Smooth Wilson Loops in $\mathcal{N} = 4$ Superspace and Yangian Symmetry

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work with J. Plefka, D. Müller and C. Vergu; to appear

Introduction and Overview

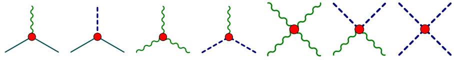
This talk is about Yangian Symmetry in the AdS/CFT duality:

- AdS/CFT duality and $\mathcal{N} = 4$ supersymmetric Yang–Mills theory,
- scattering amplitudes and duality to null polygonal Wilson loops,
- planar integrability and Yangian symmetry,
- Yangian symmetry of finite Wilson loops.

I. AdS/CFT Integrability

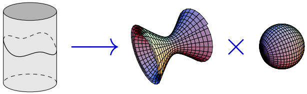
Planar AdS/CFT Correspondence

$\mathcal{N}=4$ Super Yang–Mills:



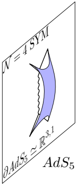
- 't Hooft coupling λ,
- rank of gauge group N_c,
- topological angle θ,
- superconformal $\widetilde{\mathrm{PSU}}(2,2|4)$

Strings on $AdS_5 \times S^5$:



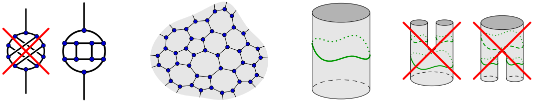
- curvature coupling λ
- string coupling: g_str,
- isometries $\widetilde{\mathrm{PSU}}(2,2|4)$.

AdS/CFT Duality:



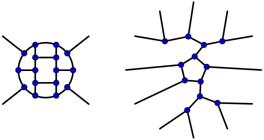
- holographic duality,
- $\mathbb{R}^{3,1} = \partial AdS_5$

Planar Limit:



- $N_\mathrm{c} \to \infty$, $g_\mathrm{str} \to 0$,
- no crossing lines, no splitting or joining.

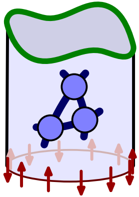
Integrability



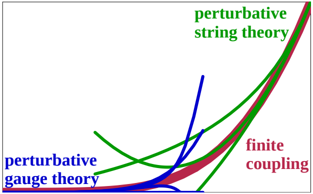
Standard QFT approach: **Feynman graphs**

- enormously difficult at higher loops . . .
- . . . but also lower loops and many legs.

Planar $\mathcal{N} = 4$ SYM is **integrable** see review collection [NB et al. 1012.3982]

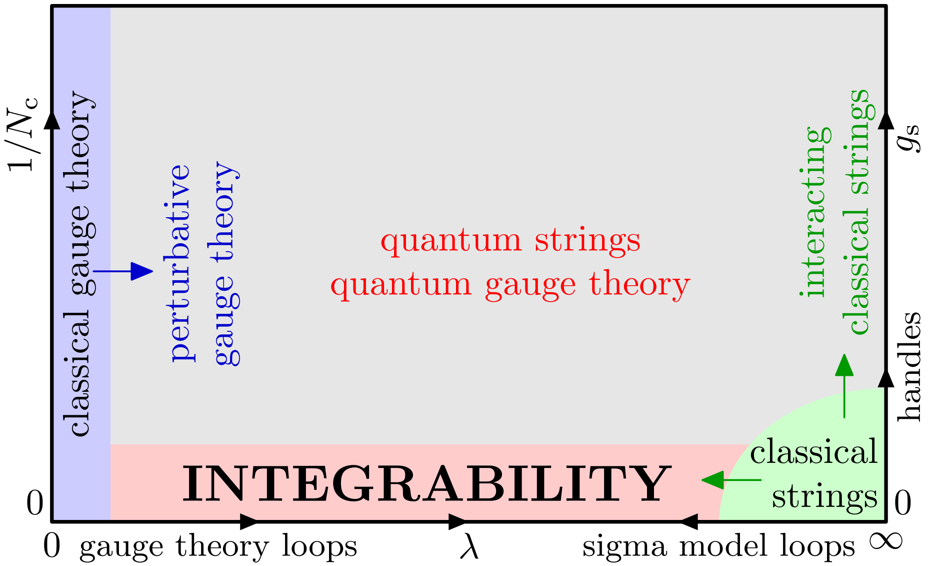


- integrability vastly simplifies calculations,
- can compute observables at finite coupling λ,



- spectral problem now largely understood,
- other observables under active investigation,
- symmetry: infinite-dimensional Yangian algebra $\mathrm{Y}(\mathrm{PSU}(2, 2|4))$.

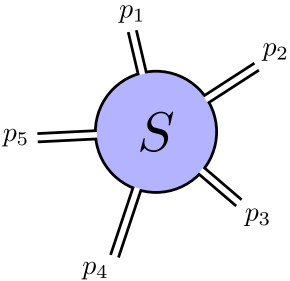
Charted Territory



II. Scattering and Wilson Loops

Planar Scattering in Gauge Theory

Consider colour-ordered **planar scattering** (ignore helicities/flavours)



Generic infrared factorisation for $S_n(\lambda, p)$:

$$S_n^{(0)}(p) \exp\left(D_{\text{cusp}}(\lambda) M_n^{(1)}(p) + R_n(\lambda, p)\right).$$

Required data:

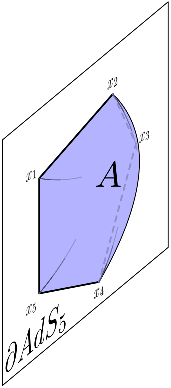
- tree level $S_n^{(0)}(p)$,
- one loop factor $M_n^{(1)}(p)$ (IR-divergent),
- cusp anomalous dimension $D_{\text{cusp}}(\lambda)$,
- remainder function $R_n(p, \lambda)$ (finite).

Intriguing observation for $n = 4, 5$ legs: $R_n = 0$! [Anastasiou, Bern, Dixon, Kosower] [Bern, Dixon, Smirnov]

- Computed/confirmed at 4 loops using unitarity. [Bern, Dixon, Smirnov] [Bern, Czakon, Dixon, Kosower, Smirnov]
- Exact result for scattering at finite λ! Why simple?
- Generalise to $n \geq 6$ legs! Compute exact R_n?!

Planar Scattering in String Theory

AdS/CFT provides a **string dual** for planar scattering. [Alday Maldacena]



Area of a minimal surface in AdS_5 ...

... ending on a null polygon on ∂AdS_5.

- Identify momenta with segments:

$$p_k = \Delta x_k = x_k - x_{k-1}$$

- on-shell particles $\to$ null segments:

$$p_k^2 = \Delta x_k^2 = 0$$

- momentum conservation $\to$ closure:

$$\sum_k p_k = \sum_k \Delta x_k = 0$$

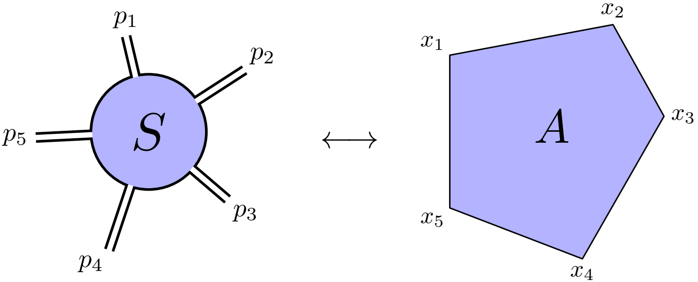
Note:

- Identification uses T-duality of $AdS_5 \times S^5$ strings.
- Functional form of exponent $M^{(1)}$ verified in string theory.

Null Polygonal Wilson Loop

AdS/CFT backwards: [Drummond Korchemsky Sokatchev] [Brandhuber Heslop Travaglini]

- Minimal surfaces correspond to Wilson loops in gauge theory.
- Amplitudes "T-dual" to null polygonal Wilson loops



Weak/weak perturbative duality. **Tested for:**

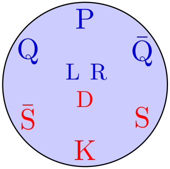
- all 1-loop amplitudes / Wilson loops [Drummond Korchemsky Sokatchev] [Brandhuber Heslop Travaglini]
- 2-loop 6-leg amplitude / hexagon Wilson loop [Bern, Dixon, Kosower Roiban, Spradlin Vergu, Volovich] [Drummond, Henn Korchemsky Sokatchev]

Dual Conformal and Yangian Symmetries

$\mathcal{N}=4$ SYM is superconformal: $\mathrm{PSU}(2,2|4)$ symmetry.

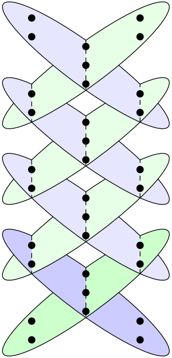
- Amplitudes are conformally invariant.*
- Wilson loops are conformally invariant.*

* IR/UV singularities break invariance (in a controllable fashion).



Two conformal symmetries: [Drummond, Henn Smirnov, Sokatchev][Drummond Korchemsky Sokatchev]

- different action on amplitudes and Wilson loops
- ordinary conformal symmetry $\updownarrow$ T-duality [Alday Maldacena]
  dual conformal symmetry
- together: generate infinite-dimensional . . . [NB, Ricci Tseytlin, Wolf]
  . . . Yangian algebra $\mathrm{Y}(\mathrm{PSU}(2,2|4))$. [Drummond Henn Plefka]



Dual conformal symmetry **explains simplicity**:

- No dual conformal cross ratios for $n=4,5$.
- Remainder function must be trivial: $R_n=0$.

Yangian Invariants

Planar scattering amplitudes appear to be Yangian invariant, but:

- Collinear singularities (at tree level) seen by free conformal action. Action can be deformed appropriately. [Bargheer, NB, Galleas Loebbert, McLoughlin]
- IR/UV divergences at loop level spoil conformal symmetry. Can we deform Yangian symmetry? [NB, Henn McLoughlin, Plefka]

Questions about Yangian symmetry for scattering amplitudes:

- Is it an exact symmetry of the S-matrix? Is it anomalous?
- Can symmetry be used to construct the S-matrix?
- Can the S-matrix be used to prove symmetry (integrability)?

Applications of Yangian symmetry for scattering amplitudes:

- Loop integrand is symmetric (up to boundary terms).
- Can construct amplitudes from invariant building blocks.

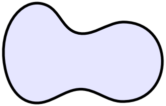
Caveat: How useful are integrands if integral is infinite?

III. Finite Wilson Loops

Maldacena–Wilson Loops

Can define finite Wilson loops in $\mathcal{N} = 4$ SYM: Couple scalars

$$W = \mathrm{P} \exp \int \big(A_\mu dx^\mu + \Phi_m q^m d\tau \big).$$



Maldacena–Wilson loops where $|dx| = |q| d\tau$:

- path is non-null in 4D;
- path is null in 10D (4 spacetime + 6 internal);
- locally supersymmetric object;
- no perimeter divergence (perimeter is null).

Yangian symmetry: [Müller, Münkler Plefka, Pollok, Zarembo]

- finite observable: could make meaningful statements;
- requires superconformal transformations; best done in superspace;
- Yangian symmetry demonstrated at leading order in θ's;
- symmetry up to subtleties regarding boundary terms.

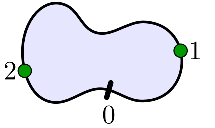
Conformal and Yangian Symmetry

Conformal action (level-zero Yangian) by path deformation.
Action equivalent to Wilson line with single insertion

$$\mathrm{J}^a W = \int d\tau\, W[1,\tau]\, \mathrm{J}^a A(\tau)\, W[\tau,0].$$

Level-one Yangian action: bi-local insertion follows coproduct

$$\widehat{\mathrm{J}}^a W = f^a_{bc} \iint\limits_{\tau_2>\tau_1} d\tau_1\, d\tau_2\, W[1,\tau_2]\, \mathrm{J}^b A(\tau_2)\, W[\tau_2,\tau_1]\, \mathrm{J}^c A(\tau_1)\, W[\tau_1,0].$$



Yangian is symmetry if (higher levels follow)

$$\big\langle \mathrm{J}^a\, \mathrm{Tr}\, W \big\rangle = 0, \qquad \big\langle \widehat{\mathrm{J}}^a\, \mathrm{Tr}\, W \big\rangle = 0.$$

Important issue: Yangian normally does not respect cyclicity.

Open Questions

[NB, Müller
Plefka, Vergu
(in progress)]

Questions:

- How about all orders in θ, full superspace?
- How about boundary terms?

Difficulties:

- How to compute perturbative corrections?
- How to define finite Wilson loop (precisely)?
- How to define Yangian action (precisely)?
- How to define superconformal action (precisely)?
- How to deal with boundary terms?
- Is the action consistent with the constraints?
- Does the Yangian algebra close (and how)?

IV. Wilson Loops in Superspace

$\mathcal{N} = 4$ Superspace

Extend spacetime (x^μ) to superspace $(x^{\beta\dot\alpha}, \theta^{\beta a}, \bar\theta_b{}^{\dot\alpha})$, $a = 1, \ldots, \mathcal{N}$.

Gauge theory on superspace:

- Extend gauge potential (A_μ) to superspace $(A_{\dot\alpha\beta}, A_{a\beta}, A_{\dot\alpha}{}^b)$.
- way too many component fields for super Yang–Mills theory.
- impose constraint: some components of F fixed (Φ scalar field)

$$F_{a\beta,\dot\gamma}{}^d = 0, \qquad F_{a\beta,c\delta} \sim \varepsilon_{\beta\delta}\varepsilon_{acef}\Phi^{ef}, \qquad F_{\dot\alpha}{}^b{}_{\dot\gamma}{}^d \sim \varepsilon_{\dot\alpha\dot\gamma}\Phi^{bd}.$$

- Remaining components of F contain fermionic fields Ψ, $\bar\Psi$:

$$F_{a\beta,\dot\gamma\delta} \sim \varepsilon_{\beta\delta}\bar\Psi_{a\dot\gamma}, \qquad F_{\dot\alpha}{}^b{}_{\dot\gamma\delta} \sim \varepsilon_{\dot\alpha\dot\gamma}\Psi_\delta{}^b.$$

Quantised field theory:

- Can derive gauge propagator $\langle A_1 A_2\rangle$ from equations of motion;
- other propagators $\langle A_1 \Phi_2\rangle$, etc., follow from $\Phi \in F \simeq dA$.

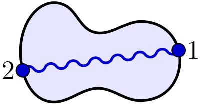
Wilson Line in Superspace

Wilson line with scalar coupling in superspace:

$$W \simeq \mathrm{P}\exp\int \big(A\cdot(dx + id\theta\bar\theta + i\theta d\bar\theta) + A\cdot d\theta + A\cdot d\bar\theta + \Phi\cdot q\,d\tau\big).$$

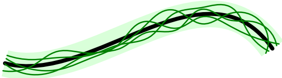
Wilson loop expectation value at $\mathcal{O}(g^2)$:

$$\langle \mathrm{Tr}\,W\rangle \sim \int\int \langle (A+\Phi)_1\,(A+\Phi)_2\rangle.$$



Only above gauge propagator needed.

Kappa-symmetry:

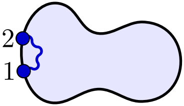


- 8 local fermionic symmetries;
- extend path reparametrisation to $1|8$ superalgebra (fat path);
- one bosonic constraint $|dx| = |q|d\tau$;
- UV finiteness expected.

Finiteness

Expand mixed chiral gauge propagator at $(\tau_1, \tau_2) = (\tau, \tau + \epsilon)$

$$\langle A_1^+ A_2^- \rangle \sim d\tau\, d\epsilon \left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \frac{2ip \cdot (\dot{\theta}\dot{\bar{\theta}})}{p^2} + \ldots \right].$$



Result depends on superspace covariant direction $p = \dot{x} + i\dot{\theta}\bar{\theta} - i\theta\dot{\bar{\theta}}$.
Sub-leading terms cancel between opposite chiralities.
Chiral and anti-chiral propagators are non-singular.

Scalar propagator contributes similar singularity

$$\langle (\Phi \cdot q)_1 (\Phi \cdot q)_2 \rangle \sim \frac{q^2}{p^2} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \frac{q \cdot \dot{q}}{q^2} - \frac{1}{\epsilon} \frac{p \cdot \dot{p}}{p^2} + \ldots \right].$$

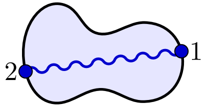
Gauge-scalar propagators are non-singular.

Singularities cancel for $p^2 + q^2 = 0$ (10D null) or $p^2 + q^2 = \text{const.}$.

Conformal Symmetry of Wilson Loops

Wilson loop expectation value at order $\mathcal{O}(g^2)$:

$$\langle \operatorname{Tr} W \rangle \sim \int\int \langle A_1\, A_2 \rangle .$$



Conformal action on propagator is non-trivial

$$\mathrm{J}\langle A_1\, A_2\rangle = \langle \mathrm{J}A_1\, A_2\rangle + \langle A_1\, \mathrm{J}A_2\rangle = d_1 f_{12} + d_2 f_{21}.$$

Total derivative terms:

- represent gauge transformations of gauge potentials;
- zero for symmetries respecting gauge fixing (e.g. Poincaré);
- non-zero for others (e.g. special conformal);
- remaining function f_{12} is finite for $1 \to 2$;
- cancel on closed Wilson loop.

V. Yangian Symmetry

Yangian Symmetry of Wilson Loops

Level-one momentum (dual conformal) $\widehat{\mathrm{P}}$ easiest:

$$\widehat{\mathrm{P}} \simeq \mathrm{P} \wedge \mathrm{D} + \mathrm{P} \wedge \mathrm{L} + \mathrm{Q} \wedge \bar{\mathrm{Q}}.$$

Action on propagator:

$$\widehat{\mathrm{J}}^a \big\langle A_1\, A_2 \big\rangle = f^a_{bc} \big\langle \mathrm{J}^b A_1\, \mathrm{J}^c A_2 \big\rangle = d_1 f_{12} - d_2 f_{21}.$$

Almost zero up to derivative terms:

- no gauge fixing that respects all of $\mathrm{P}, \mathrm{D}, \mathrm{L}, \mathrm{Q}, \bar{\mathrm{Q}}$;
- bulk-boundary terms do not cancel;
- not clear how to compensate.

Fundamental problems:

- definition not cyclic,
- definition not gauge invariant.

Improved Symmetry Action

$\mathrm{J}A$ is not gauge covariant but can rewrite conformal action

$$\mathrm{J}A = \mathrm{J}X \cdot F + D(\mathrm{J}X \cdot A).$$

Two resulting terms:

- Field strength F is covariant;
- $D(\mathrm{J}X \cdot A)$ is a boundary term.

Drop derivative term, obtain gauge-covariant conformal action:

$$\mathrm{J}^a W = \int d\tau \, W[1,\tau] \, \mathrm{J}^a X(\tau) \cdot F(\tau) \, W[\tau, 0].$$

In fact, standard superconformal action on fields.

Closure of conformal symmetry modulo gauge transformations

$$[\mathrm{J}^a, \mathrm{J}^b] = f^{ab}_c \mathrm{J}^c + \mathrm{G}[G^{ab}], \qquad G^{ab} = \mathrm{J}^a X^A \mathrm{J}^b X^B F_{AB}.$$

Improved Yangian Action

Improved level-one Yangian action:

$$\widehat{\mathrm{J}}^a W = f^a_{bc} \iint\limits_{\tau_2 > \tau_1} d\tau_1\, d\tau_2\, W[1,\tau_2]\, \mathrm{J}^b X_2 F_2\, W[\tau_2,\tau_1]\, \mathrm{J}^c X_1 F_1\, W[\tau_1,0].$$

Resolves both fundamental problems!

Curious identity needed for cyclicity: $f^c_{ab}[\mathrm{J}^a, \mathrm{J}^b] = 0$

$$f^c_{ab} f^{ab}_d = 0,$$
$$f^c_{ab} G^{ab} \sim f^c_{ab}\, \mathrm{J}^a X^A\, \mathrm{J}^b X^B\, F_{AB} = 0.$$

Is satisfied for $\mathrm{PSU}(2,2|4)$ and for fields of $\mathcal{N} = 4$ gauge theory!

New problem: Yangian algebra does not close right away.
Residual terms can be written as conformal transformations.
Sufficient for invariance of Wilson loops.

Yangian Invariance of Propagator

Level-one generators almost annihilate propagator. **Show:**

$$\begin{aligned}\widehat{\mathrm{J}}^c\big\langle dA_1A_2\big\rangle &= f^c_{ab}\big\langle \mathrm{J}^a F_1\,\mathrm{J}^b A_2\big\rangle \\ &= f^c_{ab}\mathrm{J}^a\big\langle F_1\,\mathrm{J}^b A_2\big\rangle - f^c_{ab}\big\langle F_1\,\mathrm{J}^a\mathrm{J}^b A_2\big\rangle \\ &= f^c_{ab}\mathrm{J}^a\big\langle F_1\,(\mathrm{J}^b X\cdot F)_2\big\rangle - \tfrac{1}{2}f^c_{ab}\big\langle F_1\,[\mathrm{J}^a,\mathrm{J}^b]A_2\big\rangle = 0\,.\end{aligned}$$

Therefore action on gauge propagator yields double total derivative:

$$\widehat{\mathrm{J}}^c\big\langle A_1A_2\big\rangle = d_1d_2R^c_{12}.$$

For level-one momentum $\widehat{\mathrm{P}}$ and mixed chiral gauge fields:

$$\widehat{\mathrm{P}}_{\dot\alpha\beta}\big\langle A^+_1\,A^-_2\big\rangle \sim d_1d_2\frac{1}{(x_{12}-i\theta_{12}\bar\theta_{12})^{\beta\dot\alpha}}\,.$$

Scalar fields as components of field strength $\varPhi\in F$:

$$\widehat{\mathrm{J}}^c\big\langle \varPhi_1A_2\big\rangle = \widehat{\mathrm{J}}^c\big\langle \varPhi_1\varPhi_2\big\rangle = 0.$$

Yangian Invariance of Wilson Loop

Action on Wilson line leaves a local contribution:

$$\int\int_{1<2} d_1 d_2 R^c_{12} = \int_2 (d_2 R^c_{12})|_{1=2} - \int_2 d_2 R^c_{02} = \int_2 (d_2 R^c_{12})|_{1=2}.$$

Need to adjust local action of Yangian

$$\widehat{\mathrm{J}}^c W = \widehat{\mathrm{J}}^c_{\text{bi-local}} W + \int d\tau\, W[1,\tau]\, \widehat{\mathrm{J}}^c A(\tau)\, W[\tau,0],$$
$$\widehat{\mathrm{J}}^c A_1 = -(d_2 R^c_{21})|_{2=1}.$$

Wilson loop expectation value at $\mathcal{O}(g^2)$ is Yangian invariant!

Regularisation and renormalisation:

- Local term divergent $(d_2 R^c_{21})|_{2=1} \sim \epsilon^{-2}$ for cut-off ϵ.
- Need to renormalise local and boundary part of Yangian action.

VI. Conclusions

Conclusions

Reviewed:

- AdS/CFT integrability
- Scattering and Wilson loops
- Yangian symmetry

New Results:

- Definition of Maldacena–Wilson loops in full superspace.
- Finiteness of smooth loops at $\mathcal{O}(g^2)$.
- How to act with Yangian on Wilson loops.
- Yangian symmetry of Wilson loop expectation value at $\mathcal{O}(g^2)$.