Deformation quantization of field theory models

Arthemy Kiselev

Johann Bernoulli Institute for Mathematics and Computer Science
Nijenborgh 9, 9747 AG Groningen
The Netherlands
A.V.Kiselev@rug.nl

Starting from a Poisson bi-vector $\mathcal{P}$ on a given finite-dimensional Poisson manifold $(N, \mathcal{P})$, Kontsevich's graph summation formula [1] yields the explicit deformation $\times \mapsto \star_\hbar$ of commutative product $\times$ in the algebra $C^\infty(N)$ of smooth functions. The new operation $\star_\hbar$ on the space $C^\infty(N)[[\hbar]]$ of power series is specified by the Poisson structure on N: namely, $f \star_\hbar g = f \times g + \text{const} \cdot \hbar \, \{f, g\}_{\mathcal{P}} + o(\hbar)$ such that all the bi-differential terms at higher powers of the formal parameter $\hbar$ are completely determined by the Poisson bracket $\{\,,\,\}_{\mathcal{P}}$ in the leading deformation term. (In the context of fields and strings, the constant is set to $\boldsymbol{i}/2$ so that the parameter $\hbar$ is the usual Planck constant.) The deformed product $\star_\hbar$ is no longer commutative if $\mathcal{P} \neq 0$ but it always stays associative, $\bigl(f \star_\hbar g\bigr) \star_\hbar h \doteq f \star_\hbar \bigl(g \star_\hbar h\bigr)$ all f, g, $h \in C^\infty(N)[[\hbar]]$, by virtue of bi-vector's property $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ to be Poisson.

In this talk we extend the Poisson set-up and graph summation technique in the deformation $\times \mapsto \star_\hbar$ to the jet-space (super)geometry of N-valued fields $\phi \in \Gamma(\pi)$ over a base manifold M in their bundle π and, secondly, of variational Poisson bi-vectors $\boldsymbol{\mathcal{P}}$ that encode the Poisson brackets $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ on the space of local functionals taking $\Gamma(\pi) \to \mathbf{k}$. We explain why an extension of Kontsevich's graph technique [1] is possible and how it is done by using the geometry of iterated variations [2]. For instance, we derive the variational analogue of associative Moyal's $\star$-product, $f \star g = (f)\, \exp\bigl(\overleftarrow{\partial_i} \cdot \hbar \mathcal{P}^{ij} \cdot \overrightarrow{\partial_j}\bigr)\,(g)$, in the case when the coefficients $\mathcal{P}^{ij}$ of bi-vector $\mathcal{P}$ are constant (hence the identity $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ holds trivially). By using several well-known examples of variational Poisson bi-vectors $\boldsymbol{\mathcal{P}}$, we illustrate the construction of each bi-differential term in $\star_\hbar$ in the general case, i.e., for Hamiltonian total differential operators with coefficients depending on the fields ϕ and their derivatives; we then verify that the noncommutative quantized product $\star_\hbar$ is associative by virtue of $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] = 0$. We conclude that the existing instruments for calculation of variational Poisson structures do in fact specify points in the moduli spaces of deformation quantizations for field theory models.

This research was supported in part by JBI RUG project 103511 (Groningen, The Netherlands); a part of this research was done while the author was visiting at the IHÉS (Bures-sur-Yvette, France).

References.

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