$, with leaves defined by $x^a = const.$, see [22]. Moreover the following operator

$$J_{\mathcal{W}}(\frac{\partial}{\partial y^a}) = \frac{\partial}{\partial \eta^a}\,,\; J_{\mathcal{W}}(\frac{\partial}{\partial \eta^a}) = -\frac{\partial}{\partial y^a}, \tag{2.14}$$

defines an almost complex structure on W, which is integrable. Thus $(\mathcal{W}, J_{\mathcal{W}})$ is a complex foliation on $\mathcal{T}V$ and, we can consider the holomorphic coordinates along the leaves of $\mathcal{W}$ by putting $z^a = y^a + i\eta^a$.

Let us consider now a (locally) Lagrange metric $L : TM \to \mathbb{R}$ with the fundamental metric tensor $g_{ab}(x, y) = \partial^2 L/\partial y^a \partial y^b$. It defines a nondegenerate Riemannian metric on the vertical bundle V by formula

$$g_{\mathcal{V}}(X, Y) = g_{ab}(x, y) X^a(x, y) Y^b(x, y), \tag{2.15}$$

for every $X = X^a(x, y)\partial/\partial y^a,\ Y = Y^a(x, y)\partial/\partial y^a \in \Gamma(V)$. This metric descends to a vertical pseudo-Riemannian metric on W, defined by

$$g_{\mathcal{W}}(X, Y) = g_{ab}(x, y) X_1^a(x, y, \eta) Y_1^b(x, y, \eta) - g_{ab}(x, y) X_2^a(x, y, \eta) Y_2^b(x, y, \eta), \tag{2.16}$$

for every $X = X_1^a(x, y, \eta)\partial/\partial y^a + X_2^a(x, y, \eta)\partial/\partial \eta^a$, $Y = Y_1^a(x, y, \eta)\partial/\partial y^a + Y_2^a(x, y, \eta)\partial/\partial \eta^a \in \Gamma(W)$. By direct calculus we get $g_{\mathcal{W}}(J_{\mathcal{W}}(X), J_{\mathcal{W}}(Y)) = -g_{\mathcal{W}}(X, Y)$, for every $X, Y \in \Gamma(W)$ which says that $g_{\mathcal{W}}$ is a leafwise anti-Hermitian metric on the complex foliation $(\mathcal{T}V, \mathcal{W}, J_{\mathcal{W}})$. If we consider ∇ the leafwise Levi-Civita connection associated to this metric, then it is locally given by

$$\nabla_{\frac{\partial}{\partial y^a}} \frac{\partial}{\partial y^b} = \nabla_{\frac{\partial}{\partial \eta^a}} \frac{\partial}{\partial \eta^b} = C^c_{ab} \frac{\partial}{\partial y^c}\,,\; \nabla_{\frac{\partial}{\partial y^a}} \frac{\partial}{\partial \eta^b} = \nabla_{\frac{\partial}{\partial \eta^b}} \frac{\partial}{\partial y^a} = C^c_{ab} \frac{\partial}{\partial \eta^c}, \tag{2.17}$$

where

$$C^c_{ab} = \frac{1}{2} g^{cd} \frac{\partial g_{bd}}{\partial y^a}. \tag{2.18}$$

We observe that $\nabla_{\frac{\partial}{\partial y^a}} J_{\mathcal{W}}(\frac{\partial}{\partial \eta^b}) = -J_{\mathcal{W}}\left(\nabla_{\frac{\partial}{\partial y^a}} \frac{\partial}{\partial \eta^b}\right)$, which says that $\nabla J_{\mathcal{W}} \neq 0$, so the metric $g_{\mathcal{W}}$ is not leafwise anti-Kählerian.

Example 2.7. *Sasakian-like almost contact complex Riemannian manifolds.* Let $(M, \varphi, \xi, \eta, g)$ be an almost contact complex Riemannian manifold (see [12]), that is M is a $(2m+1)$-dimensional smooth manifold endowed with a quadruple (φ, ξ, η, g) consisting of an endomorphism φ of the tangent bundle, a vector field ξ and its dual 1-form η, and g is a pseudo-Riemannian metric on M of signature $(m+1, m)$, satisfying the conditions

$$\varphi \circ \xi = 0\,,\; \varphi^2 = -\mathrm{Id} + \eta \otimes \xi\,,\; \eta \circ \varphi = 0\,,\; \eta(\xi) = 1\,,\; g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.19}$$

The associated twin metric is given by $\widetilde{g}(X,Y) = g(\varphi X,Y) + \eta(X)\eta(Y)$ and, the almost contact complex Riemannian manifold (M,φ,ξ,η,g) is said to be *normal* if the Nijenhuis tensor associated with the endomorphism φ satisfies $N_\varphi(X,Y) + 2d\eta(X,Y)\xi = 0$ for every $X,Y \in \Gamma(TM)$. Also, we consider the structure tensor F of type $(0,3)$ on (M,φ,ξ,η,g) given by $F(X,Y,Z) = g((\nabla_X\varphi)Y,Z)$, and according to [12], the almost contact complex Riemannian manifold (M,φ,ξ,η,g) is called *Sasakian-like almost contact complex Riemannian manifold* if the structure tensors φ,ξ,η,g satisfy

$$F(X,Y,Z) = F(\xi,Y,Z) = F(\xi,\xi,Z) = 0 \text{ and } F(X,Y,\xi) = -g(X,Y),\ X,Y,Z \in \Gamma(TM). \tag{2.20}$$

Moreover, it is proved (see [12]) that if (M,φ,ξ,η,g) is a Sasakian-like almost contact complex Riemannian manifold, then it is normal and the fundamental 1-form η is closed. Then, $H = \ker\eta$ is an integrable $2m$-dimensional distribution which defines a complex foliation $(H, J_H = \varphi|_H)$ on M and the leafwise metric $h = g|_H$, induced on each leaf of H, is anti-Kählerian. Indeed, from the first relation of (2.20) it follows that $h((\nabla^h_X J_H)Y,Z) = F(X,Y,Z) = 0,\ X,Y,Z \in \Gamma(H)$, where ∇^h is the leafwise Levi-Civita connection of h. For more concrete examples of such kind of structures we refer [12].

Similary to the case of complex manifolds [18, 19, 21] (see also, [2, 3]), we have the following one-to-one correspondence between the leafwise anti-Kählerian metrics and leafwise holomorphic Riemannian metrics on $(M,\mathcal{F},J_\mathcal{F})$.

Proposition 2.2. *Let M be a smooth $(2m+n)$-dimensional manifold endowed with a complex foliation $(\mathcal{F},J_\mathcal{F})$. If G is a leafwise holomorphic Riemannian metric $(M,\mathcal{F},J_\mathcal{F})$ then g defined in (2.10) is a leafwise anti-Kählerian metric on the realization $(M,\mathcal{F},J_\mathcal{F})$, and conversely if g is a leafwise anti-Kählerian metric on the underlying real foliated manifold $(M,\mathcal{F},J_\mathcal{F})$ then G defined in (2.13) is a leafwise holomorphic Riemannian metric on $(M,\mathcal{F},J_\mathcal{F})$.*

Now, let us denote by ∇ and $\widetilde{\nabla}$ the leafwise Levi-Civita connections of G and $\widetilde{G}$, respectively. Then, as usual, the leafwise Christoffel symbols of G are given by

$$\Gamma^C_{AB} = \frac{1}{2}G^{CD}\left(\frac{\partial G_{BD}}{\partial z^A} + \frac{\partial G_{AD}}{\partial z^B} - \frac{\partial G_{AB}}{\partial z^D}\right), \tag{2.21}$$

where $(G^{AB})_{m\times m}$ denotes the inverse matrix of $(G_{AB})_{m\times m}$, and similarly for the leafwise Christoffel symbols $\widetilde{\Gamma}^C_{AB}$ of $\widetilde{G}$.

Taking into account (2.6) and (2.9), we have the following relations which relates the leafwise Christoffel symbols of G and $\widetilde{G}$, respectively

$$\widetilde{\Gamma}^c_{ab} = \Gamma^c_{ab} = \frac{1}{2}G^{cd}\left(\frac{\partial G_{bd}}{\partial z^a} + \frac{\partial G_{ad}}{\partial z^b} - \frac{\partial G_{ab}}{\partial z^d}\right) \tag{2.22}$$

$$\widetilde{\Gamma}^{\overline{c}}_{ab} = -\Gamma^{\overline{c}}_{ab} = \frac{1}{2}G^{\overline{c}\,\overline{d}}\frac{\partial G_{ab}}{\partial z^{\overline{d}}}\ ,\ \widetilde{\Gamma}^c_{\overline{a}b} = \Gamma^c_{\overline{a}b} = \frac{1}{2}G^{cd}\frac{\partial G_{bd}}{\partial z^{\overline{a}}}. \tag{2.23}$$

By analogy with the case of complex manifolds [7], we define the fundamental leafwise tensor Φ of a leafwise complex Riemannian metric G by setting

$$\Phi(Z_1,Z_2) = \widetilde{\nabla}_{Z_1}Z_2 - \nabla_{Z_1}Z_2\,,\ \text{for every}\ Z_1,Z_2 \in \Gamma(T_\mathbb{C}\mathcal{F}). \tag{2.24}$$

By this definition, we deduce

$$\Phi(\overline{Z}_1,\overline{Z}_2) = \overline{\Phi(Z_1,Z_2)}\,,\ \text{for every}\ Z_1,Z_2 \in \Gamma(T_\mathbb{C}\mathcal{F}). \tag{2.25}$$

Using (2.24), (2.22), (2.23) and (2.25), it follows that the nonvanishing components of the fundamental leafwise tensor Φ are given by

$$\Phi^{\bar{c}}_{ab} = G^{\bar{c}\bar{d}}\frac{\partial G_{ab}}{\partial z^{\bar{d}}} \text{ and } \Phi^{c}_{\bar{a}\bar{b}} = \overline{\Phi^{\bar{c}}_{ab}}. \tag{2.26}$$

Also, from (2.24) and (2.26) we have

Proposition 2.3. *The fundamental leafwise tensor of a complex leafwise Riemannian metric G satisfy*

$$\Phi(Z_1, Z_2) = \Phi(Z_2, Z_1)\,,\ \Phi(J_{\mathcal{F}}Z_1, Z_2) = -J_{\mathcal{F}}\Phi(Z_1, Z_2)\,,\ \forall\, Z_1, Z_2 \in \Gamma(T_{\mathbb{C}}\mathcal{F}). \tag{2.27}$$

Remark 2.2. If $(M, \mathcal{F}, J_{\mathcal{F}}, g)$ is the realization of a complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$ we can define as in (2.24) the fundamental leafwise tensor for real leafwise vector fields, and the property (2.25) of Φ implies that Φ is the complex extension of the real fundamental leafwise tensor on $(M, \mathcal{F}, J_{\mathcal{F}}, g)$.

In the following, we extend the study from [7] to the case of complex Riemannian foliations, and we shall construct a *leafwise characteristic connection* on $(M, \mathcal{F}, J_{\mathcal{F}}, G)$.

We consider the fundamental leafwise tensor of type $(0, 3)$ defined by

$$\Psi(Z_1, Z_2, Z_3) = G(\Phi(Z_1, Z_2), Z_3)\,,\ \text{for every } Z_1, Z_2, Z_3 \in \Gamma(T_{\mathbb{C}}\mathcal{F}). \tag{2.28}$$

In an adapted coordinate system (leafwise complex) on $(M, \mathcal{F}, J_{\mathcal{F}})$, we have

$$\Psi_{AB,C} = \Phi^{D}_{AB}G_{DC}, \tag{2.29}$$

and the nonvanishing componets of $\Psi_{AB,C}$ are

$$\Psi_{ab,\bar{c}} = \frac{\partial G_{ab}}{\partial z^{\bar{c}}} \text{ and } \Psi_{\bar{a}\bar{b},c} = \overline{\Psi_{ab,\bar{c}}}. \tag{2.30}$$

We have

Theorem 2.1. *On every complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$ there exists an unique leafwise connection D with local coefficients L^{C}_{AB} such that*

(i) D is symmetric, that is $L^{C}_{AB} = L^{C}_{BA}$;

(ii) D is leafwise almost complex, that is $L^{\bar{c}}_{ab} = L^{c}_{a\bar{b}} = 0$;

(iii) The leafwise covariant derivatives $D_aG_{bc} = \partial G_{bc}/\partial z^a - L^{d}_{ab}G_{dc} - L^{d}_{ac}G_{bd}$ vanishes.

Proof. If we define the local coefficients of D by

$$L^{C}_{AB} = \Gamma^{C}_{AB} + \frac{1}{2}\Phi^{C}_{AB} - \frac{1}{2}G^{CD}(\Psi_{DA,B} + \Psi_{DB,A}), \tag{2.31}$$

where Γ^{C}_{AB} are the leafwise complex Christoffel symbols of G, then by direct calculus we obtain that D satisfies the conditions of theorem.

Also, if D' is another leafwise connection with local coefficients L'^{C}_{AB} which satisfy the all conditions of theorem, we denote by $D^{C}_{AB} = L^{C}_{AB} - L'^{C}_{AB}$ the leafwise difference tensor. Then, we easily obtain

$$D^{C}_{AB} = D^{C}_{BA}\,,\ D^{\bar{c}}_{ab} = D^{c}_{a\bar{b}} = 0\,,\ D^{d}_{ab}G_{dc} + D^{d}_{ac}G_{ab} = 0, \tag{2.32}$$

which implies $D^{C}_{AB} = 0$, that is $D = D'$, and the uniqueness then follows. □

The leafwise connection from the above theorem, will be called the *leafwise characteristic connection* of the complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$.

The defining equality (2.31) of the leafwise characteristic connection and the properties of the fundamental leafwise tensor, implies

Corollary 2.1. *On every complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$ there exists an unique leafwise connection D such that*

(i) D is symmetric;

(ii) D is leafwise almost complex;

(iii) $D_A G_{BC} = \Psi_{BC,A}$, i.e the leafwise covariant derivative of the metric G is the fundamental leafwise tensor Ψ.

Remark 2.3. The third condition of Theorem 2.1 says that the nonvanishing components of the leafwise tensor $D_A G_{BC}$ are

$$D_{\overline{a}} G_{bc} = \Psi_{bc,\overline{a}} \text{ and } D_a G_{\overline{b}\,\overline{c}} = \overline{D_{\overline{a}} G_{bc}}. \tag{2.33}$$

On the realization of a complex Riemannian foliation we have

Corollary 2.2. *If $(M, \mathcal{F}, J_{\mathcal{F}}, g)$ is the realization of a complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$, then the leafwise characteristic connection D on $(M, \mathcal{F}, J_{\mathcal{F}}, g)$ is the unique leafwise connection which satisfy the conditions*

(i) D is symmetric;

(ii) D is leafwise almost complex;

(iii) $(D_X g)(Y, Z) = (D_{J_{\mathcal{F}}X} g)(J_{\mathcal{F}}Y, Z)$, for every $X, Y, Z \in \Gamma(T\mathcal{F})$.

The defining equality (2.31) together with (2.30) imply that the nonvanishing coefficients of the leafwise characteristic connection D are

$$L^c_{ab} = \Gamma^c_{ab} \text{ and } L^{\overline{c}}_{\overline{a}\,\overline{b}} = \overline{L^c_{ab}}, \tag{2.34}$$

that is, D is completely determined on $\Gamma(T^{1,0}\mathcal{F})$.

We notice that a leafwise vector field $Z = Z^a(z, x)\partial/\partial z^a \in \Gamma(T^{1,0}\mathcal{F})$ is leafwise holomorphic if Z^a are leafwise holomorphic functions on $(M, \mathcal{F}, J_{\mathcal{F}})$. Also, using the leafwise Cauchy-Riemann equations, it is easy to see that for a given leafwise vector field $X \in \Gamma(T\mathcal{F})$, then $\widehat{X} = (1/2)(X - iJ_{\mathcal{F}}X) \in \Gamma(T^{1,0}\mathcal{F})$ is leafwise holomorphic if and only if

$$(\mathcal{L}_X J_{\mathcal{F}})Y = [X, J_{\mathcal{F}}Y] - J_{\mathcal{F}}[X, Y] = 0\,,\ \forall\, Y \in \Gamma(T\mathcal{F}). \tag{2.35}$$

In that follows we denote the set of leafwise holomorphic vector fields on $(M, \mathcal{F}, J_{\mathcal{F}})$ by $\Gamma_{\text{hol}}(T^{1,0}\mathcal{F})$.

Definition 2.6. A leafwise connection D on $(M, \mathcal{F}, J_{\mathcal{F}})$ is called *leafwise holomorphic* if $D_{Z_1} Z_2 \in \Gamma_{\text{hol}}(T^{1,0}\mathcal{F})$ for arbitrary leafwise holomorphic vector fields Z_1, Z_2.

We have

Proposition 2.4. *The leafwise characteristic connection D of a complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$ is leafwise holomorphic if and only if the leafwise complex Christoffel symbols $L^c_{ab} = \Gamma^c_{ab}$ are leafwise holomorphic functions.*

As a direct consequence of (2.30), (2.22), (2.23), Corollary 2.1 and (2.31), we get

Theorem 2.2. *For every complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$, the following assertions are equivalent:*

(i) The fundamental leafwise tensor Φ (or Ψ) is zero;

(ii) The local components G_{ab} of the leafwise metric G are leafwise holomorphic functions;

(iii) The leafwise Levi-Civita connection ∇ of G is leafwise almost complex, that is $\nabla J_{\mathcal{F}} = 0$;

(iv) The leafwise characteristic connection D is metrical with respect to G, that is $DG = 0$;

(v) The leafwise Levi-Civita connection ∇ coincides with the leafwise characteristic connection D.

Let R be the leafwise characteristic curvature tensor of the leafwise characteristic connection D, defined as usual by

$$R(X,Y)Z = [D_X, D_Y]\,Z - D_{[X,Y]}Z, \text{ for every } X,Y,Z \in \Gamma(T_{\mathbb{C}}\mathcal{F}).$$

The local components of R are given by

$$R\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right)\frac{\partial}{\partial z^C} = R^D_{C,AB}\frac{\partial}{\partial z^D}, \tag{2.36}$$

and the nonvanishing components of R are

$$R^d_{c,ab} = \frac{\partial L^d_{cb}}{\partial z^a} - \frac{\partial L^d_{ca}}{\partial z^b} + L^f_{cb}L^d_{fa} - L^f_{ca}L^d_{fb}\,,\ R^{\bar d}_{\bar c,\bar a\,\bar b} = \overline{R^d_{c,ab}}, \tag{2.37}$$

$$R^d_{c,\bar a b} = \frac{\partial L^d_{bc}}{\partial z^{\bar a}}\,,\ R^{\bar d}_{\bar c,a\bar b} = \overline{R^d_{c,\bar a b}}. \tag{2.38}$$

It is easy to see that $R^d_{c,\bar a b} = 0$ if and only if D is a leafwise holomorphic connection. Also, the leafwise characteristic Riemann curvature tensor of D is defined as usual by $\mathcal{R}(Z_1,Z_2,Z_3,Z_4) = G(R(Z_1,Z_2)Z_3,Z_4)$ and its local components are $R_{ABCD} = G_{DF}R^F_{C,AB}$. Its nonvanishing components are

$$R_{abcd} = G_{df}R^f_{c,ab} \text{ and } R_{\bar a bcd} = G_{df}R^f_{c,\bar a b}, \tag{2.39}$$

and their complex conjugates.

Moreover, every nondegenerate 2-plane in $T^{1,0}_{(z,x)}\mathcal{F}$ is called a *leafwise holomorphic* 2-plane, and the *leafwise holomorphic characteristic sectional curvature*, for a given leafwise holomorphic 2-plan $P = span\{Z_1,Z_2\}$, where $Z_1,Z_2 \in \Gamma(T^{1,0}_{(z,x)}\mathcal{F})$, $(z,x)\in M$, is defined by

$$K_{(z,x)}(P) = \frac{\mathcal{R}(Z_1,Z_2,Z_1,Z_2)}{G(Z_1,Z_1)G(Z_2,Z_2) - (G(Z_1,Z_2))^2}. \tag{2.40}$$

Then, the following Schur type theorem holds.

Theorem 2.3. *Let $(M,\mathcal{F},J_{\mathcal{F}})$ be a connected complex foliation with $m \geq 3$ endowed with a leafwise holomorphic metric G. If the leafwise holomorphic sectional curvatures does not depend on the 2-plane P, then it is a basic function $c(x)$.*

In the end of this section we describe the Einstein equations for complex Riemannian foliations. The associated leafwise characteristic Ricci tensor Ric is locally given by

$$\mathrm{Ric}\left(\frac{\partial}{\partial z^C}, \frac{\partial}{\partial z^A}\right) = \mathrm{Ric}_{CA} = R^B_{C,AB}, \tag{2.41}$$

and its nonvanishing components are

$$\mathrm{Ric}_{ca} = R^b_{c,ab}\,,\ \mathrm{Ric}_{c\bar a} = R^b_{c,\bar a b}\,,\ \mathrm{Ric}_{\bar c\,\bar a} = \overline{\mathrm{Ric}_{ca}}\,,\ \mathrm{Ric}_{\bar c a} = \overline{\mathrm{Ric}_{c\bar a}}. \tag{2.42}$$

The function ρ defined by

$$\rho = G^{CA}\mathrm{Ric}_{CA} = G^{ca}\mathrm{Ric}_{ca} + G^{\bar{c}\bar{a}}\mathrm{Ric}_{\bar{c}\bar{a}} \tag{2.43}$$

is called the *leafwise scalar curvature* of D and it is a real valued function.

The equation

$$\mathrm{Ric} - \frac{\rho}{2}G = 8\pi c T \tag{2.44}$$

is called the *Einstein equation* of the complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$. In the equation (2.44), the left hand side is called the *leafwise Einstein curvature* which is constructed using the leafwise complex Riemannian metric G, while in the right hand side we have a leafwise tensor T called the *leafwise stress-energy-momentum tensor* and represents the matter and energy that generate the gravitational field of potentials (G_{AB}). The constant c is the gravitational constant. Locally, the leafwise Einstein equation is expressed as

$$\mathrm{Ric}_{AB} - \frac{\rho}{2}G_{AB} = 8\pi c T_{AB}. \tag{2.45}$$

Remark 2.4. i) If the Einstein equation holds, then taking into account (2.6) it follows that

$$\mathrm{Ric}_{a\bar{b}} = 8\pi c T_{a\bar{b}}. \tag{2.46}$$

ii) In the empty leave space (no matter, no energy) we have $T_{AB} = 0$, and contracting (2.45) with G^{AB} one gets $\rho = 0$ and so it reduced to

$$\mathrm{Ric}_{AB} = 0. \tag{2.47}$$

Consequently, $\mathrm{Ric}_{ab} = \mathrm{Ric}_{\bar{a}\bar{b}} = 0$.

iii) Letting $E_{AB} = \mathrm{Ric}_{AB} - (\rho/2)G_{AB}$ and $E^A_B = G^{AC}E_{CB}$, the leafwise divergence of E is defined by

$$\mathrm{div}\, E = E^A_{B|A}, \tag{2.48}$$

where "$|$" denotes the leafwise covariant derivative with respect to ∇ and we have $\mathrm{div}\, E = 0$. Indeed, the assertion follows using the second leafwise Bianchi identity $\sum_{cycl}(\nabla_X R)(Y, Z) = 0$ written in a local basis $\{\partial/\partial z^A\}$ of $\Gamma(T_{\mathbb{C}}\mathcal{F})$. Assuming the Einstein equation holds, by using $\mathrm{div}\, E = 0$, we must have

$$\mathrm{div}\, T = 0, \tag{2.49}$$

which is called the *leafwise continuity condition* for complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$.

Also, by analogy with the case of complex manifolds, see [11], the following result concerning the Einstein condition for complex Riemannian foliations holds.

Definition 2.7. The complex Riemannian foliation $(M, \mathcal{F}, J_{\mathcal{F}}, G)$ is said to be *leafwise characteristic Einstein* if $\mathrm{Ric}_{c\bar{a}} = 0$ and $\mathrm{Ric}_{ca} = fG_{ca}$, where $f = f_1 + if_2$ is a complex valued function on $(M, \mathcal{F})$.

Theorem 2.4. *Let $(M, \mathcal{F}, J_{\mathcal{F}}, G)$ be a leafwise characteristic Einstein complex Riemannian foliation with $m \geq 3$. Then the leafwise characteristic scalar curvature $\rho_0 = G^{ca}\mathrm{Ric}_{ca}$ is a leafwise anti-holomorphic function on $(M, \mathcal{F}, J_{\mathcal{F}})$ and* $\mathrm{Ric}_{ca} = (\rho_0/m)G_{ca}$.

Example 2.8. Let us consider the complex foliation $(\mathcal{W}, J_{\mathcal{W}})$ defined by the double vertical bundle W on the vertical tangent manifold $\mathcal{T}V$ as in the Example 2.6 endowed with leafwise anti-hermitian metric $g_{\mathcal{W}}$ from (2.16). If we consider $z^a = y^a + i\eta^a$, $a = 1, \ldots, m$ the holomorphic coordinates along the leaves of $\mathcal{W}$ and the leafwise complex vector fields

$$\frac{\partial}{\partial z^a} = \frac{1}{2}\left(\frac{\partial}{\partial y^a} - i\frac{\partial}{\partial \eta^a}\right), \; \frac{\partial}{\partial z^{\bar{a}}} = \frac{1}{2}\left(\frac{\partial}{\partial y^a} + i\frac{\partial}{\partial \eta^a}\right), \; a = 1, \ldots, m,$$

then the corresponding leafwise complex Riemannian metric $G_{\mathcal{W}}$ induced by $g_{\mathcal{W}}$ (using (2.13)) have the local components

$$G_{ab} = G_{\mathcal{W}}\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^b}\right) = \frac{1}{2}g_{ab}\,,\; G_{\bar{a}\,\bar{b}} = G_{\mathcal{W}}\left(\frac{\partial}{\partial z^{\bar{a}}}, \frac{\partial}{\partial z^{\bar{b}}}\right) = \overline{G_{ab}} = \frac{1}{2}g_{ab} \text{ and } G_{a\bar{b}} = G_{\bar{a}b} = 0.$$

Moreover, the local coefficients of the corresponding leafwise characteristic connection are $L^c_{ab} = (1/8)C^c_{ab}$ and the local components of characteristic curvatures of $G_{\mathcal{W}}$ are given by

$$\begin{aligned} R^d_{c,ab}(G_{\mathcal{W}}) &= \frac{1}{16}\left(\frac{\partial C^d_{cb}}{\partial y^a} - \frac{\partial C^d_{ca}}{\partial y^b}\right) + \frac{1}{64}(C^f_{cb}C^d_{fa} - C^f_{ca}C^d_{fb}) \\ &= \frac{1}{64}R^d_{c,ab}(g_{\mathcal{V}}) + \frac{3}{64}\left(\frac{\partial C^d_{cb}}{\partial y^a} - \frac{\partial C^d_{ca}}{\partial y^b}\right) \end{aligned}$$

and

$$R^d_{c,\bar{a}b}(G_{\mathcal{W}}) = \frac{1}{16}\frac{\partial C^d_{cb}}{\partial y^a},$$

where $R^{\cdot}_{\cdot,\cdot\cdot}(g_{\mathcal{V}})$ denotes the local components of the curvature of vertical metric $g_{\mathcal{V}}$ from (2.15).

The corresponding leafwise Ricci tensors are

$$\mathrm{Ric}_{ca}(G_{\mathcal{W}}) = \frac{1}{64}\mathrm{Ric}_{ca}(g_{\mathcal{V}}) + \frac{3}{64}\left(\frac{\partial C^b_{cb}}{\partial y^a} - \frac{\partial C^b_{ca}}{\partial y^b}\right) \text{ and } \mathrm{Ric}_{c\bar{a}}(G_{\mathcal{W}}) = \frac{1}{16}\frac{\partial C^b_{cb}}{\partial y^a}.$$

Hence, it is easy to see that if $\partial C^b_{cb}/\partial y^a = \partial C^b_{ca}/\partial y^b = 0$ and the vertical metric $g_{\mathcal{V}}$ is Einstein (in the next section, the definition of an Einstein metric is recalled), then $(\mathcal{T}V, \mathcal{W}, J_{\mathcal{W}}, G_{\mathcal{W}})$ is leafwise characteristic Einstein.

3. Leafwise holomorphic Riemannain Einstein metrics

We recall that a (real) metric g on the (real) manifold M is said to be Einsteinian if $\mathrm{Ric}(g) = \lambda g$, where λ is a real constant and $\mathrm{Ric}(g)$ denotes the Ricci tensor of the metric g. By analogy, a (real or complex) leafwise metric G on $(M, \mathcal{F})$ is called *leafwise Einstein metric* if

$$\mathrm{Ric}(G) = \lambda G, \tag{3.1}$$

where λ is a (real or complex) constant and $\mathrm{Ric}(G)$ denotes the Ricci tensor of the leafwise metric G.

The aim of this section, is to point out that by taking the real part of a leafwise holomorphic Einstein metric on a smooth $(2m+n)$-dimensional manifold M endowed with a complex foliation $(\mathcal{F}, J_{\mathcal{F}})$ one gets a real leafwise Einstein metric on the real foliated manifold $(M, \mathcal{F})$ obtaining a result similar to Theorem 5.1 from [2] from the anti-Kählerian manifolds case.

Let $(M, \mathcal{F}, J_{\mathcal{F}}, G)$ be a complex foliation $(\mathcal{F}, J_{\mathcal{F}})$ on M endowed with a leafwise holomorphic Riemannian metric G. Then, as we already noticed in the previous section the relations (2.10) and (2.13) establishes an one-to-one correspondence between the leafwise anti-Kählerian metrics on the (real) foliated manifold $(M, \mathcal{F}, J_\mathcal{F})$ and the leafwise holomorphic Riemannian metrics on the complex foliation $(\mathcal{F}, J_\mathcal{F})$ on M.

Although we can follow an argument similar from [2, 3], for a better presentation of the notions that we use, in this section we denote the leafwise holomorphic Riemannain metric G by $\widehat{g}$ and we follow an argument similar to [19, 21] for Kähler-Norden manifolds.

Without loss of generality, we consider the real leafwise vector fields $X, Y, \ldots \in \Gamma(T\mathcal{F})$ such that $\widehat{X}, \widehat{Y}, \ldots \in \Gamma_{\rm hol}(T^{1,0}\mathcal{F})$, are leafwise holomorphic vector fields on the complex foliation $(M, \mathcal{F}, J_\mathcal{F})$, that is the relation (2.35) holds. Then, we easily obtain

$$[J_\mathcal{F}X, Y] = [X, J_\mathcal{F}Y] = J_\mathcal{F}[X, Y]\,,\; [J_\mathcal{F}X, J_\mathcal{F}Y] = [X, Y]\,,\; [\widehat{X}, \widehat{Y}] = \widehat{[X, Y]} =: [X, Y]\widehat{}\,. \tag{3.2}$$

Also, by a direct calculation, we have that for every leafwise complex function $f = Re\, f + iIm\, f$ on $(M, \mathcal{F}, J_\mathcal{F})$, and every real leafwise vector field $X \in \Gamma(T\mathcal{F})$, the following relation holds

$$f\widehat{X} = ((Re\, f)X + (Im\, f)X)\widehat{}\,, \tag{3.3}$$

and, moreover, if f is leafwise holomorphic, then the leafwise Cauchy-Riemann equations imply

$$X(Re\, f) = (J_\mathcal{F}X)(Im\, f)\,,\; (J_\mathcal{F}X)(Re\, f) = -X(Im\, f)\,,\; \widehat{X}f = X(Re\, f) + iX(Im\, f). \tag{3.4}$$

Now, for every real tangent space along the leaves $T_{(z,x),\mathbb{R}}\mathcal{F}$, $(z, x) \in M$, we can choose an adapted orthonormal (real) frame $\{e_a, J_\mathcal{F}e_a\}$, $a \in \{1, \ldots, m\}$ in $\Gamma(T\mathcal{F})$, such that

$$g(e_a, e_b) = \delta_{ab}\,,\; g(J_\mathcal{F}e_a, J_\mathcal{F}e_b) = -\delta_{ab}\,,\; g(e_a, J_\mathcal{F}e_b) = 0\,,\; a, b \in \{1, \ldots, m\}. \tag{3.5}$$

Then, we obtain an adapted leafwise complex frame $\{\widehat{e_a}\}$, $a \in \{1, \ldots, m\}$, for $\Gamma(T^{1,0}_{(z,x)}\mathcal{F})$, where $\widehat{e}_a = (1/2)(e_a - iJ_\mathcal{F}e_a)$ for which $\widehat{g}(\widehat{e}_a, \widehat{e}_b) = (1/2)\delta_{ab}$.

Let ∇ and $\widehat{\nabla}$ be the leafwise Levi-Civita connections of the leafwise anti-Kählerian metric g and of the leafwise holomorphic Riemannian metric $\widehat{g}$, respectively. We also consider the leafwise Ricci tensor fields associated to the leafwise metrics g and $\widehat{g}$, respectively, given by

$$\mathrm{Ric}(g)(X, Y) = \mathrm{Tr}\{Z \mapsto R(Z, X)Y\} \text{ and } \mathrm{Ric}(\widehat{g})(\widehat{X}, \widehat{Y}) = \mathrm{Tr}\{\widehat{Z} \mapsto \widehat{R}(\widehat{Z}, \widehat{X})\widehat{Y}\}, \tag{3.6}$$

and let us denote by Q and $\widehat{Q}$ be the associated leafwise Ricci operators, given by

$$g(QX, Y) = \mathrm{Ric}(g)(X, Y) \text{ and } \widehat{g}(\widehat{Q}\widehat{X}, \widehat{Y}) = \mathrm{Ric}(\widehat{g})(\widehat{X}, \widehat{Y}). \tag{3.7}$$

Following step by step the construction from the case of Kähler-Norden manifolds (see [21]), we have the following result which relates the leafwise Ricci tensors $\mathrm{Ric}(g)$ and $\mathrm{Ric}(\widehat{g})$.

Proposition 3.1. *The leafwise Ricci tensors* $\mathrm{Ric}(g)$*,* $\mathrm{Ric}(\widehat{g})$ *and the leafwise Ricci operators* Q*,* $\widehat{Q}$ *satisfy the following relations*

$$\mathrm{Ric}(g)(J_\mathcal{F}X, Y) = \mathrm{Ric}(g)(X, J_\mathcal{F}Y)\,,\; \mathrm{Ric}(g)(J_\mathcal{F}X, J_\mathcal{F}Y) = -\mathrm{Ric}(g)(X, Y)\,,\; QJ_\mathcal{F} = J_\mathcal{F}Q \tag{3.8}$$

and

$$\mathrm{Ric}(\widehat{g})(\widehat{X}, \widehat{Y}) = \frac{1}{2}(\mathrm{Ric}(g)(X, Y) - i\mathrm{Ric}(g)(X, J_\mathcal{F}Y))\,,\; \widehat{Q}\widehat{X} = \widehat{QX}. \tag{3.9}$$

The first relation of (3.9) leads to the announced result, that is

Theorem 3.1. *Let us suppose that g is a leafwise anti-Kählerian metric on $(M, \mathcal{F}, J_\mathcal{F})$, that is $(M, \mathcal{F}, J_\mathcal{F})$ is endowed with a leafwise holomorphic Riemannian metric $\widehat{g} \equiv (\widehat{g}_{ab}(z, x))$, $a, b \in \{1, \ldots, m\}$ and with a real leafwise metric $g \equiv (g_{\alpha\beta}(u, x))$, $\alpha, \beta \in \{1, \ldots, 2m\}$ given by $g = 2Re\,\widehat{g}$. Then, the leafwise holomorphic metric $\widehat{g}$ is Einstein with the real constant λ if and only if the real leafwise metric g is Einstein with the same constant.*

Remark 3.1. We notice that starting from the original leafwise anti-Kählerian metric g on $(M, \mathcal{F}, J_{\mathcal{F}})$, the real leafwise twin metric $\widetilde{g} = h$ can be considered, that is $h(X,Y) := (g \circ J_{\mathcal{F}})(X,Y) = g(J_{\mathcal{F}}X, Y)$, for every $X, Y \in \Gamma(T\mathcal{F})$. We find

$$h(X,Y) = 2\mathrm{Im}\widehat{g}(\widehat{X}, \widehat{Y}), \ \forall X, Y \in \Gamma(T\mathcal{F}). \tag{3.10}$$

Moreover, if we denote by ∇ the covariant differentiation of the Levi-Civita connection associated to the leafwise anti-Kählerian metric g, then we have

$$\nabla h = \nabla g \circ J_{\mathcal{F}} + g \circ \nabla J_{\mathcal{F}} = 0. \tag{3.11}$$

The above relation says that, the leafwise Levi-Civita connection of g coincides with the leafwise Levi-Civita connection of h, thus they have the same real and complex leafwise Riemann and Ricci tensors (see also the discussion from the previous section). In the real case only one of two leafwise twin metrics can be Einsteinian. In the complex case the Einstein condition $\mathrm{Ric}(\widehat{g}) = \lambda\widehat{g}$ implies $\mathrm{Ric}(\widehat{h}) = i\lambda\widehat{h}$, that is, both leafwise holomorphic metrics $\widehat{g}$ and $\widehat{h}$ are Einstein metrics at the same time. We can conclude that the leafwise metric h is an Einstein metric with an imaginary cosmological constant.

Finally, we notice that if the leafwise holomorphic metric $\widehat{g}$ is Einstein with complex constant $\widehat{\lambda}$, that is

$$\mathrm{Ric}(\widehat{g}) = \widehat{\lambda}\widehat{g}, \ \widehat{\lambda} \in \mathbb{C}, \tag{3.12}$$

then, similarly to the Kähler-Norden manifolds case (see [19]), we can describe the following generalization of Theorem 3.1.

Theorem 3.2. *The leafwise holomorphic Riemannian metric $\widehat{g}$ on $(M, \mathcal{F}, J_{\mathcal{F}})$ is leafwise holomorphic Einstein with complex constant $\widehat{\lambda} = \lambda_1 + i\lambda_2$ if and only if*

$$\mathrm{Ric}(g)(X,Y) = \lambda_1 g(X,Y) + \lambda_2 g(X, J_{\mathcal{F}}Y). \tag{3.13}$$

Moreover, in the formula (3.13)*, we have $\lambda_1 = K/2m$ and $\lambda_2 = -K^*/2m$, where $K = \mathrm{Tr}Q$ and $K^* = \mathrm{Tr}(J_{\mathcal{F}}Q)$.*

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