Supersymmetric versions and integrability of conformally parametrized surfaces

Sébastien Bertrand

Department of Mathematics and Statistics, Université de Montréal, Montréal CP 6128 (QC) H3C 3J7, Canada

Abstract. The main aim of this paper is twofold. One is the construction and analysis of supersymmetric (bosonic and fermionic) versions of the structural equations for conformally parametrized surfaces. The other is the investigation of integrability in the sense of soliton theory via a comparison of the symmetries of the system of differential equations and those of the associated linear problem. This paper consists of an overview of the results obtained in two previous works [1, 2].

1. Introduction

Over the last decades, applications of supersymmetry have expanded from particle physic to a large number of domains (see e.g. [3] and references therein). A number of supersymmetric (SUSY) extensions of classical and quantum physical models have been investigated, such as the Chaplygin gas model [4] (and references therein), the Born-Infled model [5], the Korteweg-de Vries equation (e.g. for waves in shallow water in (1+1)-dimensions) [6, 7, 8], the Kadomtsev-Petviashvili equation (e.g. for waves in shallow water in (2+1)-dimensions) [9] and the sine/sinh-Gordon equation (e.g. for crystal dislocation) [10, 11, 12, 13, 14, 15].

In differential geometry, parametrized surfaces are described in terms of a moving frame satisfying the Gauss-Weingarten (GW) equations and their compatibility conditions are the Gauss-Codazzi (GC) equations. The construction and analysis of such surfaces associated with integrable systems in several areas of mathematical physics provided new tools for the investigation of nonlinear phenomena described by these systems. The analysis of such SUSY versions of the structural equations for conformally parametrized surfaces are one of the main goals of this paper and have been studied in [1].

Methods to solve SUSY differential equations are not as well established as in the classical case. In order to establish new methods and obtain solutions, one can investigate special types of differential equations which are easier to solve in general, like integrable systems (in the sense of the inverse scattering method). In order to solve such equations, we can use the group theory approach, e.g. given in [16], and adapt it for Grassmann-valued coordinates and functions. This lead to the second objective of this paper, which is to formulate a conjecture that states that by comparing the symmetries of the (SUSY) differential equations with those of its associated linear problem, it is possible to know if the system is a candidate for integrability. This conjecture, first proposed for classical differential equations [17, 18, 19], can be extended to SUSY differential equations as stated in [2].

The paper is organized as follow. At the end of this section, an outline of the classical Gauss-Codazzi (CGC) and classical Gauss-Weingarten (CGW) equations for conformally parametrized surfaces is given. In section 2, a conjecture states that the integrability can be determinated by comparing the symmetries of a system of equations and those of the associated linear problem. This conjecture can be extented to SUSY equations and is applied to the CGC equations and to the SUSY sinh-Gordon equation in this section as examples. In section 3, the bosonic conformal parametrization of a surface is studied and its integrability is investigated via the conjecture of section 2. In section 4, it is the fermionic conformal parametrization of a surface which is studied and its integrability is also investigated. More details on this work can be found in the papers [1, 2].

1.1. Classical conformally parametrized surfaces

Let us consider a smooth orientable surface $\mathcal{S}$ immersed in the 3-dimensional Euclidean space $\mathbb{R}^3$ which is conformally parametrized by the function $F = (F_1, F_2, F_3) : \mathcal{R} \to \mathbb{R}^3$. The normalization is given by

$$\langle \partial F, \partial F\rangle = \langle \bar\partial F, \bar\partial F\rangle = 0, \quad \langle \partial F, \bar\partial F\rangle = \frac{1}{2}e^u, \quad \langle \partial F, N\rangle = \langle \bar\partial F, N\rangle = 0, \quad \langle N, N\rangle = 1, \tag{1}$$

where N is the normal unit vector and $\partial, \bar\partial$ are the partial derivatives with respect to the complex variables $z, \bar z$ $(z = x+iy)$. Here and subsequently, the form $\langle\cdot|\cdot\rangle$ is the traditionnal inner product in $\mathbb{R}^3$. Introducing the notion of the moving frame $\Omega = (\partial F, \bar\partial F, N)^T$ and assuming that the derivatives of each component of the moving frame can be written as linear combinations of ∂F, $\bar\partial F$ and N, we get the CGW equations

$$\begin{aligned}
\partial \begin{pmatrix} \partial F \\ \bar\partial F \\ N \end{pmatrix} &= \begin{pmatrix} \partial u & 0 & Q \\ 0 & 0 & \frac{1}{2}He^u \\ -H & -2e^{-u}Q & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar\partial F \\ N \end{pmatrix}, \quad \partial\Omega = V_1\Omega, \\
\bar\partial \begin{pmatrix} \partial F \\ \bar\partial F \\ N \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}He^u \\ 0 & \bar\partial u & \bar Q \\ -2e^{-u}\bar Q & -H & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar\partial F \\ N \end{pmatrix}, \quad \bar\partial\Omega = V_2\Omega,
\end{aligned} \tag{2}$$

where Q (which is associated with the Hopf differential Qdz^2) and the mean curvature H are defined by

$$Q = \langle \partial^2 F, N\rangle, \qquad H = 2e^{-u}\langle \partial\bar\partial F, N\rangle. \tag{3}$$

From the compatibility condition of the CGW equations, we obtain the CGC equations,

$$\begin{aligned}
&\partial\bar\partial u + \tfrac{1}{2}H^2e^u - 2Q\bar Q e^{-u} = 0 \quad \text{(, the Gauss equation)}, \\
&\partial\bar Q = \tfrac{1}{2}e^u\bar\partial H, \quad \bar\partial Q = \tfrac{1}{2}e^u\partial H \quad \text{(, the Codazzi equations)}.
\end{aligned} \tag{4}$$

2. Investigating integability via a comparison of symmetries

The conjecture stated further in this section was first proposed by Levy et al. [19] and then by Cieslinski [17, 18] for classical differential equations. The formulation proposed in this paper uses a new projection operator and is extended to SUSY differential equations. In order to formulate this conjecture, let us define some notation.

Let L_1 be the set of all vector fields associated with the Lie point symmetries of the original PDEs $\Delta = 0$. Let L_2 be the set of all vector fields associated with the Lie point symmetries of the linear problem (LP) $\Lambda = 0$. Let π_ρ be a projection operator acting on a set L of vector fields ω such that

$$\pi_\rho(L) = \{\omega' | \omega' = \omega\rho\}, \tag{5}$$

where ρ is in the form of a dilation generator (e.g. $\rho = x_1\partial_{x_1} + x_2\partial_{x_2} + y\partial_y$).

Here and in what follows, in order to compare the sets L_1 and L_2, we will consider a ρ involving all independent and dependent variables of the PDEs $\Delta = 0$. This choice of ρ implies that π_ρ acts as an identity on the set L_1, i.e. $\pi_\rho(L_1) = L_1$. The common symmetries of the PDEs $\Delta = 0$ and LP $\Lambda = 0$ are the vector fields which span the set

$$L_3 = L_1 \cap \pi_\rho(L_2) \neq \emptyset. \tag{6}$$

One should note that the set L_3 is not necessarily an algebra. Also, let L_4 be the set of all vector fields that generate a symmetry of the PDEs $\Delta = 0$ but do not generate a symmetry of the LP $\Lambda = 0$. The set L_4 is define as

$$L_4 = L_1 \backslash L_3. \tag{7}$$

This set does not form necessarily an algebra.

Conjecture 1

(i) If $L_1 = \pi(L_2)$ then the PDEs $\Delta = 0$ are not integrable.

(ii) If the following conditions are satisfied

(a) $\pi(L_2)$ is a proper subset of L_1, that is

$$L_1 \supset \pi(L_2). \tag{8}$$

A free parameter can be introduced into the linear system using a symmetry transformation generated by one of the vector fields appearing in L_4.

(b) The transformation given in (a) acts in a nontivial way (i.e. cannot be eliminated through an L_1-valued gauge matrix function).

Then the system of PDEs $\Delta = 0$ is a candidate to be an integrable one.

In order to apply this conjecture to the CGC equations, we must compare the symmetry Lie algebra of these PDEs,

$$\begin{aligned} &X(\eta) = \eta(z)\partial_z + \eta'(z)(-2Q\partial_Q - U\partial_U), \\ &Y(\zeta) = \zeta(\bar{z})\partial_{\bar{z}} + \zeta'(\bar{z})(-2\bar{Q}\partial_{\bar{Q}} - U\partial_U), \\ &e_0 = -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \end{aligned} \tag{9}$$

with the symmetry Lie algebra of the CGW equations,

$$\begin{aligned} &X(\eta) = \eta(z)\partial_z - \eta'(z)(U\partial_U + 2Q\partial_Q), \\ &Y(\zeta) = \zeta(\bar{z})\partial_{\bar{z}} - \zeta'(\bar{z})(U\partial_U + 2\bar{Q}\partial_{\bar{Q}}), \\ &\hat{e}_0 = -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U + F_i\partial_{F_i}, \\ &T_i = \partial_{F_i}, \qquad \mathcal{D}_i = F_i\partial_{F_{(i)}} + N_i\partial_{N_{(i)}}, \qquad i = 1,2,3 \\ &R_{ij} = (F_i\partial_{F_j} - F_j\partial_{F_i}) + (N_i\partial_{N_j} - N_j\partial_{N_i}), \qquad i < j = 2,3 \\ &S_{ij} = (F_i\partial_{F_j} + F_j\partial_{F_i}) + (N_i\partial_{N_j} + N_j\partial_{N_i}). \end{aligned} \tag{10}$$

By comparing these algebras with ρ of the form

$$\rho = z\partial_z + \bar{z}\bar{\partial} + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + H\partial_H + u\partial_u, \tag{11}$$

we get that

$$L_1 = \pi_\rho(L_2). \tag{12}$$

Hence the classical CGC equations are not integrable.

As a second example, we consider the SUSY sinh-Gordon,

$$D_+D_-\Phi = i\sin\Phi, \tag{13}$$

where the derivatives $D_\pm$ are given by

$$D_\pm = \partial_{\theta^\pm} - i\theta^\pm\partial_{x_\pm}, \tag{14}$$

$\theta^\pm$ being the fermionic independent variables and $x_\pm$ being the bosonic independent variables. The associate LP is given by

$$\begin{aligned} &D_\pm\Psi = B_\pm\Psi, \qquad \text{where} \quad \Psi = \begin{pmatrix} \psi_{11} & \psi_{12} & f_{13} \\ \psi_{21} & \psi_{22} & f_{23} \\ f_{31} & f_{32} & \psi_{33} \end{pmatrix}, \\ &B_+ = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix}, \qquad B_- = \begin{pmatrix} iD_-\Phi & 0 & -i \\ 0 & -iD_-\Phi & i \\ -1 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{15}$$

One should note that Ψ is an even-supermatrix and $B_\pm$ are odd-supermatrices.

The symmetries of the SUSY sinh-Gordon equation and its associated LP are given respectively by

$$P_\pm = \partial_{x_\pm}, \qquad J_\pm = \partial_{\theta^\pm} + i\theta^\pm\partial_{x_\pm}, \qquad \mathcal{K} = 2x_+\partial_{x_+} - 2x_-\partial_{x_-} + \theta^+\partial_{\theta^+} - \theta^-\partial_{\theta^-}, \tag{16}$$

and by

$$\begin{aligned} &P_\pm = \partial_{x_\pm}, \qquad J_\pm = \partial_{\theta^\pm} + i\theta^\pm\partial_{x_\pm}, \qquad G_1 = \psi_{11}\partial_{\psi_{11}} + \psi_{21}\partial_{\psi_{21}} + f_{31}\partial_{f_{31}}, \\ &G_2 = \psi_{12}\partial_{\psi_{12}} + \psi_{22}\partial_{\psi_{22}} + f_{32}\partial_{f_{32}}, \qquad G_3 = f_{13}\partial_{f_{13}} + f_{23}\partial_{f_{23}} + \psi_{33}\partial_{\psi_{33}}. \end{aligned} \tag{17}$$

Using conjecture 1 with ρ of the form

$$\rho = x_+\partial_{x_+} + x_-\partial_{x_-} + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + \Phi\partial_\Phi, \tag{18}$$

we get that $L_1 \supset \pi_\rho(L_2)$. So using the symmetry generator $\mathcal{K} \in L_4$, we can introduce a free parameter λ in such a way :

$$\begin{aligned} &D_+\Psi = B_+\Psi, \qquad\qquad D_-\Psi = B_-\Psi, \\ &B_+ = \tfrac{1}{2\sqrt{\lambda}}\begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix}, \quad B_- = \begin{pmatrix} iD_-\Phi & 0 & -i\sqrt{\lambda} \\ 0 & -iD_-\Phi & i\sqrt{\lambda} \\ -\sqrt{\lambda} & \sqrt{\lambda} & 0 \end{pmatrix}, \end{aligned} \tag{19}$$

where λ does not act trivially on the new linear spectral problem. Hence, the SUSY sinh-Gordon equation is a candidate to be integrable.

3. Bosonic conformal parametrization of a surface

Let the surface $\mathcal{S}$ be smooth, orientable and immersed in the superspace $\mathbb{R}^{(2,1|2)}$. We assume that the surface $\mathcal{S}$ is conformally parametrized by a bosonic superfield $F(x_+, x_-, \theta^+, \theta^-)$ with the normalization conditions

$$\langle D_\pm F, D_\pm F\rangle = 0, \qquad \langle D_+F, D_-F\rangle = \frac{1}{2}e^\phi f, \qquad \langle D_-F, D_+F\rangle = -\frac{1}{2}e^\phi f \tag{20}$$

where $D_\pm$ are defined as in equation (14), ϕ is a bosonic-valued function of $x_+, x_-, \theta^+, \theta^-$, and f is a bodiless bosonic-valued function of x_+, x_-. We can introduce a bosonic superfield N which acts as the classical normal unit vector

$$\langle D_\pm F, N\rangle = 0, \qquad \langle N, N\rangle = 1. \tag{21}$$

The tangent vectors $D_\pm F$ together with the normal superfield N form the SUSY moving frame $\Omega = (D_+F, D_-F, N)^T$. Assuming that the first derivatives of Ω can be written as linear combinations of the elements of Ω and with some calculation we get the bosonic case of the supersymmetric Gauss-Weingarten (bSUSYGW) equations

$$D_+\Omega = A_+\Omega, \qquad\qquad D_-\Omega = A_-\Omega,$$

$$A_+ = \begin{pmatrix} {\Gamma_{11}}^1 & {\Gamma_{11}}^2 & Q^+f \\ -{\Gamma_{12}}^1 & -{\Gamma_{12}}^2 & -\frac{1}{2}e^\phi Hf \\ H & 2e^{-\phi}Q^+ & 0 \end{pmatrix}, \quad A_- = \begin{pmatrix} {\Gamma_{12}}^1 & {\Gamma_{12}}^2 & \frac{1}{2}e^\phi Hf \\ {\Gamma_{22}}^1 & {\Gamma_{22}}^2 & Q^-f \\ -2e^{-\phi}Q^- & H & 0 \end{pmatrix}. \tag{22}$$

where the fermionic functions ${\Gamma_{ij}}^k$ are the Christoffel symbols of second kind and where the $Q^\pm$ and the mean curvature H are defined by

$$\langle D_+D_+F, N\rangle = Q^+f, \quad \langle D_-D_+F, N\rangle = -\langle D_+D_-F, N\rangle = \frac{1}{2}e^\phi Hf, \quad \langle D_-D_-F, N\rangle = Q^-. \tag{23}$$

The zero curvature condition of the bSUSYGW equations

$$D_+A_- + D_-A_+ - \{EA_+, EA_-\} = 0, \tag{24}$$

where

$$E = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{25}$$

constitutes the bosonic case of the supersymmetric Gauss-Codazzi (bSUSYGC) equations and corresponds to the following six linearly independent equations

$$\begin{array}{ll} (i) & D_-({\Gamma_{11}}^1) + D_+({\Gamma_{22}}^2) + D_+({\Gamma_{12}}^1) - D_-({\Gamma_{12}}^2) = 0, \\ (ii) & D_-({\Gamma_{11}}^1) - {\Gamma_{11}}^2{\Gamma_{22}}^1 + D_+({\Gamma_{12}}^1) + {\Gamma_{12}}^2{\Gamma_{12}}^1 + \frac{1}{2}H^2e^\phi f - 2Q^+Q^-e^{-\phi}f = 0, \\ (iii) & Q^+{\Gamma_{22}}^2 - {\Gamma_{11}}^2Q^- + D_-Q^+ - Q^+D_-\phi + \frac{1}{2}e^\phi D_+H = 0, \\ (iv) & Q^-{\Gamma_{11}}^1 - {\Gamma_{22}}^1Q^+ + D_+Q^- - Q^-D_+\phi - \frac{1}{2}e^\phi D_-H = 0, \\ (v) & D_-({\Gamma_{11}}^2) - {\Gamma_{12}}^1{\Gamma_{11}}^2 - {\Gamma_{11}}^2{\Gamma_{22}}^2 - {\Gamma_{11}}^1{\Gamma_{12}}^2 + D_+({\Gamma_{12}}^2) + 2Q^+Hf = 0, \\ (vi) & D_+({\Gamma_{22}}^1) + {\Gamma_{12}}^2{\Gamma_{22}}^1 - {\Gamma_{22}}^1{\Gamma_{11}}^1 + {\Gamma_{22}}^2{\Gamma_{12}}^1 - D_-({\Gamma_{12}}^1) + 2Q^-Hf = 0. \end{array} \tag{26}$$

We will now determine whether or not the bSUSYGC equations form an integrable system. In order to lighthen the Christoffel symbols's notation, we write them as

$$R^+ = {\Gamma_{11}}^1, \quad R^- = {\Gamma_{11}}^2, \quad S^+ = {\Gamma_{12}}^1, \quad S^- = {\Gamma_{12}}^2, \quad T^+ = {\Gamma_{22}}^1, \quad T^- = {\Gamma_{22}}^2. \tag{27}$$

The vector fields of the Lie point symmetries of the bSUSYGC equations are given by

$$\begin{array}{ll} C_0 = H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f, & \\ K_0 = -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, & \\ K_1^b = -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} + S^-\partial_{S^-} - T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_\phi, & \\ K_2^b = -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} + 2T^+\partial_{T^+} + T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_\phi, & \\ P_+ = \partial_{x_+}, & P_- = \partial_{x_-}, \\ J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, & J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}. \end{array} \tag{28}$$

and the symmetries of the bSUSYGW equations are spanned by

$$\begin{array}{l}
P_\pm = \partial_{x_\pm}, \qquad J_\pm = \partial_{\theta^\pm} + i\theta^\pm \partial_{x_\pm}, \\
\hat{C}_0 = H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f + N_i\partial_{N_i}, \\
\hat{K}_0 = -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi - N_i\partial_{N_i}, \\
K_1^b = -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} + S^-\partial_{S^-} - T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_\phi, \\
K_2^b = -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} + 2T^+\partial_{T^+} + T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_\phi, \\
G_i = F_i\partial_{F_i} + N_i\partial_{N_i}, \qquad B_i = \partial_{F_i}, \qquad \text{for } i = 1,2,3 \\
R_{ij} = F_i\partial_{F_j} - F_j\partial_{F_i} + N_i\partial_{N_j} - N_j\partial_{N_i}, \qquad i < j = 2,3.
\end{array} \tag{29}$$

Using conjecture 1 with ρ involving all independent and dependent variables of the bSUSYGC equations, we see that $L_1 = \pi_\rho(L_2)$. Hence the bSUSYGC equations are not integrable.

4. Fermionic conformal parametrization of a surface

Let the surface $\mathcal{S}$ be smooth, orientable and immersed in the superspace $\mathbb{R}^{(1,1|3)}$. We assume that the surface $\mathcal{S}$ is conformally parametrized by a fermionic superfield $F(x_+, x_-, \theta^+, \theta^-)$ with the normalization conditions

$$\langle D_\pm F, D_\pm F\rangle = 0, \qquad \langle D_\pm F, D_\mp F\rangle = \frac{1}{2}e^\phi f, \tag{30}$$

where ϕ is a bosonic-valued function of $x_+, x_-, \theta^+, \theta^-$ and the bosonic-valued function $f(x_+, x_-)$ may be bodiless depending on the structure of F. The normalization on the bosonic superfield N is given by

$$\langle D_\pm F, N\rangle = 0, \qquad \langle N, N\rangle = 1. \tag{31}$$

The tangent vectors $D_\pm F$ together with the normal superfield N form the SUSY moving frame $\Omega = (D_+F, D_-F, N)^T$. Assuming that the first derivatives of the moving frame Ω can be written as linear combinations of the elements of Ω and with some calculation we get the fermionic case of the supersymemtric Gauss-Weingarten (fSUSYGW) equations

$$\begin{array}{l}
D_+\begin{pmatrix} D_+F \\ D_-F \\ N\end{pmatrix} = \begin{pmatrix} {\Gamma_{11}}^1 & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^\phi Hf \\ H & -2e^{-\phi}Q^+ & 0\end{pmatrix}\begin{pmatrix} D_+F \\ D_-F \\ N\end{pmatrix}, \quad D_+\Omega = A_+\Omega, \\
D_-\begin{pmatrix} D_+F \\ D_-F \\ N\end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^\phi Hf \\ 0 & {\Gamma_{22}}^2 & Q^-f \\ -2e^{-\phi}Q^- & -H & 0\end{pmatrix}\begin{pmatrix} D_+F \\ D_-F \\ N\end{pmatrix}, \quad D_-\Omega = A_-\Omega.
\end{array} \tag{32}$$

The fermionic case of the supersymmetric Gauss-Codazzi (fSUSYGC) equations, which are equivalent to the ZCC

$$D_+A_- + D_-A_+ - \{A_+, A_-\} = 0, \tag{33}$$

reduce to the following four linearly independent equations

$$\begin{array}{rl}
(i) & D_+({\Gamma_{22}}^2) + D_-({\Gamma_{11}}^1) = 0, \\
(ii) & D_-({\Gamma_{11}}^1) + 2e^{-\phi}Q^+Q^-f = 0, \\
(iii) & D_+Q^- - \frac{1}{2}e^\phi D_-H + Q^-(D_+\phi - {\Gamma_{11}}^1) = 0, \\
(iv) & D_-Q^+ + \frac{1}{2}e^\phi D_+H + Q^+(D_-\phi - {\Gamma_{22}}^2) = 0.
\end{array} \tag{34}$$

Using the abbreviated notation (27) and conjecture 1, we seek to know if the fSUSYGC equations are integrable. The symmetries of the fSUSYGC equations are spanned by

$$\begin{aligned}
&J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \qquad W = \partial_H \\
&P_+ = \partial_{x_+}, \qquad P_- = \partial_{x_-}, \\
&C_0 = H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f, \\
&K_0 = -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\
&K_1^f = -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi, \\
&K_2^f = -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + T^-\partial_{T^-} + \partial_\phi,
\end{aligned} \tag{35}$$

and the set of vector fields which generates symmetries of the fSUSYGW equations is given by

$$\begin{aligned}
&P_\pm = \partial_{x_\pm}, \qquad J_\pm = \partial_{\theta^\pm} + i\theta^\pm\partial_{x_\pm}, \\
&\hat{C}_0 = H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f + N_i\partial_{N_i}, \\
&\hat{K}_0 = -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi - N_i\partial_{N_i}, \\
&K_1^f = -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi, \\
&K_2^f = -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + T^-\partial_{T^-} + \partial_\phi, \\
&G_i = F_i\partial_{F_i} + N_i\partial_{N_i}, \qquad B_i = \partial_{F_i}, \qquad \text{for } i,j = 1,2,3 \\
&R_{ij} = F_i\partial_{F_j} - F_j\partial_{F_i} + N_i\partial_{N_j} - N_j\partial_{N_i}, \qquad i < j = 2,3.
\end{aligned} \tag{36}$$

Comparing these sets of generators, we see that the symmetry spanned by the vector field W is not common to both the fSUSYGC equations and the fSUSYGW equations. We can use this symmetry to incorporate a free fermionic parameter λ which cannot be eliminated through a gauge transformation. The potential matrices $A_\pm$ take the form

$$\begin{aligned}
A_+ &= \begin{pmatrix} {\Gamma_{11}}^1 & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^\phi(H+\lambda)f \\ H+\lambda & -2e^{-\phi}Q^+ & 0 \end{pmatrix}, \\
A_- &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^\phi(H+\lambda)f \\ 0 & {\Gamma_{22}}^2 & Q^-f \\ -2e^{-\phi}Q^- & -H-\lambda & 0 \end{pmatrix}.
\end{aligned} \tag{37}$$

Therefore, the system of the fSUSYGC equation is a candidate to be integrable.

5. Conclusion

In this paper we have presented supersymmetric conformal parametrizations of surfaces, one for the bosonic case and another for the ferminonic case. Both the Gauss-Codazzi and Gauss-Weingarten equations have been given for the bosonic and fermionic cases. Moreover, a conjecture on the integrability of systems of differential equations in the sense of soliton theory has been presented. This conjecture allows us to know if a system of differential equations can be integrable by comparing the symmetries of the system with the symmetries of its linear problem. In this paper, examples of this conjecture have been applied to the classical Gauss-Codazzi equations, the supersymmetric sinh-Gordon equation, the bosonic case of the supersymmetric Gauss-Codazzi equations and the fermionic case of the supersymmetric Gauss-Codazzi equations.

6. Acknowledgments

The author wishes to thank Prof. A. M. Grundland (Université du Québec à Trois-Rivières & Université de Montréal) for his support and discussions on the subjet. This paper was supported by a research grant of AMG from NSERC. The author acknowledges a doctoral fellowship provided by the FQRNT of the Gouvernement du Québec.

References

[1] Bertrand S, Grundland A M and Hariton A J 2015 *J. Phys. A* **48** 175208, 37 (arXiv : 1504.08260) URL `http://stacks.iop.org/1751-8121/48/i=17/a=175208`

[2] Bertrand S, Grundland A M and Hariton A J 2015 *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* **11** 046 (arXiv : 1502.02948) URL `http://dx.doi.org/10.3842/SIGMA.2015.046`

[3] Varadarajan V S 2011 *Reflections on quanta, symmetries, and supersymmetries* (Springer, New York) ISBN 978-1-4419-0666-3 URL `http://dx.doi.org/10.1007/978-1-4419-0667-0`

[4] Jackiw R 2002 *Lectures on fluid dynamics : A particle theorist's view of supersymmetric, non-abelian, noncommutative fluid mechanics and d-branes* CRM Series in Mathematical Physics (Springer-Verlag, New York) ISBN 0-387-95422-8 URL `http://dx.doi.org/10.1007/978-1-4757-3665-6`

[5] Hariton A J 2006 *J. Phys. A* **39** 7105–7114 ISSN 0305-4470 URL `http://dx.doi.org/10.1088/0305-4470/39/22/021`

[6] Mathieu P 1988 *J. Math. Phys.* **29** 2499–2506 ISSN 0022-2488 URL `http://dx.doi.org/10.1063/1.528090`

[7] Labelle P and Mathieu P 1991 *J. Math. Phys.* **32** 923–927 ISSN 0022-2488 URL `http://dx.doi.org/10.1063/1.529351`

[8] Liu Q P and Mañas M 1998 *Phys. Lett. B* **436** 306–310 ISSN 0370-2693 URL `http://dx.doi.org/10.1016/S0370-2693(98)00852-1`

[9] Manin Y I and Radul A O 1985 *Comm. Math. Phys.* **98** 65–77 ISSN 0010-3616 URL `http://projecteuclid.org/euclid.cmp/1103942285`

[10] Coleman S 1975 *Phys. Rev. D* **11**(8) 2088–2097 URL `http://link.aps.org/doi/10.1103/PhysRevD.11.2088`

[11] Grammaticos B, Ramani A and Carstea A S 2001 *J. Phys. A* **34** 4881–4886 ISSN 0305-4470 URL `http://dx.doi.org/10.1088/0305-4470/34/23/307`

[12] Siddiq M and Hassan M 2005 *Europhys. Lett.* **70** 149–154 ISSN 0295-5075 URL `http://dx.doi.org/10.1209/epl/i2004-10475-y`

[13] Siddiq M, Hassan M and Saleem U 2006 *J. Phys. A* **39** 7313–7318 ISSN 0305-4470 URL `http://dx.doi.org/10.1088/0305-4470/39/23/010`

[14] Witten E 1984 *Comm. Math. Phys.* **92** 455–472 ISSN 0010-3616 URL `http://projecteuclid.org/euclid.cmp/1103940923`

[15] Grundland A M, Hariton A J and Šnobl L 2009 *J. Phys. A* **42** 335203, 23 ISSN 1751-8113 URL `http://dx.doi.org/10.1088/1751-8113/42/33/335203`

[16] Olver P J 1986 *Applications of Lie groups to differential equations* (*Graduate Texts in Mathematics* vol 107) (Springer-Verlag, New York) ISBN 0-387-96250-6 URL `http://dx.doi.org/10.1007/978-1-4684-0274-2`

[17] Cieśliński J, Goldstein P and Sym A 1994 *J. Phys. A* **27** 1645–1664 ISSN 0305-4470 URL `http://stacks.iop.org/0305-4470/27/1645`

[18] Cieśliński J 1992 *Nonlinear evolution equations and dynamical systems (Baia Verde, 1991)* (World Sci. Publ., River Edge, NJ) pp 260–268

[19] Levi D, Sym A and Tu G 1990 A working algorithm to isolate integrable surfaces in e^3 preprint DF-INFN, N761