Physical systems in a space with noncommutativity of coordinates

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Abstract. We consider a space with canonical noncommutativity of coordinates. The problem of rotational symmetry breaking is studied in this space. To preserve the rotational symmetry we consider the generalization of constant matrix of noncommutativity to a tensor defined with the help of additional coordinates governed by a rotationally symmetric system. The properties of physical systems are examined in the rotationally invariant space with noncommutativity of coordinates. Namely, we consider an effect of coordinate noncommutativity on the energy levels of the hydrogen atom in the rotationally invariant noncommutative space. The motion of a particle in the uniform field is also studied in the noncommutative space with preserved rotational symmetry. On the basis of exact calculations we show that there is an effect of coordinate noncommutativity on the mass of a particle and conclude that noncommutativity causes the anisotropy of mass.

1. Introduction

A lot of attention has been devoted recently to studies of properties of physical systems in a noncommutative space. The idea of noncommutative structure of space was suggested by Heisenberg. Later, Snyder formalized the idea in his paper [1]. In recent years, the interest in noncommutativity is motivated by the development of String Theory and Quantum Gravity (see, for instance, [2, 3]).

Different problems have been studied in a space with canonical noncommutativity of coordinates

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{1}$$
$$[X_i, P_j] = i\hbar\delta_{ij}, \tag{2}$$
$$[P_i, P_j] = 0, \tag{3}$$

where θ_{ij} is a constant antisymmetric matrix. Among them the hydrogen atom [4–11], the Landau problem (see, for instance, [12–16]), quantum mechanical system in a central potential [17], classical particle in a gravitational potential [18, 19], system of particles in a gravitational field [20], motion of a body in a gravitational field and the equivalence principle [21] and many others.

It is important to note that in the case of canonical noncommutativity of coordinates (1) there is a problem of rotational symmetry breaking [4, 22]. So, in order to preserve the symmetry different classes of noncommutative algebras were considered (see, for example, [23, 24] and references therein).

In previous paper [23] in order to construct rotationally invariant noncommutative algebra we considered the generalization of the constant matrix θ_{ij} to a tensor. We proposed to construct the tensor in the following form

$$\theta_{ij} = \frac{l_0}{\hbar}\varepsilon_{ijk}a_k, \tag{4}$$

where l_0 is a constant with the dimension of length and a_i are additional coordinates which are governed by a rotationally symmetric system. For simplicity, we suppose that coordinates a_i are governed by the harmonic oscillator

$$H_{osc} = \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 a^2}{2}. \tag{5}$$

It is generally believed that the parameter of noncommutativity of coordinates is of the order of the Planck scale. So, we put

$$\sqrt{\frac{\hbar}{m\omega}} = l_P, \tag{6}$$

where l_P is the Planck length. We also consider the case when the frequency of harmonic oscillator is very large. Therefore, the distance between the energy levels of harmonic oscillator tends to infinity. So, harmonic oscillator put into the ground state remains in it.

We would like to note that coordinates a_i can be treated as some internal coordinates of a particle. Quantum fluctuations of these coordinates lead effectively to a non-point-like particle, size of which is of the order of the Planck scale.

So, the rotationally invariant noncommutative algebra reads

$$[X_i, X_j] = i\varepsilon_{ijk}l_0 a_k, \tag{7}$$
$$[X_i, P_j] = i\hbar\delta_{ij}, \tag{8}$$
$$[P_i, P_j] = 0. \tag{9}$$

The coordinates a_i and momenta p_i^a satisfy the ordinary commutation relations $[a_i, a_j] = 0$, $[a_i, p_j^a] = i\hbar\delta_{ij}$, $[p_i^a, p_j^a] = 0$. Also, a_i commute with X_i and P_i. As a consequence, tensor of noncommutativity θ_{ij} given by (4) commutes with X_i and P_i too. Therefore, X_i, P_i and θ_{ij} satisfy the same commutation relations as in the case of the canonical version of noncommutativity. Besides, noncommutative algebra (7)-(9) is manifestly rotationally invariant.

It is worth noting that the rotational symmetry is preserved in the case of another way of generalization of the tensor of noncommutativity $\theta_{ij} = \alpha(a_i b_j - a_j b_i)/\hbar$ where a_i, b_i are additional coordinates governed by a rotationally symmetric system and α is a dimensionless constant [23]. In previous papers [23, 25] we studied the hydrogen atom in the rotationally invariant noncommutative space $[X_i, X_j] = i\alpha(a_i b_j - a_j b_i)$, $[X_i, P_j] = i\hbar\delta_{ij}$, $[P_i, P_j] = 0$.

In this paper we consider physical systems in rotationally invariant space with noncommutativity of coordinates (7). We study the motion of a particle in the uniform field in the space. On the basis of exact calculations we show that there is an effect of noncommutativity of coordinates on the mass of a particle and noncommutativity causes the anisotropy of mass. Also we consider the hydrogen atom in rotationally invariant noncommutative space (7)-(9) and study the effect of coordinate noncommutativity on the energy levels of the atom.

The paper is organized as follows. In Section 2, we consider the energy levels of the hydrogen atom in rotationally invariant noncommutative space (7)-(9). In Section 3, the motion of a particle in the uniform field in rotationally invariant noncommutative space is studied. Conclusions are presented in Section 4.

2. Energy levels of hydrogen atom in noncommutative space with preserved rotational symmetry

Let us consider the hydrogen atom in noncommutative space (7)-(9). The Hamiltonian of the hydrogen atom reads

$$H_h = \frac{P^2}{2M} - \frac{e^2}{R}, \tag{10}$$

where $R = \sqrt{\sum_i X_i^2}$ and X_i satisfy (7).

Because of definition of the tensor of noncommutativity (4) in rotationally invariant noncommutative space we have to take into account additional terms that correspond to the harmonic oscillator (5). Therefore, we consider the total Hamiltonian as follows

$$H = H_h + H_{osc}. \tag{11}$$

Let us use the following representation

$$X_i = x_i - \frac{1}{2}\theta_{ij} p_j, \tag{12}$$

$$P_i = p_i, \tag{13}$$

where θ_{ij} is given by (4). Coordinates x_i and momenta p_i satisfy the ordinary commutation relations

$$[x_i, x_j] = 0, \tag{14}$$

$$[p_i, p_j] = 0, \tag{15}$$

$$[x_i, p_j] = i\hbar\delta_{ij}, \tag{16}$$

and commute with a_i, p_i^a, namely $[x_i, a_j] = 0$, $[x_i, p_j^a] = 0$, $[p_i, a_j] = 0$, $[p_i, p_j^a] = 0$. It is worth mentioning that coordinates X_i do not commute with p_j^a. Taking into account (4) and (12), the coordinates X_i can be written as follows

$$X_i = x_i + \frac{l_0}{2\hbar}[\mathbf{a} \times \mathbf{p}]_i. \tag{17}$$

Therefore, we have

$$[X_i, p_j^a] = i\varepsilon_{ijk}\frac{l_0}{2} p_k. \tag{18}$$

Let us write the expansion for H up to the second order in $\boldsymbol{\theta} = l_0\mathbf{a}/\hbar$. Using (17), we have

$$R = \sqrt{\sum_i X_i^2} = \sqrt{r^2 - \frac{l_0}{\hbar}(\mathbf{a}\cdot\mathbf{L}) + \frac{l_0^2}{4\hbar^2}[\mathbf{a}\times\mathbf{p}]^2}, \tag{19}$$

with $r = \sqrt{\sum_i x_i^2}$ and $\mathbf{L} = [\mathbf{r} \times \mathbf{p}]$. It is important to note that the operators under the square root do not commute. Therefore, the expansion for R can be written as follows

$$R = r - \frac{l_0}{2\hbar r}(\mathbf{a}\cdot\mathbf{L}) - \frac{l_0^2}{8\hbar^2 r^3}(\mathbf{a}\cdot\mathbf{L})^2 + \frac{l_0^2}{16\hbar^2}\left(\frac{1}{r}[\mathbf{a}\times\mathbf{p}]^2 + [\mathbf{a}\times\mathbf{p}]^2\frac{1}{r} + a^2 f(\mathbf{r})\right), \tag{20}$$

where $f(\mathbf{r})$ is unknown function. Squaring left- and right-hand sides of equation (20) we obtain

$$\frac{\hbar^2}{r^4}[\mathbf{a}\times\mathbf{r}]^2 - ra^2 f(\mathbf{r}) = 0. \tag{21}$$

Finally, from (21) we have

$$a^2 f(\mathbf{r}) = \frac{\hbar^2}{r^5}[\mathbf{a}\times\mathbf{r}]^2. \tag{22}$$

So, using (20) and (22), it is easy to write expansion for the inverse distance R^{-1}

$$\frac{1}{R} = \frac{1}{r} + \frac{l_0}{2\hbar r^3}(\mathbf{a}\cdot\mathbf{L}) + \frac{3l_0^2}{8\hbar^2 r^5}(\mathbf{a}\cdot\mathbf{L})^2 - \frac{l_0^2}{16\hbar^2}\left(\frac{1}{r^2}[\mathbf{a}\times\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\mathbf{a}\times\mathbf{p}]^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}[\mathbf{a}\times\mathbf{r}]^2\right). \tag{23}$$

Therefore, the Hamiltonian (11) can be rewritten as follows

$$H = H_0 + V, \tag{24}$$

with

$$H_0 = H_h^{(0)} + H_{osc}. \tag{25}$$

Here $H_h^{(0)} = \frac{p^2}{2M} - \frac{e^2}{r}$ is the Hamiltonian of the hydrogen atom in the ordinary space and perturbation V caused by the noncommutativity of coordinates is given by

$$V = -\frac{l_0 e^2}{2\hbar r^3}(\mathbf{a}\cdot\mathbf{L}) - \frac{3l_0^2 e^2}{8\hbar^2 r^5}(\mathbf{a}\cdot\mathbf{L})^2 + \frac{l_0^2 e^2}{16\hbar^2}\left(\frac{1}{r^2}[\mathbf{a}\times\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\mathbf{a}\times\mathbf{p}]^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}[\mathbf{a}\times\mathbf{r}]^2\right). \tag{26}$$

Let us find the corrections to the energy levels of the hydrogen atom caused by the noncommutativity of coordinates (7). Note that $H_h^{(0)}$ commutes with H_{osc}. So, the eigenvalues and the eigenstates which correspond to H_0 (25) read

$$E^{(0)}_{n,\{n^a\}} = -\frac{e^2}{2a_B n^2} + \hbar\omega\left(n_1^a + n_2^a + n_3^a + \frac{3}{2}\right), \tag{27}$$

$$\psi^{(0)}_{n,l,m,\{n^a\}} = \psi_{n,l,m}\psi^a_{n_1^a,n_2^a,n_3^a}, \tag{28}$$

where $\psi_{n,l,m}$ are the eigenfunctions of the hydrogen atom in the ordinary space, $\psi^a_{n_1^a,n_2^a,n_3^a}$ are the eigenfunctions of three-dimensional harmonic oscillator, and a_B is the Bohr radius. In the case when harmonic oscillator is in the ground state, according to the perturbation theory, in the first order in V we have

$$\Delta E^{(1)}_{n,l} = \langle\psi^{(0)}_{n,l,m,\{0\}}|V|\psi^{(0)}_{n,l,m,\{0\}}\rangle =$$
$$= -\frac{\hbar^2 e^2\langle\theta^2\rangle}{a_B^5 n^5}\left(\frac{1}{6l(l+1)(2l+1)} - \frac{6n^2 - 2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)} + \right.$$
$$\left. + \frac{5n^2 - 3l(l+1) + 1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)} - \frac{5}{6}\frac{5n^2 - 3l(l+1) + 1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}\right), \tag{29}$$

where $\langle\theta^2\rangle$ is given by

$$\langle\theta^2\rangle = \frac{l_0^2}{\hbar^2}\langle\psi^a_{0,0,0}|a^2|\psi^a_{0,0,0}\rangle = \frac{3l_0^2}{2\hbar}\left(\frac{1}{m\omega}\right) = \frac{3}{2}\left(\frac{l_0 l_P}{\hbar}\right)^2. \tag{30}$$

The details of calculation of the corresponding integrals can be found in our previous paper [23].

In the second order of the perturbation theory we obtain

$$\Delta E^{(2)}_{n,l,m,\{0\}} = \sum_{n',l',m',\{n^a\}} \frac{\left|\left\langle\psi^{(0)}_{n',l',m',\{n^a\}}\,|V|\,\psi^{(0)}_{n,l,m,\{0\}}\right\rangle\right|^2}{E^{(0)}_n - E^{(0)}_{n'} - \hbar\omega(n^a_1+n^a_2+n^a_3)}, \tag{31}$$

here the set of numbers n', l', m', $\{n^a\}$ does not coincide with the set n, l, m, $\{0\}$, and $E^{(0)}_n = -e^2/(2a_B n^2)$ is the unperturbed energy of the hydrogen atom. Note that matrix elements $\left\langle\psi^{(0)}_{n',l',m',\{n^a\}}\,|V|\,\psi^{(0)}_{n,l,m,\{0\}}\right\rangle$ do not depend on ω because of our assumption (6). We consider the frequency of the harmonic oscillator ω to be very large. In the case of $\omega\to\infty$ we have

$$\lim_{\omega\to\infty}\Delta E^{(2)}_{n,l,m,\{0\}} = 0. \tag{32}$$

So, we obtain the following corrections up to the second order in the parameter of noncommutativity

$$\Delta E_{n,l} = \Delta E^{(1)}_{n,l}. \tag{33}$$

It is worth noting that in the case of $l=0$ or $l=1$ corrections (33) are divergent. In order to find corrections to the ns energy levels let us write the perturbation V in the following form

$$V = -\frac{e^2}{R} + \frac{e^2}{r} = -\frac{e^2}{\sqrt{r^2 - \frac{l_0}{\hbar}(\mathbf{a}\cdot\mathbf{L}) + \frac{l_0^2}{4\hbar^2}[\mathbf{a}\times\mathbf{p}]^2}} + \frac{e^2}{r}. \tag{34}$$

Consequently for the corrections to the ns energy levels we have

$$\Delta E_{ns} = \left\langle \psi^{(0)}_{n,0,0,\{0\}} \left| \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 - \frac{l_0}{\hbar}(\mathbf{a}\cdot\mathbf{L}) + \frac{l_0^2}{4\hbar^2}[\mathbf{a}\times\mathbf{p}]^2}} \right| \psi^{(0)}_{n,0,0,\{0\}} \right\rangle. \tag{35}$$

It is important that $(\mathbf{a}\cdot\mathbf{L})$ commutes with $[\mathbf{a}\times\mathbf{p}]^2$ and r^2. Also, it can be shown that $(\mathbf{a}\cdot\mathbf{L})\psi^{(0)}_{n,0,0,\{0\}} = 0$. So, the corrections (35) can be written as follows

$$\Delta E_{ns} = \left\langle \psi^{(0)}_{n,0,0,\{0\}} \left| \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 + \frac{l_0^2}{4\hbar^2}[\mathbf{a}\times\mathbf{p}]^2}} \right| \psi^{(0)}_{n,0,0,\{0\}} \right\rangle = \frac{\chi^2 e^2}{a_B} I_{ns}(\chi), \tag{36}$$

where we use the following notation

$$I_{ns}(\chi) = \int d\mathbf{a}'\tilde{\psi}^a_{0,0,0}(\mathbf{a}') \int d\mathbf{r}'\tilde{\psi}_{n,0,0}(\chi\mathbf{r}') \left(\frac{1}{r'} - \frac{1}{\sqrt{(r')^2 + [\mathbf{a}'\times\mathbf{p}']^2}}\right) \tilde{\psi}_{n,0,0}(\chi\mathbf{r}')\tilde{\psi}^a_{0,0,0}(\mathbf{a}'), \tag{37}$$

with

$$\chi=\sqrt{\frac{l_0 l_P}{2a_B^2}}. \tag{38}$$

Here $\tilde{\psi}_{n,0,0}(\chi\mathbf{r}')=\sqrt{\frac{1}{\pi n^5}}e^{-\frac{\chi r'}{n}}L^1_{n-1}\left(\frac{2\chi r'}{n}\right)$ are the dimensionless eigenfunctions of the hydrogen atom, $L^1_{n-1}\left(\frac{2\chi r'}{n}\right)$ are the generalized Laguerre polynomials, $\tilde{\psi}^a_{0,0,0}(\mathbf{a}')=\pi^{-\frac{3}{4}}e^{-\frac{(a')^2}{2}}$ are the dimensionless eigenfunctions corresponding to the harmonic oscillator, $\mathbf{a}'=\mathbf{a}/l_P$ and $\mathbf{r}'=\mathbf{r}\sqrt{2}/\sqrt{l_0 l_P}$.

It is important to mention that in the case of $\chi=0$ integral (37) has a finite value. Therefore, for $\chi\to 0$ the asymptotic of ΔE_{ns} can be written as follows

$$\Delta E_{ns}=\frac{\chi^2 e^2}{a_B}I_{ns}(0). \tag{39}$$

In order to find $I_{ns}(0)$, let us first consider the following integral

$$I_{ns}(\chi,\mathbf{a}')=\int d\mathbf{r}'\tilde{\psi}_{n,0,0}(\chi\mathbf{r}')\left(\frac{1}{r'}-\frac{1}{\sqrt{(r')^2+[\mathbf{a}'\times\mathbf{p}']^2}}\right)\tilde{\psi}_{n,0,0}(\chi\mathbf{r}'). \tag{40}$$

In the case of $\chi=0$ the integral reads

$$I_{ns}(0,\mathbf{a}')\simeq 1.72\frac{\pi a'}{4n^3}, \tag{41}$$

here $a'=|\mathbf{a}'|$. The details of calculation of integral (41) can be found in our previous paper [25].

It is clear that

$$I_{ns}(0)=\langle I_{ns}(0,\mathbf{a}')\rangle_{\mathbf{a}'}, \tag{42}$$

where $\langle ...\rangle_{\mathbf{a}'}$ denotes $\langle\tilde{\psi}^a_{0,0,0}(\mathbf{a}')|...|\tilde{\psi}^a_{0,0,0}(\mathbf{a}')\rangle$. So, taking into account (38), (39), (41), (42) and returning to $\mathbf{a}=l_P\mathbf{a}'$, the leading term in the asymptotic expansion of the corrections to the ns energy levels over the small parameter of noncommutativity reads

$$\Delta E_{ns}\simeq 1.72\frac{\hbar\langle\theta\rangle\pi e^2}{8a_B^3 n^3}, \tag{43}$$

where

$$\langle\theta\rangle=\frac{l_0}{\hbar}\langle\psi^a_{0,0,0}|\sqrt{\sum_i a_i^2}|\psi^a_{0,0,0}\rangle=\frac{2l_0 l_P}{\sqrt{\pi}\hbar}. \tag{44}$$

Note that corrections to the ns energy levels (43) are proportional to $\langle\theta\rangle$. In the case of $l>1$ we found that corrections (33) are proportional to $\langle\theta^2\rangle$. So, we can conclude that ns energy levels are more sensitive to the noncommutativity of coordinates (7).

3. A particle in the uniform field in rotationally invariant noncommutative space
Let us consider the motion of a particle in the uniform field in rotationally invariant noncommutative space (7)-(9). In the case when the field is pointed in the X_3 direction and is characterized by the factor κ the Hamiltonian of the particle reads

$$H_p = \frac{P^2}{2m} + \kappa X_3, \tag{45}$$

where m is the mass of a particle. For example, in a particular case of motion of a charged particle q in the uniform electric field E directed along the X_3 axis, we have $\kappa = qE$. In the case of motion of a particle of mass m in the uniform gravitational field g directed along the X_3 axis factor κ reads $\kappa = -mg$.

Taking into account the additional terms which correspond to the harmonic oscillator (5), we have

$$H = H_p + H_{osc} = \frac{P^2}{2m} + \kappa X_3 + \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 a^2}{2}. \tag{46}$$

It is convenient to use representation (12), (13). Therefore, we can write Hamiltonian (46) in the following form

$$H = \frac{p^2}{2m} + \kappa x_3 + \frac{\kappa l_0}{2\hbar}(a_1 p_2 - a_2 p_1) + \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 a^2}{2}. \tag{47}$$

After algebraic transformations, Hamiltonian (47) can be rewritten as

$$H = \left(1 - \frac{\kappa^2 l_0^2 m}{4\hbar^2\omega^2 m_{osc}}\right)\frac{p_1^2}{2m} + \left(1 - \frac{\kappa^2 l_0^2 m}{4\hbar^2\omega^2 m_{osc}}\right)\frac{p_2^2}{2m} + \frac{p_3^2}{2m} + \kappa x_3 + \frac{(p^a)^2}{2m_{osc}}$$
$$+\frac{m_{osc}\omega^2}{2}\left(a_1 + \frac{\kappa l_0}{2\hbar\omega^2 m_{osc}}p_2\right)^2 + \frac{m_{osc}\omega^2}{2}\left(a_2 - \frac{\kappa l_0}{2\hbar\omega^2 m_{osc}}p_1\right)^2 + \frac{m_{osc}\omega^2 a_3^2}{2}. \tag{48}$$

So, we can represent Hamiltonian (48) as follows

$$H = \tilde{H}_p + \tilde{H}_{osc}. \tag{49}$$

Here we use the notations

$$\tilde{H}_p = \frac{p_1^2}{2m_{eff}} + \frac{p_2^2}{2m_{eff}} + \frac{p_3^2}{2m} + \kappa x_3, \tag{50}$$

where m_{eff} is an effective mass which is defined as

$$m_{eff} = m\left(1 - \frac{\kappa^2 l_0^2 m}{4\hbar^2\omega^2 m_{osc}}\right)^{-1}, \tag{51}$$

and

$$\tilde{H}_{osc} = \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 q^2}{2}. \tag{52}$$

The components of $\mathbf{q}$ read

$$q_1 = a_1 + \frac{\kappa l_0}{2\hbar\omega^2 m_{osc}}p_2, \tag{53}$$

$$q_2 = a_2 - \frac{\kappa l_0}{2\hbar\omega^2 m_{osc}}p_1, \tag{54}$$

$$q_3 = a_3. \tag{55}$$

It is worth mentioning that q_i satisfy the ordinary commutation relations

$$[q_i, q_j] = 0, \tag{56}$$
$$[q_i, p_j^a] = i\hbar\delta_{ij}. \tag{57}$$

Therefore, Hamiltonian $\tilde{H}_{osc}$ corresponds to the tree-dimensional harmonic oscillator in the ordinary space. Also the following commutation relations are satisfied $[q_i, x_j] = -i\varepsilon_{ij3}\kappa l_0/(2m_{osc}\omega^2)$, $[q_i, p_j] = 0$.

It is important to note that

$$\tilde{H}_1 = \frac{p_1^2}{2m_{eff}}, \tag{58}$$
$$\tilde{H}_2 = \frac{p_2^2}{2m_{eff}}, \tag{59}$$
$$\tilde{H}_3 = \frac{p_3^2}{2m} + \kappa x_3, \tag{60}$$

and $\tilde{H}_{osc}$ which is given by (52) commute with each other. The eigenfunctions of $H = \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3 + \tilde{H}_{osc}$ (49) can be written as follows

$$\psi(\mathbf{x}, \tilde{\mathbf{q}}) = Ce^{ik_1x_1}e^{ik_2x_2}\psi^{(3)}(x_3)\psi^{\tilde{q}}(\tilde{\mathbf{q}}). \tag{61}$$

where C is a constant, k_1 and k_2 are the components of the wave vector corresponding to the free motion of a particle in the perpendicular directions to the field direction, $\psi^{(3)}(x_3)$ are well known eigenfunctions of $\tilde{H}_3$ which correspond to the motion of a particle in the field direction and can be written in terms of the Airy function, and $\psi^{\tilde{q}}(\tilde{\mathbf{q}})$ are eigenfunctions of tree-dimensional harmonic oscillator with parameters m_{osc} and ω. The components of $\tilde{\mathbf{q}}$ are the following

$$\tilde{q}_1 = a_1 + \frac{\kappa l_0 k_2}{2\omega^2 m_{osc}}, \tag{62}$$
$$\tilde{q}_2 = a_2 - \frac{\kappa l_0 k_1}{2\omega^2 m_{osc}}, \tag{63}$$
$$\tilde{q}_3 = a_3. \tag{64}$$

The eigenvalues of H (49) read

$$E = \frac{\hbar^2 k_1^2}{2m_{eff}} + \frac{\hbar^2 k_2^2}{2m_{eff}} + E_3 + \frac{1}{2}\hbar\omega, \tag{65}$$

where E_3 corresponds to the motion of a particle in the field direction. Here we take into account that the harmonic oscillator is in the ground state.

So, from (65) and (51) we can conclude that there is an effect of noncommutativity (7) on the mass of a particle in the uniform field. Note that the motion of a particle in the field direction described by $\tilde{H}_3$ (60) is the same as in the ordinary space. Noncommutativity has an effect on the motion of a particle in perpendicular directions to the direction of uniform field (see first two terms in (65)). So, noncommutativity of coordinates (7) causes the anisotropy of mass.

In this Section we have considered large but finite limit for ω. Note that in the limit $\omega \to \infty$ effect of noncommutativity on the mass of a particle tends to zero.

It is worth mention that because of the rotationally invariance obtained results can be easy generalized to the case of an arbitrary direction of the uniform field.

4. Conclusion

In this paper we have studied physical systems in rotationally invariant space with noncommutativity of coordinates. We have considered rotationally invariant noncommutative algebra (7)-(9) proposed in [23]. The algebra is constructed with the help of generalization of constant matrix of noncommutativity to the tensor defined by additional coordinates which are governed by harmonic oscillator.

The hydrogen atom has been considered in rotationally invariant noncommutative space (7)-(9). We have studied the corrections to the energy levels of the atom caused by the noncommutativity of coordinates (7). We have found that corrections to the ns energy levels (43) are proportional to $\langle\theta\rangle$, whereas the corrections to the energy levels with $l > 1$ (33) are proportional to $\langle\theta^2\rangle$. Therefore, we have concluded that ns energy levels are more sensitive to the noncommutativity of coordinates (7). The motion of a particle in the uniform field in rotationally invariant noncommutative space (7)-(9) has been also examined. On the basis of exact calculations we have concluded that there is an effect of coordinate noncommutativity on the mass of a particle. The motion of a particle in the perpendicular to the field directions can be described with the help of effective mass whereas the motion of a particle in the field direction is the same as in the ordinary space. So, noncommutativity of coordinates (7) causes the anisotropy of mass.

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