Low energy dynamics of slender monopoles in non-Abelian superconductor

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Abstract. Low energy dynamics of magnetic monopoles and anti-monopoles in the $U(2)_{\rm C}$ gauge theory is studied in the Higgs (non-Abelian superconducting) phase. The monopoles in this phase are slender ellipsoids, pierced by a vortex string. We investigate scattering of monopole with anti-monopole and find that they do not always decay into radiation, contrary to our naive intuition. They can repel, make bound states (magnetic mesons) or resonances. We point out that some part of solutions in 1 + 3 dimensions can be mapped exactly onto the sine-Gordon system in 1+1 dimensions in the first non-trivial order of rigid-body approximation and we provide analytic formulas for such solutions there.

1. Introduction

Among various topological solitons, magnetic monopoles are one of the most fascinating ones in high energy physics. If a monopole exists, electric charges in the universe are quantized by the Dirac's quantization condition [2]: product of electric charge $Q_e e$ and magnetic charge $Q_m g$ must be proportional to integers: $Q_e e Q_m g \propto n$. This implies that a weak electric coupling ecorresponds to a strong magnetic coupling g. This strong-weak coupling duality is a powerful tool to understand strong coupling physics. Furthermore, monopoles are expected to play an important role to explain the confinement in QCD. It has been proposed that the vacuum is in dual color superconductor where magnetic monopoles condense [3, 4, 5]. It is, however, very difficult to verify this idea, since QCD is strongly coupled at low energies because of the asymptotic freedom.

In contrast, QCD becomes weakly coupled at high baryon density, and enters into the colorsuperconducting phase where di-quarks condense [6, 7]. Then magnetic monopoles should be confined and color magnetic fields form flux tubes. A distinctive feature of non-Abelian vortex compared to the usual Nielsen-Olesen Abelian vortex is that the non-Abelian vortex breaks a non-Abelian global symmetry of the vacuum state and non-Abelian orientational moduli [8, 9, 10] emerge. This orientational moduli has been found to give a confining state of monopole and anti-monopole, namely magnetic meson was predicted in high density QCD [11, 12, 13].

If we choose couplings to critical values which enable us to embed the theory into a supersymmetric one, we can have Bogomolnyi-Prasad-Sommerfield (BPS) solitons [14], which preserve a part of supersymmetry [15]. The dynamics of slowly moving BPS solitons are well-approximated by a geodesic motion on the moduli space [16]. This is called the moduli

approximation. Although the moduli approximation is useful, one should note that it can be applied neither for scattering of BPS solitons with high momentum nor for non-BPS systems. Among non-BPS solutions, an interesting non-BPS "bound state" of a monopole and an antimonopole in the Coulomb phase has been rigorously established [17]. Although these "bound states" eventually decay due to unstable modes, they can play a significant role in understanding the dynamics of monopole and anti-monopole system in the Coulomb phase.

If the non-Abelian gauge symmetry is completely broken, we are in the Higgs phase, namely the non-Abelian superconducting phase. In contrast to the Coulomb phase, monopoles in the Higgs phase have several distinctive features. Firstly, they are pierced by a vortex string. In fact, the static BPS monopole in the Higgs phase has been found as a static BPS kink solution in the 1 + 1-dimensional low-energy effective field theory on the vortex [18]. Secondly, shape of a monopole is not spherical. There are two fundamental length scales: One is a transverse size $L_{\rm T}$ of the flux tube and the other is a length $L_{\rm L}$ of the monopole. Since a monopole resides on a vortex, its shape is generally not spherical depending on the ratio of the two scales. If we assume

$$L_{\rm T} \ll L_{\rm L},$$
 (1)

monopoles are of a slender ellipsoidal shape. This approximation has been used previously [19] to obtain an effective action of 1/4 BPS non-Abelian monopole-vortex complex. Another work to obtain effective action of monopole-vortex complex appeared recently [20].

In this work, we will consider a straight vortex string and we study head-on collisions of monopoles and anti-monopoles. In contrast to lots of studies on scattering of the BPS monopoles, there is very few studies about the dynamics of BPS monopoles and anti-monopoles, especially in the Higgs phase. This is mainly due to the inapplicability of the moduli approximation to the non-BPS monopoles and anti-monopoles system, which necessitates other methods such as numerical analysis. Instead of numerical methods, we consider a systematic expansion in powers of the ratio $L_{\rm T}/L_{\rm L}$ of length scales of the model, which allows us to obtain analytic solutions. At the first order of the expansion, we obtain the rigid-body approximation, where we can provide analytic solutions, which are very useful to understand the dynamics of monopole and anti-monopole system. The monopole and anti-monopole dynamics here is essentially 1 + 1-dimensional. We will observe that a part of the dynamics can be mapped on to the integrable sine-Gordon model. We provide exact mapping between 1 + 3-dimensional field configurations of gauge theory and 1 + 1-dimensional field configurations of the sine-Gordon model.

2. Model & topological solitons

Let us consider a $U(2)_{\rm C}$ Yang-Mills-Higgs system

$$\mathcal{L} = \operatorname{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + D_{\mu}H(D^{\mu}H)^{\dagger} + \frac{1}{g^2}D_{\mu}\Sigma D^{\mu}\Sigma\right] - V, \qquad (2)$$

$$V = \operatorname{Tr}\left[\frac{g^2}{4}\left(HH^{\dagger} - v^2\mathbf{1}_2\right)^2 + (\Sigma H - HM)\left(\Sigma H - HM\right)^{\dagger}\right],\tag{3}$$

where the field strength and the covariant derivatives are defined by

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + i [A_{\mu}, A_{\nu}], \qquad (4)$$

$$D_{\mu}H = \partial_{\mu}H + iA_{\mu}H, \qquad (5)$$

$$D_{\mu}\Sigma = \partial_{\mu}\Sigma + i [A_{\mu}, \Sigma].$$
(6)

The N_F species of Higgs fields in the fundamental representation of the $U(2)_{\rm C}$ gauge group is denoted by a $2 \times N_{\rm F}$ matrix H. We concentrate on $N_{\rm F} = 2$ case in the following. Another Higgs field Σ in the adjoint representation of the $U(2)_{\rm C}$ gauge group is denoted by a real 2×2 matrix. The quartic scalar coupling is given in terms of the gauge coupling constant g, which allows our model to be embedded in a supersymmetric theory. The parameter v giving the vacuum expectation value of the Higgs field H comes from the so-called Fayet-Illiopoulos (FI) parameter in the supersymmetric context. We assume v > 0 in what follows. We take the mass matrix Min the following form

$$M = \frac{m}{2}\sigma_3. \tag{7}$$

Global symmetry of the model depends on the mass parameter m. If m = 0, the flavor symmetry is $SU(2)_{\rm F}$. If $m \neq 0$, the flavor symmetry reduces to $U(1)_{\rm F} \subset SU(2)_{\rm F}$ generated by the third component of $SU(2)_{\rm F}$. In this work, we consider $m \neq 0$ case unless stated otherwise.

The vacuum of the model is determined by the condition V = 0:

$$HH^{\dagger} = v^2 \mathbf{1}_2, \quad \Sigma = M. \tag{8}$$

The vacuum manifold is topologically a torus T^2 but we can make $U(2)_{\rm C}$ gauge transformations to bring the fields to the following representative value

$$H = v \mathbf{1}_2, \quad \Sigma = M. \tag{9}$$

Therefore all the points in the vacuum manifold are physically equivalent, and the vacuum moduli space consists of only one point. We call this vacuum the color-flavor locking (CFL) vacuum. The vacuum is in the Higgs phase, where the gauge symmetry is completely broken.

The model (2) admits rich topological excitations; vortex strings and magnetic monopoles. In the Higgs vacuum, magnetic field can only exist by having an unbroken normal vacuum in a small neighbourhood of the zero of the Higgs field. Hence magnetic field is squeezed into a vortex, which we call a vortex string. The vortex string is topologically stable due to a non-trivial fundamental homotopy group in the massive case

$$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}.$$
(10)

There are two kinds of vortex quantum numbers, corresponding to two kinds of vortex strings, which we call the N-vortex and S-vortex. A magnetic monopole is a source of the conserved magnetic fluxes which are squeezed into vortex strings in this Higgs vacuum. Therefore a stable magnetic monopole is possible only as a composite soliton in the middle of a vortex string, but cannot exist as an isolated soliton, which can also be understood from the trivial homotopy

$$\pi_2(T^2) = 0. \tag{11}$$

In the $U(2)_{\rm C}$ Yang-Mills Higgs theory, solutions for the magnetic monopole pierced by vortex strings have been found [18, 21], which preserve a quarter of supersymmetry charges when embedded into the supersymmetric theory.

The total energy E for 1/4 BPS configuration can be written as the sum of two topological charges representing the monopole energy M_{mono} and the vortex energy M_{vor}

$$E \geq M_{\text{mono}} + M_{\text{vor}},$$
 (12)

$$M_{\rm mono} = \frac{1}{g^2} \int d^3x \, \operatorname{Tr}[\epsilon_{ijk} \partial_i \left(\Sigma F_{jk}\right)], \qquad (13)$$

$$M_{\rm vor} = -v^2 \int d^3x \, {\rm Tr}[F_{12}].$$
 (14)

This bound is saturated when the following BPS equations are satisfied

$$F_{12} - D_3 \Sigma - \frac{g^2}{2} \left(H H^{\dagger} - v \mathbf{1}_2 \right) = 0, \tag{15}$$

$$F_{23} - D_1 \Sigma = 0, \tag{16}$$

$$F_{31} - D_2 \Sigma = 0, \tag{11}$$

$$(D_1 + iD_2)H = 0, (18)$$

$$D_3H + \Sigma H - HM = 0. (19)$$

If we define a 2×2 matrix field S taking values in $GL(2,\mathbb{C})$ whose elements are functions of $x^{1,2,3}$ as

$$\bar{A} = -iS^{-1}\bar{\partial}S, \quad A_3 - i\Sigma = -iS^{-1}\partial_3S, \tag{20}$$

we can solve the equations (16) - (19) in terms of a holomorphic matrix $H_0(z)$

$$H = vS^{-1}(x^1, x^2, x^3)H_0(z)e^{Mx^3},$$
(21)

where we have defined

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2, \quad \partial = \frac{\partial_1 - i\partial_2}{2}, \quad \bar{\partial} = \frac{\partial_1 + i\partial_2}{2}, \quad (22)$$

$$A = \frac{A_1 - iA_2}{2}, \quad \bar{A} = \frac{A_1 + iA_2}{2}, \quad D = \frac{D_1 - iD_2}{2}, \quad \bar{D} = \frac{D_1 + iD_2}{2}.$$
 (23)

This method to solve the BPS equation is called the moduli matrix formalism [21, 1]. The following V-transformations leave the physical fields H in Eq. (21) and A_i and Σ in Eq. (20) unchanged.

$$S(x^1, x^2, x^3) \to V(z)S(x^1, x^2, x^3), \quad H_0(z) \to V(z)H_0(z), \qquad V(z) \in GL(2, \mathbb{C}),$$
 (24)

where elements of the $GL(2, \mathbb{C})$ matrix V(z) are holomorphic functions in z. Therefore the moduli space of the monopole vortex complex becomes the moduli matrices divided by the V-equivalence relation.

The $U(2)_{\rm C}$ gauge transformations act on S^{-1} from the left as

$$S^{-1} \to U_{\rm C} S^{-1}.\tag{25}$$

By defining $U(2)_{\rm C}$ gauge invariant matrices Ω and Ω_0 ,

$$\Omega = SS^{\dagger}, \quad \Omega_0 = H_0 e^{2Mx^3} H_0^{\dagger}, \tag{26}$$

we can cast the remaining BPS equation (15) into the following master equation

$$\frac{1}{g^2 v^2} \left[4\bar{\partial} \left(\partial \Omega \Omega^{-1} \right) + \partial_3 \left(\partial_3 \Omega \Omega^{-1} \right) \right] = \mathbf{1}_2 - \Omega_0 \Omega^{-1}.$$
(27)

This master equation should be solved with the boundary condition

$$\Omega \to \Omega_0 \qquad \text{as} \qquad |\vec{x}| \to \infty.$$
 (28)

The $U(2)_{\rm C}$ gauge invariants Ω and Ω_0 are covariant under the V-transformations.

3. Vortex strings

Before describing the monopole-vortex complex, let us first explain a simpler configuration of vortex strings without monopoles. We have two different types of Abelian vortices. If the complex zero is placed in the upper-left corner in the moduli matrix we obtain

$$H_0 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} e^{\frac{\psi(x^1, x^2)}{2}} & 0 \\ 0 & 1 \end{pmatrix} e^{Mx^3},$$
(29)

which is called an N-vortex, We find that the master equation for the N-vortex (29) reduces to the master equation for the Abelian vortex

$$\frac{4}{g^2 v^2} \partial \bar{\partial} \psi = 1 - |z|^2 e^{-\psi}.$$
(30)

This equation has no known analytic solution, but can easily be solved numerically. The asymptotic behavior for ψ is given by

$$\psi \to \log |z|^2 + qK_0(gv|z|), \quad \text{as} \quad |z| \to \infty,$$
(31)

where K_0 is the modified Bessel function of the second kind and a constant q can be obtained numerically.

For later convenience, let us decompose the magnetic field F_{12} in the $U(2)_{\rm C} = (U(1)_{\rm C0} \times SU(2)_{\rm C})/\mathbb{Z}_2$ gauge group into the field F_{12}^0 for the overall $U(1)_{\rm C0}$ and the field F_{12}^{Σ} projected along the adjoint field Σ (this is identical to the third component $U(1)_{\rm C3}$ of $SU(2)_{\rm C}$ in the present case) as

$$F_{ij}^{0} = \operatorname{Tr}\left[F_{ij}\frac{\mathbf{1}_{2}}{2}\right], \quad F_{ij}^{\Sigma} = \operatorname{Tr}\left[F_{ij}\frac{\Sigma}{m}\right].$$
(32)

We will call F_{12}^0 as Abelian magnetic field and F_{12}^{Σ} as non-Abelain magnetic field. Note that these two magnetic fields are associated with $(U(1)_{C0} \times U(1)_{C3})/\mathbb{Z}_2 \subset U(2)_C$ (asymptotically) which is not broken by the adjoint scalar field $\Sigma = (m/2)\sigma_3$. A linear combination of these U(1) gauge symmetries is restored inside vortices. Therefore they are precisely the appropriate magnetic fields to measure the magnetic flux flowing to infinity through vortices. For the N-vortex, we obtain Abelian and non-Abelian magnetic fields as

$$F_{12}^0 = -\partial\bar{\partial}\psi, \quad F_{12}^\Sigma = -\partial\bar{\partial}\psi.$$
(33)

We see that only the sum $F_{12}^0 + F_{12}^{\Sigma}$ has nonvanishing magnetic field inside the N-vortex. This linear combination precisely corresponds to the restored U(1) gauge symmetry inside the Nvortex.

Another possibility to place the zero of the Higgs field is at the lower-right corner of the moduli matrix as

$$H_0 = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & e^{\psi/2} \end{pmatrix} e^{Mx^3}, \tag{34}$$

where ψ is the same function as the N-vortex. We call this the S-vortex. The Abelian F_{12}^0 and non-Abelian F_{12}^3 magnetic fields of the S-vortex is given as

$$F_{12}^0 = -\partial\bar{\partial}\psi, \quad F_{12}^\Sigma = +\partial\bar{\partial}\psi. \tag{35}$$

We see that only the difference $F_{12}^0 - F_{12}^{\Sigma}$ has nonvanishing magnetic field inside the S-vortex. This linear combination is the restored U(1) gauge symmetry inside the S-vortex.

4. Rigid-body approximation

In this section, we consider monopoles in the non-Abelian superconducting phase. We use a systematic expansion up to the next-to-leading order within an approximation, which we call the rigid-body approximation. The transverse size of the vortex string $L_{\rm T} = 1/(gv)$ is associated with a large mass scale gv, and the longitudinal monopole size $L_{\rm L} = 1/m$ is associated with a small mass scale. Therefore the condition $gv \gg m$ introduces hierarchal mass scales in the system: the thin vortex-string is generated at the high energy scale $\sim gv$, and the slender monopole is generated at the lower energy scale $\sim m$.

This picture allows us to understand the slender monopole as a kink in the 1+1 dimensional theory on the vortex world-sheet [18]. Assuming

$$\epsilon = \frac{m}{gv} \sim \frac{\partial_{\alpha}}{\partial_i} \ll 1, \quad (\alpha = 0, 3 \text{ and } i = 1, 2),$$
(36)

we expand the fields in power series of ϵ

$$H = H^{(0)} + H^{(2)} + \cdots, (37)$$

$$A_i = A_i^{(0)} + A_i^{(2)} + \cdots, \qquad (i = 1, 2),$$
(38)

$$A_{\alpha} = A_{\alpha}^{(1)} + A_{\alpha}^{(3)} + \cdots, \qquad (\alpha = 0, 3), \tag{39}$$

$$\Sigma - M = \Sigma^{(1)} + \Sigma^{(3)} + \cdots,$$
 (40)

where the superscript (n) indicates the *n*-th order in powers of ϵ . Note that H and A_i start from the zeroth order because they are nontrivial in the background vortex-string configuration. On the other hand, since A_{α} and $\Sigma - M$ vanish in the background vortex-string configuration, they start from the first order.

Zero-th order: background vortex string for m = 0 Retaining only the zero-th order fields in Eqs. (37) - (40), we find the following zero-th order reductions of full equations of motion

$$2(D_z D_{\bar{z}} H^{(0)} + D_{\bar{z}} D_z H^{(0)}) - \frac{g^2}{2} (H^{(0)} H^{(0)\dagger} - v^2 \mathbf{1}_2) H^{(0)} = 0,$$
(41)

$$-\frac{4}{g^2}D_{\bar{z}}F_{z\bar{z}}^{(0)} + i(H^{(0)}D_{\bar{z}}H^{(0)\dagger} - D_{\bar{z}}H^{(0)}H^{(0)\dagger}) = 0.$$
(42)

The zero-th order solutions can be compactly expressed in the moduli matrix formalism as

$$H^{(0)} = vS^{(0)-1}H_0(z), \quad \bar{A}^{(0)} = -iS^{(0)-1}\bar{\partial}S^{(0)}, \tag{43}$$

with the master equation for the vortex

$$\frac{4}{g^2 v^2} \bar{\partial} \left(\partial \Omega^{(0)} \Omega^{(0)-1} \right) = \mathbf{1}_2 - \Omega_0^{(0)} \Omega^{(0)-1}, \quad \Omega^{(0)} = S^{(0)} S^{(0)\dagger}, \quad \Omega_0^{(0)} = H_0^{(0)} H_0^{(0)\dagger}. \tag{44}$$

When M = 0, the flavor symmetry is enhanced from $U(1)_{\rm F}$ to $SU(2)_{\rm F}$ and the symmetry of the vacuum becomes $SU(2)_{\rm C+F}$. A single vortex spontaneously breaks this symmetry to $U(1)_{\rm C+F}$. Therefore, the Nambu-Goldstone zero modes ϕ appear as a moduli

$$\mathbb{C}P^1 = \frac{SU(2)_{C+F}}{U(1)_{C+F}} \simeq S^2.$$
(45)

By introducing the moduli parameter ϕ as an inhomogeneous coordinate of the moduli space $\mathbb{C}P^1 \simeq S^2$, we can express the generic moduli matrix H_0 with the moduli parameter $\phi \in \mathbb{C}$

as a color-flavor $SU(2)_{C+F}$ rotation of the N-vortex solution together with an accompanying V-transformation as

$$H_0^{(0)} = \begin{pmatrix} z & 0 \\ -\phi & 1 \end{pmatrix} = V \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} U,$$
(46)

$$S^{(0)} = \begin{pmatrix} \frac{e^{\frac{\psi}{2}} + z|\phi|^2}{1+|\phi|^2} & \frac{(e^{\frac{\psi}{2}} - z)\bar{\phi}}{1+|\phi|^2} \\ -\phi & 1 \end{pmatrix} = V \begin{pmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & 1 \end{pmatrix} U,$$
(47)

with

$$U = \frac{1}{\sqrt{1+|\phi|^2}} \begin{pmatrix} 1 & \bar{\phi} \\ -\phi & 1 \end{pmatrix} \in SU(2)_{\mathrm{C+F}}, \tag{48}$$

$$V = \frac{1}{\sqrt{1+|\phi|^2}} \begin{pmatrix} 1 & -\bar{\phi}z \\ 0 & 1+|\phi|^2 \end{pmatrix}.$$
 (49)

Here we need the V-transformation V(z), in order for $H_0^{(0)}$ to be a holomorphic function of the moduli parameter ϕ . The single vortex solution with the generic moduli ϕ can be obtained explicitly by inserting the N-vortex solution ψ into Eqs. (47) and (43).

At the zero-th order in $\epsilon \ll 1$, we obtained a moduli parameter ϕ in Eqs. (48) and (49) as a constant. However, our approximation allows the weak dependence of ϕ on x^0, x^3 from the beginning. Therefore, we should consider $\phi(x^0, x^3)$ to be a slowly varying function of x^0, x^3 . Then, the vortex background configuration depends on x^0 and x^3 only through the moduli field.

$$H^{(0)}(x^1, x^2; \phi(x^0, x^3)), \quad A^{(0)}_i(x^1, x^2; \phi(x^0, x^3)) \qquad (i = 1, 2).$$
(50)

We can determine $A_{\alpha}^{(1)}$ and $\Sigma^{(1)}$ as

$$A_{\alpha}^{(1)} = i \left[(\delta_{\alpha} S^{(0)\dagger}) S^{(0)\dagger-1} - S^{(0)-1} \delta_{\alpha}^{\dagger} S^{(0)} \right], \quad (\alpha = 0, 3),$$
(51)

$$\Sigma^{(1)} = M + i \left[(\delta_{\phi} S^{(0)\dagger}) S^{(0)\dagger-1} - S^{(0)-1} \delta_{\phi}^{\dagger} S^{(0)} \right],$$
(52)

with

$$\delta_{\alpha} = \partial_{\alpha}\phi \frac{\delta}{\delta\phi}, \quad \delta^{\dagger}_{\alpha} = \partial_{\alpha}\bar{\phi}\frac{\delta}{\delta\bar{\phi}}, \quad \delta_{\phi} = -im\phi\frac{\delta}{\delta\phi}, \quad \delta^{\dagger}_{\phi} = im\bar{\phi}\frac{\delta}{\delta\bar{\phi}}.$$
(53)

The remaining task is to look for the appropriate configurations of $\phi(x^0, x^3)$ which minimize the energy of the solution. To this end, we plug $A_{\alpha}^{(1)}$ and $\Sigma^{(1)}$ into the original Lagrangian (2) and pick up terms up to the second order in ϵ . After a tedious calculation, one obtains the following expression, where the $x^{1,2}$ and $x^{0,3}$ dependence are factorized as

$$\mathcal{L} = -v^2 F_{12}(x^1, x^2) + \frac{F(x^1, x^2)}{g^2} \times \frac{|\partial_\alpha \phi(x^0, x^3)|^2 - m^2 |\phi(x^0, x^3)|^2}{(1 + |\phi(x^0, x^3)|^2)^2} + \mathcal{O}(\epsilon^4),$$
(54)

where we ignore unessential total derivative terms. The prefactor in the second term depends on only x^1 and x^2 and it is given by

$$F(x^1, x^2) = 4\partial\bar{\partial}\psi(x^1, x^2).$$
(55)

Hence, in order to minimize the action to the second order, we need to find a stationary point of

$$\mathcal{L}^{(2)} = \frac{F(x^1, x^2)}{g^2} \times \frac{|\partial_\alpha \phi(x^0, x^3)|^2 - m^2 |\phi(x^0, x^3)|^2}{(1 + |\phi(x^0, x^3)|^2)^2}.$$
(56)

Since the prefactor $F(x^1, x^2)$ is determined at the zero-th order, our task is basically to solve the massive non-linear sigma model in two dimensions with the target space $\mathbb{C}P^1$. Note that the process here is essentially the same as a well known derivation of a low energy effective action in the moduli approximation. To obtain the effective action, one just needs to integrate the Lagrangian over x^1 and x^2 . The resulting overall coefficient is $4\pi = \int dx^1 dx^2 F(x^1, x^2)$ and thus

$$\mathcal{L}_{\text{eff}} = \frac{4\pi}{g^2} \frac{|\partial_\alpha \phi(x^0, x^3)|^2 - m^2 |\phi(x^0, x^3)|^2}{(1 + |\phi(x^0, x^3)|^2)^2}.$$
(57)

In summary, in order to solve the equations of motion to the first order, we just need to solve the equations of motion of the effective theory, and to plug the solution $\phi(x^0, x^3)$ into $H(x^1, x^2; \phi(x^0, x^3))$ and $A_{1,2}(x^1, x^2; \phi(x^0, x^3))$. The remaining fields $A_{0,3}^{(1)}(x^0, x^3)$ and $\Sigma^{(1)}(x^0, x^3)$ to the first order are obtained through Eqs. (51) and (52).

For later convenience, let us introduce another parametrization of $\mathbb{C}P^1$ in terms of polar angles $0 \le \Theta \le \pi$ and $0 \le \Phi \le 2\pi$ as

$$\phi = -e^{i\Phi} \tan \frac{\Theta}{2}.$$
(58)

The effective Lagrangian is rewritten as

$$\mathcal{L}_{\text{eff}} = \frac{\pi}{g^2} \left[\partial_\alpha \Theta \partial^\alpha \Theta + \sin^2 \Theta \partial_\alpha \Phi \partial^\alpha \Phi - m^2 \sin^2 \Theta \right].$$
(59)

The scalar potential $(\pi m^2/g^2) \sin^2 \Theta$ is minimized at $\Theta = 0$ and $\Theta = \pi$. Clearly, these correspond to the N-vortex ($\phi = 0$) and the S-vortex ($\phi = \infty$).

4.1. Monopole

We are now ready to reconsider the slender magnetic monopole in the Higgs phase in our rigidbody approximation. Let us first look for an appropriate moduli configuration which minimizes the action in 1+3 dimensions by solving the equations of motion in the low energy effective theory:

$$\partial_{\alpha}\partial^{\alpha}\Theta + (m^2 - \partial_{\alpha}\Phi\partial^{\alpha}\Phi)\sin\Theta\cos\Theta = 0, \tag{60}$$

$$\partial^{\alpha} \left(\sin^2 \Theta \partial_{\alpha} \Phi \right) = 0. \tag{61}$$

Eq. (61) admits a constant solution for Φ , say $\Phi = \eta$. In this study, we focus our attention to this class of solutions. Then the equation of motion reduces to the sine-Gordon equation

$$-\Theta'' + m^2 \sin \Theta \cos \Theta = 0, \tag{62}$$

where the prime stands for the derivative in terms of x^3 . The sine-Gordon model admits nontrivial topological excitations, kinks. The kinks interpolating $\Theta = 0$ and $\Theta = \pi$ are given by

$$\Theta = 2 \arctan \exp\left(\pm m(x^3 - X_m)\right).$$
(63)

The solution with the plus sign is the kink connecting $\Theta = 0$ at $x^3 \to -\infty$ and $\Theta = \pi$ at $x^3 \to +\infty$, while that with minus sign is the anti-kink which connects $\Theta = \pi$ at $x^3 \to -\infty$ and 0 at $x^3 \to +\infty$.

From now on, we consider $X_m = \eta = 0$ case for simplicity. Combining this with Eqs. (43), (46), (47), and (48), we find

$$H^{(0)}(x^1, x^2, \phi(x^3)) = vU^{\dagger}(x^3) \begin{pmatrix} ze^{-\frac{\psi}{2}} & 0\\ 0 & 1 \end{pmatrix} U(x^3),$$
(64)

$$\bar{A}^{(0)}(x^1, x^2, \phi(x^3)) = U^{\dagger}(x^3) \begin{pmatrix} -\frac{i}{2}\bar{\partial}\psi & 0\\ 0 & 0 \end{pmatrix} U(x^3),$$
(65)

with

$$U(x^3) = \frac{1}{\sqrt{1 + |\phi(x^3)|^2}} \begin{pmatrix} 1 & \bar{\phi}(x^3) \\ -\phi(x^3) & 1 \end{pmatrix}.$$
 (66)

Furthermore, plugging $S^{(0)}$

$$S^{(0)} = \begin{pmatrix} \frac{e^{\frac{\psi}{2}} + z|\phi|^2}{1+|\phi|^2} & \frac{(e^{\frac{\psi}{2}} - z)\bar{\phi}}{1+|\phi|^2} \\ -\phi & 1 \end{pmatrix}, \qquad \phi = -\exp(\pm mx^3), \tag{67}$$

into the solutions $A_{\alpha}^{(1)}$ and $\Sigma^{(1)}$ given in Eqs. (51) and (52), we obtain the induced fields

$$A_3 \simeq \pm \frac{im}{2} \operatorname{sech} mx^3 U^{\dagger}(x^3) \begin{pmatrix} 0 & 1 - ze^{-\frac{\psi}{2}} \\ \bar{z}e^{-\frac{\psi}{2}} - 1 & 0 \end{pmatrix} U(x^3),$$
(68)

$$\Sigma \simeq \frac{m}{2} \operatorname{sech} mx^3 U^{\dagger}(x^3) \begin{pmatrix} \mp \sinh mx^3 & ze^{-\frac{\psi}{2}} \\ \bar{z}e^{-\frac{\psi}{2}} & \pm \sinh mx^3 \end{pmatrix} U(x^3).$$
(69)

The electric fields are given as:

$$F_{12}^0 \simeq -\partial \bar{\partial} \psi, \quad F_{23}^0 \simeq 0, \quad F_{31}^0 \simeq 0,$$
 (70)

and the magnetic fields are given as:

$$B_3^{\Sigma} = F_{12}^{\Sigma} \simeq \pm \partial \bar{\partial} \psi \tanh m x^3, \tag{71}$$

$$B_1^{\Sigma} = F_{23}^{\Sigma} \simeq \pm \frac{m}{4} \partial_1 (r^2 e^{-\psi}) \operatorname{sech}^2 m x^3,$$
 (72)

$$B_2^{\Sigma} = F_{31}^{\Sigma} \simeq \pm \frac{m}{4} \partial_2 (r^2 e^{-\psi}) \operatorname{sech}^2 m x^3.$$
 (73)

The solution with the lower sign connects the N-vortex as $x^3 \to +\infty$ and the S-vortex as $x^3 \to -\infty$, which is opposite to the configuration with upper sign. The corresponding monopole has the magnetic field F_{ij}^{Σ} pointing toward monopole, namely it is an anti-monopole in the Higgs phase.

Magnetic charges of the above solutions can be easily calculated

$$Q_{\rm m} = \frac{1}{g} \int d^3x \, \operatorname{div} \vec{B}^{\Sigma} = \frac{1}{g} \left[\int_{x^3 \to \infty} dx^1 dx^2 \, (\pm \partial \bar{\partial} \psi) - \int_{x^3 \to -\infty} dx^1 dx^2 \, (\mp \partial \bar{\partial} \psi) \right] = \pm \frac{4\pi}{g}, \qquad (74)$$

where we used $\int dx^1 dx^2 \ \partial \bar{\partial} \psi = \pi$ and $r^2 e^{-\psi} \to 1$ as $r \to \infty$. Here the factor 1/g is needed due to our notation that the gauge coupling is absorbed in the gauge field, see Eqs. (4) – (6). This magnetic charge precisely coincides with one of the 't Hooft-Polyakov monopole in the Coulomb phase [18].

5. Dynamics of slender monopoles and anti-monopoles

In the following, we make full use of the similarity between our system and the sine-Gordon model. Let us denote another choice of the range of angles as

$$\tilde{\Theta} \in \mathbb{R} \pmod{2\pi}, \qquad \tilde{\Phi} \in [0,\pi),$$
(75)

to parametrize the $\mathbb{C}P^1$ moduli ϕ

$$\phi(x^0, x^3) = -e^{i\tilde{\Phi}(x^0, x^3)} \tan \frac{\Theta(x^0, x^3)}{2}.$$
(76)

The equations of motion for $\tilde{\Theta}, \tilde{\Phi}$ are the same as those for Θ, Φ . Therefore, $\tilde{\Phi} = \text{const.}$ is a solution, to which we restrict ourselves in the following. Without loss of generality, the value of the constant Φ can be chosen as

$$\tilde{\Phi} = 0. \tag{77}$$

Then the equation of motion for $\tilde{\Theta}$ is reduced to

$$\partial_{\alpha}\partial^{\alpha}\tilde{\Theta} + m^{2}\sin\tilde{\Theta}\cos\tilde{\Theta} = 0, \qquad \tilde{\Theta} \in \mathbb{R} \pmod{2\pi}.$$
 (78)

This is nothing but the sine-Gordon equation with a periodicity π in 1+1 dimensions.

Now we can compute all field configurations in 1+3 dimensions with the help of the sine-Gordon field $\tilde{\Theta}$

$$F_{12}^0 = -\partial \bar{\partial} \psi, \quad F_{23}^0 = F_{31}^0 = F_{01}^0 = F_{02}^0 = F_{03}^0 = 0,$$
 (79)

and

$$F_{12}^{\Sigma} = -\partial\bar{\partial}\psi\cos\tilde{\Theta}, \qquad (80)$$

$$F_{23}^{\Sigma} = \frac{1}{4} \partial_1 (r^2 e^{-\psi}) \partial_3 \tilde{\Theta} \sin \tilde{\Theta}, \qquad (81)$$

$$F_{31}^{\Sigma} = \frac{1}{4} \partial_2 (r^2 e^{-\psi}) \partial_3 \tilde{\Theta} \sin \tilde{\Theta}, \qquad (82)$$

$$F_{01}^{\Sigma} = \frac{1}{4} \partial_2 (r^2 e^{-\psi}) \partial_0 \tilde{\Theta} \sin \tilde{\Theta}, \qquad (83)$$

$$F_{02}^{\Sigma} = -\frac{1}{4}\partial_1(r^2 e^{-\psi})\partial_0\tilde{\Theta}\sin\tilde{\Theta}, \qquad (84)$$

$$F_{03}^{\Sigma} = 0. (85)$$

Here we define Abelian and non-Abelian electric fields in the same spirit as in Eq. (32)

$$F_{0i}^{0} = \operatorname{Tr}\left[F_{0i}\frac{\mathbf{1}_{2}}{2}\right], \quad F_{0i}^{\Sigma} = \operatorname{Tr}\left[F_{0i}\frac{\Sigma}{m}\right].$$
(86)

Note that the electric field and magnetic field are orthogonal

$$\epsilon^{ijk}F_{ij}^{\Sigma}F_{0k}^{\Sigma} = 0. \tag{87}$$

Therefore, there is no energy dissipation.

The Hamiltonian density is decomposed into two parts: the energy density of the rigid vortexstring $\mathcal{H}_{vortex}^{(0)}$ and that of the dressed monopole $\mathcal{H}_{dress}^{(2)}$

$$\mathcal{H} = \mathcal{H}_{\text{vortex}}^{(0)} + \mathcal{H}_{\text{dress}}^{(2)} + \mathcal{O}(\epsilon^4).$$
(88)

The rigid vortex-string Hamiltonian density does not depend on x^0 and x^3

$$\mathcal{H}_{\text{vortex}}^{(0)} = \operatorname{Tr} \left[\frac{1}{g^2} (F_{12}^{(0)})^2 + (D_i H)^{(0)} (D_i H)^{(0)\dagger} + \frac{g^2}{4} \left(H^{(0)} H^{(0)\dagger} - v^2 \mathbf{1}_2 \right)^2 \right]$$

$$= \operatorname{Tr} \left[\frac{1}{g^2} \left(F_{12}^{(0)} - \frac{g^2}{2} \left(H^{(0)} H^{(0)\dagger} - v^2 \mathbf{1}_2 \right) \right)^2 + 4(\bar{D}H)^{(0)} (\bar{D}H)^{(0)\dagger} - v^2 F_{12}^{(0)} + i \left\{ \partial_1 (H^{(0)} (D_2 H)^{(0)\dagger}) - \partial_2 (H^{(0)} (D_1 H)^{(0)\dagger}) \right\} \right]$$

$$= 2v^2 \mathcal{V}.$$
(89)

where

$$\mathcal{V}(x^1, x^2) = \partial \bar{\partial} \psi - \frac{4}{g^2 v^2} (\partial \bar{\partial})^2 \psi \tag{90}$$

The dressed Hamiltonian density which depends on x^0 and x^3 is given by

$$\mathcal{H}_{dress}^{(2)} = \operatorname{Tr} \left[\frac{1}{g^2} \left\{ (F_{23}^{(1)})^2 + (F_{31}^{(1)})^2 + (F_{01}^{(1)})^2 + (F_{02}^{(1)})^2 + (D_1 \Sigma^{(1)})^2 + (D_2 \Sigma^{(1)})^2 \right\} \\ + D_0 H^{(0)} (D_0 H^{(0)})^{\dagger} + D_3 H^{(0)} (D_3 H^{(0)})^{\dagger} + (\Sigma^{(1)} H^{(0)} - H^{(0)} M) (\Sigma^{(1)} H^{(0)} - H^{(0)} M)^{\dagger} \right] \\ = \frac{\mathcal{V}}{g^2} \left((\partial_0 \tilde{\Theta})^2 + (\partial_3 \tilde{\Theta})^2 + m^2 \sin^2 \tilde{\Theta} \right),$$
(91)

where we used the master equation (30).

The topological charge density is given as

$$Q_{\rm m} = \frac{2\mathcal{V}}{g} \partial_3 \tilde{\Theta} \sin \tilde{\Theta}. \tag{92}$$

From this expression, one can easily compute the magnetic charge as

$$Q_{\rm m} = \int d^3x \ \mathcal{Q}_{\rm m} = \frac{2}{g} \int dx^1 dx^2 \ \mathcal{V} \quad \int dx^3 \ \partial_3 \tilde{\Theta} \sin \tilde{\Theta} = \frac{2\pi}{g} \left[-\cos \tilde{\Theta} \right]_{x^3 = -\infty}^{x_3 = +\infty}.$$
 (93)

Here we used $\int dx^1 dx^2 \partial \bar{\partial} \psi = \pi$. As a check, one can compute the energy of the magnetic monopoles from the solutions given in Eq. (63)

$$Q_{\rm m} = \pm \frac{4\pi}{g}.\tag{94}$$

Similarly, one may introduce a electric charge density by

$$\mathcal{Q}_{\rm e} = \frac{1}{g} \partial^i F_{0i}^{\Sigma}.$$
(95)

But this is identically zero for any $\tilde{\Theta}(x^0, x^3)$. This matches with a naive intuition that the fixed azimuthal angle $\tilde{\Phi}$ does not generate any electric charges. Note, however, that this does not mean the electric fields themselves are zeros. One can easily find that rotation of $\vec{E}^{\Sigma} = (F_{10}^{\Sigma}, F_{20}^{\Sigma}, F_{30}^{\Sigma})$ are non-zero.

$$(\vec{\nabla} \times \vec{E}^{\Sigma})_3 = -\frac{4}{g^2 v^2} (\partial \bar{\partial})^2 \psi \ \partial_0 \tilde{\Theta} \sin \tilde{\Theta}.$$
(96)

The other components are of higher order, so we ignore them.

5.1. Two different species of slender monopoles

The zenith angle Θ takes values between 0 and 2π (= $\mathbb{R} \mod 2\pi$), and the sine-Gordon equation (78) is periodic with a period π . Therefore, there exist two sine-Gordon kinks: the one interpolates from 0 to π as $x^3 = -\infty \to +\infty$, and the other interpolates from π to 2π as $x^3 = -\infty \to +\infty$. Here we need to pay some attention to our terminology in translating the sine-Gordon kinks into monopoles in 1 + 3 dimensions. Although these two configurations are both to be called kinks in the sense of the sine-Gordon model, the former connects the N-vortex and S-vortex from left to right, while the latter connects them from right to left. Namely, the former kink is the monopole (denoted as \mathcal{M}_0) and the latter kink is the anti-monopole ($\overline{\mathcal{M}}_{\pi}$). Similarly, the anti-kink interpolating from π to 0 as $x^3 = -\infty \to +\infty$ is the monopole (\mathcal{M}_{π}). Correspondence between the sine-Gordon (anti-)kinks and the slender (anti-)monopoles are depicted in Fig. 1. The configurations are given by



Figure 1. Correspondence between the sine-Gordon (anti-)kinks and the slender (anti-)monopoles.

$$\mathcal{M}_0$$
 : $\tilde{\Theta} = 2 \arctan \exp(mx^3) + 2n\pi,$ (97)

$$\bar{\mathcal{M}}_0$$
 : $\tilde{\Theta} = 2 \arctan \exp(-mx^3) + 2n\pi,$ (98)

$$\mathcal{M}_{\pi} : \quad \tilde{\Theta} = 2 \arctan \exp(-mx^3) + (2n+1)\pi, \tag{99}$$

$$\bar{\mathcal{M}}_{\pi} : \quad \tilde{\Theta} = 2 \arctan \exp(mx^3) + (2n+1)\pi, \tag{100}$$

with n being an integer.

5.2. Magnetic meson

It is well-known that the sine-Gordon model admits a bound state of kink and anti-kink, the so-called breather solution. In our case, it is nothing but a bound state of the slender monopole and anti-monopole, which we call the magnetic meson. The configuration is given by

$$\tilde{\Theta}(x^0, x^3) = 2 \arctan\left(\frac{\eta \sin \omega x^0}{\cosh \eta \omega x^3}\right), \quad \eta = \sqrt{\frac{m^2}{\omega^2} - 1}, \quad \omega < m, \tag{101}$$

where ω is the frequency and $(\eta \omega)^{-1} = 1/\sqrt{m^2 - \omega^2}$ is the typical size of the magnetic meson. The mass of the meson depends on ω as

$$M_{\rm meson} = 2M_{\rm mono} \times \sqrt{1 - \frac{\omega^2}{m^2}} < 2M_{\rm mono}.$$
 (102)

The mass of the mesonic bound state is smaller than the sum of the masses of isolated monopole and anti-monopole.

We show how the magnetic meson varies in one period $T = \frac{2\pi}{\omega}$ in Fig. 2. The sources of outgoing magnetic field are identified as slender monopoles and those of incoming magnetic field as slender anti-monopoles. It is interesting to observe that the meson is made of \mathcal{M}_0 and $\bar{\mathcal{M}}_0$ at an instance (for example t = T/4), and that it transforms into a different meson made of \mathcal{M}_{π} and $\bar{\mathcal{M}}_{\pi}$ at another instance (for example t = 3T/4). In Fig. 2, we also show the topological charge density $\mathcal{Q}_{\rm m}$ given in Eq. (92) together with the energy density of the electric field

$$\mathcal{E} = \frac{1}{g^2} \operatorname{Tr} \left[(F_{01})^2 + (F_{02})^2 \right] = \frac{1}{g^2} \left| 1 - z \partial \psi \right|^2 e^{-\psi} (\partial_t \tilde{\Theta})^2.$$
(103)

As the monopole and anti-monopole approach each other, the magnetic energy density \mathcal{M} decreases and the electric field energy density \mathcal{E} grows. At the very instance of collision, the magnetic energy disappears and is transferred into the electric energy completely. The electric field is generated by the time variation (decrease) of the magnetic field as monopole and anti-monopole merge.

5.3. Scattering of the slender monopole and anti-monopole

Let us next study the head-on collision of the slender monopole and anti-monopole. There are two types of collisions: one type is the collision of \mathcal{M}_0 and $\bar{\mathcal{M}}_0$, $(\mathcal{M}_{\pi} \text{ and } \bar{\mathcal{M}}_{\pi})$ and the other type is that of \mathcal{M}_0 and $\bar{\mathcal{M}}_{\pi}$ (\mathcal{M}_{π} and $\bar{\mathcal{M}}_0$).

Scattering of \mathcal{M}_0 and $\overline{\mathcal{M}}_0$ (\mathcal{M}_{π} and $\overline{\mathcal{M}}_{\pi}$) The exact solution for the moduli field for the collision of a monopole and anti-monopole of the same species (\mathcal{M}_0 or \mathcal{M}_{π}) is given by

$$\tilde{\Theta} = 2 \arctan\left(\frac{\sinh mu\gamma x^0}{u\cosh m\gamma x^3}\right), \quad \gamma = \frac{1}{\sqrt{1-u^2}}.$$
(104)

The parameter u corresponds to the velocity of the monopole. However, we should keep in mind that our approximation holds only for small velocities, that is

$$u \ll 1 \qquad (\gamma \simeq 1). \tag{105}$$

Since we are using the rigid-body approximation we cannot faithfully describe Lorentz boosted monopoles. Thus even though we can solve the 1+1-dimensional effective dynamics for arbitrary velocities, the full 1+3-dimensional dynamics is correctly represented only within the restriction of Eq. (105). A typical configuration is shown in Fig. 3. The slender magnetic monopole \mathcal{M}_{π} comes from the left infinity and the anti-monopole $\bar{\mathcal{M}}_{\pi}$ comes from the right infinity. As they approach to the collision point, the magnetic energy decreases while the electric energy grows. After the collision, the magnetic energy grows as the monopole \mathcal{M}_0 (anti-monopole $\bar{\mathcal{M}}_0$) goes toward the left (right) infinity. Thus we find that the species of the monopole and the antimonopole changes after the collision.

Scattering of \mathcal{M}_0 and $\overline{\mathcal{M}}_{\pi}$ (or \mathcal{M}_{π} and $\overline{\mathcal{M}}_0$) The solution for the scattering of a monopole and an anti-monopole of different species is given by

$$\tilde{\Theta} = 2 \arctan\left(\frac{u \sinh m\gamma x^3}{\cosh mu\gamma x^0}\right).$$
(106)

A typical configuration is shown in Fig. 4. In contrast to the previous type of scattering the species of monopoles (\mathcal{M}_0 or \mathcal{M}_π) do not change into different species during the collision. As shown in Fig. 4, the anti-monopole $\bar{\mathcal{M}}_\pi$ comes from the left infinity and reflects back toward the left infinity, while the monopole \mathcal{M}_0 comes from the right infinity and reflects back toward the right infinity.



Figure 2. Snapshots of a single period of the magnetic meson. The top is at t = 0 and the bottom is at $t = T - \delta t$ with $\delta t = T/16$. The left panel shows the magnetic field $(F_{12}^{\Sigma}, F_{23}^{\Sigma})$ by blue streamlines and the topological charge densities, $\mathcal{M} = \pm 0.017$, electric energy density, $\mathcal{E} = 0.012$, and the dressed energy density $\mathcal{H}_{dress} = 0.02$ by red/green/grey contours. In the right figures, $\tilde{\Theta}(x^3, t)$ is plotted. The curves are piecewise colored by black, red, blue and green for \mathcal{M}_0 , $\bar{\mathcal{M}}_0$, \mathcal{M}_{π} and $\bar{\mathcal{M}}_{\pi}$, respectively. We set gv = 1, m = 1/5 and $\omega = 1/10$. $x^1 \in [-3, 3]$ and $x^3 \in [-30, 30]$.



Figure 3. Snapshots (from t = -42 to t = 42 with $\delta t = 6$ interval) of scattering of the slender monopole and anti-monopole. The red/green/gray contours are $(\mathcal{M}, \mathcal{E}, \mathcal{H}_{dress}) = (\pm 0.017, 0.01, 0.02)$, see the caption of Fig. 2 for explanation. We set gv = 1, m = 1/5 and u = 1/10. $x^1 \in [-3, 3]$ and $x^3 \in [-30, 30]$.



Figure 4. Snapshots (from t = -70 to t = 70 with $\delta t = 10$ interval) of scattering of the slender monopole and anti-monopole. The red/gray contours are $(\mathcal{M}, \mathcal{H}_{dress}) = (\pm 0.017, 0.02)$. See the caption of Fig. 2 for explanation. We set gv = 1, m = 1/3 and u = 1/3. $x^1 \in [-3, 3]$ and $x^3 \in [-30, 30]$.

6. Conclusion

In this paper we have investigated the low energy dynamics of monopoles and anti-monopoles in the non-Abelian superconductor. We have restricted ourselves to the parameter region $m \ll gv$ where the monopoles are of slender ellipsoidal shape, confined on a vortex string, with the crosssection comparable to that of the monopole. For that reason, the scattering problem becomes essentially 1 + 1 dimensional. Indeed, we have found that at least a part of the low energy dynamics is identical to the sine-Gordon system in 1 + 1 dimensions up to the first order of the expansion in $\epsilon = m/(gv)$, when $\{m, \partial_0, \partial_3\} \ll \{gv, \partial_1, \partial_2\}$ holds. This observation is very useful because the sine-Gordon system is solvable. In the literature, only the static kink was identified with the monopole. In this paper, we have dealt with all the sine-Gordon solutions and have constructed the dictionary with which one can easily translate the dynamics of sine-Gordon kinks in 1+1 dimensions into the dynamics of monopoles in 1+3 dimensions. A surprising fact is that the monopole and anti-monopole do not always decay into radiation when they make a head-on collision, although they are not protected by topology. We have studied three concrete examples: (1) the magnetic meson which is the bound state of the slender monopole and antimonopole, (2) the scattering of the monopole and anti-monopole of the same species, and (3) the scattering of the monopole and anti-monopole of the different species. All these three examples show that the monopole and anti-monopole do not always annihilate. This observation may be counter-intuitive and remarkable.

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