Functions of positive type on phase space, between classical and quantum, and beyond

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Abstract. Functions of positive type on locally compact abelian groups can be defined as positive functionals on group algebras, and play a remarkable role in probability theory and in classical statistical mechanics. By Bochner's celebrated theorem, indeed, they are Fourier-Stieltjes transforms of finite positive measures. Hence, a properly normalized nonzero function of positive type on (the group of translations on) phase space provides a realization of a classical state, so it may be called a function of *classical* positive type. A similar result holds in the quantum setting as well, where a generalized kind of functions of positive type on phase space — the so-called functions of *quantum* positive type — are related, via the Fourier-Plancherel transform, to the Wigner quasi-probability distributions. In this paper, we will argue that, as in the classical setting, the notion of function of quantum positive type is of a group-theoretical nature. Exploiting an interesting interplay between functions of classical and quantum positive type, we will then provide an interesting characterization of a class of semi-groups of operators that describe the evolution of certain open quantum systems which are of interest in quantum information science. Finally, a suitable extension of this framework to generalized phase spaces that are relevant for current applications will be proposed.

1. Introduction

In the standard Hilbert space formulation of quantum mechanics, the physical states are realized as normalized, positive trace class operators (the so-called density or statistical operators) [1]. In the — more abstract — algebraic formulation, on the other hand, states can be defined as positive linear functionals on a C*-algebra of observables which, by the celebrated Gelfand-Naimark theorem, will be isomorphic to a C*-algebra of bounded operators in a Hilbert space [2,3]. This more abstract approach to physical states allows one, among other things, to encompass in a unique framework both the quantum and the *classical* case (i.e., classical statistical mechanics), where the complex Radon phase-space measures provide a realization of the bounded functionals on the commutative algebra of continuous complex functions on phase space that vanish at infinity [4].

More precisely, in this context the classical states are the probability measures on phase space, while the classical observables are the real functions of the relevant algebra (the selfadjoint part of the $*$-algebra of observables). Probability measures, being set functions, are often rather awkward objects to deal with, so it may be convenient to replace them with their Fourier — i.e., Fourier-Stieltjes — transforms, which are 'ordinary' bounded continuous functions. Actually, this is a standard technique in probability theory, where the Fourier-Stieltjes transforms of probability measures are usually called *characteristic functions* [5].

The Fourier transform is in a natural way a group-theoretical object [6,7] — clearly, in the case at hand, the group of phase-space translations is involved — hence it is not surprising that characteristic functions on phase space admit an *intrinsic* group-theoretical characterization. Indeed, by Bochner's theorem, they coincide with the properly normalized, nonzero (i.e., not identically zero) continuous *functions of positive type* on phase space [6,7]. More generally, functions of positive type on locally compact groups play a remarkable role in abstract harmonic analysis [7,8]. They can be defined as positive functionals on a suitable group algebra.

The circle of ideas between classical and quantum mechanics outlined above becomes even more intriguing by switching to the phase-space formulation of quantum mechanics [9–20]. In this approach, the physical states are implemented by *quasi*-probability distributions — the Wigner functions — and the Fourier-Plancherel transform of these distributions — the so-called *quantum characteristic functions* — turn out to be suitably normalized positive functionals on a certain noncommutative $*$-algebra of square integrable functions [21]. We will call the positive functionals on this algebra *functions of quantum positive type.*

It is known that quantum mechanics on phase space has an underlying group-theoretical framework [15,19–21], based on the theory of square integrable representations [22–24]. This theoretical framework is actually at the root of the notion of function of quantum positive type [21], and the link — via the Fourier-Plancherel transform — between functions of this kind and quantum quasi-distributions can be regarded as a quantum version of Bochner's theorem [21,25–29].

However, beside these striking analogies between the classical and the quantum setting, it is worth noting a few significant differences. Indeed, on the one hand, in the classical case there is a one-to-one correspondence between the physical states and the normalized nonzero functions of positive type, and it turns out that these functions are continuous (modulo unessential modifications on sets of zero measure). On the other hand, a nonzero function of *quantum* positive type may not be a quantum characteristic function, not even up to normalization, although the reverse implication is always true; otherwise stated, not every nonzero function of quantum positive type represents, up to normalization, a quantum state. One can show that *continuity* is precisely the condition that allows one to select, up to a positive constant factor, the quantum characteristic functions among all nonzero functions of quantum positive type. Moreover, whereas in the classical case functions of positive type are normalized in the sense of functionals, in the quantum case a different kind of normalization is associated with the physical states and normalization in the sense of functionals has the meaning of the square root of *purity* [30]. Therefore, the two normalization criteria coincide for *pure* states only.

The peculiar properties of functions of positive type allow one to note an interesting interplay between classical and quantum states. In fact, probability measures on phase space are a semigroup, with respect to the convolution product, and, equivalently, the corresponding functions of positive type form a semigroup with respect to the ordinary point-wise multiplication. One can prove, moreover, that taking the point-wise product of a function of positive type on phase space by a continuous function of *quantum* positive type one gets a function of the *latter* type. This result implies that one can obtain a semigroup of operators, acting on functions of quantum positive type (more precisely, on a Banach space generated by these functions) — a so-called *classical-quantum semigroup* [21,31] — out of a one-parameter multiplication semigroup of positive definite functions.

A straightforward way to unveil the physical meaning of a classical-quantum semigroup is to 'quantize' it (a group-covariant quantization procedure strictly related to quantization *à la* Weyl is involved), i.e., to pass from phase-space functions to Hilbert space operators. This procedure yields a new semigroup of operators acting in the Banach space of trace class operators; more precisely, a *quantum dynamical semigroup.* As is well known, semigroups of operators of this kind describe the evolution of open quantum systems [32, 33]. The quantized version of a classical-quantum semigroup belongs, more specifically, to the class of *twirling semigroups* [34–37]. Because of the role that they play in quantum information, this type of twirling semigroups can be called *classical-noise semigroups* [21, 31].

The paper is organized as follows. In sect. 2, we will introduce the notions of function of classical and quantum positive type, and we will briefly comment about their physical meaning. In sect. 3, the interplay between these two kinds of functions of positive type will be analyzed. We will first show — see subsect. 3.1 — that with every one-parameter multiplication semigroup of functions of positive type is associated in a natural way a classical-quantum semigroup. Then, in subsect. 3.2, we will argue that group-covariant quantization/dequantization establishes a precise relation between functions of quantum positive type and Hilbert space operators. At that point, we will be able to show that a classical-quantum semigroup is mapped, via quantization, to a suitable quantum dynamical semigroup; see subsect. 3.3. Finally, in sect. 4, possible extensions of these results will be proposed.

2. Functions of positive type on phase space

The notion of function of positive type is group-theoretical; hence, we will consider, at first, the case of an abstract locally compact group G. Let $\mathrm{L}^1(G)$ be the Banach space of complex (i.e., $\mathbb{C}$-valued) functions on G, integrable with respect to a left-invariant Haar measure ν_G. $\mathrm{L}^1(G)$ becomes a Banach $*$-algebra if endowed with the convolution product $(\cdot) \circledcirc (\cdot)$,

$$(\varphi_1 \circledcirc \varphi_2)(g) := \int_G \varphi_1(h)\, \varphi_2(h^{-1}g)\, \mathrm{d}\nu_G(h), \tag{1}$$

and with the involution $\mathsf{I}\colon \varphi \mapsto \varphi^*$,

$$\varphi^*(g) := \Delta_G(g^{-1})\, \overline{\varphi(g^{-1})}, \tag{2}$$

where Δ_G denotes the modular function on G. Functions of positive type can be defined as (linear) functionals [7, 21].

Definition 1 *A* function of positive type *on G is a positive bounded functional on the Banach algebra $\big(\mathrm{L}^1(G), \circledcirc, \mathsf{I}\big)$, implemented by a function belonging to the Banach space of ν_G-essentially bounded functions $\mathrm{L}^\infty(G)$. Otherwise stated, a complex function χ on G is said to be of positive type if it belongs to $\mathrm{L}^\infty(G)$ and, for every $\varphi \in \mathrm{L}^1(G)$, satisfies the inequality*

$$\int_G \chi(g)\, (\varphi^* \circledcirc \varphi)(g)\, \mathrm{d}\nu_G(g) \geq 0. \tag{3}$$

It turns out that a function of positive type $\chi \in \mathrm{L}^\infty(G)$ agrees ν_G-almost everywhere with a continuous function (it goes without saying that functions agreeing ν_G-almost everywhere are identified) and

$$\|\chi\|_\infty = \chi(e), \tag{4}$$

where $\|\chi\|_\infty := \nu_G\text{-ess}\sup_{g\in G} |\chi(g)|$ and $\chi(e)$ — with a slight abuse of notation — denotes the value at the identity $e \in G$ of the 'continuous version' of χ.

It can be shown, moreover, that for a bounded continuous function $\chi\colon G \to \mathbb{C}$ the following properties are equivalent:

P1) χ is of positive type;

P2) χ satisfies the inequality (3), for all $\varphi \in \mathsf{C_c}(G)$ (with $\mathsf{C_c}(G)$ denoting the linear space of continuous complex functions on G, with a compact support);

P3) χ satisfies the inequality

$$\int_G \int_G \chi(g^{-1}h)\,\overline{\varphi(g)}\,\varphi(h)\;\mathrm{d}\nu_G(g)\,\mathrm{d}\nu_G(h) \geq 0, \quad \forall \varphi \in \mathsf{C_c}(G); \tag{5}$$

P4) χ is a *positive definite function*, i.e., it is such that

$$\sum_{j,k} \chi(g_j^{-1} g_k)\,\overline{c_j}\,c_k \geq 0, \tag{6}$$

for every finite subset $\{g_1, \ldots, g_m\}$ of G and arbitrary complex numbers $c_1, \ldots, c_m$.

We will now consider the case where the group G is *abelian* (e.g., a vector group). The Pontryagin dual of G [6, 7] — the group of all irreducible (necessarily one-dimensional) unitary representations of G, the unitary characters — will be denoted by $\widehat{G}$, and $\mathrm{CM}(\widehat{G})$ will indicate the Banach space of complex Radon measures on $\widehat{G}$. In this case, by Bochner's theorem, the previous list of equivalent properties will admit a further entry:

P5) χ is the Fourier-Stieltjes transform of a positive measure $\mu \in \mathrm{CM}(\widehat{G})$.

If the positive measure $\mu \in \mathrm{CM}(\widehat{G})$ is, in particular, a probability measure, the normalization condition $\mu(\widehat{G}) = 1$ translates into the constraint

$$\chi(0) = 1, \tag{7}$$

for the associated function of positive type (here 0 denotes the identity of the abelian group G). Clearly, by relation (4), the latter condition is precisely the normalization of χ as a functional on $\mathrm{L}^1(G)$. In probability theory, χ is often called the *characteristic function* of μ [5].

We will now focus on the case where G is the group of translations on the $(n+n)$-dimensional phase space — namely, the vector group $\mathbb{R}^n \times \mathbb{R}^n$. In this case, we will obviously identify the dual group $\widehat{G}$ with G itself, and it will be convenient to use the *symplectic* Fourier transform [13], instead of the ordinary one.

The physical relevance of functions of positive type is related to the fact that probability measures on phase space provide a realization of physical *states* in classical statistical mechanics [3]. In fact, a classical state can be defined as a normalized positive functional on the commutative C*-algebra of (classical) observables. By Gelfand theory — see, e.g., [38] — such an algebra is isomorphic to an algebra of continuous functions vanishing at infinity. Therefore, a natural choice is $\mathsf{C}_0(\mathbb{R}^n \times \mathbb{R}^n)$, the Banach space of continuous complex functions on $\mathbb{R}^n \times \mathbb{R}^n$ that vanish at infinity, endowed with the point-wise product and with the ordinary complex conjugation (involution). Clearly, the selfadjoint part $\mathsf{C}_0(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ of the whole algebra — i.e., the real functions — contains the *true* observables of the theory. The dual space of $\mathsf{C}_0(\mathbb{R}^n \times \mathbb{R}^n)$ is the Banach space $\mathrm{CM}(\mathbb{R}^n \times \mathbb{R}^n)$ of complex Radon measures on $\mathbb{R}^n \times \mathbb{R}^n$ and, in particular, the convex set of physical states consists of the probability measures on $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, the expectation value of an observable $f \in \mathsf{C}_0(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ in the state $\mu \in \mathrm{CM}(\mathbb{R}^n \times \mathbb{R}^n)$ is given by the 'pairing'

$$\langle f \rangle_\mu = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(q,p)\,\mathrm{d}\mu(q,p). \tag{8}$$

As already mentioned, it is often convenient to replace a probability measure $\mu \in \mathrm{CM}(\mathbb{R}^n \times \mathbb{R}^n)$ (a classical state) with its symplectic Fourier-Stieltjes transform $\widetilde{\mu}$,

$$\widetilde{\mu}(q,p) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathrm{e}^{\mathrm{i}\omega((q,p),(q',p'))}\;\mathrm{d}\mu(q',p') \tag{9}$$

— with ω denoting the standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$:

$$\omega((q,p),(q',p')) := (q,p)^{\intercal}\,\Omega\,(q',p') = q\cdot p' - p\cdot q', \quad \Omega = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \tag{10}$$

— which is a continuous function of positive type on phase space.

Remark 1 As is well known [7], given a unitary representation V of a locally compact group G and a vector Ψ in the Hilbert space $\mathcal{H}$ where V acts, the map

$$G \ni g \mapsto \langle \Psi, V(g)\Psi \rangle \tag{11}$$

is a function of positive type on G. Conversely, every function of positive type on G is of the form (11), for some unitary representation V and some vector Ψ. Keeping in mind this fact and considering the case where $G = \mathbb{R}^n \times \mathbb{R}^n$, it is not surprising that formula (9) admits the following interesting interpretation. Let $\mathsf{p}\colon \mathsf{B} \to \mathbb{R}^n \times \mathbb{R}^n$ be the constant Hilbert bundle over $\mathbb{R}^n \times \mathbb{R}^n$ such that $\mathsf{p}^{-1}(q,p) = \mathbb{C} \equiv \mathcal{H}_{q,p}$, for all $(q,p) \in \mathbb{R}^n \times \mathbb{R}^n$. Of course, the direct integral Hilbert space associated with this Hilbert bundle and with a probability measure μ on $\mathbb{R}^n \times \mathbb{R}^n$ is given by

$$\mathcal{H} := \int_{\mathbb{R}^n\times\mathbb{R}^n}^{\oplus} \mathcal{H}_{q,p}\,\mathrm{d}\mu(q,p) = \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n, \mu). \tag{12}$$

Let V be the unitary representation of $\mathbb{R}^n \times \mathbb{R}^n$ in $\mathcal{H}$ defined as a *direct integral* [39]

$$V := \int_{\mathbb{R}^n\times\mathbb{R}^n}^{\oplus} V_{q',p'}\,\mathrm{d}\mu(q',p') \tag{13}$$

of unitary characters

$$V_{q',p'}(q,p)\,z = \mathrm{e}^{\mathrm{i}(q\cdot p' - p\cdot q')}\,z, \quad z \in \mathbb{C} \equiv \mathcal{H}_{q',p'}. \tag{14}$$

We then have that $(V(q,p)\Phi)(q',p') = \exp(\mathrm{i}(q\cdot p' - p\cdot q'))\,\Phi(q',p')$ and

$$\widetilde{\mu}(q,p) = \langle \Psi, V(q,p)\,\Psi \rangle, \tag{15}$$

where the matrix element on the rhs of (15) is computed with respect to the constant function $\Psi \equiv 1$ in $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n, \mu)$.

At this point, before going back to the role of functions of positive type in classical statistical mechanics, it is worth giving two simple examples. Consider first the *Dirac measure* δ_{q_0,p_0} concentrated at $(q_0,p_0) \in \mathbb{R}^n \times \mathbb{R}^n$. This is a pure — namely, an extremal — classical state. It is mapped by the symplectic Fourier-Stieltjes transform into the function of positive type $(q,p) \mapsto \mathrm{e}^{\mathrm{i}(q\cdot p_0 - p\cdot q_0)}$. Actually, *every* pure state on the C*-algebra $\mathsf{C}_0(\mathbb{R}^n \times \mathbb{R}^n)$ is a Dirac measure, and the associated function of positive type is characterized by the property that the corresponding unitary representation V — see the rhs of (15) — is irreducible; hence, a unitary character of the form (14).

Consider next a *Gaussian measure* γ on $\mathbb{R}^n \times \mathbb{R}^n$,

$$\mathrm{d}\gamma(q,p) = \pi^{-n}\det(\mathtt{M})^{-1/2}\,\exp\big(-(q-q_0,p-p_0)^{\intercal}\mathtt{M}^{-1}(q-q_0,p-p_0)\big)\,\mathrm{d}^n q\,\mathrm{d}^n p, \tag{16}$$

where $\mathtt{M}$ is a positive definite, symmetric, $2n \times 2n$ real matrix, the so-called *covariance matrix*. The symplectic Fourier-Stieltjes transform of this probability measure is the function of positive type

$$(q,p) \mapsto \widetilde{\gamma}(q,p) = \exp\big(-(q,p)^{\intercal}\,\widetilde{\mathtt{M}}\,(q,p)/4 + \mathrm{i}\,(q,p)^{\intercal}\,\Omega\,(q_0,p_0)\big), \tag{17}$$

where $\widetilde{\mathtt{M}} = -\,\Omega\,\mathtt{M}\,\Omega$.

Now, to complete our reasoning recall that the symplectic Fourier transform maps $\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$ into a dense linear subspace $\mathcal{F}\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$ of the Banach space $\mathsf{C}_0(\mathbb{R}^n\times\mathbb{R}^n)$:

$$\varphi \mapsto \check{\varphi}, \quad \check{\varphi}(q,p) := \int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi(q',p')\, \mathrm{e}^{\mathrm{i}(q'\cdot p - p'\cdot q)}\, \mathrm{d}^n q'\, \mathrm{d}^n p'. \tag{18}$$

Thus, for every observable $f \in \mathcal{F}\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$ the expectation value $\langle f\rangle_\mu$ admits the following expression:

$$\langle f\rangle_\mu = \int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi(q,p)\,\chi(q,p)\;\mathrm{d}^n q\,\mathrm{d}^n p =: \langle \varphi, \chi\rangle\,, \tag{19}$$

with $\check{\varphi} = f$ and $\chi = \widetilde{\mu}$. Namely, the expectation value $\langle f\rangle_\mu$ can be expressed as a pairing $\langle\varphi,\chi\rangle$ between $\varphi \in \mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$ and the function of positive type $\chi \in \mathrm{L}^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, representing respectively an observable and a state.

Remark 2 The condition that $f \in \mathcal{F}\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$ be an observable — i.e., $f = \overline{f}$ — translates into the condition that φ be invariant with respect to the involution of the algebra $\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$: $\varphi = \mathsf{I}\,\varphi$; namely, that

$$\varphi(q,p) = \overline{\varphi(-q,-p)}. \tag{20}$$

The main ideas underlying the picture sketched above can be condensed as follows:

(i) A function of positive type can be defined, for every locally compact group G, as a positive functional on the group algebra $\mathrm{L}^1(G)$, and gives rise to various equivalent characterizations. In particular, in the case of an abelian group, a nonzero function of positive type can be characterized, up to normalization, as the Fourier-Stieltjes transform of a probability measure.

(ii) Therefore, there is a link between the notion of function of positive type and classical statistical mechanics, where the physical states are realized as probability measures on phase space, while the observables form the selfadjoint part of the C*-algebra $\mathsf{C}_0(\mathbb{R}^n\times\mathbb{R}^n)$.

(iii) Adopting a *characteristic function approach*, one can replace probability measures with their Fourier-Stieltjes transforms, i.e., with the corresponding normalized functions of positive type. The standard algebra of classical observables is then 'densely' replaced with the Banach $*$-algebra $\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$.

(iv) It is worth remarking once again that the *positive* functionals on the algebra $\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$ are precisely the Fourier-Stieltjes transform of the positive functionals on the C*-algebra $\mathsf{C}_0(\mathbb{R}^n\times\mathbb{R}^n)$. Hence, in the characteristic function approach $\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$ may be thought of, *directly*, as the algebra of physical observables and, in principle, there is no point in considering a larger algebra (say, the the universal C*-completion of $\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n)$).

The functions of positive type on phase space form a convex cone $\mathsf{P}_n \equiv \mathsf{P}(\mathbb{R}^n\times\mathbb{R}^n)$ in $\mathrm{L}^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, and the characteristic functions — i.e., the normalized (nonzero) functions of positive type — form a convex subset $\check{\mathsf{P}}_n$ of P_n. For reasons that will be clear soon, the elements of P_n will be also called in the following *functions of classical positive type*.

In fact, a quantum analogue of functions of classical positive type emerges by undertaking a phase-space approach to quantum mechanics. In particular, among the various phase-space formalisms proposed in the literature [19, 40–47], we will adopt here the classical approach developed by Weyl, Wigner, Groenewold and Moyal [9–12] that in the following will be referred to as the *WWGM formulation* of quantum mechanics.

A *pure* state $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$, $\psi \in \mathrm{L}^2(\mathbb{R}^n)$, with $\|\psi\| = 1$ — a one-dimensional projection — is replaced in the WWGM formulation with a function $\varrho_\psi \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ defined by the celebrated classical formula ($\hbar = 1$) [10]:

$$\varrho_\psi(q,p) := (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}p\cdot x}\, \overline{\psi\Big(q - \frac{x}{2}\Big)}\, \psi\Big(q + \frac{x}{2}\Big)\ \mathrm{d}^n x\,. \tag{21}$$

This definition extends in a natural way to any density operator in $\mathrm{L}^2(\mathbb{R}^n)$ — the associated phase-space functions are usually called *Wigner functions* — and then to any trace class operator (recall that every trace class operator is a linear superposition of four density operators). This construction gives rise to a complex Banach space of functions which will be denoted by LW_n. The linear space LW_n contains a convex cone W_n consisting of those functions that are associated with *positive* trace class operators, and W_n contains a convex set $\breve{\mathsf{W}}_n$ consisting of the Wigner functions, characterized as those functions in W_n satisfying the normalization condition

$$\lim_{r\to+\infty} \int_{|q|^2+|p|^2<r} \varrho(q,p)\ \mathrm{d}^n q\, \mathrm{d}^n p = 1 = \mathrm{tr}(\hat{\rho}) \tag{22}$$

— see, e.g., [48] — where we have denoted by $\varrho \in \breve{\mathsf{W}}_n$ the Wigner function associated with the density operator $\hat{\rho}$.

We stress that the limit on the lhs of (22) is necessary for taking into account the fact that a Wigner function ϱ may *not* be integrable with respect to the Lebesgue measure [48]; accordingly, although always real, ϱ in general is *not* a genuine probability distribution, because it may take both positive and negative values. On the other hand, a Wigner function can be regarded as a *quasi-probability distribution*, since one can express the expectation value $\langle \hat{A}\rangle_{\hat{\rho}} = \mathrm{tr}(\hat{A}\,\hat{\rho})$ of an observable $\hat{A}$ in the state $\hat{\rho}$ as a 'classical' phase-space integral, i.e.,

$$\langle \hat{A}\rangle_{\hat{\rho}} = \int_{\mathbb{R}^n\times\mathbb{R}^n} \alpha(q,p)\, \varrho(q,p)\ \mathrm{d}^n q\, \mathrm{d}^n p, \tag{23}$$

where α is a real function — or generalized function [49] — suitably associated with the selfadjoint operator $\hat{A}$.

Remark 3 The characterization of those Wigner functions that are genuine probability distributions is an interesting problem [50–52]. By results due to Hudson [50] and Littlejohn [51], the Wigner functions of *pure* states assuming non-negative values only are those probability distributions associated with Gaussian measures, see (16), whose covariance matrix $\mathtt{M}$, besides being a positive definite symmetric matrix, is also symplectic: $\mathtt{M}^{\mathsf{T}}\Omega\,\mathtt{M} = \mathtt{M}\,\Omega\,\mathtt{M} = \Omega$.

Similarly to the classical setting, it is often useful to replace a quasi-probability distribution with its symplectic Fourier transform. In this regard, it is worth recalling that a Wigner function is always *square* integrable [19, 20]. Therefore, in this case the natural tool is the (symplectic) *Fourier-Plancherel transform* $\hat{\mathcal{F}}_{\mathrm{sp}}$, namely, the selfadjoint unitary operator in the Hilbert space $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ determined by

$$\big(\hat{\mathcal{F}}_{\mathrm{sp}} f\big)(q,p) = (2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} f(q',p')\, \mathrm{e}^{\mathrm{i}(q\cdot p' - p\cdot q')}\ \mathrm{d}^n q'\, \mathrm{d}^n p', \tag{24}$$

with $f \in \mathrm{L}^1(\mathbb{R}^n \times \mathbb{R}^n) \cap \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$. The linear space LW_n is mapped via $\hat{\mathcal{F}}_{\mathrm{sp}}$ onto a (dense) subspace LQ_n of $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ and the convex cone $\mathsf{W}_n \subset \mathsf{LW}_n$ is mapped onto $\mathsf{Q}_n \subset \mathsf{LQ}_n$. Since every function in LQ_n agrees almost everywhere with a continuous function, it will be convenient to regard LQ_n as a linear space of continuous functions.

As it will be clear soon, the convex cone Q_n can indeed be considered as a quantum analogue of P_n. By this analogy, we will call a function $\widetilde{\varrho}$ in Q_n, such that

$$\widetilde{\varrho} = (2\pi)^n \, \hat{\mathcal{F}}_{\mathrm{sp}} \, \varrho, \tag{25}$$

for some quasi-probability distribution $\varrho \in \breve{\mathsf{W}}_n$, the *quantum characteristic function* associated with ϱ. Hence, the quantum characteristic functions form a convex subset $\breve{\mathsf{Q}}_n = (2\pi)^n \, \hat{\mathcal{F}}_{\mathrm{sp}} \, \breve{\mathsf{W}}_n$ of the linear space LQ_n.

Remark 4 The factor $(2\pi)^n$ in (25) has been fixed in such a way that the quantum characteristic functions are those functions in Q_n satisfying the normalization condition

$$\widetilde{\varrho}(0) = 1, \tag{26}$$

with $0 \equiv (0,0)$ denoting the origin in $\mathbb{R}^n \times \mathbb{R}^n$; compare with (7).

A rather obvious problem arising in the WWGM formulation is to achieve an *intrinsic* characterization of the convex set $\breve{\mathsf{W}}_n$ of quasi-probability distributions or, equivalently, of the convex set $\breve{\mathsf{Q}}_n$ of quantum characteristic functions. Analyzing this problem one is lead to the notion of *function of quantum positive type*. Indeed, keeping in mind the classical case, it is natural to consider a suitable $*$-algebra of functions, and then to define the functions of positive type as positive functionals on this algebra. It is clear, however, that a *noncommutative* algebra should be involved in this case.

Recall that the Hilbert space $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ becomes a H*-algebra [20] once endowed with the *twisted convolution* $(\cdot) \circledast (\cdot)$, where

$$\big(\mathcal{A}_1 \circledast \mathcal{A}_2\big)(q,p) := (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{A}_1(q',p') \, \mathcal{A}_2(q-q',p-p') \, \mathrm{e}^{\frac{\mathrm{i}}{2}(q \cdot p' - p \cdot q')} \, \mathrm{d}^n q' \, \mathrm{d}^n p', \tag{27}$$

$\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$, and with the involution $\mathsf{J} \colon \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n) \ni \mathcal{A} \mapsto \mathcal{A}^*$,

$$\mathcal{A}^*(q,p) := \overline{\mathcal{A}(-q,-p)}. \tag{28}$$

The involution J acts precisely as in the classical case, compare with (20), but of course in a different space. An element $\mathcal{A}$ of $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\mathcal{A} = \mathsf{J}\,\mathcal{A}$ is said to be *selfadjoint*.

To justify our choice of the twisted convolution as the relevant algebra operation, it is worth anticipating that it can be considered as an expression of the product of Hilbert space operators in terms of phase-space functions, see subsect. 3.2. A 'nonlocal' product of functions of this kind is often called a *star product* [19, 20].

We are now ready to give a formal definition of a function of positive type in the quantum setting. Clearly, the Hilbert space $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ is selfdual. Hence, in our approach the functions of quantum positive type, defined as functionals, are assumed to be square integrable [21]:

Definition 2 *A* function of quantum positive type *on $\mathbb{R}^n \times \mathbb{R}^n$ is a positive bounded functional on the H*-algebra $\big(\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n), \circledast, \mathsf{J}\big)$, implemented by a square integrable function. Otherwise stated, a complex function $\mathcal{Q}$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be of quantum positive type if it belongs to $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ and, for every $\mathcal{A} \in \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$, satisfies the inequality*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{Q}(q,p) \, (\mathcal{A}^* \circledast \mathcal{A})(q,p) \, \mathrm{d}^n q \, \mathrm{d}^n p \geq 0. \tag{29}$$

As already mentioned, the twisted convolution of functions 'mimics' the product of operators, hence, it is not surprising that one should regard a function $\mathcal{Q}$ defined as above as a 'quantum' object. Nevertheless, the precise consequences of the previous definition are not immediately clear. We will now argue that, similarly to the classical setting, a *continuous* function of quantum positive type can be characterized by various equivalent properties, and one of these has a precise physical meaning.

It will be convenient to adopt the following concise notation: we set $z \equiv (q,p) \in \mathbb{R}^n \times \mathbb{R}^n$, $\mathrm{d}z \equiv \mathrm{d}^n q\, \mathrm{d}^n p$ and, for the symplectic form, $\omega(z,z') \equiv \omega((q,p),(q',p')) = q\cdot p' - p\cdot q'$.

One can prove the following facts; see [21] and references therein. If a *continuous* function $\mathcal{Q}$ is of quantum positive type, then it is bounded and

$$\|\mathcal{Q}\|_\infty = \mathcal{Q}(0). \tag{30}$$

This relation is the quantum analogue of (4). For a *continuous* function $\mathcal{Q}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ the following properties are equivalent:

`Q1)` $\mathcal{Q}$ is of quantum positive type;

`Q2)` $\mathcal{Q}$ satisfies the inequality (29), for all $\mathcal{A} \in \mathsf{C_c}(\mathbb{R}^n \times \mathbb{R}^n)$;

`Q3)` $\mathcal{Q}$ satisfies the inequality

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathcal{Q}(z-z')\,\overline{\mathcal{A}(z')}\,\mathcal{A}(z)\,\mathrm{e}^{\mathrm{i}\omega(z',z)/2}\,\mathrm{d}z\,\mathrm{d}z' \geq 0, \quad \forall \mathcal{A} \in \mathsf{C_c}(\mathbb{R}^n \times \mathbb{R}^n); \tag{31}$$

`Q4)` $\mathcal{Q}$ is a *quantum positive definite function*, i.e.,

$$\sum_{j,k} \mathcal{Q}(z_k - z_j)\,\mathrm{e}^{\mathrm{i}\omega(z_j,z_k)/2}\,\overline{c_j}\,c_k \geq 0, \tag{32}$$

for every finite subset $\{z_1,\ldots,z_m\}$ of $\mathbb{R}^n \times \mathbb{R}^n$ and arbitrary complex numbers $c_1,\ldots,c_m$;

`Q5)` $\mathcal{Q}$ is contained in the convex cone $\mathsf{Q}_n \subset \mathsf{LQ}_n$, i.e., it is a non-negative real multiple of the Fourier-Plancherel transform of a quasi-probability distribution.

The equivalence between the first and the last property is a quantum version of Bochner's theorem [21,25–29]. By this equivalence, Q_n can be regarded as the set of continuous functions of quantum positive type, and LQ_n as the complex vector space generated by linear superpositions of functions of this kind. By (26) and (30), the convex set $\breve{\mathsf{Q}}_n$ of quantum characteristic functions coincides with the set of those continuous functions of quantum positive type normalized in such a way that

$$\|\mathcal{Q}\|_\infty = \mathcal{Q}(0) = 1. \tag{33}$$

One can show moreover — see subsect. 3.2 — that the norm of every $\mathcal{Q} \in \breve{\mathsf{Q}}_n$, regarded as a functional (i.e., as an element of $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$), verifies the inequality

$$\|\mathcal{Q}\|_2 \leq 1, \tag{34}$$

for a suitable normalization of the Haar measure (a multiple of the Lebesgue measure on $\mathbb{R}^n\times\mathbb{R}^n$). Indeed, in subsect. 3.2 it will be argued that the square of $\|\mathcal{Q}\|_2$ is nothing but the purity of the density operator associated with $\mathcal{Q}$. Therefore, the inequality (34) is saturated if and only if the state associated with $\mathcal{Q}$ is *pure*.

Remark 5 The analogy with the classical setting cannot be pushed too far. On the one hand, a function of positive type on a locally compact group is automatically continuous (modulo modifications on ν_G-null sets). On the other hand, a function of quantum positive type $\mathcal{Q}$ is *not*, in general, a continuous function, and continuity, together with the normalization condition (33), must be *imposed* in order to select the quantum characteristic functions. Moreover, the normalization of $\mathcal{Q}$ as a functional in general differs from (33) (its normalization as a state).

Remark 6 A comparison of relations (3), (5) and (6) with relations (29), (31) and (32), shows the central role played by the function $(\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \ni (z, z') \mapsto \exp(\mathrm{i}\omega(z, z')/2)$. Of course, this is the *non-exact* $\mathbb{T}$-multiplier, for the abelian group $\mathbb{R}^n \times \mathbb{R}^n$, associated with the Weyl system and with the canonical commutation relations (expressed in the Weyl form) [53]. Accordingly, whereas an ordinary function of positive type on the vector group $\mathbb{R}^k$ is defined for every $k = 1, 2, \ldots$, the *quantum* notion of function of positive type involves the symplectic form ω — namely, a non-degenerate skew-symmetric bilinear form; hence, in this case the dimension k of the vector group must be even.

3. Interplay between classical and quantum

The notion of function of positive type on phase space — classical and quantum — allows one to note a remarkable interplay between classical and quantum states. To clarify this point, we first recall that the *convolution* $\mu_1 \circledcirc \mu_2$ of two positive measures $\mu_1, \mu_2 \in \mathrm{CM}(G)$ on a locally compact group G — defined by

$$(\mu_1 \circledcirc \mu_2)(\mathscr{E}) := \int_G \int_G \varkappa_{\mathscr{E}}(gh)\, \mathrm{d}\mu_1(g) \mathrm{d}\mu_2(h), \tag{35}$$

where $\varkappa_{\mathscr{E}}$ is the characteristic function of a Borel set $\mathscr{E} \subset G$ — is again a positive measure in $\mathrm{CM}(G)$ (a probability measure if μ_1, μ_2 are normalized) [7]. The convex subset of $\mathrm{CM}(G)$ formed by the probability measures, endowed with convolution, becomes a semigroup, whose identity is δ_e (the Dirac measure at the identity e of G). Assuming now that G is abelian, the convolution of two probability measures is mapped, via the Fourier-Stieltjes transform, to the point-wise multiplication of the corresponding characteristic functions [6]. Hence, by Bochner's theorem, the point-wise product $\chi_1 \chi_2$ of two (normalized) continuous functions of positive type χ_1 and χ_2 on G is a (normalized) continuous function of positive type too. In particular, in the case where $G = \mathbb{R}^n \times \mathbb{R}^n$, the set $\breve{\mathsf{P}}_n$ of normalized functions of classical positive type, endowed with the point-wise product, is a semigroup, the identity being the constant function $\chi \equiv 1$.

3.1. Classical-quantum semigroups

As we have seen, point-wise multiplication is a legitimate operation between functions of classical positive type. What happens if one multiplies a function of *classical* positive type by a continuous function of *quantum* positive type? The answer is contained in the following result [21]:

Theorem 1 *The point-wise product $\chi\, \mathcal{Q}$ of a function $\chi \in \mathsf{P}_n$ by a function $\mathcal{Q} \in \mathsf{Q}_n$ is contained in Q_n. In particular, if χ and $\mathcal{Q}$ are normalized, $\chi\, \mathcal{Q}$ belongs to $\breve{\mathsf{Q}}_n$.*

This result, which may be regarded at first sight as a mere mathematical curiosity, actually establishes a link between the notion of function of positive type and the theory of open quantum systems. Consider, indeed, a set $\{\chi_t\}_{t \in \mathbb{R}^+}$ of normalized continuous functions of positive type on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\chi_t\, \chi_s = \chi_{t+s}, \ \forall t, s \geq 0, \quad \chi_0 \equiv 1, \tag{36}$$

where $\chi_t\, \chi_s$ is again a point-wise product; namely, a (one-parameter) *multiplication semigroup of functions of positive type*. One should also assume continuity, with respect to a suitable topology on $\breve{\mathsf{P}}_n$ [21], for the homomorphism $\mathbb{R}^+ \ni t \mapsto \chi_t \in \breve{\mathsf{P}}_n$. One-parameter semigroups of this kind can be classified because they are related, via the Fourier-Stieltjes transform, with the convolution semigroups of probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ [34, 54].

Observe now that for every function of positive type $\chi \in \breve{\mathsf{P}}_n$ one can define a linear operator $\hat{\mathfrak{C}}_\chi$ in $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ by setting

$$\big(\hat{\mathfrak{C}}_\chi\, f\big)(q, p) := \chi(q, p)\, f(q, p). \tag{37}$$

It is clear that $\hat{\mathfrak{C}}_\chi$ is a (well defined) bounded operator because χ is a bounded continuous function. It will be shown — see Remark 8 below — that it maps the convex cone of functions of quantum positive type into itself. For every multiplication semigroup of functions of positive type $\{\chi_t\}_{t\in\mathbb{R}^+} \subset \breve{\mathsf{P}}_n$ and for every $t \geq 0$, one can set

$$\big(\hat{\mathfrak{C}}_t\, f\big)(q,p) \equiv \big(\hat{\mathfrak{C}}_{\chi_t}\, f\big)(q,p) := \chi_t(q,p)\, f(q,p), \quad f \in \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n). \tag{38}$$

The set $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ is a (one-parameter) semigroup of operators [55]:

(i) $\hat{\mathfrak{C}}_t\, \hat{\mathfrak{C}}_s = \hat{\mathfrak{C}}_{t+s}$, $t, s \geq 0$;

(ii) $\hat{\mathfrak{C}}_0 = I$ (with I denoting the identity operator).

We will call the semigroups of operators $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ a *classical-quantum semigroup*.

Taking into account the contents Theorem 1, we will now consider a natural restriction of the semigroup of operators $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ to a linear subspace of $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$. In fact, as mentioned in sect. 2, one can extend the convex cone Q_n of continuous functions of quantum positive type on $\mathbb{R}^n \times \mathbb{R}^n$, by taking (complex) linear superpositions, to a dense subspace LQ_n of $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$. Since the point-wise product of a continuous function of classical positive type by a continuous function quantum positive type is again a function of the latter type, a semigroup of operators $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ in LQ_n is defined by setting

$$\big(\mathfrak{C}_t\, \mathcal{Q}\big)(q,p) := \chi_t(q,p)\, \mathcal{Q}(q,p), \quad \mathcal{Q} \in \mathsf{LQ}_n. \tag{39}$$

Note that here $\mathcal{Q}$, with a slight abuse with respect to our previous convention, denotes a *linear superposition* of four functions of quantum positive type, namely, $\mathcal{Q} = \mathcal{Q}_1 - \mathcal{Q}_2 + \mathrm{i}\big(\mathcal{Q}_3 - \mathcal{Q}_4\big)$, with $\mathcal{Q}_1, \ldots, \mathcal{Q}_4 \in \mathsf{Q}_n$. Obviously, by construction we have that $\mathfrak{C}_t\, \mathsf{Q}_n \subset \mathsf{Q}_n$ and $\mathfrak{C}_t\, \breve{\mathsf{Q}}_n \subset \breve{\mathsf{Q}}_n$.

Thus, it is now clear that the semigroup of operators $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ — which will be called a *proper classical-quantum semigroup* — maps the convex set of quantum characteristic functions into itself. Nevertheless, the precise connection with the theory of open quantum systems needs to be further clarified. It will be unveiled by means of a suitable quantization procedure mapping phase-space functions to Hilbert space operators.

3.2. Group-theoretical quantization/dequantization

The procedure that allows one to associate with a Hilbert space operator a phase-space function (e.g., a Wigner function) involves a suitable *dequantization map*. Of course the reverse arrow is a *quantization map*, which transforms functions into operators. By these maps, one is able to set a (at least implicit) correspondence between the product of operators and a *star product* [19,20,56] of functions — the twisted convolution (27), in the case where quantum characteristic functions are involved. The star product is an essential ingredient for a self-consistent formulation of quantum mechanics in terms of phase-space functions, together with the related notion of function of quantum positive type. Our aim, now, is to briefly illustrate the meaning of this notion in terms of Hilbert space operators. As a byproduct, we will be able to highlight the precise connection of classical-quantum semigroups with quantum dynamical semigroups.

A fundamental tool for constructing a pair of 'group-covariant' quantization/dequantization maps is a *square integrable* (in general, projective) representation U of a locally compact group G in a Hilbert space $\mathcal{H}$; see [15,23,24]. One may think of $\mathcal{H}$ as the Hilbert space of a quantum-mechanical system and of G as a symmetry group. Let $\mathcal{B}_2(\mathcal{H})$ be the Hilbert space of Hilbert-Schmidt operators in $\mathcal{H}$. The representation U allows one to define the dequantization map as a linear isometry $\mathscr{D}\colon \mathcal{B}_2(\mathcal{H}) \to \mathrm{L}^2(G)$, see [15,19,20]. If the group G is unimodular and $\hat{\rho} \in \mathcal{B}_2(\mathcal{H})$ is a trace class operator, the function $\mathscr{D}\hat{\rho}$ associated with $\hat{\rho}$ is of the simple form

$$(\mathscr{D}\hat{\rho})(g) = d_U^{-1}\ \mathrm{tr}(U(g)^* \hat{\rho}), \tag{40}$$

where $d_U > 0$ is a constant which depends on U and on the normalization of the Haar measure on G. The quantization map $\mathscr{Q}$ associated with U is nothing but the adjoint of the map $\mathscr{D}$; i.e., it is the partial isometry $\mathscr{Q}$ defined by $\mathscr{Q} := \mathscr{D}^*\colon \mathrm{L}^2(G) \to \mathcal{B}_2(\mathcal{H})$. It is clear that $\mathrm{Ker}(\mathscr{Q}) = \mathrm{Ran}(\mathscr{D})^\perp$ (in general, the closed subspace $\mathrm{Ker}(\mathscr{Q})$ of $\mathrm{L}^2(G)$ is not trivial). The star product associated with U is then defined implicitly by

$$f_1 \star f_2 := \mathscr{D}((\mathscr{Q}f_1)(\mathscr{Q}f_2)), \quad f_1, f_2 \in \mathrm{L}^2(G), \tag{41}$$

were $(\mathscr{Q}f_1)(\mathscr{Q}f_2)$ is the ordinary composition of the operators $\mathscr{Q}f_1$ and $\mathscr{Q}f_2$. Note that for a pair of functions belonging to $\mathrm{Ran}(\mathscr{D})$ the star product can be thought of as the dequantized version of the product of operators. Explicit formulae for the star products are derived in [20].

Let now G be, in particular, the group $\mathbb{R}^n \times \mathbb{R}^n$ of phase-space translations (see [19,20] for the details). In this case, $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^n)$, $\mathrm{L}^2(G) = \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n) \equiv \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n, (2\pi)^{-n}\mathrm{d}^n q\, \mathrm{d}^n p; \mathbb{C})$ (the Haar measure is normalized in such a way that $d_U = 1$ in (40)) and the representation U is a *Weyl system* [20, 53], namely, $U(q,p) := \exp(\mathrm{i}(p \cdot \hat{q} - q \cdot \hat{p}))$, where $\hat{q} = (\hat{q}_1, \ldots, \hat{q}_n)$, $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)$, with $\hat{q}_j$, $\hat{p}_j$ denoting the standard j-th coordinate position and momentum operators in $\mathrm{L}^2(\mathbb{R}^n)$. The Weyl system U is a square integrable, genuinely projective representation,

$$U(q+\tilde{q}, p+\tilde{p}) = \mathsf{m}(q,p;\tilde{q},\tilde{p})\, U(q,p)\, U(\tilde{q},\tilde{p}), \quad \mathsf{m}(q,p;\tilde{q},\tilde{p}) \in \mathbb{T}, \tag{42}$$

where the (non-exact) multiplier m is of the form $\mathsf{m}(q,p;\tilde{q},\tilde{p}) := \exp(\mathrm{i}(q \cdot \tilde{p} - p \cdot \tilde{q})/2)$. It turns out that for every density operator $\hat{\rho}$ in $\mathrm{L}^2(\mathbb{R}^n)$ the function $\mathscr{D}\hat{\rho} = \mathrm{tr}(U(\cdot)^*\hat{\rho})$ coincides with the quantum characteristic function $\widetilde{\varrho}$ defined by (25), where ϱ is the Wigner distribution associated with $\hat{\rho}$; see [15, 19, 20]. Thus, as $\hat{\mathcal{F}}_{\mathrm{sp}} = \hat{\mathcal{F}}_{\mathrm{sp}}^*$, we have that $\varrho = (2\pi)^{-n}\hat{\mathcal{F}}_{\mathrm{sp}}\mathscr{D}\hat{\rho}$, and quantization *à la* Weyl is obtained composing $\mathscr{Q} = \mathscr{D}^*$ with the symplectic Fourier-Plancherel operator.

Moreover, the following facts hold [19, 20]:

- $\mathrm{Ran}(\mathscr{D}) = \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$, so that in this case the partial isometries $\mathscr{Q}$ and $\mathscr{D}$ are actually unitary operators. The unitary operator $\mathscr{Q}$ intertwines the involution $\mathsf{J}\colon \mathcal{A} \mapsto \mathcal{A}^*$ in $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$, defined by (28), with the adjoining map $\hat{A} \mapsto \hat{A}^*$ in $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}^n))$, i.e.,

$$\mathscr{Q}\,\mathsf{J}\,\mathcal{A} = (\mathscr{Q}\,\mathcal{A})^*, \quad \forall \mathcal{A} \in \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n). \tag{43}$$

- The star product (41) is implemented by the twisted convolution, see (27).

Taking into account these points and the contents of sect. 2, one can also argue that

- Translated in terms of Hilbert space operators, the quantum positivity condition (29) amounts to requiring that $\hat{B} = \mathscr{Q}\,\mathcal{Q} \in \mathcal{B}_2(\mathrm{L}^2(\mathbb{R}^n))$ satisfies the inequality $\mathrm{tr}(\hat{B}\,\hat{A}^*\hat{A}) \geq 0$, for all $\hat{A} \in \mathcal{B}_2(\mathrm{L}^2(\mathbb{R}^n))$; hence, equivalently, that $\hat{B} \geq 0$.
- For the linear space LQ_n and for the convex cone $\mathsf{Q}_n \subset \mathsf{LQ}_n$, we have that

$$\mathsf{LQ}_n = \mathscr{D}\,\mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n)) \ \text{ and } \ \mathsf{Q}_n = \mathscr{D}\,\mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n))^+, \tag{44}$$

  where $\mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n))$ is the Banach space of trace class operators in $\mathrm{L}^2(\mathbb{R}^n)$ and $\mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n))^+$ the convex cone of *positive* trace class operators. It is clear, moreover, that for every $\hat{\rho} \in \mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n))^+$, $\|\mathscr{D}\hat{\rho}\|_\infty = (\mathscr{D}\hat{\rho})(0) = \mathrm{tr}(\hat{\rho})$.
- It follows that the unitary operator $\mathscr{Q} = \mathscr{D}^*$ maps a function of quantum positive type into a positive Hilbert-Schmidt operator and a *continuous* function of quantum positive type into a positive trace class operator (in particular, a *normalized continuous* function of quantum positive type into a density operator). By (43), a function of quantum positive type $\mathcal{Q}$ is selfadjoint: $\mathcal{Q} = \mathsf{J}\,\mathcal{Q}$.

- For every density operator $\hat{\rho}$ in $\mathrm{L}^2(\mathbb{R}^n)$, denoting by $\|\cdot\|_2$ the norm of the Hilbert space $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n, (2\pi)^{-n}\mathrm{d}^n q\, \mathrm{d}^n p; \mathbb{C})$, we have that $\|\mathscr{D}\hat{\rho}\|_2 = \sqrt{\mathrm{tr}(\hat{\rho}^2)} \le 1$. Hence, as mentioned in sect. 2 (see (34)), with a suitable choice of the Haar measure the norm, as a functional, of a normalized continuous function of quantum positive type coincides with the square root of the purity of the associated state.

Remark 7 Taking into account the previous facts, the contents of Theorem 1 can be slightly extended by including the following assertion: *for every $\chi \in \mathsf{P}_n$ and for every function of quantum positive type $\mathcal{Q}$ in* $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ (not necessarily contained in Q_n) *the point-wise product $\chi\,\mathcal{Q}$ is again function of the latter type.* We now give a sketch of the proof. Let us set $\hat{A} \equiv \mathscr{Q}\,\mathcal{Q} \in \mathcal{B}_2(\mathrm{L}^2(\mathbb{R}^n))$, $\hat{A} \ge 0$. Assuming that $\hat{A} \notin \mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n))^+$ (i.e., $\mathcal{Q} \notin \mathsf{Q}_n$), consider the (necessarily infinite) spectral decomposition $\hat{A} = \sum_j \alpha_j\, \hat{\Pi}_j$ — where $\alpha_1 > \alpha_2 > \cdots > 0$ and $\hat{\Pi}_j$ is a (finite rank) projection operator — which converges in the Hilbert-Schmidt norm. Then, the sequence $\{\mathcal{Q}_m\}_{m\in\mathbb{N}}$, with $\mathcal{Q}_m \equiv \mathscr{D}(\sum_{j=1}^m \alpha_j\,\hat{\Pi}_j)$, is contained in Q_n and $\mathcal{Q}_m \to \mathcal{Q}$ in $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore, we have that $\{\chi\,\mathcal{Q}_m\}_{m\in\mathbb{N}} \subset \mathsf{Q}_n$, $\chi\,\mathcal{Q}_m \to \chi\,\mathcal{Q}$ and $0 \le \int (\chi\,\mathcal{Q}_m)(q,p)\, (\mathcal{A}^* \circledast \mathcal{A})(q,p)\, \mathrm{d}^n q\, \mathrm{d}^n p \to \int (\chi\,\mathcal{Q})(q,p)\, (\mathcal{A}^* \circledast \mathcal{A})(q,p)\, \mathrm{d}^n q\, \mathrm{d}^n p$.

Remark 8 By Remark 7, as anticipated in subsect. 3.1, for every function of positive type $\chi \in \breve{\mathsf{P}}_n$ the bounded operator $\hat{\mathfrak{C}}_\chi$ defined by (37) maps the convex cone of functions of quantum positive type into itself, and obviously the same property holds for each member of the semigroup of operators $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ defined by (38).

3.3. Unveiling the physical meaning of classical-quantum semigroups

We are now ready to unveil the physical meaning of a classical-quantum semigroup. Observe that with the projective representation U is associated an isometric representation $U \vee U$ of the group $\mathbb{R}^n \times \mathbb{R}^n$ acting in the Banach space $\mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n))$; i.e.,

$$U \vee U(q,p)\, \hat{\rho} := U(q,p)\, \hat{\rho}\, U(q,p)^*, \quad \forall\, \hat{\rho} \in \mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n)). \tag{45}$$

This is, of course, the canonical symmetry action of the group of phase-space translations on trace class operators, according to Wigner's theorem on symmetry transformations. Given a convolution semigroup $\{\mu_t\}_{t\in\mathbb{R}^+}$ of probability measures on $\mathbb{R}^n \times \mathbb{R}^n$, one can define a semigroup of operators $\{\mu_t[U \vee U]\}_{t\in\mathbb{R}^+}$ in $\mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n))$ by setting

$$\mu_t[U \vee U]\, \hat{\rho} := \int_{\mathbb{R}^n \times \mathbb{R}^n} U \vee U(q,p)\, \hat{\rho}\; \mathrm{d}\mu_t(q,p). \tag{46}$$

It can be shown that the *twirling semigroup* $\{\mu_t[U \vee U]\}_{t\in\mathbb{R}^+}$ is a quantum dynamical semigroup (namely, a completely positive, trace-preserving semigroup of operators in $\mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n))$); see [34–37]. More specifically, $\{\mu_t[U \vee U]\}_{t\in\mathbb{R}^+}$ belongs to the class of *classical-noise semigroups*; see [31] and references therein. The following result [21] establishes a precise connection between the semigroup of operators (46) and a classical-quantum semigroup:

Theorem 2 *Let $\{\chi_t\}_{t\in\mathbb{R}^+}$ be the one-parameter multiplication semigroup of functions of positive type related, via the symplectic Fourier-Stieltjes transform, to the convolution semigroup of probability measures $\{\mu_t\}_{t\in\mathbb{R}^+}$ — namely, $\chi_t(q,p) = \int \exp(\mathrm{i}(q \cdot p' - p \cdot q'))\, \mathrm{d}\mu_t(q',p')$ — and let $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ be the proper classical-quantum semigroup associated with $\{\chi_t\}_{t\in\mathbb{R}^+}$, see (39). Then, the unitary operator $\mathscr{Q}$ intertwines the semigroup of operators $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ with the quantum dynamical semigroup $\{\mu_t[U \vee U]\}_{t\in\mathbb{R}^+}$; i.e., for every $\mathcal{Q} \in \mathsf{LQ}_n$, we have:*

$$\mathscr{Q}\, \mathfrak{C}_t\, \mathcal{Q} = \mu_t[U \vee U]\, \mathscr{Q}\, \mathcal{Q}, \quad \forall\, t \ge 0. \tag{47}$$

Remark 9 The isometric representation $U \vee U$ in $\mathcal{B}_1(\mathrm{L}^2(\mathbb{R}^n))$ admits a natural extension to a unitary representation in $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}^n))$. By this extension, one can define a new semigroup of operators in the Hilbert space $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}^n))$ acting in a way completely analogous to (46). The new semigroup of operators is unitarily equivalent, via the dequantization map $\mathscr{D}$, to the classical-quantum semigroup $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ in $\mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$.

4. . . . and beyond

Functions of positive type on phase space play a remarkable role in classical statistical mechanics and in the WWGM formulation of quantum mechanics since they are related to probability measures and to quasi-probability distributions. Both the 'classical' and the 'quantum' functions of positive type admit a simple and elegant group-theoretical characterization as positive functionals on suitable group algebras. In the classical case, on the one hand, the standard commutative convolution algebra $\big(\mathrm{L}^1(\mathbb{R}^n\times\mathbb{R}^n), \circledcirc, \mathsf{I}\big)$ is involved, and continuity of the associated positive type functions is a byproduct of their definition. On the other hand, the *continuous* functions of quantum positive type — equivalently, the non-negative real multiples of those functions representing physical states (the *quantum characteristic functions*) — are embedded in the convex cone of positive functionals on the twisted convolution algebra $\big(\mathrm{L}^2(\mathbb{R}^n\times\mathbb{R}^n), \circledast, \mathsf{J}\big)$. As we have seen, this embedding corresponds to the natural inclusion of the positive trace class operators in the positive Hilbert-Schmidt operators. It is also worth observing that the physically relevant normalization condition for the continuous functions of quantum positive type differs, in general, from their normalization as functionals; the coincidence of the two normalization criteria is indeed a distinguishing feature of the *pure* states.

It is clear that in the group-theoretical approach to quantization/dequantization outlined in subsect. 3.2 a natural problem is to achieve a generalized notion of function of positive type, suitable for an extension of the results valid for the group of translations on phase space to other groups. Consider, e.g., the case of a *unimodular* locally compact group G admitting a square integrable projective representation U, with multiplier m. It can be shown that, for every pair of functions f_1, f_2 *living in the range of the dequantization map* $\mathscr{D}$ (associated with U), and for a suitable normalization of the Haar measure ν_G, the star product (41) can be expressed as a m-twisted convolution product [20]:

$$(f_1 \star f_2)(g) = \int_G f_1(h)\, f_2(h^{-1}g)\, \overline{\mathsf{m}(h, h^{-1}g)}\; \mathrm{d}\nu_G(h). \tag{48}$$

Of course, this is a natural generalization of the standard twisted convolution (27). One can define a notion of function of positive type relatively to the algebra structure determined by this star product. Note, however, that in general the closed subspace $\mathrm{Ran}(\mathscr{D})^\perp = \mathrm{Ker}(\mathscr{Q})$ of $\mathrm{L}^2(G)$ is not trivial, and this aspect has to be taken into account when formulating the positivity condition. Another nontrivial aspect is related to the characterization of those functions of positive type (in the aforementioned generalized sense) that represent quantum states. The case of non-unimodular groups presents further delicate aspects, due to more complicated expressions of quantization/dequantization maps and of the associated star products [20], and the case of certain semidirect product groups that do not admit square integrable representations (e.g., the Poincaré group) entails other intricacies [64]. These natural developments of our work will be analyzed in detail elsewhere. Let us only mention, here, that an interesting and reasonably simple case is that of a group of the form $G = \mathbb{A} \times \widehat{\mathbb{A}}$, where $\mathbb{A}$ is a locally compact abelian group and $\widehat{\mathbb{A}}$ its Pontryagin dual. Like $\mathbb{R}^n \times \mathbb{R}^n$, also $\mathbb{A} \times \widehat{\mathbb{A}}$ may be thought of, simultaneously, as a (generalized) phase space and as the group of translations on this space. Actually, phase spaces of this sort — where $\mathbb{A}$ is, e.g., a cyclic groups or a finite field — and the associated representations of quantum states are of current interest in quantum information, quantum tomography etc.; see [57–63] and further references therein.

In this circle of ideas is also inscribed the notion of classical-quantum semigroup. We have first introduced this notion in [31], with the aim of highlighting connections with other classes of semigroups of operators and with quantum information science. Then, in [21] we have studied the link of classical-quantum semigroups with functions of positive type on phase space. As shown in subsect. 3.3, by quantizing a classical-quantum semigroup one obtains a *twirling semigroup*, generated in a natural way by a a Weyl system and by a convolution semigroup on the vector group $\mathbb{R}^n \times \mathbb{R}^n$. Actually, every pair formed by a projective representation of a locally compact group and by a convolution semigroup of probability measures on that group gives rise in a natural way to a twirling semigroup [34–36]. Thus, it would be interesting to study the construction — e.g., for groups of the form $G = \mathbb{A} \times \widehat{\mathbb{A}}$ — of suitable classical-quantum semigroups, a further development of our work that we plan to pursue.

References

[1] Prugovečki E 1981 *Quantum Mechanics in Hilbert Space* 2nd edn (New York: Academic)

[2] Emch G G 1984 *Mathematical and Conceptual Foundations of 20th-Century Physics* (Amsterdam: North-Holland)

[3] Strocchi F 2005 *An Introduction to the Mathematical Structure of Quantum Mechanics* (Singapore: World Scientific)

[4] Folland G B 1984 *Real Analysis* (New York: John Wiley & Sons)

[5] Klenke A 2014 *Probability Theory* 2nd edn (London: Springer)

[6] Rudin W 1962 *Fourier Analysis on Groups* (New York: Interscience)

[7] Folland G B 1995 *A Course in Abstract Harmonic Analysis* (Boca Raton, FL: CRC)

[8] Godement R 1948 Les fonctions de type positif et la theorie des groupes *Trans. Am. Math. Soc.* **63** 1

[9] Weyl H 1927 Quantenmechanik und Gruppentheorie *Z. Phys.* **46** 1

[10] Wigner E 1932 On the quantum correction for thermodinamic equilibrium *Phys. Rev.* **40** 749

[11] Groenewold H J 1946 On the principles of elementary quantum mechanics *Physica* **12** 405

[12] Moyal J E 1949 Quantum mechanics as a statistical theory *Proc. Camb. Phil. Soc.* **45** 99

[13] Folland G B 1989 *Harmonic Analysis in Phase Space* (Princeton, NJ: Princeton University Press)

[14] Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 Distribution functions in physics: fundamentals *Phys. Rep.* **106** 121

[15] Ali S T, Antoine J P and Gazeau J P 2000 *Coherent States, Wavelets and Their Generalizations* (New York: Springer)

[16] Schleich W P 2001 *Quantum Optics in Phase Space* (Berlin: Wiley-VCH)

[17] Dodonov V V and Man'ko V I eds 2003 *Theory of Nonclassical States of Light* (London: Taylor & Francis)

[18] Aniello P, Man'ko V, Marmo G, Solimeno S and Zaccaria F 2000 On the coherent states, displacement operators and quasidistributions associated with deformed quantum oscillators *J. Opt. B: Quantum Semiclass. Opt.* **2** 718

[19] Aniello P, Man'ko V I and Marmo G 2008 Frame transforms, star products and quantum mechanics on phase space *J. Phys. A: Math. Theor.* **41** 285304

[20] Aniello P 2009 Star products: a group-theoretical point of view *J. Phys. A: Math. Theor.* **42** 475210

[21] Aniello P 2015 Playing with functions of positive type, classical and quantum *Phys. Scr.* **90** 074042

[22] Duflo M and Moore C C 1976 On the regular representation of a nonunimodular locally compact group *J. Funct. Anal.* **21** 209

[23] Aniello P, Cassinelli G, De Vito E and Levrero A 1998 Square-integrability of induced representations of semidirect products *Rev. Math. Phys.* **10** 301

[24] Aniello P 2006 Square integrable projective representations and square integrable representations modulo a relatively central subgroup *Int. J. Geom. Methods Mod. Phys.* **3** 233

[25] Kastler D 1965 The C*-algebras of free boson field *Commun. Math. Phys.* **1** 14

[26] Loupias G and Miracle-Sole S 1966 C*-algèbre des systèmes canoniques. I *Commun. Math. Phys.* **2** 31

[27] Loupias G and Miracle-Sole S 1967 C*-algèbre des systèmes canoniques. II *Ann. Inst. Henri Poincaré* A **6** 39

[28] Narcowich F J and O'Connell F R 1986 Necessary and sufficient conditions for a phase-space function to be a Wigner dstribution *Phys. Rev.* A **34** 1

[29] Narcowich F J 1989 Distributions of $\hbar$-positive type and applications *J. Math. Phys.* **30** 2565

[30] Lupo C, Aniello P and Scardicchio A 2008 Bipartite quantum systems: on the realignment criterion and beyond *J. Phys. A: Math. Theor.* **41** 415301

[31] Aniello P 2014 Classical-quantum semigroups, *J. Phys.: Conf. Ser.* **563** 012002

[32] Holevo A S 2001 *Statistical Structure of Quantum Theory* (Berlin: Springer)
[33] Breuer H-P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
[34] Aniello P, Kossakowski A, Marmo G and Ventriglia F 2010 Brownian motion on Lie groups and open quantum systems *J. Phys. A: Math. Theor.* **43** 265301
[35] Aniello P 2011 On a certain class of semigroups of operators *Open Syst. Inf. Dyn.* **18** 129
[36] Aniello P 2013 Quantum dynamical semigroups, group representations and convolution semigroups *Phys. Scr.* **T153** 014003
[37] Aniello P 2013 Operators versus functions: from quantum dynamical semigroups to tomographic semigroups *J. Phys.: Conf. Ser.* **474** 012005
[38] Murphy G J 1990 C^**-algebras and Operator Theory* (San Diego, CA: Academic)
[39] Dixmier J 1981 *Von Neumann Algebras* (Amsterdam: North-Holland)
[40] Husimi K 1940 Some formal properties of the density matrix *Proc. Phys. Math. Soc. Japan* **22** 264
[41] Kano Y 1965 A new phase-space distribution function in the statistical theory of the electromagnetic field *J. Math. Phys.* **6** 1913
[42] Sudarshan E C G 1963 Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams *Phys. Rev. Lett.* **10** 277
[43] Mehta C L and Sudarshan E C G 1965 Relation between quantum and semiclassical description of optical coherence *Phys. Rev.* **138** B274
[44] Glauber R J 1963 Photon correlations *Phys. Rev. Lett.* **10** 84
[45] Cahill K E and Glauber R J 1969 Ordered expansions on boson amplitude operators *Phys. Rev.* **177** 1857
[46] Cahill K E and Glauber R J 1969 Density operators and quasi-probability distributions *Phys. Rev.* **177** 1882
[47] Mancini S, Man'ko V I and Tombesi P 1996 Symplectic tomography as classical approach to quantum systems *Phys. Lett.* A **213** 1
[48] Daubechies I 1983 Continuity statements and counterintuitive examples in connection with Weyl quantization *J. Math. Phys.* **24** 1453
[49] Daubechies I 1980 On the distributions corresponding to bounded operators in the Weyl quantization *Commun. Math. Phys.* **75** 229
[50] Hudson R L 1974 When is the Wigner quasi-probability density non-negative? *Rep. Math. Phys.* **6** 249
[51] Littlejohn R G 1986 The semiclassical evolution of wave packets *Phys. Rep.* **138** 193
[52] Bröcker T and Werner R F 1995 Mixed states with positive Wigner functions *J. Math. Phys.* **36** 62
[53] Aniello P 2010 On the notion of Weyl system *J. Russ. Laser Res.* **31** 102
[54] Heyer H 1977 *Probability Measures on Locally Compact Groups* (Berlin: Springer)
[55] Yosida K 1968 *Functional Analysis* (Berlin: Springer)
[56] Aniello P, Ibort A, Man'ko V I and Marmo G 2009 Remarks on the star product of functions on finite and compact groups *Phys. Lett.* A **373** 401
[57] Gibbons K S, Hoffman M J and Wootters W K 2004 Discrete phase space based on finite fields *Phys. Rev.* A **70** 062101
[58] Paz J P, Roncaglia A J and Saraceno M 2005 Qubits in phase space: Wigner-function approach to quantum-error correction and the mean-king problem *Phys. Rev.* A **72** 0122309
[59] Marchiolli M A, Ruzzi M and Galetti D 2005 Extended Cahill-Glauber formalism for finite-dimensional spaces. II. Applications in quantum tomography and quantum teleportation *Phys. Rev.* A **72** 042308
[60] Vourdas A 2008 Quantum mechanics on p-adic numbers *J. Phys. A: Math. Theor.* **41** 455303
[61] Vourdas A and Banderier C 2010 Symplectic transformations and quantum tomography in finite quantum systems *J. Phys. A: Math. Theor.* **43** 042001
[62] Patra M K and Braunstein S L 2011 Quantum Fourier transform, Heisenberg groups and quasi-probability distributions *New J. Phys.* **13** 063013
[63] Shalaby M and Vourdas A 2013 Mutually unbiased projectors and duality between lines and bases in finite quantum systems *Ann. Phys.* **337** 208
[64] Aniello P 2006 Extended wavelet transforms *Int. J. Geom. Methods Mod. Phys.* **3** 341