Cotangent bundles over all the Hermitian symmetric spaces

Masato Arai[1] **and Kurando Baba**[2]

[1]Faculty of Science, Yamagata University, Kojirakawa-machi 1-4-12, Yamagata, Yamagata 990-8560, Japan,

[2]Department of General Education, Fukushima National College of Technology, Fukushima 970-8034, Japan

E-mail: `arai@sci.kj.yamagata-u.ac.jp`

Abstract. We construct the $\mathcal{N}=2$ supersymmetric nonlinear sigma models on the cotangent bundles over all the compact and non-compact Hermitian symmetric spaces. In order to construct them we use the projective superspace formalism which is an $\mathcal{N}=2$ off-shell superfield formulation in four-dimensional space-time. This formalism allows us to obtain the explicit expression of $\mathcal{N}=2$ supersymmetric nonlinear sigma models on the cotangent bundles over any Hermitian symmetric spaces in terms of the $\mathcal{N}=1$ superfields, once the Kähler potentials of the base manifolds are obtained. Starting with $\mathcal{N}=1$ supersymmetric Kähler nonlinear sigma models on the Hermitian symmetric spaces, we extend them into the $\mathcal{N}=2$ supersymmetric models by using the projective superspace formalism and derive the general formula for the cotangent bundles over all the compact and non-compact Hermitian symmetric spaces. We apply to the formula for the non-compact Hermitian symmetric space $E_7/E_6 \times U(1)$[1].

1. Introduction

Supersymmetry (SUSY) has an intimate relation to complex geometry in mathematics. Indeed, it is well known that target spaces of $\mathcal{N}=1$ and $\mathcal{N}=2$ SUSY nonlinear sigma models (NLSMs) must be Kähler [1] and hyperkähler manifolds [2], respectively. It is important to construct these manifolds because they frequently appear in field theories with/without supersymmetry, supergravity and superstring theories.

Recently there have been developments to construct $\mathcal{N}=2$ SUSY NLSMs in the projective superspace formalism [3, 4, 5, 6, 7], which is an $\mathcal{N}=2$ off-shell superfield formulation in four-dimensional space-time. In this formalism, $\mathcal{N}=2$ SUSY NLSMs on cotangent bundles over Kähler manifolds have been constructed [8, 9, 10, 11, 12, 13, 14]. A key observation in the developments is that once a certain $\mathcal{N}=1$ SUSY NLSM is obtained, this model can be extended into the $\mathcal{N}=2$ SUSY NLSM with use of the projective superspace formalism. In other words, if we have the $\mathcal{N}=1$ SUSY NLSM on the Kähler manifold, we can obtain $\mathcal{N}=2$ SUSY NLSM on the cotangent bundle over the Kähler manifold. The target space of the $\mathcal{N}=2$ SUSY NLSM is shown to be an open domain of the zero section of the cotangent bundle [8, 9]. Namely it is hyperkähler. Based on the observation in [8, 9], the $\mathcal{N}=2$ SUSY NLSMs on the cotangent bundles over the irreducible Hermitian symmetric spaces (HSSs) except the

[1] This talk is based on [16].

	classical type				exceptional type	
compact type	$\frac{U(n+m)}{U(n)\times U(m)}$	$\frac{SO(2n)}{U(n)}$	$\frac{Sp(n)}{U(n)}$	$\frac{SO(n+2)}{SO(n)\times U(1)}$	$\frac{E_6}{SO(10)\times U(1)}$	$\frac{E_7}{E_6\times U(1)}$
non-compact type	$\frac{U(n,m)}{U(n)\times U(m)}$	$\frac{SO^*(2n)}{U(n)}$	$\frac{Sp(n,\mathbf{R})}{U(n)}$	$\frac{SO_0(n,2)}{SO(n)\times U(1)}$	$\frac{E_{6(-14)}}{SO(10)\times U(1)}$	$\frac{E_{7(-25)}}{E_6\times U(1)}$

Table 1. Irreducible Hermitian symmetric spaces

non-compact exceptional types of them have been constructed [8, 9, 10, 11, 12, 13, 14]. The irreducible HSSs classified by Cartan [15] consist of compact type and non-compact type. They are summarized in Table 1.

In this talk we construct the $\mathcal{N}=2$ SUSY NLSM on the cotangent bundle over the non-compact HSS $E_{7(-25)}/E_6\times U(1)$. First we show how to derive general formula for the $\mathcal{N}=2$ SUSY NLSMs on cotangent bundles over all the compact and the non-compact exceptional HSSs in the framework of the projective superspace formalism [13, 14]. The method to construct the models in [8, 9, 10, 11, 12] is model-dependent and furthermore it is difficult to apply for the exceptional types of the HSS such as the compact type $E_7/E_6\times U(1)$ and the non-compact type $E_{7(-25)}/E_6\times U(1)$. Accordingly, the other method to construct the $\mathcal{N}=2$ SUSY NLSMs on cotangent bundles over the HSS has been developed in [13] and [14]. In this talk, we apply the method to the case for the $E_{7(-25)}/E_6\times U(1)$. Combined with the application to $\frac{E_{6(-14)}}{SO(10)\times U(1)}$ given in [16], we complete construction of the $\mathcal{N}=2$ SUSY NLSM on the cotangent bundle over all the HSSs.

2. $\mathcal{N}=2$ sigma models and the projective superspace

Projective superspace is described as $(x_\mu,\theta_{\alpha i},\bar\theta^i_{\dot\alpha},\zeta)$, where $\mu=0,1,2,3$ is a space-time index, $\alpha,\dot\alpha=1,2$ are spinor indices, $i=1,2$ is an $SU(2)_R$ index and ζ is the projective coordinate. Superfields are functions on its subspace, which are defined by the so-called projective condition [3] similar to the chiral condition in the four-dimensional $\mathcal{N}=1$ SUSY field theory. This condition makes a number of the Grassmann coordinates be half and integration measure for SUSY invariant action reduces to one on the full $\mathcal{N}=1$ superspace $z_M=(x_\mu,\theta_\alpha,\bar\theta_{\dot\alpha})$ with the projective coordinate ζ. A certain class of four-dimensional $\mathcal{N}=2$ NLSM is described in terms of $\mathcal{N}=1$ language as [17, 8, 9]

$$S[\Upsilon,\breve\Upsilon]=\frac{1}{2\pi i}\oint\frac{\mathrm{d}\zeta}{\zeta}\int\mathrm{d}^8z\,K\big(\Upsilon^I(z,\zeta),\breve\Upsilon^{\bar J}(z,\zeta)\big),\tag{1}$$

where I,J are indices of fields[2]. The contour encircles the origin of the ζ-plane in anti-clockwise direction. The action is written by the function of the superfields representing the polar multiplets Υ and $\breve\Upsilon$, which are called an arctic superfield and an antarctic superfield respectively. They are expanded with respect to ζ as

$$\Upsilon(z,\zeta)=\sum_{n=0}^{\infty}\Upsilon_n\zeta^n=\Phi+\Sigma\,\zeta+\mathcal{A}\,,\qquad\breve\Upsilon(\zeta)=\sum_{n=0}^{\infty}\bar\Upsilon_n(-\zeta)^{-n}\,,\tag{2}$$

where $\Upsilon_0\equiv\Phi$ is a chiral superfield ($\bar D_{\dot\alpha}\Phi=0$) and $\Upsilon_1=\Sigma$ is a complex linear superfield ($\bar D^2\Sigma=0$). An infinite set of unconstrained auxiliary fields is expressed as $\mathcal{A}$ which contains terms with an order higher than ζ. The antarctic superfield $\breve\Upsilon$ is a conjugate of Υ, which is the combination of the ordinary complex conjugate and the antipodal map $\zeta\mapsto-1/\zeta$ on the

[2] More general type of action has the form $K(\Upsilon,\breve\Upsilon,\zeta)$ [5, 6].

Riemann sphere. Note that the action (1) is an $\mathcal{N}=2$ extension of the general $\mathcal{N}=1$ SUSY NLSM [1].

The action (2) is invariant under

$$K(\Upsilon^I, \breve{\Upsilon}^{\bar{J}}) \to K(\Upsilon^I, \breve{\Upsilon}^{\bar{J}}) + \Lambda(\Upsilon^I) + \breve{\Lambda}(\breve{\Upsilon}^{\bar{J}}), \tag{3}$$

and

$$\Upsilon^I \to f^I(\Upsilon^J), \tag{4}$$

respectively. The latter implies the reparametrization of the manifold, yielding the transformation law for Σ^I as

$$\Sigma^I = \left.\frac{d\Upsilon^I}{d\zeta}\right|_{\zeta=0} \to \left.\frac{df^I}{d\zeta}\right|_{\zeta=0} = \left.\frac{d\Upsilon^J}{d\zeta}\frac{df^I}{d\Upsilon^J}\right|_{\zeta=0} = \Sigma^J \left.\frac{df^I}{d\Upsilon^J}\right|_{\zeta=0}. \tag{5}$$

It is seen that Σ^I transforms as a tangent vector.

In order to represent the action (1) in terms of physical fields (Φ^I, Σ^J) only, we need to eliminate the auxiliary fields by using their equations of motion

$$\oint \frac{\mathrm{d}\zeta}{\zeta}\zeta^n \frac{\partial K(\Upsilon, \breve{\Upsilon})}{\partial \Upsilon^I} = \oint \frac{\mathrm{d}\zeta}{\zeta}\zeta^{-n}\frac{\partial K(\Upsilon, \breve{\Upsilon})}{\partial \breve{\Upsilon}^{\bar{I}}} = 0, \qquad n \geq 2\,. \tag{6}$$

Let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ be a unique solution of the equation (6) with the initial conditions

$$\Upsilon_*(0) = \Phi, \qquad \left.\frac{d\Upsilon(\zeta)}{d\zeta}\right|_{\zeta=0} = \Sigma. \tag{7}$$

For a general Kähler manifold, it is possible to eliminate $\Upsilon_n (n \geq 2)$ and their conjugates by solving (6) perturbatively [18]. After eliminating all the auxiliary fields, the following form of the action is obtained

$$S_{\rm tb}[\Phi, \Sigma] = \int \mathrm{d}^8 z \Big\{ K\big(\Phi, \bar{\Phi}\big) + \mathcal{L}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big)\Big\}, \tag{8}$$

where $\mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ is the part describing the tangent space:

$$\begin{aligned}\mathcal{L}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big) &= \sum_{n=1}^{\infty} \mathcal{L}^{(n)}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big) \\ &= \sum_{n=1}^{\infty} \mathcal{L}_{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}\big(\Phi, \bar{\Phi}\big)\Sigma^{I_1} \dots \Sigma^{I_n}\bar{\Sigma}^{\bar{J}_1} \dots \bar{\Sigma}^{\bar{J}_n}.\end{aligned} \tag{9}$$

Here $\mathcal{L}_{I\bar{J}} = -g_{I\bar{J}}(\Phi, \bar{\Phi})$ while the tensors $\mathcal{L}_{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n} (n \geq 2)$ are functions of the metric $g_{I\bar{J}}(\Phi, \bar{\Phi})$, the Riemann tensor $R_{I\bar{J}K\bar{L}}(\Phi, \bar{\Phi})$ and its covariant derivative. The action (8) is written by the base manifold coordinate Φ and the tangent vector Σ. Therefore this action represents the $\mathcal{N}=2$ SUSY model on the tangent bundle over the Kähler manifold.

The rest of the work is to derive the Kähler potential of the cotangent bundle over the Kähler manifold. It is carried out by changing the tangent vectors Σ's in (8) into chiral one-forms, cotangent vectors Ψ's. It can be performed by the generalized Legendre transformation [5] as follows.

$$\begin{aligned}&S_{\rm tb} = \int d^8 z\left(K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})\right) \\ &\rightsquigarrow S = \int \mathrm{d}^8 z \Big\{ K\big(\Phi, \bar{\Phi}\big) + \mathcal{L}\big(\Phi, \bar{\Phi}, U, \bar{U}\big) + \Psi_I U^I + \bar{\Psi}_{\bar{I}}\bar{U}^{\bar{I}}\Big\},\end{aligned} \tag{10}$$

where U is a complex unconstrained superfield and Ψ is a chiral superfield. This action goes back to the tangent bundle action (8) after eliminating the chiral superfields Ψ and $\bar{\Psi}$ by their equations of motion. On the other hand, eliminating U and $\bar{U}$ with the aid of their equations of motion, the action is written only in terms of Φ, Ψ and their conjugates:

$$S_{\rm ctb}[\Phi,\Psi] \;=\; \int \mathrm{d}^8 z \left\{ K\big(\Phi,\bar{\Phi}\big) + \mathcal{H}\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big) \right\}, \tag{11}$$

where

$$\begin{aligned} \mathcal{H}\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big) \;&=\; \sum_{n=1}^{\infty} \mathcal{H}^{(n)}(\Phi,\bar{\Phi},\Psi,\bar{\Psi}) \\ &=\; \sum_{n=1}^{\infty} \mathcal{H}^{I_1\cdots I_n \bar{J}_1\cdots \bar{J}_n}\big(\Phi,\bar{\Phi}\big)\Psi_{I_1}\ldots\Psi_{I_n}\bar{\Psi}_{\bar{J}_1}\ldots\bar{\Psi}_{\bar{J}_n}\,, \end{aligned} \tag{12}$$

with $\mathcal{H}^{I\bar{J}}\big(\Phi,\bar{\Phi}\big) = g^{I\bar{J}}\big(\Phi,\bar{\Phi}\big)$. Here $g^{I\bar{J}}$ is the inverse metric of $g_{I\bar{J}}$. The variables (Φ^I, Ψ_J) parameterize the cotangent bundle over the Kähler manifold and therefore the action gives the Kähler potential of the cotangent bundle over the Kähler manifold.

The explicit forms of $\mathcal{L}$ and $\mathcal{H}$ are obtained in [12] and [13, 14] for the case that the base manifold is the HSS respectively. Here we focus only on the derivation of $\mathcal{H}$, which we are interested in. The cotangent bundle action (11) has to be invariant under the following second SUSY transformations [12]

$$\begin{aligned} \delta\Phi^I \;&=\; \frac{1}{2}\bar{D}^2\big\{\bar{\varepsilon}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,\Sigma^I\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)\big\}\,, & (13)\\ \delta\Psi_I \;&=\; -\frac{1}{2}\bar{D}^2\Big\{\bar{\varepsilon}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,K_I\big(\Phi,\bar{\Phi}\big)\Big\} + \frac{1}{2}\bar{D}^2\Big\{\bar{\varepsilon}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,\Gamma^K_{IJ}\big(\Phi,\bar{\Phi}\big)\,\Sigma^J\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)\Big\}\,\Psi_K, & (14) \end{aligned}$$

with

$$\Sigma^I\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big) = \frac{\partial}{\partial\Psi_I}\,\mathcal{H}\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big) := \mathcal{H}^I. \tag{15}$$

The requirement of invariance under such transformations can be shown to be equivalent to the following nonlinear equation [12]:

$$\mathcal{H}^I g_{I\bar{J}} - \frac{1}{2}\mathcal{H}^K\mathcal{H}^L R_{K\bar{J}L}{}^I\Psi_I = \bar{\Psi}_{\bar{J}}. \tag{16}$$

Eq. (16) implies that

$$\Psi_I\mathcal{H}^I - \mathcal{H}^K\mathcal{H}^L(R_\Psi)_{KL} = g^{I\bar{J}}\Psi_I\bar{\Psi}_{\bar{J}}, \quad (R_\Psi)_{KL} := \frac{1}{2}R_K{}^I{}_L{}^J\Psi_I\Psi_J. \tag{17}$$

By using the identities

$$\Psi_I\mathcal{H}^I = \bar{\Psi}_{\bar{I}}\mathcal{H}^{\bar{I}} = \sum_{n=1}^{\infty} n\mathcal{H}^{(n)}, \tag{18}$$

(17) is rewritten as

$$\mathcal{H}^{(1)} = g^{I\bar{J}}\Psi_I\bar{\Psi}_{\bar{J}}, \quad n\mathcal{H}^{(n)} - \sum_{p=1}^{n-1}\mathcal{H}^{(p)K}(R_\Psi)_{KL}\mathcal{H}^{(n-p)L} = 0, \quad n \geq 2. \tag{19}$$

In order to solve this, it is useful to define

$$\mathbf{R}_{\Psi,\bar{\Psi}} = \begin{pmatrix} 0 & (R_\Psi)_I{}^{\bar{J}} \\ (R_{\bar{\Psi}})_{\bar{I}}{}^{J} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}R_I{}^{K\bar{J}L}\Psi_K\Psi_L \\ \frac{1}{2}R_{\bar{I}}{}^{\bar{K}J\bar{L}}\bar{\Psi}_{\bar{K}}\bar{\Psi}_{\bar{L}} & 0 \end{pmatrix} \tag{20}$$

and

$$G^{(2k+2)} = \Psi_I g^{I\bar{J}} \left((R_{\bar{\Psi}} R_\Psi)^k \right)_{\bar{J}}{}^{\bar{K}} (R_{\bar{\Psi}})_{\bar{K}}{}^{L} \Psi_L = \Psi^\dagger_{\bar{I}} g^{\bar{I}J} \left((R_\Psi R_{\bar{\Psi}})^k \right)_J{}^K (R_\Psi)_K{}^{\bar{L}} \bar{\Psi}_{\bar{L}}, \tag{21}$$

$$G^{(2k+1)} = \Psi_I g^{I\bar{J}} \left((R_{\bar{\Psi}} R_\Psi)^k \right)_{\bar{J}}{}^{\bar{K}} \bar{\Psi}_{\bar{K}} = \Psi^\dagger_{\bar{I}} g^{\bar{I}J} \left((R_\Psi R_{\bar{\Psi}})^k \right)_J{}^K \Psi_K, \tag{22}$$

with $k = 0, 1, 2 \cdots$. They satisfy the identities

$$\Psi_I \frac{\partial G^{(n)}}{\partial \Psi_I} = \bar{\Psi}_{\bar{I}} \frac{\partial G^{(n)}}{\partial \bar{\Psi}_{\bar{I}}} = n G^{(n)}, \tag{23}$$

$$\begin{aligned} \frac{\partial G^{(2k+2)}}{\partial \Psi_I} &= (2k+2) g^{I\bar{J}} \left((R_{\bar{\Psi}} R_\Psi)^k \right)_{\bar{J}}{}^{\bar{K}} (R_{\bar{\Psi}})_{\bar{K}}{}^{L} \Psi_L \\ &= (2k+2) \Psi_J g^{J\bar{K}} \left((R_{\bar{\Psi}} R_\Psi)^k \right)_{\bar{K}}{}^{\bar{L}} (R_{\bar{\Psi}})_{\bar{L}}{}^{I}, \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\partial G^{(2k+1)}}{\partial \Psi_I} &= (2k+1) g^{I\bar{J}} \left((R_{\bar{\Psi}} R_\Psi)^k \right)_{\bar{J}}{}^{\bar{K}} \bar{\Psi}_{\bar{K}} \\ &= (2k+1) \Psi^\dagger_{\bar{J}} g^{\bar{J}K} \left((R_\Psi R_{\bar{\Psi}})^k \right)_K{}^I. \end{aligned} \tag{25}$$

Now if we introduce the ansatz

$$\mathcal{H} = \sum_{n=1}^{\infty} c_n G^{(n)}, \tag{26}$$

where c_n is a constant, we find that the differential equation (19) turns out to be the algebraic equation

$$n c_n - \sum_{p=1}^{n-1} p(n-p) c_p c_{n-p} = 0, \quad c_1 = 1. \tag{27}$$

The equation (27) is universal and independent of the Hermitian symmetric space. Therefore, their solution can be deduced by considering any choice of the Hermitian symmetric space. For instance, the projective complex space $\mathbf{C}P^1$ can be used. These considerations lead to the solution [13]

$$\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \frac{1}{2} \mathbf{\Psi}^{\mathrm{T}} \mathbf{g}^{-1} \mathcal{F}(-\mathbf{R}_{\Psi,\bar{\Psi}}) \mathbf{\Psi}, \tag{28}$$

where

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_I \\ \bar{\Psi}_{\bar{I}} \end{pmatrix}, \quad \mathbf{g}^{-1} = \begin{pmatrix} 0 & g^{I\bar{J}} \\ g^{\bar{I}J} & 0 \end{pmatrix}, \tag{29}$$

$$\mathcal{F}(x) = \frac{1}{x} \left\{ \sqrt{1+4x} - 1 - \ln \left(\frac{1+\sqrt{1+4x}}{2} \right) \right\}, \quad \mathcal{F}(0) = 1. \tag{30}$$

For our purpose to construct the $\mathcal{N}=2$ supersymmetric NLSMs on the cotangent bundles over exceptional Hermitian symmetric spaces, we shall rewrite (28) into a more convenient form [14]. First performing the Taylor expansion for (28) one can see that $\mathcal{H}$ has the same form as (26) with the coefficient c_n given by

$$c_n = \frac{(-1)^{n-1}\mathcal{F}^{(n-1)}(0)}{(n-1)!}. \tag{31}$$

Second we introduce the differential operators

$$\mathcal{R}_{\Psi,\bar{\Psi}} = -(R_{\Psi})_I{}^{\bar{J}}\bar{\Psi}_{\bar{J}}\frac{\partial}{\partial\Psi_I}, \quad \bar{\mathcal{R}}_{\Psi,\bar{\Psi}} = -(R_{\bar{\Psi}})_{\bar{I}}{}^{J}\Psi_J\frac{\partial}{\partial\bar{\Psi}_{\bar{I}}}, \tag{32}$$

which satisfy the following identity

$$\mathcal{R}_{\Psi,\bar{\Psi}}\mathcal{H}^{(n)} = \bar{\mathcal{R}}_{\Psi,\bar{\Psi}}\mathcal{H}^{(n)}. \tag{33}$$

With the use of (32), we can prove that (21) and (22) are compactly written as

$$G^{(n+1)} = \frac{(-\mathcal{R}_{\Psi,\bar{\Psi}})^n}{n!}|\Psi|^2, \quad |\Psi|^2 := g^{I\bar{J}}\Psi_I\bar{\Psi}_{\bar{J}}, \quad n \geq 1. \tag{34}$$

Substituting (34) with (31) into (26), we find

$$\mathcal{H} = \sum_{n=0}^{\infty}\frac{\mathcal{F}^{(n)}(0)}{n!}\frac{(\mathcal{R}_{\Psi,\bar{\Psi}})^n}{n!}|\Psi|^2. \tag{35}$$

Making use of the following formula

$$\frac{x^n}{n!} = \oint_C \frac{d\xi}{2\pi i}\frac{e^{\xi x}}{\xi^{n+1}}, \tag{36}$$

where the contour C encircles the origin of the complex ξ-plane in the counterclockwise direction, we have

$$\mathcal{H} = \sum_{n=0}^{\infty}\frac{\mathcal{F}^{(n)}(0)}{n!}\oint_C \frac{d\xi}{2\pi i}\frac{e^{\xi\mathcal{R}_{\Psi,\bar{\Psi}}}}{\xi^{n+1}}|\Psi|^2. \tag{37}$$

Here the contour C must be chosen such that the function $e^{\xi\mathcal{R}_{\Psi,\bar{\Psi}}}|\Psi|^2$ is analytic. Keeping this in mind, one finds that (37) can be transformed into

$$\mathcal{H} = \oint_C \frac{d\xi}{2\pi i}\frac{\mathcal{F}(1/\xi)}{\xi}e^{\xi\mathcal{R}_{\Psi,\bar{\Psi}}}|\Psi|^2. \tag{38}$$

The function $\mathcal{F}(1/\xi)/\xi$ gives a branch cut between -4 and 0. Since the $\mathcal{H}$ is regular and analytic, the contour C has to be chosen so that it does not cross the branch cut. The resultant contour encircles $\xi = -4, 0$ without crossing the cut and is bounded by poles of the factor $e^{\xi\mathcal{R}_{\Psi,\bar{\Psi}}}|\Psi|^2$. One can transform the contour C as $C' + \tilde{C}$ where C' encircles the poles that may arise from $e^{\xi\mathcal{R}_{\Psi,\bar{\Psi}}}|\Psi|^2$ in the counterclockwise direction and $\tilde{C}$ encircles those poles together with the branch cut in the clockwise direction. One can check that contribution from the contour $\tilde{C}$ is just a constant by substituting $\xi = Re^{i\theta}$ with $R \to \infty$. Therefore, this does not contribute to the Kähler metric and can be neglected. We finally have

$$\mathcal{H} = -\oint_{-C'} \frac{d\xi}{2\pi i}\frac{\mathcal{F}(1/\xi)}{\xi}e^{\xi\mathcal{R}_{\Psi,\bar{\Psi}}}|\Psi|^2, \tag{39}$$

where $-C'$ goes in the counterclockwise direction, yielding the minus sign in front of the integration.

3. Cotangent bundle

In this section, we derive the cotangent bundle action of $E_{7(-25)}/E_6 \times U(1)$ by using (39). To this end, we first derive the Kähler potential on $E_{7(-25)}/E_6 \times U(1)$, which was firstly obtained in [16]. The Kähler potential is written by the coordinates which are representation of the broken generator for the case when $E_{7(-25)}$ is broken down to $E_6 \times U(1)$. We shall write them as

$$\Phi^I \to \phi^i, \quad \bar{\Phi}^{\bar{I}} \to \bar{\phi}_i. \tag{40}$$

The transformation law for ϕ is obtained from the commutation relations:

$$[E^i, \phi^j] = \phi^i \phi^j - \frac{1}{2}\Gamma^{ijk}\Gamma_{klm}\phi^l\phi^m, \tag{41}$$
$$[E^i, \bar{\phi}_j] = \delta^i_j, \quad [\bar{E}_i, \phi^j] = \delta^j_i, \tag{42}$$
$$[\bar{E}_i, \bar{\phi}_j] = \bar{\phi}_i\bar{\phi}_j - \frac{1}{2}\Gamma_{ijk}\Gamma^{klm}\bar{\phi}_l\bar{\phi}_m, \tag{43}$$
$$[T_A, \phi^i] = -2i\rho(T_A)^i{}_j\phi^j, \quad [T_A, \bar{\phi}_i] = 2i\bar{\phi}_j\rho(T_A)^j{}_i, \tag{44}$$
$$[T, \phi^i] = -i\sqrt{\frac{2}{3}}\phi^i, \quad [T, \bar{\phi}_i] = i\sqrt{\frac{2}{3}}\bar{\phi}_i. \tag{45}$$

Here Γ^{ijk} is the invariant tensor of E_6. This is symmetric with respect to the indices (i, j, k) and the complex conjugate is defined as $(\Gamma^{ijk})^\dagger = \Gamma_{ijk}$. This satisfies the following identity

$$\Gamma_{ijk}\Gamma^{ljk} = 10\delta^l_i, \tag{46}$$

and the Springer relation [19]

$$\Gamma^{ijk}\Gamma_{jl(m}\Gamma_{pq)k} = \delta^i_{(l}\Gamma_{mpq)}. \tag{47}$$

One can see the closure of the algebra by checking the Jacobi identity.

The commutation relations (41)–(43) lead to the infinitesimal transformation law for ϕ^i:

$$\begin{aligned} \delta\phi^i &= \epsilon^j[\bar{E}_j, \phi^i] + \bar{\epsilon}_j[E^j, \phi^i] \\ &= \epsilon^i - (\bar{\epsilon}_j\phi^j)\phi^i + \frac{1}{2}\Gamma^{ijk}\bar{\epsilon}_j\Gamma_{klm}\phi^l\phi^m, \end{aligned} \tag{48}$$

where ϵ^i is the complex transformation parameter.

Now we look for the function being invariant under (48). To this end, we introduce the $E_6 \times U(1)$ invariants:

$$I_1 = \bar{\phi}_i\phi^i, \tag{49}$$
$$I_2 = (\Gamma_{ijk}\phi^j\phi^k)(\Gamma^{ilm}\bar{\phi}_l\bar{\phi}_m), \tag{50}$$
$$I_3 = \frac{1}{9}(\Gamma_{ijk}\phi^i\phi^j\phi^k)(\Gamma^{lmn}\bar{\phi}_l\bar{\phi}_m\bar{\phi}_n). \tag{51}$$

They transform under (48) as

$$\delta I_1 = (1 - I_1)(\bar{\epsilon}_i\phi^i) + \frac{1}{2}(\Gamma_{ijk}\phi^j\epsilon^k)(\Gamma^{ilm}\bar{\phi}_l\bar{\phi}_m) + \text{c.c.} \tag{52}$$

$$\delta I_2 = 2(\Gamma_{ijk}\phi^j\phi^k)(\Gamma^{ilm}\bar{\phi}_l\bar{\phi}_m) - (\bar{\epsilon}_i\phi^i)I_2 + \frac{1}{3}(\Gamma_{ijk}\phi^i\phi^j\phi^k)(\Gamma^{lmn}\bar{\phi}_l\bar{\phi}_m\bar{\phi}_n) + \text{c.c.} \tag{53}$$

$$\delta I_3 = -(\bar{\epsilon}_i\phi^i)I_3 + \frac{1}{3}(\Gamma_{ijk}\phi^i\phi^j\phi^k)(\Gamma^{lmn}\bar{\epsilon}_l\bar{\phi}_m\bar{\phi}_n) + \text{c.c.} \tag{54}$$

By using (52)–(54), one can check that the function

$$K = -\ln\left(1 - I_1 + \frac{1}{4}I_2 - \frac{1}{4}I_3\right) \tag{55}$$

transforms under the infinitesimal transformation (48) as

$$\delta K = (\bar{\epsilon}_i \phi^i) + \text{c.c.} \tag{56}$$

This shows that (55) is invariant under (48) up to the Kähler transformation (56). Thus we conclude that (55) is the Kähler potential of $E_{7(-25)}/E_6 \times U(1)$. The sign in front of the logarithm in (55) cannot be determined by the invariance under (48). It is just chosen so that positivity of the metric is ensured.

We are now ready to construct the cotangent bundle over $E_{7(-25)}/E_6 \times U(1)$. First we introduce the cotangent vectors:

$$\Psi \to \Psi_i, \quad \bar{\Psi}^{\bar{I}} \to \bar{\Psi}^i \tag{57}$$

Since we are considering the symmetric space, we set $\phi = \bar{\phi} = 0$ in calculations. The metric and the Riemann tensor at $\phi = \bar{\phi} = 0$ are derived from (55):

$$g_i{}^j\Big|_{\phi=\bar{\phi}=0} = \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}_j}\Big|_{\phi=\bar{\phi}=0} = \delta_i{}^j, \tag{58}$$

$$R_i{}^j{}_k{}^l\Big|_{\phi=\bar{\phi}=0} = \delta_i{}^l \delta_k{}^j - \Gamma_{mik}\Gamma^{mjl} + \delta_k{}^l \delta_i{}^j. \tag{59}$$

Then the differential operator (32) at $\phi = \bar{\phi} = 0$ is obtained as

$$\mathcal{R}_{\Psi,\bar{\Psi}}\Big|_{\phi=\bar{\phi}=0} = -|\psi|^2 \psi_i \frac{\partial}{\partial \psi_i} + \frac{1}{2}(\Gamma^{mik}\psi_i\psi_k)\Gamma_{mlj}\bar{\psi}^j \frac{\partial}{\partial \psi_l}, \quad |\psi|^2 := \psi_i \bar{\psi}^i, \tag{60}$$

where ψ and $\bar{\psi}$ are coordinates of the cotangent space at $\phi = \bar{\phi} = 0$. If we define the $E_6 \times U(1)$ invariants in terms of the cotangent vector

$$x := \psi_i \bar{\psi}^i, \tag{61}$$

$$y := (\Gamma^{ijk}\psi_j\psi_k)(\Gamma_{ilm}\bar{\psi}^l\bar{\psi}^m), \tag{62}$$

$$z := (\Gamma^{ijk}\psi_i\psi_j\psi_k)(\Gamma_{lmn}\bar{\psi}^l\bar{\psi}^m\bar{\psi}^n), \tag{63}$$

the differential operator (60) is rewritten as

$$\mathcal{R}_{\Psi,\bar{\Psi}}\Big|_{\phi=\bar{\phi}=0} = xD - \frac{1}{2}y\frac{\partial}{\partial x} - \frac{1}{3}z\frac{\partial}{\partial y}, \quad D := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{64}$$

The factor $e^{\xi \mathcal{R}_{\Psi,\bar{\Psi}}}x$ in (39) is calculated by using the Baker-Campbell-Hausdorff formula

$$\begin{aligned} e^{\xi \mathcal{R}_{\Psi,\bar{\Psi}}}x\Big|_{\phi=\bar{\phi}=0} &= e^{-\xi x D} e^{\frac{1}{2}\xi y \frac{\partial}{\partial x}} e^{\frac{1}{3}\xi z \frac{\partial}{\partial y}} e^{\frac{1}{12}\xi^2 z \frac{\partial}{\partial x}} e^{-\frac{1}{4}\xi^2 y D} e^{\frac{1}{18}\xi^3 z D} x \\ &= \frac{\partial}{\partial \xi} \ln\left(1 + \xi x + \frac{1}{4}\xi^2 y + \frac{1}{36}\xi^3 z\right). \end{aligned} \tag{65}$$

The poles arising from this factor contribute to the integral in (39), which are obtained from the equation

$$1 + \xi x + \frac{1}{4}\xi^2 y + \frac{1}{36}\xi^3 z = \frac{z}{36}(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3) = 0. \tag{66}$$

The explicit expressions of the poles are given in [16]. The cotangent bundle part (39) is led to the form:

$$\begin{aligned}\mathcal{H} &= -\oint_{-C'} \frac{d\xi}{2\pi i}\frac{\mathcal{F}(1/\xi)}{\xi}\frac{\partial}{\partial\xi}\ln\left(1+\xi x+\frac{1}{4}\xi^2 y+\frac{1}{36}\xi^3 z\right) \\ &= -\left(\frac{\mathcal{F}(1/\xi_1)}{\xi_1}+\frac{\mathcal{F}(1/\xi_2)}{\xi_2}+\frac{\mathcal{F}(1/\xi_3)}{\xi_3}\right). \end{aligned} \quad (67)$$

The result at an arbitrary point of Φ can be obtained by the following replacements

$$x \to (g^{-1})_i{}^j \Psi_j \bar{\Psi}^i\,, \quad (68)$$

$$\frac{1}{4}y \to \frac{1}{2}((g^{-1})_i{}^j \Psi_j \bar{\Psi}^i)^2 - \frac{1}{4}\tilde{R}_i{}^j{}_k{}^l \bar{\Psi}^i \Psi_j \bar{\Psi}^k \Psi_l\,, \quad (69)$$

$$\begin{aligned}-\frac{1}{36}z \to &-\frac{1}{6}((g^{-1})_i{}^j \Psi_j \bar{\Psi}^j)^3 + \frac{1}{4}((g^{-1})_i{}^j \Psi_j \bar{\Psi}^i)(\tilde{R}_k{}^l{}_m{}^n \bar{\Psi}^k \Psi_l \bar{\Psi}^m \Psi_n) \\ &- \frac{1}{12}|(g^{-1})_i{}^j \tilde{R}_j{}^k{}_l{}^m \Psi_k \bar{\Psi}^l \Psi_m|^2\,, \end{aligned} \quad (70)$$

where $\tilde{R}_i{}^j{}_k{}^l = (g^{-1})_i{}^m (g^{-1})_n{}^j (g^{-1})_k{}^p (g^{-1})_q{}^l R_m{}^n{}_p{}^q$.

4. Conclusion

We have constructed the $\mathcal{N}=2$ SUSY NLSM on the cotangent bundle over the non-compact exceptional HSS $\mathcal{M} = E_{7(-25)}/E_6 \times U(1)$ by using the results elaborated in [13] and [14]. The point is to use the projective superspace formalism which is an $\mathcal{N}=2$ off-shell superfield formulation. Once an $\mathcal{N}=1$ SUSY NLSM on a certain Kähler manifold is obtained, it is possible to extend it to the $\mathcal{N}=2$ SUSY model containing the corresponding $\mathcal{N}=1$ SUSY NLSM. We first have derived the transformation law of the fields parameterizing $\mathcal{M}$ and have constructed the $\mathcal{N}=1$ SUSY NLSMs on $\mathcal{M}$ invariant under the derived transformation laws. Second we have extended the $\mathcal{N}=1$ SUSY NLSM to one with the $\mathcal{N}=2$ SUSY model by using the explicit formula of the cotangent bundle over any HSS developed in [13, 14]. We have also constructed the $\mathcal{N}=2$ SUSY NLSM on the cotangent bundle over $E_{6(-14)}/SO(10) \times U(1)$ in [16] (where more detailed derivation for $E_{7(-25)}/E_6 \times U(1)$ is given). By the series of works [8, 9, 11, 12, 13, 14, 16], we have completed constructing the Kähler potentials of the cotangent bundles over *all* the compact and non-compact HSSs listed in Table 1.

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