# Contact symmetries of constrained quadratic Lagrangians

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**Abstract.** The conditions for the existence of (polynomial in the velocities) contact symmetries of constrained systems that are described by quadratic Lagrangians is presented. These Lagrangians mainly appear in mini-superspace reductions of gravitational plus matter actions. In the literature, one usually adopts a gauge condition (mostly for the lapse $N$) prior to searching for symmetries. This, however, is an unnecessary restriction which may lead to a loss of symmetries and consequently to the respective integrals of motion. A generalization of the usual procedure rests in the identification of the lapse function $N$ as an equivalent degree of freedom and the according extension of the infinitesimal generator. As a result, conformal Killing tensors (with appropriate conformal factors) can define integrals of motion (instead of just Killing tensors used in the regular gauge fixed case). Additionally, rheonomic integrals of motion - whose existence is unique in this type of singular systems - of various orders in the momenta can be constructed. An example of a relativistic particle in a pp-wave space-time and under the influence of a quadratic potential is illustrated.

## 1. Introduction

Symmetries have always played a fundamental role in physical theories. In recent years there exists an increased interest on the subject of Noether point and/or generalized symmetries of physical systems and their geometric nature [1]-[7].

In general, one can discern two approaches regarding the treatment of symmetries in mini-superspace systems: The first entails an a priori fixation of the gauge (usualy $N = 1$) at the Lagrangian level and the treatment of the ensuing system as if it were regular ([8]-[12]). The second method ([6], [7], [13], [14]) utilizes the gauge invariance of parametrization invariant systems, resulting in the emergence of larger symmetry groups.

The general form of a mini-superspace Lagrangian is

$$L = \frac{1}{2N} G_{\mu\nu}(q)\dot{q}^\mu \dot{q}^\nu - NV(q), \quad \mu,\nu = 1,...,d, \quad \det(G_{\mu\nu}) \neq 0 \tag{1.1}$$

where $q^\mu(t)$ and $N(t)$ are the $d+1$ degrees of freedom of the system while the dot symbolizes the total time derivative $\frac{d}{dt}$. The Lagrangian denoted by (1.1) is singular, due to the quadratic constraint equation for $N$ ($\frac{\partial L}{\partial N} = 0$). This constraint can be used to generalize what is considered as an integral of motion, since the time derivative of such a quantity need not be strictly zero, but just a multiple of the constraint.

Let us first review some basic facts from the theory of symmetries of the action for regular systems [16]. A contact transformation whose infinitesimal generator is $X = \Xi^\kappa(t,q,\dot{q})\frac{\partial}{\partial q^\kappa}$, constitutes a symmetry of the action if the condition

$$pr^{(1)}X(L) = \frac{dF}{dt} \tag{1.2}$$

is satisfied, where $pr^{(1)}X$ is the first prolongation of the generator (that expresses the change induced by $X$ in the velocities $\dot{q}$'s) and $F(t,q,\dot{q})$ a gauge function. Given the form of the generator, a conserved quantity can be constructed as $\Xi^\kappa \frac{\partial L}{\partial q^\kappa} - F$.

For singular systems it has been shown [6] that the form of $X$ should include dependence in $N$, since the latter is also a degree of freedom for the system. Thus, one is led to consider

$$X = \Xi^\kappa(t,N,q,\dot{q})\frac{\partial}{\partial q^\kappa} + \Omega(t,N,q,\dot{q})\frac{\partial}{\partial N}. \tag{1.3}$$

Notice that when the additional term $\Omega(t,N,q,\dot{q})\frac{\partial}{\partial N}$ is acting on the Lagrangian through (1.2), creates a multiple of the quadratic constraint. Thus, (1.2) becomes an equation that holds modulo the constraint.

## 2. General results for polynomial in the velocities contact symmetries

In our case we shall restrict ourselves to the consideration of symmetries with corresponding generators polynomial in the velocities. This means that $\Xi^\kappa$ in (1.3) is taken as

$$\Xi^\kappa = \xi^\kappa(t,N,q) + S^\kappa{}_{\alpha_1}(t,N,q)\dot{q}^{\alpha_1} + ... + S^\kappa{}_{\alpha_1...\alpha_n}(t,N,q)\dot{q}^{\alpha_1}...\dot{q}^{\alpha}_n. \tag{2.1}$$

This automatically restricts the forms of $\Omega(t,N,q,\dot{q})$ and $F(t,N,q,\dot{q})$. The first has to be polynomial of first order in $\dot{N}$ and of $n$ order in $\dot{q}$'s, while the second of zero order in $\dot{N}$ and of $n+1$ order in $\dot{q}$'s. However, the existence of the gauge function can be proven to be trivial [7], so we won't refer to it any longer.

The procedure in order to derive the symmetry generator in these cases is the following: After the application of the symmetry criterion (1.2), the coefficients of terms involving velocities $\dot{N}$ and $\dot{q}^\mu$ of various orders are gathered and set to zero. By solving these partial differential equations, the general form for the symmetry generator is obtained:

$$X = \left(\xi^\kappa(q) + \frac{1}{N}S^\kappa_{\alpha_1}(q)\dot{q}^{\alpha_1} + \frac{1}{N^2}S^\kappa_{\alpha_1\alpha_2}(q)\dot{q}^{\alpha_1}\dot{q}^{\alpha_2} + ... + \frac{1}{N^n}S^\kappa_{\alpha_1...\alpha_n}(q)\dot{q}^{\alpha_1}...\dot{q}^{\alpha_n}\right)\frac{\partial}{\partial q^\kappa}$$
$$- \left(\frac{N}{V}\tilde{\xi}^\sigma V_{,\sigma} + \frac{2V^{,\sigma}}{V}S_{\sigma\alpha_1}\dot{q}^{\alpha_1} + \frac{n+1}{N^{n-1}}\frac{V^{,\sigma}}{V}S_{\sigma\alpha_1...\alpha_n}\dot{q}^{\alpha_1}...\dot{q}^{\alpha_n}\right)\frac{\partial}{\partial N}$$

together with the subsequent conditions:

$$\mathcal{L}_\xi G_{\mu\nu} = -\frac{\mathcal{L}_\xi V}{V}G_{\mu\nu} \tag{2.2a}$$

$$S_{(\mu\alpha_1;\nu)} = -\frac{V^{,\sigma}}{V}S_{\sigma(\alpha_1}G_{\mu\nu)},\ ...\ , S_{(\mu\alpha_1...\alpha_n;\nu)} = -\frac{n+1}{2}\frac{V^{,\sigma}}{V}S_{\sigma(\alpha_1...\alpha_n}G_{\mu\nu)}. \tag{2.2b}$$

From the derived symmetry generator, one can construct integrals of motion of several orders in the momenta. It is clear that equations (2.2) already contain the possibility of $\xi$ (the $S$'s) being a Killing field (Killing tensors) of the metric and the potential ($S^{\sigma}\alpha_1...\alpha_n V_{,\sigma} = 0$), which are of course the known cases of symmetries for regular systems. Equations (2.2) indicate that the situation in constraint systems is more general. What is more, it can be easily verified that for systems described by (1.1), rheonomic integrals of motion (conserved quantities that have an explicit time dependence) can also be constructed by any conformal Killing field of $G_{\mu\nu}$ [6]. If there is a conformal Killing vector $\xi(q)$ that satisfies

$$\mathcal{L}_{\xi} G_{\mu\nu} = \omega(q) G_{\mu\nu} \tag{2.3}$$

then the quantity

$$\tilde{Q} = \xi^{\alpha} p_{\alpha} + \int N(t) \left(\omega(q(t)) + f(q(t))\right) V(q(t)) dt, \tag{2.4}$$

where $f = \frac{\mathcal{L}_{\xi} V}{V}$, is an integral of motion of the system, since

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \{Q, H\} = \omega(q)\mathcal{H} \approx 0. \tag{2.5}$$

In the case where $\omega = -f$ the integral of motion is autonomous and one returns to the case described by (2.2a).

An analogous construction can be made for any conformal Killing tensor of arbitrary order. Let us suppose that

$$S^{(\kappa\alpha_1...\alpha_n;\mu)} = \frac{1}{2}\omega^{(\alpha_1...\alpha_n} G^{\mu\nu)} \tag{2.6}$$

then one can we can consider the quantity

$$\tilde{Q} = S^{\kappa\alpha_1...\alpha_n} p_{\kappa} p_{\alpha_1}...p_{\alpha_n} + \int N \left(\omega^{\alpha_1...\alpha_n} + (n+1) f^{\alpha_1...\alpha_n}\right) V \frac{\partial L}{\partial \dot{q}^{\alpha_1}} ... \frac{\partial L}{\partial \dot{q}^{\alpha_n}} dt \tag{2.7}$$

where everything inside the integral is to be regarded strictly as a function of time and $f^{\alpha_1...\alpha_n}$ is an $n$-rank tensor defined as

$$f^{\alpha_1...\alpha_n} = S^{\kappa\alpha_1...\alpha_n} \frac{V_{,\kappa}}{V}. \tag{2.8}$$

It is easy to check that, in the same manner as previously, the time derivative of $\tilde{Q}$ results in

$$\frac{d\tilde{Q}}{dt} = N\omega^{\alpha_1...\alpha_n} p_{\alpha_1}...p_{\alpha_n} \mathcal{H} \approx 0 \tag{2.9}$$

and thus $\tilde{Q}$ is a rheonomic integral of motion due to the constraint.

Now we have all the tools necessary to apply into a simple example that illustrates the significance of this generalization.

## 3. Example: A particle in a curved space-time under the influence of a quadratic potential

We have already mentioned that Lagrangians of the form (1.1) are mainly encountered in cosmology. However, here we shall consider an example of a relativistic particle moving in a type Biv pp-wave space-time described by the line element

$$ds^2 = -2dvdu - \frac{2}{z^2}du^2 + dx^2 + dy^2 \tag{3.1}$$

under the influence of a quadratic potential of the form $V(u) = \frac{\mu}{2}u^2$. In the case where $\mu > 0$ the latter can be considered to represent an oscillator in the $u$ direction. The space-time described by (3.1) has been examined in [17] with respect to its conformal algebra and its Killing tensors.

By using (1.1), the Lagrangian of the system reads

$$L = \frac{1}{N}\left(-\dot{u}\,\dot{v} - \frac{\dot{u}^2}{z^2} + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\dot{z}^2\right) - N\frac{\mu\,u^2}{2} \tag{3.2}$$

and, as we can observe, it possesses five degrees of freedom. However, due to its constrained nature, four of them can be expressed as functions of the fifth on the solution space.

There exist three vector fields satisfying condition (2.2a) (which in this case happen also to be Killing fields of both the metric $G_{\mu\nu}$ and the potential $V(u)$). At the same time, there also exists a homothetic vector of both the metric and the potential. With their help the following four linear integrals of motion can be written:

$$Q_1 = -\frac{\dot{u}}{N}, \quad Q_2 = \frac{\dot{y}}{N}, \quad Q_3 = \frac{u\,\dot{y}}{N} - \frac{y\,\dot{u}}{N} \tag{3.3a}$$

$$Q_h = -\frac{2\,u\,\dot{u}}{N\,z^2} - \frac{u\,\dot{v}}{N} + \frac{y\,\dot{y}}{2\,N} + \frac{z\,\dot{z}}{2\,N} + \frac{3}{2}\,\mu\int N\,u^2\,dt. \tag{3.3b}$$

that consequently define the equations $Q_i = \kappa_i$ and $Q_h = \kappa_h$.

It can also be verified that there exist seven second rank non reducible conformal Killing tensors satisfying criterion (2.2b), all of which, albeit one, are also Killing tensors for the space-time under consideration. In our example we are going to make use of the following two

$$S_1^{\mu\nu} = \begin{pmatrix} -\frac{8}{u^2 z^2 \mu^2} & \frac{4}{u^2\mu^2} & 0 & 0 \\ \frac{4}{u^2\mu^2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{4}{u^2\mu^2} & 0 \\ 0 & 0 & 0 & -\frac{4}{u^2\mu^2} \end{pmatrix}, \quad S_2^{\mu\nu} = \begin{pmatrix} \frac{8}{z^2\mu^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{\mu^2} \end{pmatrix},$$

with $S_1^{\mu\nu}$ being a proper CKT. Thus, we are supplemented with two additional quadratic in the momenta integrals of motion $Q_4 = S_1^{\mu\nu} p_\mu p_\nu = \kappa_4$ and $Q_5 = S_2^{\mu\nu} p_\mu p_\nu = \kappa_5$. We can now proceed with the derivation of the solution.

### *3.1. Case $\kappa_1 \neq 0$ and $\kappa_5 \neq 0$*

In order to completely integrate the equations, we choose to write $N(t) = \dot{f}(t)$, where $f(t)$ is considered to be a non-constant (scalar under time re-parameterizations) function. We first integrate equations $Q_1 = \kappa_1$ and $Q_2 = \kappa_2$ with respect to $u(t)$ and $y(t)$ respectively, and get

$$u(t) = c_u - \kappa_1\, f(t) \quad \text{and} \quad y(t) = c_y + \kappa_2\, f(t) \tag{3.4}$$

where $c_u$ and $c_y$ are constants of integration. With the help of the first, equation $Q_5 = \kappa_5$, yields the following quartet of solutions

$$z(t) = z_{\pm\pm}(t) = \pm\frac{\sqrt{\pm 4\,c_z\,\kappa_5^2\,\mu^4\,f(t) + 4\,c_z^2\,\kappa_5^2\,\mu^4 + \kappa_5^2\,\mu^4\,f(t)^2 + 32\,\kappa_1^2}}{2\sqrt{\kappa_5}\,\mu}, \tag{3.5}$$

where $c_z$ is a constant of integration; of course we must also require $\kappa_5 \neq 0$ (the case $\kappa_5 = 0$ leads to a complex solution). At this point, equation $Q_4 = \kappa_4$ can be integrated to give

$$\begin{aligned} v(t) = v_\pm(t) = & c_v - \frac{f(t)\left(3\left(\mu^2\left(c_u^2\kappa_4+\kappa_5\right)+4\kappa_2^2\right)-3c_u\kappa_1\kappa_4\mu^2 f(t)+\kappa_1^2\kappa_4\mu^2 f(t)^2\right)}{24\kappa_1} \\ & + \sqrt{2}\tan^{-1}\left(\frac{\kappa_5\mu^2(f(t)\pm 2c_z)}{4\sqrt{2}\kappa_1}\right), \end{aligned} \tag{3.6}$$

with $c_v$ being constant and $\kappa_1 \neq 0$; the plus sign $v_+(t)$ corresponds to solutions $z = z_{++}(t)$ and $z = z_{-+}(t)$, while the minus $v_-(t)$ is for the cases $z = z_{--}(t)$ and $z = z_{+-}(t)$.

By using the solutions (3.4), (3.5) and (3.6) in equation $Q_h = \kappa_h$ we see that, in order for $Q_h$ to be constant for any function $f(t)$, one needs to set $\kappa_4 = \frac{4}{\mu}$. It is easy to check that substitution of the same solutions in the equations of motion together with the constraint $\kappa_4 = \frac{4}{\mu}$ satisfies the system. It is noteworthy that, we were able to obtain the general solution without even turning to the Euler-Lagrange equations of the system.

*3.2. Case $\kappa_1 = 0$*

Equations $Q_1 = 0$, $Q_2 = \kappa_2$ and $Q_5 = \kappa_5$ lead to the same result with (3.4) and (3.5) by simply setting $\kappa_1 = 0$. One can check that all the other equations regarding autonomous integrals of motion reduce to relations between constants. However, the one defined by the rheonomous integral of motion $Q_h = \kappa_h$ yields, upon integration,

$$v(t) = v_\pm(t) = c_v + \frac{\frac{1}{2}f(t)^2\left(12c_u^2\mu+4\kappa_2^2+|\kappa_5|\mu^2\right)+2f(t)\left(2c_y\kappa_2 \pm c_z\sqrt{|\kappa_5|}\mu - 4\kappa_h\right)}{8c_u}, \tag{3.7}$$

where the $v_+(t)$ corresponds to $z_+(t)$, while $v_-(t)$ to $z_-(t)$ and of course in each case $c_u \neq 0$ (the subcase where $c_u = 0$ will be explored later on).

At this point we have used all equations available by the integrals of motion, substitution of their solutions into the Euler - Lagrange equations yields the constraint relation between constants $\kappa_5 = -\frac{4\left(c_u^2\mu+\kappa_2^2\right)}{\mu^2}$.

As we previously remarked, apart from this solution we must also check the case where $c_u = 0$. The application of the relations (3.4) and (3.5) under the condition $\kappa_1 = 0$, leads to the observation that $Q_h$ is constant for every $f(t)$, if and only if $\kappa_5 = -\frac{4\kappa_2^2}{\mu^2}$. Substitution of all these results into the equations of motion leads to a final ordinary differential equation for $v(t)$ that by integration yields

$$v(t) = c_1 f(t) + c_2 \tag{3.8}$$

and thus, we have obtained the complete solution space.

## 4. Discussion

In this work we have considered, polynomial in the velocities, contact variational symmetries of the action. We derived that, for constrained systems described by (1.1), the infinitesimal criterion (1.2) allows for more symmetries than in the case of regular systems. The main reason for this result is the consideration of gauge invariance not as a redundant degree of freedom that must be fixed out, but rather as an essential part of the system under consideration.

The existence of the constraint gives rise to larger symmetry groups. Of course, this does not mean that one must never choose a gauge. The optimum procedure would be to first calculate all symmetries and then apply any gauge fixing, if necessary, in order to simplify the equations. The important thing is not to do so prior to the determination of the integrals of motion, since their existence may depend on the constraint itself.

In order to exhibit the importance of the method, we examined an example of motion of a relativistic particle in a curved space-time. Due to the consideration of conformal Killing tensors instead of just Killing, we were able to obtain the general solution with no need of gauge fixing and without even using the equations of motion themselves.

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