Quantizing the classical non-cooperative symmetric games with arbitrary number of strategies

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Abstract. Two-player N-strategy symmetric noncooperative games are considered. A general form of gate operators is found using group-theoretical methods. The stability group of initial state is introduced and calculated in the case of three strategies games. Its role in determining the Nash equilibria for maximally entangled games is stressed.

1. Introduction

We present here the review of results obtained in our papers [1] and [2]. The aim of the paper is to describe the generalization of the ELW game to the case of two-player and N-strategy games, in particular the construction of the gate operator using group-theoretical methods.

The first scheme of quantization of classical noncooperative symmetric games was proposed by Eisert, Wilkens and Lewenstein [3], [4]. Their method concerns two-player two-strategy (C-cooperate and D-defect) game. Classical game is defined in Table 1, where the letters r, p, s, t are the possible outcomes for the classical strategies C and D. The game is uniquely defined if the value of r, p, s, t are selected. For example, if the classical payoffs obey $t > r > p > s$, the Prisoner Dilemma emerges on the classical level. The Prisoner Dilemma is one of the most interesting game in economy and is set up in such a way that both players choose to protect themselves at the expense of the other player. As a result of a purely logical thought process to help oneself, both players find themselves in a worse state than if they had cooperated with each other in the decision-making process.

Table 1. The payoffs resulting from different ELW strategies.

Strategies		Payoffs	
player A	player B	player A	player B
C	C	r	r
C	D	s	t
D	C	t	s
D	D	p	p

The procedure of quantization proceeds as follows. Firstly, we assign the classical strategies C and D to the basis vectors $|1\rangle$ and $|2\rangle$ of 2D complex Hilbert spaces, one ascribed to each player. The initial state of the game is given by

$$|\Psi_{in}\rangle = J\left(|1\rangle \otimes |1\rangle\right), \tag{1}$$

where J is an unitary operator known to both players. The J operator, called gate operator, is very important because it represents entanglement allowing for the genuinely quantum correlations. Strategic moves of both players are associated with unitary 2×2 operators U_A, U_B operating on their own qubits. The resulting final state of the game is given by

$$|\Psi_{out}\rangle = J^{+}\left(U_A \otimes U_B\right)|\Psi_{in}\rangle = J^{+}\left(U_A \otimes U_B\right) J\left(|1\rangle \otimes |1\rangle\right). \tag{2}$$

The expected payoffs are computed according to

$$\begin{aligned} \$_A &= rP_{11} + pP_{22} + tP_{21} + sP_{12} \\ \$_B &= rP_{11} + pP_{22} + sP_{21} + tP_{12}. \end{aligned} \tag{3}$$

where

$$P_{kk'} \equiv \left|\left\langle k \otimes k' | \Psi_{out}\right\rangle\right|^2, \qquad k, k' = 1, 2 \tag{4}$$

denotes the probability of obtaining one of the four outcomes.

The form of the gate operator J results from two assumptions: (a) to preserve the symmetry of the game J is symmetric with respect to the interchange of players; (b) the quantum game entails a faithful representation of its classical counterpart.
In the case of ELW the game J has form

$$J = \exp\left(-\frac{i\gamma}{2}\sigma_2 \otimes \sigma_2\right), \tag{5}$$

where $\gamma \in [0, \frac{\pi}{2}]$ is a real parameter and σ_2 is the second Pauli matrix. In fact, γ is a measure of the game's entanglement. Eisert et al. considered the particular case of the Prisoners' Dilemma [3]. They chose the following values of payoffs $r = 3$, $p = 1$, $t = 5$, $s = 0$ and restricted the strategic space to the 2-parameter set of unitary matrices

$$U\left(\theta, \phi\right) = \begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix} \tag{6}$$

with $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \frac{\pi}{2}$. This attitiude was criticized by Benjamin and Hayden [5]. They pointed out that there are no compelling reasons to impose such a restriction on the strategic space. In our considerations the set of allowed strategies is the whole $SU(N)$ group.

In the case of maximal entangled game $\gamma = \frac{\pi}{2}$, Eisert et al. constructed quantum strategy $Q \equiv U\left(0, \frac{\pi}{2}\right)$ which is optimal strategy for both players. What's more, $Q \times Q$ is not only Nash equilibrium but also Pareto optimal. So we could say that the players escape the dilemma. Since the papers [3] and [4] appeared quantum game theory has been a subject of intensive research [5]÷[54].

The paper is organized as follows. In Sect. 2, we introduce the generalization of the ELW game to the case of N strategies and we construct a wide class of entanglers for arbitrary N. The case of three strategies is discussed in Sect. 3, as an example. In Section 4 the stability group of initial state of the game is introduced and its role in determining the Nash equilibria is stressed. The last Section 5 is devoted to some conclusions.

2. Non-cooperative symmetric games with N strategies

The generalized ELW game can be described as follows. We have some classical non-cooperative two-player N-strategy game defined by a relevant payoff table. In order to quantize it, one ascribes to any player an N dimensional complex Hilbert space spanned by the vectors

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad |N\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{7}$$

The initial state is defined in the same way as in the $SU(2)$ case

$$|\Psi_{in}\rangle \equiv J\left(|1\rangle \otimes |1\rangle\right). \tag{8}$$

In order to preserve the symmetry of the game we assume that the gate operator J is symmetric with respect to the permutation of the factors entering the tensor product; further, we assume that the classical game is faithfully represented in its quantum counterpart. We demand that the set of allowed strategies is the whole $SU(N)$ group, unlike Eisert et. al who assumed that the set of strategies, in the case of 2-strategy game, belongs to the 2D submanifold of $SU(2)$, which itself is not a group. The strategic moves of the players are associated with unitary operators U_A and U_B operating on their own qubits and the final state of the game is given by

$$|\Psi_{out}\rangle = J^{+}\left(U_A \otimes U_B\right) J\left(|1\rangle \otimes |1\rangle\right). \tag{9}$$

The player's expected payoffs can be computed according to

$$\$^{A,B} = \sum_{k,k'=1}^{N} p^{A,B}_{k,k'} \left|\langle k, k'|\Psi_{out}\rangle\right|^2, \tag{10}$$

where $|k,k'\rangle \equiv |k\rangle \otimes |k'\rangle$, $k,k' = 1,...,N$ and $p^{A,B}_{k,k'}$ are classical payoffs of Alice and Bob, respectively. There exists one difference between original and generalized ELW game. The set of allowed gate operators J is parametrized by one real parameter γ (cf. Eq. (5)) in the case of SU(2) group. For general N we showed in [2] that J depends on a number of free parameters growing proportionally to N^2. As we have mentioned above, in order to find the form of gate operators we make two assumptions: the gate operator is symmetric under the exchange of the factors in the tensor product of Hilbert spaces ascribed to both players and all classical pure strategies are contained in the set of pure quantum ones. The second assumption is fulfilled if N matrices $U_k \in SU(N)$, $k = 1,...,N$ exist which satisfy

$$U_k\,|1\rangle = e^{i\phi_k}\,|k\rangle\,, \quad k = 1,...,N \tag{11}$$

$$[J, U_k \otimes U_l] = 0, \quad k,l = 1,...,N. \tag{12}$$

We want to leave as much freedom as possible in the choice of gate operator so we assume that $[U_k, U_l] = 0$, $k,l = 1,...,N$. We define the matrices U_k in form

$$U_k \equiv e^{\frac{i\pi(N-1)(k-1)}{N}} U^{k-1} \in SU(N), \qquad k = 1,...,N \tag{13}$$

where

$$U = \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 & 1 \\ 1 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 1 & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & 1 & 0 \end{pmatrix} \tag{14}$$

is the matrix representing the cyclic permutation of $12\dots N$. All matrices U_k commute and eq. (11) is satisfied. In order to diagonalize the matrices U_k we must diagonalize the matrix U. The elements of matrix V diagonalizing U have the form

$$V_{ik} = \frac{1}{\sqrt{N}} \overline{\varepsilon}^{(i-1)(k-1)}, \qquad i,k = 1,...,N. \tag{15}$$

ε being the primitive N-th root of unity. Next we define

$$\widetilde{J} \equiv \left(V^{+} \otimes V^{+}\right) J \left(V \otimes V\right). \tag{16}$$

The matrix $V^{+}UV$ is diagonal matrix with different elements on the diagonal and operator $\widetilde{J}$ commute with $V^{+}UV$; therefore $\widetilde{J}$ must be diagonal and can be written as

$$\widetilde{J} = \exp\left(i \sum_{k=1}^{N-1} \lambda_k \left(\Lambda_k \otimes \Lambda_k\right) + i \sum_{k \neq l = 1}^{N-1} \mu_{kl} \left(\Lambda_k \otimes \Lambda_l + \Lambda_l \otimes \Lambda_k\right)\right) \tag{17}$$

where Λ_i, $i = 1,...,N-1$ is any basis in Cartan subalgebra of $SU(N)$; λ_k and $\mu_{kl} = \mu_{lk}$ are real parameters and determine the degree of entanlglement. The gate operator defined by eq. (17) depends on $N - 1 + \binom{N-1}{2} = \binom{N}{2}$ free parameters.

3. Two-player three-strategy game

In this section we present the example of quantization scheme presented above. In the special case $N = 3$ one can construct the gate operator J in full generality. Considering the game with three strategies; we assume that the matrices representing classical strategies satisfy the conditions

$$\begin{array}{ll} U_k \left|1\right\rangle = e^{i\varphi_k} \left|k\right\rangle, & k = 1,2,3 \\ \left[J, U_j \otimes U_k\right] = 0, & j,k = 1,2,3 \end{array} \tag{18}$$

$$\left[U_j, U_k\right] = 0. \tag{19}$$

Next, we assume that $U_1 = I$ and from eqs. (18) and (19) we find the final form of matrices U_2 and U_3

$$\begin{array}{l} U_2 = \begin{pmatrix} 0 & 0 & \varepsilon e^{-i\varphi_3} \\ e^{i\varphi_2} & 0 & 0 \\ 0 & \overline{\varepsilon} e^{i(\varphi_3 - \varphi_2)} & 0 \end{pmatrix} \\ U_3 = \begin{pmatrix} 0 & \varepsilon e^{-i\varphi_2} & 0 \\ 0 & 0 & \overline{\varepsilon} e^{i(\varphi_2 - \varphi_3)} \\ e^{i\varphi_3} & 0 & 0 \end{pmatrix}, \end{array} \tag{20}$$

where ε is any cubic root from unity, we assume $\varepsilon \neq 1$.
All matrices U_k can be diagonalized [2]

$$\begin{array}{l} \widetilde{U}_1 = V^{+} U_1 V = I \\ \widetilde{U}_2 = V^{+} U_2 V = diag\left(1, \varepsilon, \varepsilon^2\right) . \\ \widetilde{U}_3 = V^{+} U_3 V = diag\left(\varepsilon, 1, \varepsilon^2\right). \end{array} \tag{21}$$

using the matrix V defined by

$$V=\frac{1}{\sqrt{3}}\left(\begin{array}{ccc}1 & 1 & 1\\ e^{i\varphi_2} & \overline{\varepsilon}e^{i\varphi_2} & \varepsilon e^{i\varphi_2}\\ \overline{\varepsilon}e^{i\varphi_3} & e^{i\varphi_3} & \varepsilon e^{i\varphi_3}\end{array}\right),\qquad VV^{+}=I. \tag{22}$$

The operator $\widetilde{J}$, defined by eq. (16), commutes with $\widetilde{U}_i$, $i=1,2,3$ and has form

$$\widetilde{J}=\exp i\left(\tau\left(\Lambda\otimes\Lambda\right)+\rho\left(\Lambda\otimes\Delta+\Delta\otimes\Lambda\right)+\sigma\left(\Delta\otimes\Delta\right)\right), \tag{23}$$

where τ, ρ and σ are real parameters, while $\Lambda=\lambda_3$, $\Delta=\frac{1}{2}\left(\lambda_3+\sqrt{3}\lambda_8\right)$ (λ_3 and λ_8 are Gell-Mann matrices).

4. The stability subgroup

It should be noted that different strategies may lead to the same outcome. For that reason we want to find the stability subgroup $G_s\in SU(N)\times SU(N)$ of the initial state $|\Psi_{in}\rangle$, i.e. the set of elements (pairs of strategies) $g\in SU(N)\times SU(N)$ such that

$$g\left|\Psi_{in}\right\rangle=\left|\Psi_{in}\right\rangle. \tag{24}$$

Then two games, (U_A,U_B) and (U'_A,U'_B), differing by an element $g\in G_s$,

$$\left(U'_A,U'_B\right)=\left(U_A,U_B\right)\cdot g \tag{25}$$

share the same final result. Consequently, the coset space $SU(N)\times SU(N)/G_s$ is the effective set of strategies.

The stability group depends on the degree of entanglement of the initial state. We consider the most interesting case of maximal entanglement. In order to find the form of elements of the stability group we write the initial state of the game as

$$\left|\Psi_{in}\right\rangle\equiv J\left(|1\rangle\otimes|1\rangle\right)\equiv F_{ij}\left|i\right\rangle\otimes\left|j\right\rangle, \tag{26}$$

where the summation over repeated indices is understood and $F_{ij}=F_{ji}$.
The corresponding density matrix reads

$$\rho_{in}=\left|\Psi_{in}\right\rangle\left\langle\Psi_{in}\right|. \tag{27}$$

The initial state is maximally entangled if the corresponding reduced density matrix is proportional to the unit matrix [2]

$$Tr_A\rho_{in}=\frac{1}{N}I,\qquad Tr_B\rho_{in}=\frac{1}{N}I. \tag{28}$$

Equations (28) imply that the matrix $\widetilde{F}\equiv\sqrt{N}F$ is unitary. After some calculations [2] we find that G_s consists of the elements of the form

$$\left(U,\widetilde{F}\overline{U}\widetilde{F}^{+}\right). \tag{29}$$

Therefore, the stability group G_s of $|\Psi_{in}\rangle$ is, up to a group automorphism, the diagonal subgroup of $SU(N)\times SU(N)$. Its Lie algebra induces the symmetric Cartan decomposition of $sU(N)\oplus sU(N)$.

Let (U_1, U_2) be a pair of strategies leading to the expected payoffs desired by one of the player, let's say Bob. Assume that Alice has chosen the strategy U_A. The pair of strategies $(U_1, U_2) \in SU(N) \times SU(N)$ can be written as [1],[2]

$$(U_1, U_2) = \left(U_A, U_2 \widetilde{F}\,\overline{U}_1^{+}\,\overline{U}_A \widetilde{F}^{+}\right)\left(U_A^{+} U_1, \widetilde{F}\,\overline{U}_A^{+}\,\overline{U}_1 \widetilde{F}^{+}\right). \tag{30}$$

The second element on the right hand side of eq. (30) belongs to the stability group of initial state; therefore, the strategies (U_1, U_2) and $\left(U_A, U_2 \widetilde{F}\,\overline{U}_1^{+}\,\overline{U}_A \widetilde{F}^{+}\right)$ lead to the same payoffs. We conclude that $U_2 \widetilde{F}\,\overline{U}_1^{+}\,\overline{U}_A \widetilde{F}^{+}$ is the relevant counterstrategy to Alice's strategy U_A. As a result, there is no pure Nash equilibrium unless, in the table of payoff, there exists a pair of strategies leading to the optimal outcomes for both players. Then, we have only trivial pure Nash equilibria.

As an example we present the stability group in the case of three strategies. We have to find the values of parameters ρ, σ, τ for which the initial state is maximal entangled. Then, the reduced density matrix defined by the initial state $|\Psi_{in}\rangle$

$$\mathrm{Tr}_B \rho_{in} = \frac{1}{9}\left(\begin{array}{c|c|c} 3 & \begin{array}{c} e^{i(3\rho+\sigma+2\tau)} + \\ +e^{-i(\rho+2\tau)} + \\ +e^{-i(2\rho+\sigma)} \end{array} & \begin{array}{c} e^{i(3\rho+2\sigma+\tau)} + \\ +e^{-i(2\rho+\tau)} \\ +e^{-i(\rho+2\sigma)} \end{array} \\ \hline \begin{array}{c} e^{-i(3\rho+\sigma+\tau)} + \\ +e^{i(\rho+2\tau)} + \\ +e^{2\rho+\sigma} \end{array} & 3 & \begin{array}{c} e^{i(\sigma-\tau)} + \\ +e^{-i(\rho-\tau)} + \\ +e^{i(\rho-\sigma)} \end{array} \\ \hline \begin{array}{c} e^{-i(3\rho+2\sigma+\tau)} + \\ +e^{i(2\rho+\tau)} + \\ +e^{i(\rho+2\sigma)} \end{array} & \begin{array}{c} e^{-i(\sigma-\tau)} + \\ +e^{i(\rho-\tau)} + \\ +e^{-i(\rho-\sigma)} \end{array} & 3 \end{array}\right). \tag{31}$$

must be proportional to the unit matrix, eq. (28). Therefore all off-diagonal elements of reduced density matrix defined by (31) vanish. From this condition, we find the following sets of parameters [2]

$$\begin{aligned}
&\begin{cases} \tau = \rho = \sigma - \frac{2\pi}{3} \\ \sigma = \frac{2\pi}{3}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{4\pi}{3}, \frac{14\pi}{9}, \frac{16\pi}{9}, 2\pi \end{cases} \\
&\begin{cases} \tau = \rho = \sigma + \frac{2\pi}{3} \\ \sigma = 0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{2\pi}{3}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{4\pi}{3}, \frac{14\pi}{9}, \frac{16\pi}{9} \end{cases} \\
&\begin{cases} \tau = \sigma - \frac{2\pi}{3} \\ \rho = \sigma = \frac{2\pi}{3}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{4\pi}{3}, \frac{14\pi}{9}, \frac{16\pi}{9}, 2\pi \end{cases} \\
&\begin{cases} \tau = \sigma + \frac{2\pi}{3} \\ \rho = \sigma = 0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{2\pi}{3}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{4\pi}{3}, \frac{14\pi}{9}, \frac{16\pi}{9} \end{cases} \\
&\begin{cases} \rho = \sigma - \frac{2\pi}{3} \\ \tau = \sigma = \frac{2\pi}{3}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{4\pi}{3}, \frac{14\pi}{9}, \frac{16\pi}{9}, 2\pi \end{cases} \\
&\begin{cases} \rho = \sigma + \frac{2\pi}{3} \\ \tau = \sigma = 0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{2\pi}{3}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{4\pi}{3}, \frac{14\pi}{9}, \frac{16\pi}{9}. \end{cases}
\end{aligned} \tag{32}$$

According to eq. (29), the generators of the stability subgroup have form

$$Y \otimes I - I \otimes \widetilde{F}\overline{Y}\widetilde{F}^{+}, \tag{33}$$

where Y runs over all generators of $SU(3)$. Using a fact that $\widetilde{F}$ is a symmetric matrix and substituting

$$Y \to Y \mp \widetilde{F}\overline{Y}\widetilde{F}^{+} \equiv X \tag{34}$$

one easily finds that the generators can be put in the form

$$X \otimes I \pm I \otimes X. \tag{35}$$

As an example we write out the explicit form of generators of stability subgroup for one of the solutions (32), i.e. $\rho = \frac{2\pi}{3}$, $\sigma = \tau = 0$:

$$\begin{aligned}
G_1 &= \left(\lambda_1 - \sqrt{3}\lambda_2 + \frac{2}{\sqrt{3}}\lambda_8\right) \otimes I - I \otimes \left(\lambda_1 - \sqrt{3}\lambda_2 + \frac{2}{\sqrt{3}}\lambda_8\right) \\
G_2 &= \left(\sqrt{3}\lambda_2 + \lambda_3 + \lambda_4 - \frac{1}{\sqrt{3}}\lambda_8\right) \otimes I - I \otimes \left(\sqrt{3}\lambda_2 + \lambda_3 + \lambda_4 - \frac{1}{\sqrt{3}}\lambda_8\right) \\
G_3 &= \left(\lambda_3 + 2\lambda_6 + \frac{1}{\sqrt{3}}\lambda_8\right) \otimes I - I \otimes \left(\lambda_3 + 2\lambda_6 + \frac{1}{\sqrt{3}}\lambda_8\right) \\
G_4 &= (\lambda_2 + \lambda_5) \otimes I - I \otimes (\lambda_2 + \lambda_5) \\
G_5 &= \left(4\lambda_2 + \sqrt{3}\lambda_3 + 2\lambda_7 - 3\lambda_8\right) \otimes I - I \otimes \left(4\lambda_2 + \sqrt{3}\lambda_3 + 2\lambda_7 - 3\lambda_8\right) \\
G_6 &= \left(\lambda_1 - \frac{1}{2}\lambda_4 + \frac{1}{4}\lambda_6 - \frac{3\sqrt{3}}{4}\lambda_7 - \frac{\sqrt{3}}{2}\lambda_8\right) \otimes I + \\
&\quad + I \otimes \left(\lambda_1 - \frac{1}{2}\lambda_4 + \frac{1}{4}\lambda_6 - \frac{3\sqrt{3}}{4}\lambda_7 - \frac{\sqrt{3}}{2}\lambda_8\right) \\
G_7 &= \left(\lambda_2 - \frac{\sqrt{3}}{2}\lambda_4 - \lambda_5 - \frac{\sqrt{3}}{4}\lambda_6 + \frac{1}{4}\lambda_7 + \frac{3}{2}\lambda_8\right) \otimes I + \\
&\quad + I \otimes \left(\lambda_2 - \frac{\sqrt{3}}{2}\lambda_4 - \lambda_5 - \frac{\sqrt{3}}{4}\lambda_6 + \frac{1}{4}\lambda_7 + \frac{3}{2}\lambda_8\right) \\
G_8 &= \left(\lambda_3 - \lambda_4 - \frac{1}{2}\lambda_6 - \frac{\sqrt{3}}{2}\lambda_7\right) \otimes I + I \otimes \left(\lambda_3 - \lambda_4 - \frac{1}{2}\lambda_6 - \frac{\sqrt{3}}{2}\lambda_7\right).
\end{aligned} \tag{36}$$

5. Conclusions

We have generalized the ELW game to the case of N-strategies. The gate operator J is one of the most important elements of the quantum game because it introduces quantum correlations. In order to construct the operator J we assume that it preserves the symmetry of the classical game and the classical game is faithfully represented in its quantum counterpart.

The maximal entanglement game corresponds to the maximal stability subgroup of initial state. The elements differing by an element of the stability subgroup lead to the same outcome. The effective manifold of games is the coset space $SU(N) \times SU(N)/G_s$.

In the $N = 3$ case we found the most general form of the gate operator and we determined the values of parameters leading to the maximal entanglement of the game. As an example, the explicit form of generators of stability group G_s was given for one case.

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