Solvable structures of a simple model of earthquakes

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Abstract.

This paper propose an extension of Random Domino Automaton by introducing extra parameter related to a very short-distance interactions between clusters which allows to isolate influence two mechanisms of clusters grow, namely enlarging and coalescence. In this setting statistically stationary state under assumption of independence of clusters is investigated and respective set of equations is derived.

Moreover, a special solvable case is studied in detail and it is shown how to derive Motzkin numbers out of the automaton for a family of parameters without considering a limit.

1. Introduction

Random Domino Automaton (RDA) [6, 7, 8] is a toy model of earthquakes — an extension of well known Drossel-Schwabl model of forest fires [10]. RDA is a slowly driven system, and their rules for accumulation and abrupt release of energy result in various types of frequency-size distributions of produced avalanches.

Such simple models, as for example [13, 14] are useful for analysis of specific properties of earthquakes. The Random Domino Automaton may give a possible mechanistic explanation of an universal earthquake recurrence time distribution [9]. So far, a finite version of RDA [4] was studied, and preliminary results [5] are promising. To explain the recurrence time distribution, it is necessary to solve an inverse problem for the model, which is already done for finite version of RDA [3]. The solution for more realistic cases will be published soon.

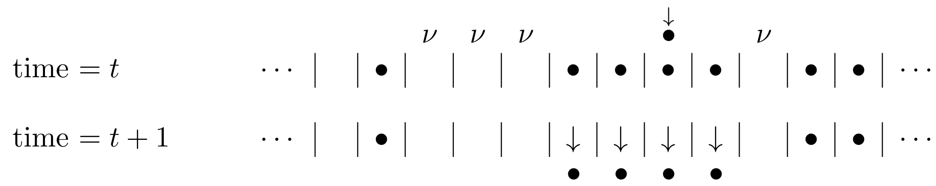
Moreover, the RDA model possess a interesting mathematical structure which is fully solvable for a specific choice of parameters and this choice leads to relation to Motzkin numbers [2]. Here we discuss the structure of the automaton and propose an extension of RDA in order to obtain Motzkin numbers for wider family of parameters, without considering a limit, as has been done originally.

2. The Random Domino Automaton

Random Domino Automaton comes from very simplified view of earthquakes. The space - one dimensional lattice - corresponds to boundaries of two tectonic plates moving with relative constant velocity. Due to irregularities of materials, relative motion can be locked at some places producing stress accumulation. Beyond some threshold of the stress, a relaxation takes place. A size of the relaxation depends on the nearby accumulated stress.

Energy in the automaton is represented by balls added to the randomly chosen cell (each one is equally possible) with a constant rate - one ball in one time step. If the chosen cell is empty, it becomes occupied with probability ν or the ball is scattered with probability $(1-\nu)$.

If the chosen place is already occupied, there are also two possibilities: the ball is scattered with probability $(1-\mu)$ or with probability μ the incoming ball triggers a relaxation, i.e. balls from the chosen cell and all adjacent occupied cells are removed. An example of relaxation of size **four** is presented in the diagram below. The up most line contains also probability of occupation of respective empty cells.



The stationary state of the system (it exists, see [4]) may be described by the distribution of clusters. The number of clusters of the length i, for $i = 1, 2, \ldots$, is denoted by n_i; the number of empty clusters of length 1 is denoted by n_1^0. Then the number of all clusters n and and the density ρ are

$$n = \sum_{i\geq 1} n_i, \qquad \rho = \frac{1}{N}\sum_{i\geq 1} i n_i. \tag{1}$$

The following set of equations for RDA is derived [7]

$$n_1 = \frac{1}{\frac{\mu_1}{\nu}+2}\left((1-\rho)N - 2n + n_1^0\right), \tag{2}$$

$$n_2 = \frac{2}{2\frac{\mu_2}{\nu}+2}\left(1-\frac{n_1^0}{n}\right) n_1, \tag{3}$$

$$n_i = \frac{1}{\frac{\mu_i}{\nu} i + 2}\left(2n_{i-1}\left(1-\frac{n_1^0}{n}\right) + \frac{n_1^0}{n^2}\sum_{k=1}^{i-2} n_k n_{i-1-k}\right) \qquad for \quad i \geq 3, \tag{4}$$

and

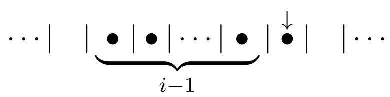
$$n_1^0 = \frac{2n}{\left(3 + \frac{2}{\nu n}\sum_{i\geq 1}\mu_i i n_i\right)}. \tag{5}$$

The balance equation for the total number of clusters n and for the density ρ reads

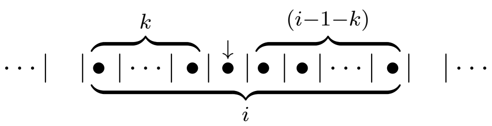
$$(1-\rho)N - 2n = \sum_{i\geq 1}\frac{\mu_i}{\nu} n_i i, \tag{6}$$

$$(1-\rho)N = \sum_{i\geq 1}\frac{\mu_i}{\nu} n_i i^2. \tag{7}$$

The linear term on the righ hand side of equation (4), i.e. the term containing n_{i-1} stems from enlarging of a cluster of the size $i-1$ to the size i, like, for example, presented below.



The nonlinear (quadratic in n_j) term of equation (4) results from merging of two clusters: one of size $k \in \{1, 2, \ldots, (i-2)\}$ and the second of the size $((i-1)-k)$, like presented below.



The equations above are derived under the assumption of the independence for the order of clusters. In other words, we neglect correlations.

Now we consider a case $\mu = \mu_i = \delta/i$, which is related to equal probability of triggering relaxation for each cluster. For $\mu_i = \frac{\delta}{i}$, where $\delta = const$, the balance for ρ takes the following form

$$\rho = \frac{1}{(\theta + 1)}, \tag{8}$$

where $\theta = \frac{\delta}{\nu}$, which relates the density with the ratio of coefficients δ and ν only. Since $\theta \in (0, \infty)$, any density $\rho \in (0, 1)$ may be realized. The balance for n is reduced to $(1 - \rho)N = (2 + \theta)n$. Together with the balance equation for ρ it gives

$$n = N\frac{\theta}{(1 + \theta)(2 + \theta)}. \tag{9}$$

We define new variables c_i for $i = 0, 1, \dots,$ by

$$c_i = \frac{\beta}{\alpha^{i+1}} n_{i+1}, \tag{10}$$

where

$$\alpha = \frac{2}{2 + \theta}\left(\frac{2\theta + 1}{2\theta + 3}\right), \qquad \beta = \frac{(\theta + 1)(\theta + 2)}{N\theta(2\theta + 1)}. \tag{11}$$

Then, the set of equations for n_i can be rewritten in the form

$$c_{m+2} = c_{m+1} + \sum_{k=0}^{m} c_k c_{m-k}, \tag{12}$$

which is valid for $m \geq 0$ $(m = i - 2)$ and initial data c_0 and c_1 are

$$c_0 = c_1 = \frac{1 + \frac{3}{2}\theta + \theta^2}{(1 + 2\theta)^2}. \tag{13}$$

The above recurrence generate Motzkin numbers [1], if it starts with $c_0 = c_1 = 1$. Thus, to obtain Motzkin numbers it is necessary to consider a limit $\theta \longrightarrow 0$, while θN is constant [2]. The generating function formalism gives explicit solution for any value of θ, namely for $m \geq 0$ given by

$$c_m = \frac{1}{2} \sum_{j=0}^{[\frac{m+2}{2}]} \frac{(2c_0 - \frac{1}{2})^j}{(m - j + 2)2^{m-j}} \binom{2(m - j) + 1}{m - j + 1} \binom{m - j + 2}{j} \tag{14}$$

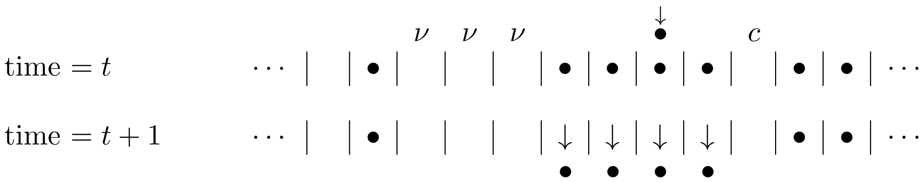
3. An extension of Random Domino Automaton

Now we propose generalisation of RDA to interacting RDA (iRDA), namely we introduce a parameter "c" controlling efficiency of some sort of short range interaction. In the evolution rule we change occupation rule for empty cell as follows.

- If the chosen cell is empty and two adjacent cells are occupied, it becomes occupied with probability c or the ball is scattered with probability $(1 - c)$ leaving the state of the automaton unchanged.
- If the chosen cell is empty and at least one adjacent cell is also empty, it becomes occupied with probability ν or the ball is scattered with probability $(1 - \nu)$.

All the other rules remain valid.

In the new setting an example of relaxation of size **four** looks as in the diagram below. Like previously, the upmost line present probability of occupation of respective empty cells.



The set of equations for iRDA is in the form remarkably similar to the original set for RDA. It is of the form

$$
\begin{aligned}
n_1 &= \frac{1}{\frac{\mu_1}{\nu} + 2 + 2(\xi-1)\frac{n_1^0}{n}} \left((1-\rho)N - 2n + n_1^0\right), && (15)\\
n_2 &= \frac{2}{2\frac{\mu_2}{\nu} + 2 + 2(\xi-1)\frac{n_1^0}{n}} \left(1 - \frac{n_1^0}{n}\right) n_1, && (16)\\
n_i &= \frac{1}{\frac{\mu_i}{\nu} i + 2 + 2(\xi-1)\frac{n_1^0}{n}} \left(2n_{i-1}\left(1 - \frac{n_1^0}{n}\right) + \xi\frac{n_1^0}{n^2}\sum_{k=1}^{i-2} n_k n_{i-1-k}\right) && (17)
\end{aligned}
$$

for $i \geq 3$, where $\xi = \frac{c}{\nu} \in [0, \infty)$ and

$$
n_1^0 = \frac{2n}{\left(2 + \xi + \frac{2}{\nu n}\sum_{i\geq 1}\mu_i i n_i\right)}. \tag{18}
$$

The balance equation for the total number of clusters n and for the density ρ reads

$$
(1-\rho)N - 2n - (\xi-1)n_1^0 = \sum_{i\geq 1}\frac{\mu_i}{\nu} n_i i, \tag{19}
$$

$$
(1-\rho)N + (\xi-1)n_1^0 = \sum_{i\geq 1}\frac{\mu_i}{\nu} n_i i^2. \tag{20}
$$

Naturally, the above equations for $\xi = 1$ reproduce former RDA equations.

The balance equations (20) and (19) (as well as equations for higher weighted moments of n_i defined below) can be also obtained from the above set (15)-(17). For n_i, $i = 1, 2, \ldots$, we define a moment of order γ and a weighted moment $\widehat{m}_\gamma$ of order γ by

$$
m_\gamma = \frac{1}{N}\sum_{i\geq 1} n_i i^\gamma, \qquad \widehat{m}_\gamma = \frac{1}{N}\sum_{i\geq 1}\frac{\mu_i}{\nu} n_i i^\gamma. \tag{21}
$$

Then, one obtains

$$
\begin{aligned}
\widehat{m}_{z+1} &= 1 - m_1 - 2(\xi-1)\frac{n_1^0}{N m_0} m_z - 2m_0 - 2m_z + \\
&+ \frac{n_1^0}{N} + 2\left(1 - \frac{n_1^0}{N m_0}\right)\sum_{k=0}^{z}\binom{z}{k} m_k + \xi\frac{n_1^0}{N m_0}\sum_{l,p=1}^{l+p\leq z}\binom{z}{l+p}\binom{l+p}{l} m_l m_p. && (22)
\end{aligned}
$$

4. Special case: $\mu = \delta/i$, where $\delta =$ constant

In the case which refers to equal probability of triggering avalanche for each cluster, the parameters are fixed as follows: $\nu =$ const and $\mu = \frac{\delta}{i}$, where $\delta =$ const and $\theta = \frac{\delta}{\nu}$. Equations (20) and (19) give

$$\rho = \frac{1}{(\theta+1)}\left(1 + \frac{n}{N}\frac{2(\xi-1)}{2(\theta+1)+\xi}\right), \tag{23}$$

$$n = N\frac{(1-\rho)}{2+\theta+\frac{2(\xi-1)}{2(\theta+1)+\xi}}, \tag{24}$$

where $\theta = \frac{\delta}{\nu} \in (0, \infty)$. Hence

$$\rho = 1 - \frac{\theta}{\theta+1+K}, \tag{25}$$

$$n = \frac{N\theta}{(\theta+1+K)\left(\theta+2+\frac{2(\xi-1)}{2(\theta+1)+\xi}\right)}, \tag{26}$$

where

$$K = \frac{2(\xi-1)}{(2+\theta)\,(2(\theta+1)+\xi)+2(\xi-1)}. \tag{27}$$

Together with simple form of

$$n_1^0 = \frac{2n}{2(\theta+1)+\xi} \tag{28}$$

it allows to reduce the set of equations (15)-(17) to the following recurrence:

$$n_1 = \frac{1}{2+\theta+\frac{4(\xi-1)}{2(\theta+1)+\xi}}\left(\theta+\frac{2\xi}{2(\theta+1)+\xi}\right)n, \tag{29}$$

$$n_2 = \frac{2}{2+\theta+\frac{4(\xi-1)}{2(\theta+1)+\xi}}\left(\frac{2\theta+\xi}{2(\theta+1)+\xi}\right)n_1, \tag{30}$$

$$n_{i+1} = \frac{2}{2+\theta+\frac{4(\xi-1)}{2(\theta+1)+\xi}} \times \tag{31}$$

$$\times\left(\left(\frac{2\theta+\xi}{2(\theta+1)+\xi}\right)n_i + \xi\frac{\sum_{k=1}^{i-1} n_k n_{i-k}}{n(2(\theta+1)+\xi)}\right)$$

for $i \geq 2$ and where n is an explicit function of θ and ξ given by formula (26).

We define new variables c_i for $i = 0, 1, \ldots,$ by

$$c_i = \frac{\beta}{\alpha^{i+1}} n_{i+1}, \tag{32}$$

where

$$\alpha = \frac{2}{2+\theta+\frac{4(\xi-1)}{2(\theta+1)+\xi}}\left(\frac{2\theta+\xi}{2(\theta+1)+\xi}\right), \tag{33}$$

$$\beta = \xi\frac{1}{(2\theta+\xi)n}. \tag{34}$$

Then, the equation (31) can be rewritten in the form of Motzkin numbers recurrence (12). Initial data c_0 and c_1 are easily obtained from equations (29)-(30), when it is transformed according the rule of equation (32), namely

$$c_0 = c_1 = \xi \frac{\xi + (1 + \frac{\xi}{2})\theta + \theta^2}{(\xi + 2\theta)^2}. \tag{35}$$

Contrary to the previous case, equations (35) allow to obtain value 1 without considering a limit. Condition $c_0 = c_1 = 1$ is fulfiled for a family of parameters satisfying

$$\theta = \frac{\xi}{\xi - 4}(3 - \frac{\xi}{2}) \quad \text{or} \quad \theta = 0. \tag{36}$$

Thus for $\xi \in (4, 6)$ and respective value of θ iRDA corresponds exactly to the Motzkin numbers. For iRDA, a solution of the recurrence (12) is given by formula (14) but with $c_0 = c_1$ given by equations (35). Together with the formula (32), it gives explicit solution of equations (15)-(17) for the distribution n_is for any value of θ.

5. Conclusions

We have proposed an extension of Random Domino Automaton by introducing extra parameter related to very short distance interactions between clusters. Investigating statistically stationary state under assumption of independence of clusters, we have derived respective set of equations for clusters size distribution as well as balance equations for density and the total number of clusters.

We have also shown how to derive Motzkin Numbers out of the Automaton for a family of parameters without considering a limit.

By adjusting a value of the parameter related to very short distance interactions between clusters it is possible to separate of influence of two mechanisms of clusters grow. Thus it is straightforward to deduce other results presented in [7] accordingly and all the relations to other disciplines mentioned there are still applicable.

We point out that considering of hierarchical clustering in the spirit of [11] based on Random Domino Automaton, and in particular relation to branching numbers would be of interest and we intend to investigate this topic in future.

Acknowledgments

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