Symmetries of the Pais-Uhlenbeck oscillator on the Hamiltonian level

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Abstract. In this survey we report on the symmetry properties of the Pais-Uhlenbeck oscillator. Especially, basing on the paper [Nucl. Phys. B 889 (2014) 333] we describe the Hamiltonian formalisms and its symmetry realizations on the Hamiltonian level.

1. Introduction

We present here a review of the results obtained in the paper [1] concerning the Hamiltonian approaches to the Pais-Uhlenbeck (PU) oscillator. We describe the three main Hamiltonian formalisms and relation between them as well as the realization of symmetries of the PU model in these frameworks.

We start with a very brief discussion of the ordinary mechanics: the free motion and the harmonic oscillator and their symmetries (the Schrödinger group). Next, we move, on the Lagrangian level, to their generalizations to the case of higher derivatives theory, especially to the PU model. The main part is devoted to the relations between the Hamiltonian formalisms for the PU model: the original due to Pais and Uhlenbeck, the Ostrogradski one and the third one based on the free higher derivatives theory. We also discuss the realization of the symmetries of the PU oscillator in those formalisms.

It is well known that the maximal symmetry group of the free Lagrangian

$$L = \frac{m}{2}\dot{\vec{x}}^2, \tag{1}$$

is the Schrödinger group

$$(SO(3,\mathbb{R}) \times SL(2,\mathbb{R})) \uplus A_6. \tag{2}$$

Its Lie algebra is generated by: $J^\alpha(SO(3,\mathbb{R}))$, $H, D, K(SL(2,\mathbb{R}))$ and $P^\beta, B^\gamma(A_6)$ $\alpha,\beta,\gamma = 1,2,3$; together with the following commutation rules

$$\begin{gathered} [H,D] = H, \quad [H,K] = 2D, \quad [D,K] = K, \\ [D,\vec{B}] = \frac{1}{2}\vec{B}, \quad [D,\vec{P}] = -\frac{1}{2}\vec{P}, \quad [K,\vec{P}] = \vec{B}, \quad [H,\vec{B}] = \vec{P}, \\ [J^\alpha, J^\beta] = \epsilon_{\alpha\beta\gamma}J^\gamma, \quad [J^\alpha, P^\beta] = \epsilon_{\alpha\beta\gamma}P^\gamma, \quad [J^\alpha, B^\beta] = \epsilon_{\alpha\beta\gamma}B^\gamma. \end{gathered} \tag{3}$$

Moreover, it turns out that the maximal symmetry group of the harmonic oscillator Lagrangian

$$L = \frac{m}{2}\dot{\vec{x}}^2 - \frac{m\omega^2}{2}\vec{x}^2, \tag{4}$$

is isomorphic to the Schrödinger group. This isomorphism is related to the different choice of the generator of the time translation (Hamiltonian)

$$H \to H + \omega^2 K. \tag{5}$$

Consequently, for the harmonic oscillator the symmetry algebra enjoys the following commutation relations

$$\begin{aligned}
&[H, D] = H - 2\omega^2 K, \quad [H, K] = 2D, \quad [D, K] = K,\\
&[D, \vec{B}] = \frac{1}{2}\vec{B}, \quad [D, \vec{P}] = -\frac{1}{2}\vec{P}, \quad [K, \vec{P}] = \vec{B},\\
&[H, \vec{P}] = -\omega^2\vec{B}, \quad [H, \vec{B}] = \vec{P},\\
&[J^\alpha, J^\beta] = \epsilon_{\alpha\beta\gamma}J^\gamma, \quad [J^\alpha, P^\beta] = \epsilon_{\alpha\beta\gamma}P^\gamma, \quad [J^\alpha, B^\beta] = \epsilon_{\alpha\beta\gamma}B^\gamma.
\end{aligned} \tag{6}$$

Although both algebra are isomorphic the change of the Hamiltonian alters the dynamics. Moreover, on the Hamiltonian level (and quantum level), the Schrödinger algebra is enlarged by the central charge M

$$[P^\alpha, B^\beta] = M\delta_{\alpha\beta}. \tag{7}$$

On the other hand, we observe the increasing interest in theories containing higher order time derivatives. Originally, these theories were proposed as a method for dealing with ultraviolet divergences [2] then they appeared in some physical models (see, e.g., [3]-[6]). The most basic theory involving higher time derivatives is the one defined by the following Lagrangian (generalization of the free motion)

$$\tilde{L} = \frac{1}{2}\left(\frac{d^n\vec{x}}{dt^n}\right)^2, \quad n = 1, 2, \ldots \tag{8}$$

The dynamical equation of motion, following from the action principle, takes the form

$$\frac{d^{2n}\vec{x}}{dt^{2n}} = 0. \tag{9}$$

It was shown (see [7]) that the maximal symmetry group of the Lagrangian (8) is a generalization of the Schrödinger group which is called the l-conformal Galilei group (see, e.g., [8]-[10]) with $l = n - \frac{1}{2}$; (for $n = 1$ we obtain the ordinary free dynamics and the Schrödinger symmetry). The structure of the Lie algebra of this group is as follows

$$\begin{aligned}
&[H, D] = H, \quad [H, K] = 2D, \quad [D, K] = K,\\
&[D, \vec{C}_p] = (p - \frac{2n-1}{2})\vec{C}_p, \quad [K, \vec{C}_p] = (p - 2n + 1)\vec{C}_{p+1},\\
&[H, \vec{C}_p] = p\vec{C}_{p-1},\\
&[J^{\alpha\beta}, J^{\gamma\delta}] = \delta^{\alpha\delta}J^{\gamma\beta} + \delta^{\alpha\gamma}J^{\beta\delta} + \delta^{\beta\gamma}J^{\delta\alpha} + \delta^{\beta\alpha}J^{\alpha\gamma},\\
&[J^{\alpha\beta}, C_p^\gamma] = \delta^{\alpha\gamma}C_p^\beta - \delta^{\beta\gamma}C_p^\alpha.
\end{aligned} \tag{10}$$

There exists also a generalization of the harmonic oscillator to the case of higher derivatives. Such model was proposed by Pais and Uhlenbeck in their classical paper [12] (and then quite extensively studied – see among others, [13]-[25]). This model is described by the following Lagrangian,

$$L=-\frac{1}{2}\vec{x}\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\vec{x}, \tag{11}$$

$n=1,2,\ldots$. which implies the following equation of motion

$$\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\vec{x}=0. \tag{12}$$

The general solution is of the form

$$\vec{x}(t)=\sum_{k=1}^{n}(\vec{\alpha}_k\cos\omega_k t+\vec{\beta}_k\sin\omega_k t), \tag{13}$$

where $\vec{\alpha}'s$ and $\vec{\beta}'s$ are some arbitrary constants.

In contrast to the ordinary harmonic oscillator the structure of the symmetry group of Lagrangian (11) depends on the values of $\omega's$ (see, [26]).

1. If the frequencies of oscillation are proportional to odd integers,

$$\omega_k=(2k-1)\omega, \quad \omega\neq 0, \quad k=1,\ldots,n \tag{14}$$

then the maximal group of Noether symmetries of the system (11) is the l-conformal Newton-Hooke group with $l=n-\frac{1}{2}$. The Lie algebra of this group is spanned by $H, D, K, J^{\alpha\beta}$ and C^α_p, $\alpha,\beta=1,2,3$, $p=0,1,\ldots,2n-1$, satisfying the following commutation rules

$$\begin{aligned}
&[H,D]=H-2\omega^2K, \quad [H,K]=2D, \quad [D,K]=K,\\
&[D,\vec{C}_p]=(p-\frac{2n-1}{2})\vec{C}_p, \quad [K,\vec{C}_p]=(p-2n+1)\vec{C}_{p+1},\\
&[H,\vec{C}_p]=p\vec{C}_{p-1}+(p-2n+1)\omega^2\vec{C}_{p+1},\\
&[J^{\alpha\beta},J^{\gamma\delta}]=\delta^{\alpha\delta}J^{\gamma\beta}+\delta^{\alpha\gamma}J^{\beta\delta}+\delta^{\beta\gamma}J^{\delta\alpha}+\delta^{\beta\alpha}J^{\alpha\gamma},\\
&[J^{\alpha\beta},C^\gamma_p]=\delta^{\alpha\gamma}C^\beta_p-\delta^{\beta\gamma}C^\alpha_p.
\end{aligned} \tag{15}$$

So, similarly to the ordinary case it is isomorphic to the symmetry group of the free higher derivative theory i.e., l-conformal Galilei group.

2. In the *generic* case the maximal symmetry group is simpler. Its Lie algebra consists of $H, J^{\alpha\beta}$ and $\vec{C}^{\pm}_k$, $k=1,\ldots,n$ and the action of $J^{\alpha\beta}$ remains unchanged, only the commutation rules between H and $\vec{C}'s$ are modified

$$\begin{aligned}
&[H,\vec{C}^+_k]=-\omega_k\vec{C}^-_k,\\
&[H,\vec{C}^-_k]=\omega_k\vec{C}^+_k.
\end{aligned} \tag{16}$$

Moreover, both symmetry algebras posses the central extension:

– in the case of odd frequencies

$$[C^\alpha_p,C^\beta_q]=(-1)^p p!q!\delta_{\alpha\beta}\delta_{2n-1,p+q}, \tag{17}$$

– and in the generic case

$$[C^{+\alpha}_k C^{-\beta}_j]=\frac{\omega_k}{\rho_k}\delta_{kj}\delta^{\alpha\beta}, \tag{18}$$

which turns out to be relevant on the Hamiltonian level.

2. Hamiltonian approaches to PU oscillator

Much attention has been paid to the Hamiltonian formulations of the PU oscillator. There exists a few approaches.

1 Decomposition into the sum of the alternating harmonic oscillators proposed by Pais and Uhlenbeck in their original paper,

2 Ostrogradski Hamiltonian for theories with higher time derivatives,

3 An alternative one, applicable in the case of odd frequencies, which exhibits the group structure of the model.

Consequently, the question arises concerning the relations between them as well as the realization of the symmetry on the Hamiltonian level.

2.1. The Pais-Uhlenbeck approach

The first Hamiltonian approach was proposed by Pais and Uhlenbeck in Ref. [12]. The Hamiltonian turns into the sum of the Hamiltonians of decoupled harmonic oscillators with alternating sign. To show this we introduce new variables

$$\vec{x}_k = \Pi_k \vec{x}, \quad k = 1, \ldots, n; \tag{19}$$

where Π_k is the projector operator:

$$\Pi_k = \sqrt{|\rho_k|} \prod_{\substack{i=1 \\ i \neq k}}^{n} \left(\frac{d^2}{dt^2} + \omega_i^2 \right), \tag{20}$$

and

$$\rho_k = \frac{1}{\prod\limits_{\substack{i=1 \\ i \neq k}}^{n} (\omega_i^2 - \omega_k^2)}, \quad k = 1, 2, \ldots, n. \tag{21}$$

Then one finds

$$\vec{x} = \sum_{k=1}^{n} (-1)^{k-1} \sqrt{|\rho_k|} \vec{x}_k, \tag{22}$$

and

$$L = -\frac{1}{2} \sum_{k=1}^{n} (-1)^{k-1} \vec{x}_k \left(\frac{d^2}{dt^2} + \omega_k^2 \right) \vec{x}_k = \frac{1}{2} \sum_{k=1}^{n} (-1)^{k-1} (\dot{\vec{x}}_k^2 - \omega_k^2 \vec{x}_k^2) + t.d. \tag{23}$$

The corresponding Hamiltonian reads

$$H = \frac{1}{2} \sum_{k=1}^{n} (-1)^{k-1} (\vec{p}_k^2 + \omega_k^2 \vec{x}_k^2), \tag{24}$$

while the canonical equations of motion are of the form

$$\dot{\vec{x}}_k = (-1)^{k-1} \vec{p}_k, \quad \dot{\vec{p}}_k = (-1)^k \omega_k^2 \vec{x}_k. \tag{25}$$

Taking into account the form of the general solution (13) we see that

$$\begin{aligned} \vec{x}_k &= \frac{(-1)^{k-1}}{\sqrt{|\rho_k|}} (\vec{\alpha}_k \cos(\omega_k t) + \vec{\beta}_k \sin(\omega_k t)), \\ \vec{p}_k &= \frac{\omega_k}{\sqrt{|\rho_k|}} (\vec{\beta}_k \cos(\omega_k t) - \vec{\alpha}_k \sin(\omega_k t)). \end{aligned} \tag{26}$$

Consequently we have the correspondence between the set of solutions of the Lagrange equation (12) and the set of solutions of the canonical equations (25). Thus, we can transform the action of the group symmetry from the Lagrangian level to the Hamiltonian one As we will see soon the generators, obtained in this way, form the algebra which is the central extension of the symmetry algebra on the Lagrangian level.

In the generic case it is easy to find the form of the symmetry generators. First, we find that the infinitesimal action of $\vec{\mu}_k \vec{C}_k^+$ and $\vec{\nu}_k \vec{C}_k^-$, $k = 1, \dots, n$, on the Lagrangian level, takes the form

$$\vec{x}'(t) = \vec{x}(t) + \sum_{k=1}^{n} (\vec{\mu}_k \cos \omega_k t + \vec{\nu}_k \sin \omega_k t). \tag{27}$$

Acting with Π_k and applying eq. (25) we find the infinitesimal action of $\vec{C}_k^{\pm}$ on the phase space; by virtue of

$$\delta(\cdot) = \epsilon\{\cdot, \text{Generator}\}, \tag{28}$$

we obtain the following generators:

$$\begin{aligned} \vec{C}_k^+ &= \frac{(-1)^{k-1}}{\sqrt{|\rho_k|}} \cos(\omega_k t) \vec{p}_k + \frac{\omega_k}{\sqrt{|\rho_k|}} \sin(\omega_k t) \vec{x}_k, \\ \vec{C}_k^- &= \frac{(-1)^{k-1}}{\sqrt{|\rho_k|}} \sin(\omega_k t) \vec{p}_k - \frac{\omega_k}{\sqrt{|\rho_k|}} \cos(\omega_k t) \vec{x}_k, \end{aligned} \tag{29}$$

which commute to the central charge according to eq. (18). Similarly, the angular momentum generators read

$$J^{\alpha\beta} = \sum_{k=1}^{n} (x_k^\alpha p_k^\beta - p_k^\alpha x_k^\beta). \tag{30}$$

Consequently, we obtain the desired centrally extended algebra (18).

In the odd case the symmetry group is reacher and, therefore, this case is much more interesting. Let us assume that the frequencies are odd , i.e., $\omega_k = (2k-1)\omega$, $k = 1, \dots, n$. In this case the main point is that the numbers ρ_k can be explicitly computed; the final result reads

$$\rho_k = \frac{(-1)^{k-1}(2k-1)}{(4\omega^2)^{n-1}(n-k)!(n+k-1)!}, \quad k = 1, \dots, n. \tag{31}$$

Next, the following Fourier expansion

$$\sin^p \omega t \cos^{2n-1-p} \omega t = \begin{cases} \sum\limits_{k=1}^{m} \gamma_{kp}^+ \cos(\omega_k t), & p \text{ - even;} \\ \sum\limits_{k=1}^{m} \gamma_{kp}^- \sin(\omega_k t), & p \text{ - odd;} \end{cases} \tag{32}$$

enables us to rewrite the infinitesimal action (27) in the equivalent form

$$\vec{x}'(t) = \vec{x}(t) + \frac{1}{\omega^p} \vec{\epsilon}_p \sin^p \omega t \cos^{2n-1-p} \omega t, \tag{33}$$

which gives suitable family of the generators $\vec{C}_p$, $p = 0, 1, 2, \dots, 2n-1$ on the Lagrangian level, i.e., satisfying commutation rules of the l-conformal Newton-Hooke algebra (8). The Fourier coefficients (32) enjoy some useful identities (see [1] for more details) which are crucial for further consideration, Here, we write out two of them

$$\gamma_{kp}^+ = (-1)^{k-1} \gamma_{k,2n-1-p}^-, \qquad \beta_{pk}^+ = (-1)^{k-1} \beta_{2n-1-p,k}^-, \tag{34}$$

$$2p!(2n-1-p)!\beta^{\pm}_{pk} = 2^{2n-1}(n-k)!(n+k-1)!\gamma^{\pm}_{kp}, \tag{35}$$

where β^+, β^- are the inverse matrices to γ^+, γ^-.

In order to find the action of $\vec{C}_p$ in this form of Hamiltonian formalism, we use eqs. (32) together with (19) and (25), which yields

$$\vec{x}'_k = \vec{x}_k + \frac{(-1)^{k-1}\vec{\epsilon}_p}{\omega^p\sqrt{|\rho_k|}} \begin{cases} \gamma^+_{kp}\cos(2k-1)\omega t, & p \text{ - even;} \\ \gamma^-_{kp}\sin(2k-1)\omega t, & p \text{ - odd;} \end{cases} \tag{36}$$

$$\vec{p}'_k = \vec{p}_k + \frac{(2k-1)\omega\vec{\epsilon}_p}{\omega^p\sqrt{|\rho_k|}} \begin{cases} -\gamma^+_{kp}\sin(2k-1)\omega t, & p \text{ - even;} \\ \gamma^-_{kp}\cos(2k-1)\omega t, & p \text{ - odd.} \end{cases} \tag{37}$$

Using eq. (28) we derive the explicit expression for the generators $\vec{C}_p$

$$\vec{C}_p = \sum_{k=1}^{n} \frac{\omega^{-p}\gamma^+_{kp}}{\sqrt{|\rho_k|}} \left((2k-1)\omega\sin((2k-1)\omega t)\vec{x}_k - (-1)^k\cos((2k-1)\omega t)\vec{p}_k\right) \tag{38}$$

for p even, and

$$\vec{C}_p = \sum_{k=1}^{n} \frac{\omega^{-p}\gamma^-_{kp}}{\sqrt{|\rho_k|}} \left((2k-1)\omega\cos((2k-1)\omega t)\vec{x}_k - (-1)^k\sin((2k-1)\omega t)\vec{p}_k\right), \tag{39}$$

for p odd. Eqs. (38) and (39) can be inverted to yield $\vec{x}_k$ and $\vec{p}_k$ in terms of the generators $\vec{C}_p$

$$\begin{aligned} \vec{p}_k = {} & (-1)^{k-1}\sqrt{|\rho_k|}\cos((2k-1)\omega t)\sum_{p=0}^{2n-1}{}'' \beta^+_{pk}\omega^p\vec{C}_p \\ & + (-1)^{k-1}\sin((2k-1)\omega t)\sum_{p=0}^{2n-1}{}' \beta^-_{pk}\omega^p\vec{C}_p, \end{aligned} \tag{40}$$

$$\begin{aligned} \vec{x}_k = {} & \frac{\sqrt{|\rho_k|}}{(2k-1)\omega}\sin((2k-1)\omega t)\sum_{p=0}^{2n-1}{}'' \beta^+_{pk}\omega^p\vec{C}_p \\ & - \frac{\sqrt{|\rho_k|}}{(2k-1)\omega}\cos((2k-1)\omega t)\sum_{p=0}^{2n-1}{}' \beta^-_{pk}\omega^p\vec{C}_p, \end{aligned} \tag{41}$$

one and two primes $'$,$''$ denote the sum over odd and even indices, respectively. Let us consider the dilatation generator. We find that the infinitesimal action of dilatation on coordinates is of the form ([26])

$$\vec{x}'(t) = \vec{x}(t) - \frac{\epsilon}{2\omega}\left((2n-1)\omega\cos(2\omega t)\vec{x}(t) - \sin(2\omega t)\dot{\vec{x}}(t)\right). \tag{42}$$

Substituting (22) and acting with Π_k we obtain the infinitesimal dilatation transformation on the phase space. According to eq. (28) infinitesimal canonical transformation is generated by

$$D = \frac{-1}{2\omega}\left(\omega A\cos(2\omega t) + B\sin(2\omega t)\right), \tag{43}$$

where

$$A = -\sum_{k=1}^{n}\left(\sqrt{\left|\frac{\rho_{k-1}}{\rho_k}\right|}(n-k+1)\vec{x}_{k-1} + \sqrt{\left|\frac{\rho_{k+1}}{\rho_k}\right|}(n+k)\vec{x}_{k+1}\right)\vec{p}_k + n\vec{x}_1\vec{p}_1,$$

$$B = -\sum_{k=1}^{n}(-1)^k\frac{n-k+1}{2k-3}\sqrt{\left|\frac{\rho_{k-1}}{\rho_k}\right|}(\vec{p}_k\vec{p}_{k-1} - (2k-1)(2k-3)\omega^2\vec{x}_k\vec{x}_{k-1}) + \frac{1}{2}n(\omega^2\vec{x}_1^2 - \vec{p}_1^2), \tag{44}$$

and, by definition, $\vec{x}_0 = \vec{p}_0 = 0$. The meaning of the components A and B will become more clear later (see (63)). Similar calculations can be done for the conformal generator K. Namely, the infinitesimal conformal transformation, on the Lagrangian level, reads [26]

$$\vec{x}'(t) = \vec{x}(t) - \frac{\epsilon}{2\omega^2}\Big((2n-1)\omega\sin(2\omega t)\vec{x}(t) + (\cos(2\omega t)-1)\dot{\vec{x}}(t)\Big). \tag{45}$$

and the induced generator K on phase space takes form

$$K = \frac{1}{2\omega^2}\left(B\cos(2\omega t) - \omega A\sin(2\omega t) + H\right). \tag{46}$$

Finally, the angular momentum takes the same form as in the generic case

$$J^{\alpha\beta} = \sum_{k=1}^{n}(x_k^\alpha p_k^\beta - p_k^\alpha x_k^\beta). \tag{47}$$

It remains to verify that the obtained generators, do yield integrals of motion and define the centrally extended l-conformal Newton-Hooke algebra. First, we compute the commutators of $\vec{C}$'s. The only nontrivial case is $[C_p^\alpha, C_q^\beta]$ with p even and q odd. By virtue of eqs. (31), (34) and (35) we have

$$\begin{aligned}[C_p^\alpha, C_q^\beta] &= \frac{\omega(2\omega)^{2(n-1)}\delta^{\alpha\beta}}{\omega^{p+q}}\sum_{k=1}^{n}(-1)^{k-1}(n-k)!(n+k-1)!\gamma_{kp}^+\gamma_{kq}^- \\ &= \frac{p!(2n-1-p)!\omega^{2n-1}\delta^{\alpha\beta}}{\omega^{p+q}}\sum_{k=1}^{n}\beta_{2n-1-p,k}^-\gamma_{kq}^- = \\ &= p!q!\delta_{\alpha\beta}\delta_{2n-1,p+q}.\end{aligned} \tag{48}$$

In order to find the remaining commutators we observe that

$$\begin{aligned}[][A,B] &= -2H, \\ [B,H] &= 2\omega^2 A, \\ [A,H] &= -2B.\end{aligned} \tag{49}$$

Finally, it is not hard to find the adjoint action of $H, D, K, J^{\alpha\beta}$ on $\vec{C}_p$ as well as the commutators involving angular momentum.

Having all the commutation rules one can verify that the obtained generators are constants of motion. This concludes the proof that for odd frequencies the symmetry generators form, on the Hamiltonian level, the centrally extended l-conformal Newton-Hooke algebra.

2.2. Ostrogradski approach

Now we use the Hamiltonian formalism proposed by Ostrogradski for higher derivatives theories [27]. To this end let us expand Lagrangian (11) in the sum of higher derivatives terms (here, $\vec{Q} = \vec{x}$)

$$L = -\frac{1}{2}\vec{Q}\prod_{k=1}^{n}\left(\frac{d^2}{dt^2} + \omega_k^2\right)\vec{Q} = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k-1}\sigma_k(\vec{Q}^{(k)})^2 + t.d., \tag{50}$$

where

$$\sigma_k = \sum_{i_1<\ldots<i_{n-k}} \omega_{i_1}^2 \cdots \omega_{i_{n-k}}^2, \quad k = 0, \ldots, n; \quad \sigma_n = 1. \tag{51}$$

Now, we introduce the Ostrogradski variables

$$\begin{aligned} \vec{Q}_k &= \vec{Q}^{(k-1)}, \\ \vec{P}_k &= \sum_{j=0}^{n-k}\left(-\frac{d}{dt}\right)^j \frac{\partial L}{\partial \vec{Q}^{(k+j)}} = (-1)^{k-1}\sum_{j=k}^{n}\sigma_j \vec{Q}^{(2j-k)}, \end{aligned} \tag{52}$$

for $k = 1, \ldots, n$. Then the Ostrogradski Hamiltonian takes the form

$$H = \frac{(-1)^{n-1}}{2}\vec{P}_n^2 + \sum_{k=2}^{n}\vec{P}_{k-1}\vec{Q}_k - \frac{1}{2}\sum_{k=1}^{n}(-1)^k\sigma_{k-1}\vec{Q}_k^2. \tag{53}$$

By virtue of eqs. (25) and (52), for $k = 1, \ldots, n$, we find that

$$\begin{aligned} \vec{Q}_k &= (-1)^{\frac{k-1}{2}}\sum_{j=1}^{n}\sqrt{|\rho_j|}(-1)^{j-1}\omega_j^{k-1}\vec{x}_j, \quad k - \text{odd}, \\ \vec{Q}_k &= (-1)^{\frac{k}{2}-1}\sum_{j=1}^{n}\sqrt{|\rho_j|}\omega_j^{k-2}\vec{p}_j, \quad k - \text{even}, \end{aligned} \tag{54}$$

and

$$\begin{aligned} \vec{P}_k &= (-1)^{\frac{k}{2}-1}\sum_{i=1}^{n}(-1)^{i-1}\sqrt{|\rho_i|}\left(\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k}\right)\vec{x}_i, \quad k - \text{ even}; \\ \vec{P}_k &= (-1)^{\frac{k-3}{2}}\sum_{i=1}^{n}\sqrt{|\rho_i|}\left(\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k-1}\right)\vec{p}_i, \quad k - \text{ odd}. \end{aligned} \tag{55}$$

define a canonical transformation. The inverse transformation is of the form

$$\begin{aligned} \vec{x}_i &= {\sum_{k=1}^{n}}'(-1)^{\frac{k-3}{2}}\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k-1}\sqrt{|\rho_i|}\vec{Q}_k + {\sum_{k=1}^{n}}''(-1)^{\frac{k}{2}}\sqrt{|\rho_i|}\omega_i^{k-2}\vec{P}_k, \\ \vec{p}_i &= {\sum_{k=1}^{n}}''(-1)^{\frac{k}{2}+i-1}\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k}\sqrt{|\rho_i|}\vec{Q}_k + {\sum_{k=1}^{n}}'(-1)^{\frac{k+1}{2}+i}\sqrt{|\rho_i|}\omega_i^{k-1}\vec{P}_k. \end{aligned} \tag{56}$$

Now, we can find the symmetry generators in terms of Ostrogradski variables. First, we can check that the Hamiltonian (24) is transformed into the Ostrogradski one. Similarly, we check that the angular momentum transforms under (56) into Ostrogradski angular momentum

$$J^{\alpha\beta} = \sum_{k=1}^{n}(Q_k^\alpha P_k^\beta - Q_k^\beta P_k^\alpha). \tag{57}$$

As far as the generators $\vec{C}'s$ are concerned we obtain the following expressions:
– in the case of generic frequencies

$$\begin{aligned} \vec{C}_k^+ &= \sum_{k=1}^{n} (\cos\omega_i t)^{(k-1)} \vec{P}_k - \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{j=k}^{n} \sigma_j (\cos\omega_i t)^{2j-k} \right) \vec{Q}_k, \\ \vec{C}_k^- &= \sum_{k=1}^{n} (\sin\omega_i t)^{(k-1)} \vec{P}_k - \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{j=k}^{n} \sigma_j (\sin\omega_i t)^{2j-k} \right) \vec{Q}_k, \end{aligned} \tag{58}$$

– and in the case of odd frequencies

$$\begin{aligned} \vec{C}_p = \frac{1}{\omega^p} \sum_{k=1}^{n} \Big(& \vec{P}_k (\sin^p \omega t \cos^{2n-1-p} \omega t)^{(k-1)} \\ & + (-1)^k \vec{Q}_k \sum_{j=k}^{n} \sigma_j (\sin^p \omega t \cos^{2n-1-p} \omega t)^{(2j-k)} \Big), \end{aligned} \tag{59}$$

which agrees with the definitions of the Ostrogradski canonical variables (52) and the action of $\vec{C}'s$ on Q (eqs. (27) and (33)). Similar reasoning can be done for the remaining two generators D and K in the odd case. Then, they become bilinear forms in the Ostrogradski variables; however the explicit form of the coefficients is less transparent.

2.3. Algebraic approach to the odd case

Since the free higher derivatives theory possesses also the $SL(2,R)$ symmetry it is instructive to consider, as in the case of the ordinary oscillator, the new Hamiltonian by adding the conformal generator to the Hamiltonian of the free higher derivatives theory. We show that the new Hamiltonian, obtained in this way, corresponds to the PU model with odd frequencies.

Denoting by $\vec{q}_m, \vec{\pi}_m,\ m = 0,\ldots,n-1$ the phase space coordinates of the free higher derivatives theory, the generators $\tilde{H}, \tilde{K}$ (at $t=0$ in the free case) take the form (see, [28, 29])

$$\begin{aligned} \tilde{H} &= \frac{(-1)^{n+1}}{2} \pi_{n-1}^2 - \sum_{m=1}^{n-1} \vec{q}_m \vec{\pi}_{m-1}, \\ \tilde{K} &= (-1)^{n+1} \frac{n^2}{2} \vec{q}_{n-1}^{\,2} + \sum_{m=0}^{n-2} (2n-1-m)(m+1) \vec{q}_m \vec{\pi}_{m+1}. \end{aligned} \tag{60}$$

Thus

$$\begin{aligned} H = \tilde{H} + \omega^2 \tilde{K} = & \frac{(-1)^{n+1}}{2} \pi_{n-1}^2 - \sum_{m=1}^{n-1} \vec{q}_m \vec{\pi}_{m-1} \\ & + (-1)^{n+1} \frac{n^2 \omega^2}{2} \vec{q}_{n-1}^{\,2} + \sum_{m=0}^{n-2} (2n-1-m)(m+1) \omega^2 \vec{q}_m \vec{\pi}_{m+1}. \end{aligned} \tag{61}$$

Now, we show that eq. (61) gives the PU Hamiltonian and we find the remaining generators in terms of q_m, π_m. To this end let us define

$$\begin{aligned}
\vec{x}_k &= (-1)^k \left({\sum_{m=0}^{n-1}}'' \frac{\omega^{-m}}{m!\sqrt{|\rho_k|}} \gamma^+_{km} \vec{q}_m + {\sum_{m=0}^{n-1}}' \frac{m!\omega^m \sqrt{|\rho_k|}}{(2k-1)\omega} \beta^+_{2n-1-m,k} \vec{\pi}_m \right), \\
\vec{p}_k &= (-1)^k \left({\sum_{m=0}^{n-1}}' \frac{\omega^{-m}(2k-1)\omega}{m!\sqrt{|\rho_k|}} \gamma^+_{k,2n-1-m} \vec{q}_m + {\sum_{m=0}^{n-1}}'' m!\omega^m \sqrt{|\rho_k|} \beta^+_{mk} \vec{\pi}_m \right),
\end{aligned} \tag{62}$$

for $k = 1, \ldots, n$. By direct calculations one can check that eqs. (62) define a canonical transformation and that the PU Hamiltonian (24) (with odd frequencies) is transformed into the one given by eq. (61). The remaining generators can be also transformed

$$\begin{aligned}
A &= -2\tilde{D}, \\
B &= -\tilde{H} + \omega^2 \tilde{K},
\end{aligned} \tag{63}$$

and, consequently, we obtain an interpretation of A and B. Using eqs. (63) one checks that H, D, K take the form

$$\begin{aligned}
H &= \tilde{H} + \omega^2 \tilde{K}, \\
D &= \tilde{D} \cos 2\omega t + \frac{1}{2\omega} (\tilde{H} - \omega^2 \tilde{K}) \sin 2\omega t, \\
K &= \frac{1}{2}(1 + \cos 2\omega t) \tilde{K} + \frac{1}{2\omega^2}(1 - \cos 2\omega t) \tilde{H} + \frac{\sin 2\omega t}{\omega} \tilde{D}.
\end{aligned} \tag{64}$$

The angular momentum reads

$$J^{\alpha\beta} = \sum_{m=0}^{n-1} (q^\alpha_m \pi^\beta_m - q^\beta_m \pi^\alpha_m), \tag{65}$$

i.e., takes the same form as the one for the free theory. Finally,

$$\vec{C}_p = \sum_{r=0}^{n-1} \left((-1)^{r-1} r! a_{pr}(t) \vec{\pi}_r + (2n-1-r)! a_{p,2n-1-r}(t) \vec{q}_r \right), \tag{66}$$

where

$$a_{pr}(t) = \omega^{r-p} \sum_{k=1}^{n} \beta^\pm_{rk} \gamma^\pm_{kp} \cos{(2k-1)\omega t}, \tag{67}$$

upper (lower) sign corresponds to p, r even (odd); and

$$a_{pr}(t) = \mp \omega^{r-p} \sum_{k=1}^{n} \beta^\mp_{rk} \gamma^\pm_{kp} \sin{(2k-1)\omega t}, \tag{68}$$

upper (lower) sign corresponds to p even and r odd (p odd and r even).
So, we expressed the PU symmetry generators in terms of the ones for free theory.

Summarizing, we focused on the Hamiltonian approaches to the PU model and its symmetries. We described the form of the symmetry generators in the original Pais and Uhlenbeck approach. We have shown that the resulting algebra is the central extension of the one obtained on the

Lagrangian level: in the case of odd frequencies it is the centrally extended l-conformal Newton-Hooke algebra. Next, we wrote out the canonical transformation leading from the decoupled oscillators approach to the Ostrogradski Hamiltonian formalism, which enables us to find the symmetry generators in the Ostrogradski approach. Let us note that both the above approaches, do not distinguish the odd frequencies. A deeper insight is attained by noting that for odd frequencies an alternative Hamiltonian formalism can be constructed: to the Hamiltonian of the free higher derivatives theory we added the conformal generator. As a result, we obtain the new Hamiltonian, which turns out to be related, by a canonical transformation, to the PU one with odd frequencies. Then the symmetry generators can be directly expressed in terms of the variables (integrals of motion) of the free higher derivatives theory.

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