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Problém minimálního pokrytí pro konvexní množiny

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Contents

Notation

Throughout this work the following notation concerning points and sets is used. The d-dimensional euclidean space is denoted by $\mathbb{R}^d$. Its dimension d will very often be $= 2$. Points of $\mathbb{R}^d$ are always denoted by x, y, z, a, b and other lowercase Latin letters and employ lower indices. The coordinates of a vector are denoted by the same letter, with an upper index $1, ..., d$. Thus $x = (x^1, ..., x^d)$. Vector x^T means the transposition of the vector x, i.e. x^T is a column vector. Integers are denoted by i, j, k, l, m, n, s with or without lower indices. For real numbers $\lambda, \delta, \varepsilon, \rho, \nu, \alpha$ and other lowercase Greek letters are used. The letters f, g, L, A are used to denote functions, mappings. The origin is denoted by o. The scalar product of vectors x, y is

$$(x, y) = \sum_{i=1}^{d} x^i y^i.$$

The Gramm matrix of the set of vectors $(x_1, ..., x_d)$ is

$$(x_i, x_j)_{i,j=1}^d = \begin{pmatrix} (x_1, x_1) & (x_1, x_2) & ... & (x_1, x_d) \\ (x_2, x_1) & (x_2, x_2) & ... & (x_2, x_d) \\ . & & & \\ . & & & \\ . & & & \\ (x_d, x_1) & (x_d, x_2) & ... & (x_d, x_d) \end{pmatrix}.$$

The length of a vector x is

$$|x| = \sqrt{(x, x)} = \sqrt{(x^1)^2 + ... + (x^d)^2}.$$

Arbitrary sets in $\mathbb{R}^d$ are denoted by capital Latin and Greek letters. The set of points for which some given property $P(x)$ holds, is denoted by $\{x | P(x)\}$. $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0$ denotes the set of non-negative integers, i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Furthermore, the following notations are used

$$x + M = \{x + y \bigm| y \in M\},$$

$$\alpha M = \{\alpha x \bigm| x \in M\},$$

$$M_1 + M_2 = \{x + y \bigm| x \in M_1, y \in M_2\},$$

$$M_1 \oplus M_2 = \{z \bigm| (\exists_1 x \in M_1)(\exists_1 y \in M_2)(z = x + y)\}$$

$$M_1 - M_2 = \{x - y \bigm| x \in M_1, y \in M_2\},$$

$$M_1 \setminus M_2 = \{x \in M_1 \bigm| x \notin M_2\}.$$

In general, the set $M + M$ and $2M$ are not identical. The set $M - M$ is also called the *difference set* of M. The symbols $\cup, \subset, \supset$ are used to denote the set-theoretical union and inclusion relations, respectively. Further, $\emptyset$ denotes an empty set. The interior of a set M is denoted by M°, the closure of M by $\overline{M}$ and the boundary of M by ∂M. A set H is called *o-symmetric* if $H = -H$, i.e. $(\forall x \in H)(-x \in H)$.

Chapter 1

Introduction

In this work we deal with mathematical models of quasicrystals, i.e. non-cystallographic solids with long range aperiodic order. Therefore it is useful to briefly introduce mathematical models for quasicrystals at first, indicate connection of these models with the contents of the work and set the main targets of the work.

A suitable mathematical representation of quasicrystals are the so-called Meyer sets. They were first introduced by Meyer [5] as sets $\Sigma \subset \mathbb{R}^d$ which fulfil the Delone property (Definition 1.1) and the property of almost lattices,

$$\Sigma - \Sigma \subset \Sigma + F \tag{1.1}$$

for a finite set $F \subset \mathbb{R}^d$. The former property shows that there is only a finite number of local configurations in the considered model of quasicrystal. The knowledge of local configurations is essential for example for the study of interatomic bonds in quasicrystalline materials. Consequently, there arises a question how to define 'neighbours' in a point set that is not a lattice. A natural definition is provided by the notion of Voronoi tiling (Definition 1.5). A Delone set $\Sigma \subset \mathbb{R}^d$ uniquely determines a perfect Voronoi tiling of the space $\mathbb{R}^d$ consisting of polytops which do not overlap and fill the entire space. If Σ is a Meyer set, it has finitely many local configurations, which implies also only finitely many types of Voronoi tiles. One of the conditions which determine the number of different Voronoi tiles is the cardinality of the set F in the property(1.1).

We are interested in sets $\Sigma = \Sigma(\Omega)$ obtained by cut-and-project method. Their definition requires a bounded set Ω called the acceptance window (Definition 1.4). The choice of Ω strongly influences the properties of the cut-and-project set $\Sigma(\Omega)$. In this work we treat especially convex compact sets. Moody [6] has shown that cut-and-project sets are Meyer sets. The main aim of this work is to explore the cardinality of the set F from the Meyer property (1.1) for different acceptance windows Ω. The problem can be transformed into investigation of the function f defined by

$$f(\Omega) = \text{the minimal number of translated copies of } \Omega^\circ \text{ needed for covering of } \overline{\Omega - \Omega}. \tag{1.2}$$

The main result of this work is that the function f is bounded on the space of convex compact sets Ω (Theorem 3.2 in Section 3.4). We further show that convexity is an essential assumption, since we can construct a sequence of general non-convex compact sets $(\Omega_n)_{n\in\mathbb{N}}$ such that $f(\Omega_n)$ tends to infinity with growing n (Chapter 5). In other words, there exists a universal upper bound on the cardinality of F for all convex acceptance windows Ω, but this is not the case of non-convex Ω.

In Sections 4.2 and 4.3 we provide estimations of the upper bound on the cardinality of the set F for convex and convex centrally symmetric sets Ω in $\mathbb{R}^2$. We further determine the value of the function f on sets $\Omega \subset \mathbb{R}^2$ of special types, namely regular polygons (Section 4.1). As a concrete example how to apply the obtained upper bounds we estimate the number of Voronoi cells for the acceptance window Ω being a rhombus in $\mathbb{R}^2$ (Chapter 6).

1.1 Cut-and-project sets as models for quasicrystals

Let us introduce mathematical definitions which describe the previous mentioned models correctly. For more details about this topic see [4] and [8]. In general, a mathematical object representing atomic positions in a material is a point set $\Sigma \subset \mathbb{R}^d$ constrained by two simple physically reasonable properties - discreteness and homogenity. The common basis for all the constructions is that Σ should satisfy the so-called *Delone property.*

Definition 1.1. *A set $\Sigma \subset \mathbb{R}^d$ is called Delone, if it satisfies two conditions*

1. *Σ is uniformely discrete, i.e. there exists $r > 0$, such that $|x-y| > r$ for any $x, y \in \Sigma$, $x \neq y$. This condition assures that Σ has no accumulation points. The maximal r with this property is called minimal distance in Σ*

$$r_\Sigma = \sup\{r > 0 \bigm| |x-y| \geq r, x, y \in \Sigma, x \neq y\}.$$

2. *Σ is relatively dense, i.e. there exists $R > 0$, such that $B(x, R) \cap \Sigma \neq \emptyset$, for any $x \in \mathbb{R}^d$. This condition tells that in Σ there are no unbounded gaps. The union of balls with radius R centered at points of Σ cover the space $\mathbb{R}^d$. The minimal R with this property is called the covering radius of Σ*

$$R_\Sigma = \inf\{R > 0 \bigm| B(x, R) \cap \Sigma \neq \emptyset, x \in \mathbb{R}^d\}.$$

A simple example of a Delone set in $\mathbb{R}^d$ is a *lattice.*

Definition 1.2. *Denote an arbitrary base of $\mathbb{R}^d$ by $(e_1, e_2, ..., e_d)$. A lattice Σ in $\mathbb{R}^d$ is a set $\Sigma = \{\sum_{i=1}^{d} a_i e_i \mid \forall i \in \{1, 2, ..., d\},\ a_i \in \mathbb{Z}\}$.*

Lattices are characterized by the property

$$\Sigma - \Sigma \subset \Sigma,$$

which corresponds to the fact that they have translational symmetries and therefore serve for models of periodic crystals.

For models of quasicrystalline structures Meyer [5] proposed a concept that generalizes lattices. He calls a 'quasicrystal' a Delone set Σ which satisfies the property of almost-lattices

$$\Sigma - \Sigma \subset \Sigma + F$$

for some finite set F.

A rich class of *Meyer sets* can be obtained by the so-called *cut-and-project method* [2]. Roughly speaking, one projects points of a higherdimensional lattice to a lowerdimensional subspace and then chooses projections which have their projection to the complementary subspace in a given bounded region, one obtain a *cut-and-project set.*

Let us step up to a correct definition. Let V_1, V_2 be subspaces of $\mathbb{R}^{c+d}$, such that $V_1 \oplus V_2 = \mathbb{R}^{c+d}$. For any $x \in \mathbb{R}^{c+d}$ there exists a unique decomposition $x = x_1 + x_2$, where $x_1 \in V_1, x_2 \in V_2$. Therefore projections $\pi_1 : \mathbb{R}^{c+d} \to V_1$ such that $\pi_1(x) := x_1$ and $\pi_2 : \mathbb{R}^{c+d} \to V_2$ such that $\pi_2(x) := x_2$ are well defined.

Definition 1.3. *The cut-and-project scheme is a triplet (V_1, V_2, L), where V_1, V_2 are subspaces of $\mathbb{R}^{c+d}$ satisfying $V_1 \oplus V_2 = \mathbb{R}^{c+d}$ and L is a lattice in $\mathbb{R}^{c+d}$, which fulfils*

1. *restriction π_1 to L is an injection,*

2. *$\pi_2(L)$ is dense in V_2.*

Definition 1.4. *Let (V_1, V_2, L) be a cut-and-project scheme and let $\Omega \subset V_2$ such that Ω is bounded, $\Omega^\circ \neq \emptyset$ and $\overline{\Omega^\circ} = \overline{\Omega}$. Then the set*

$$\Sigma(\Omega) := \{\pi_1(x) \mid x \in L, \pi_2(x) \in \Omega\}$$

is called a cut-and-project set with the acceptance window Ω.

If we take into account that our considerations concern physical quasicrystals, we call $\Sigma(\Omega)$ a *d-dimensional quasicrystal* and Ω the *acceptance window of the quasicrystal.*

Let us mention some properties of cut-and-project sets

1. $\Sigma(\Omega)$ is a Delone set.

2. $\Sigma(\Omega)$ is locally finite, i.e. for every $\rho > 0$ there exist only finitely many ρ-neighbourhoods

$$N_\rho(x) = \{y - x \bigm| y \in \Sigma(\Omega), |y - x| < \rho\}.$$

3. $\Sigma(\Omega)$ is a Meyer set, i.e. it is Delone and there exists a finite set F such that

$$\Sigma(\Omega) - \Sigma(\Omega) \subset \Sigma(\Omega) + F.$$

1.2 Voronoi and Delone tiling

In the previous section we got acquainted with mathematical models of quasicrystals. It was stated, for some purposes one needs to describe the local configurations of points in these models, in particular, one needs to define neighbours in a general Delone set. A natural definition is provided by the following description of a *Voronoi cell.*

Every Delone set in $\mathbb{R}^d$ determines two different perfect tilings of $\mathbb{R}^d$, the so-called *Voronoi and Delone tiling.*

Definition 1.5. *Let $x \in \Sigma$. The Voronoi cell of x is defined by*

$$V(x) = \{y \in \mathbb{R}^d \bigm| |x - y| \leq |z - y| \text{ for all } z \in \Sigma\}.$$

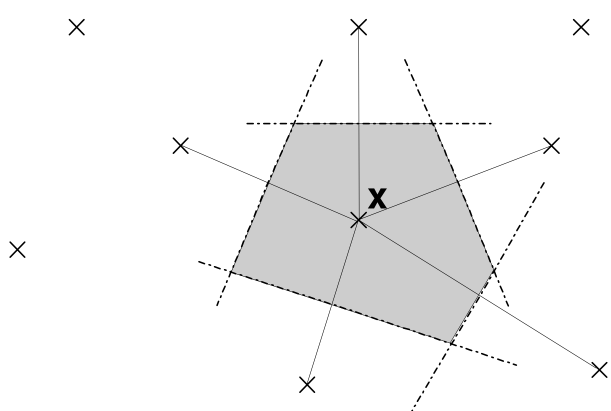


Figure 1.1: Illustration of a Voronoi tile in $\mathbb{R}^2$.

Voronoi cells are convex polytops in $\mathbb{R}^d$ that cover the entire $\mathbb{R}^d$ without thick overlap and without gaps. In such a way they form a perfect tiling of $\mathbb{R}^d$, called the Voronoi tiling of Σ. Now, neighbours can be described as points $x, y \in \Sigma$ such that Voronoi cells $V(x), V(y)$ share a face

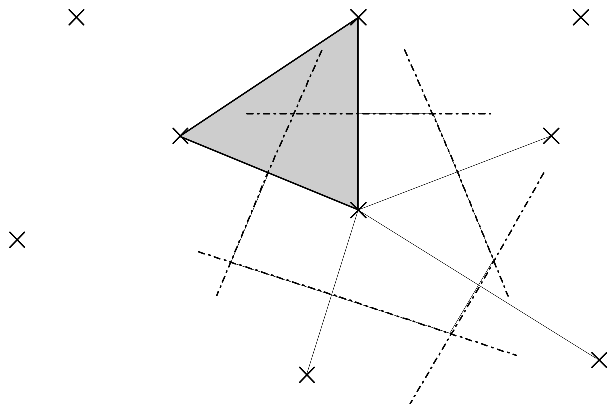


Figure 1.2: Illustration of a Delone tile in $\mathbb{R}^2$.

of dimension $d-1$. If we connect the neighbours in Σ by line segments, we obtain edges of the so-called Delone tiling of Σ.

Tiles in the Delone tiling are again convex polytopes whose all vertices lie on a sphere. It can be shown that for determining the Voronoi cell $V(x)$ it suffices to study only the local configuration $\Sigma \cap B(x, 2R_\Sigma)$, where R_Σ is the covering radius of Σ. We would like to estimate the number of such local configurations for cut-and-project sets $\Sigma(\Omega)$. We will obtain an estimate using the Meyer property of $\Sigma(\Omega)$.

1.3 The Meyer property and local configurations

We want to study local configurations of cut-and-project sets, therefore we consider all points of $\Sigma(\Omega)$ that lie in a given distance of a chosen element x. The set ρ-neighbourhood $N_\rho(x)$ of a point x in a Delone set $\Sigma(\Omega)$ satisfies

$$B(x,\rho) \cap \Sigma(\Omega) = x + N_\rho(x).$$

As a consequence of the Meyer property (1.1), any cut-and-project set has only finitely many ρ-neighbourhoods. The more different ρ-neighbourhoods there are, the more different shapes of tiles are there in the Voronoi tiling. Since in a lattice ρ-neighbourhoods of all elements are the same, there is only one type of Voronoi cell.

Let us show how to estimate the number of neighbourhoods using the Meyer property of cut-and-project sets. For every ρ-neighbourhood we have

$$N_\rho(x) \subset (\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0,\rho).$$

We can notice that the number of different ρ-neighbourhoods in $\Sigma(\Omega)$ depends on the cardinality $\#(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0,\rho)$. At this moment we will use the Meyer property

$$\Sigma(\Omega) - \Sigma(\Omega) \subset \Sigma(\Omega) + F, \tag{1.3}$$

where F is a finite set. We obtain

$$\#(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0,\rho) \leq \#(\Sigma(\Omega) + F) \cap B(0,\rho) \leq \#F \frac{\text{vol } B(0,\rho)}{\text{vol } B(0, \frac{1}{2} r_\Sigma)}, \tag{1.4}$$

where r_Σ is the minimal distance in the set $\Sigma(\Omega)$. For the estimate of the number of shapes of Voronoi cells in $\Sigma(\Omega)$ we use $\rho = 2R_\Sigma$, where R_Σ is the covering radius of $\Sigma(\Omega)$.

The main topic of this work is the estimate of the cardinality of F. Let us explain how it is connected with the searching for an upper bound on values of the function f, which was defined in (1.2). It is obvious from the definition of $\Sigma(\Omega)$ that the finite set F in the Meyer property (1.3) of $\Sigma(\Omega)$ satisfies $F \subset \pi_1(L)$. For the finite set $G := \pi_2\pi_1^{-1}(F)$ we have

$$\Omega - \Omega \subset \Omega + G\,. \tag{1.5}$$

Thus the Meyer property of $\Sigma(\Omega)$ implies the relation (1.5) for its acceptance window. The converse is however not that simple. Having a finite set $G \subset V_2$ which satisfies (1.5), it is not always possible to find F of the same cardinality, so that (1.3) holds. This comes from the fact that G may not be subset of $\pi_2(L)$. However, this inconvenience can be avoided if instead of (1.5) we study covering of the difference set $\Omega - \Omega$ by the copies of the interior Ω°,

$$\Omega - \Omega \subset \Omega^\circ + G\,. \tag{1.6}$$

Having such G and due to the fact that $\pi_2(L)$ is dense in V_2 (Definition 1.3), we can clearly find a set $\tilde{G} \subset \pi_2(L)$ of the same cardinality as G and satisfying (1.5). Therefore we may set $F = \pi_1\pi_2^{-1}(\tilde{G})$ to obtain (1.3) with $|F| = |G|$. We have therefore explained that the number of Voronoi cells in the Voronoi tiling of $\Sigma(\Omega)$ is directly related to the value of the function f defined as

$$f(\Omega) = \text{the minimal number of translated copies of } \Omega^\circ \text{ needed for covering of } \overline{\Omega - \Omega}.$$

The aim of this work is to study its topologic properties and to determine the values of f on certain special types of acceptance windows Ω.

Chapter 2

Topology of spaces of convex sets

Before we start description of spaces of convex sets, which play an essential role in our main researches, it is useful to sum up basic properties of topological and metric spaces.

2.1 Summary of general knowledge of topology

In this part basic topological knowledge is summarized. In the following sections we will refer to points of this summary or we will even use them without any reference considering them as self-evident facts. For more details see [9].

2.1.1 Compactness of topological spaces

Definition 2.1. *A topological space $\mathcal{X}$ is called compact if every open covering of $\mathcal{X}$ has a finite open subcovering, i.e.*

$$(\mathcal{X} \subset \cup_{i \in I} S_i,\ S_i \text{ open}) \Rightarrow (\exists k \in \mathbb{N})(\exists j_1, ..., j_k \in I)(\mathcal{X} \subset \cup_{i=1}^{k} S_{j_i}).$$

Theorem 2.1. *Let $\mathcal{X}$ be a compact space and let A be a set closed in $\mathcal{X}$. Then A is compact.*

Theorem 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. Moreover, let $\mathcal{X}$ be a compact space and let F be a continuous mapping: $\mathcal{X} \to \mathcal{Y}$. Then $F(\mathcal{X})$ is compact in $\mathcal{Y}$.*

Proof. Let $F(\mathcal{X}) \subset \cup_{i \in I} S_i$. Then $(F^{-1}(S_i))_{i \in I}$ is an open covering of $\mathcal{X}$. As $\mathcal{X}$ is compact there exist indices $i_1, ..., i_k$ so that $(F^{-1}(S_{i_j}))_{j=1}^{k}$ is an open subcovering of $\mathcal{X}$. It implies that $F(\mathcal{X}) \subset \cup_{j=1}^{k} S_{i_j}$. □

Theorem 2.3. *Let $\mathcal{X}$ be a compact space and let f be a continuous function: $\mathcal{X} \to \mathbb{R}$. Then f reaches its supremum and infimum on $\mathcal{X}$.*

Proof. As $f(\mathcal{X})$ is compact in $\mathbb{R}$, it is closed and bounded. Therefore $\sup f(\mathcal{X}) \in f(\mathcal{X})$ and $\inf f(\mathcal{X}) \in f(\mathcal{X})$. □

2.1.2 Compactness of metric spaces

Definition 2.2. *Let $\mathcal{X}$ be a topological space. Real function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a metric on $\mathcal{X}$ if ρ satisfies for all $x, y, z \in \mathcal{X}$ the following three properties*

1. $\rho(x, y) \geq 0$, *moreover* $\rho(x, y) = 0 \Leftrightarrow x = y$,
2. $\rho(x, y) = \rho(y, x)$ *(symmetry)*,
3. $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ *(triangle inequality)*.

Theorem 2.4. *Let A be a subset of the metric space $\mathcal{X}$. Then A is closed if and only if every sequence in A has all its limit points in A.*

Theorem 2.5 (Weierstrass)**.** *Let $\mathcal{X}$ be a metric space. Then $\mathcal{X}$ is compact if and only if every sequence in $\mathcal{X}$ has a convergent subsequence, i.e. a subsequence, which convergs to a point of $\mathcal{X}$.*

Definition 2.3. *Let $(x_n)_{n=1}^{\infty}$ be a sequence in the metric space $\mathcal{X}$ with the metric ρ. $(x_n)_{n=1}^{\infty}$ is called Cauchy sequence if*

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall m, n > n_0)(\rho(x_n, x_m) < \varepsilon).$$

Definition 2.4. *A metric space $\mathcal{X}$ is called complete if every Cauchy sequence has its limit in $\mathcal{X}$.*

Definition 2.5. *The metric space $\mathcal{X}$ with the metric ρ is called totally bounded if for every $\varepsilon > 0$ there exists a finite ε-net, i.e. for every $\varepsilon > 0$ there exists a finite set $N(\varepsilon)$ such that*

$$(\forall x \in \mathcal{X})(\exists y \in N(\varepsilon))(\rho(x, y) \leq \varepsilon).$$

Theorem 2.6. *A metric space $\mathcal{X}$ is compact if and only if $\mathcal{X}$ is complete and totally bounded.*

Corollary 2.6.1. *Let A be a set on linear normed space $\mathbb{R}^d$. A is bounded if and only if A is totally bounded. Therefore A is compact if and only if A is closed and bounded.*

2.2 Topology of spaces of convex sets

This section is immediately connected with the main subject, which deals with the set $\Omega - \Omega$, where Ω is a convex compact set with non-empty interior in $\mathbb{R}^d$. Therefore it is necessary to get acquainted with topology of spaces of convex sets at first.

Definition 2.6. *A set H in $\mathbb{R}^d$ is called convex, if, for any two points x,$y \in H$, it contains all points of the line segment joining x and y, i.e.*

$$(\forall x, y \in H)(\forall \lambda \in [0, 1])(\lambda x + (1 - \lambda)y \in H).$$

Consider $\mathbb{R}^d$ with the euclidean norm $|\cdot|$.
First, we define topological structures which will be used to describe properties of spaces of convex sets. One defines the distance of a set $A \subset \mathbb{R}^d$ from a point x as

$$\rho(x, A) := \inf\{|x - y| \bigm| y \in A\}.$$

An open ball of radius ε, centered at a is defined by

$$B(a, \varepsilon) := \{x \in \mathbb{R}^d \bigm| |x - a| < \varepsilon\}.$$

An ε-neighbourhood of the set A for $\varepsilon > 0$ is the set

$$A_\varepsilon := \cup_{a \in \mathrm{A}} B(a, \varepsilon) = \{x \in \mathbb{R}^d | \rho(x, A) < \varepsilon\}.$$

Clearly, A_ε is an open set in $\mathbb{R}^d$.
An $(-\varepsilon)$-neighbourhood of a bounded set A is defined by

$$A_{-\varepsilon} := \{x \in A \ \mid \rho(x, \partial A) > \varepsilon\}.$$

Let us show two properties of $(-\varepsilon)$-neighbourhood of a set in $\mathbb{R}^d$.

Lemma 2.1. *Let $0 < a < b$. $\Omega_{-b} \subset \Omega_{-a}$ and $\Omega_a \subset \Omega_b$.*

Proof. Both inclusions are clear from the corresponding definitions. $\square$

Lemma 2.2. *Let $\alpha > 0$. $\Omega_{-\alpha} = \Omega^{\circ}_{-\alpha}$.*

Proof. $\Omega_{-\alpha} = \{x \in \Omega | \rho(x, \partial\Omega) > \alpha\}$ is an open set. ☐

Theorem 2.7. *Let $H_1, H_2 \subset \mathbb{R}^d$ be non-empty closed convex sets which have no common points and at least one of them is compact. Then there exists a hyperplane separating H_1, H_2. More precisely, there exist a vector $c \in \mathbb{R}^d$ and a number α such that for every $x \in H_1$ and for every $y \in H_2$ it holds*

$$cx^T > \alpha > cy^T.$$

Then $\{z \in \mathbb{R}^d \mid cz^T = \alpha\}$ is the searched hyperplane.

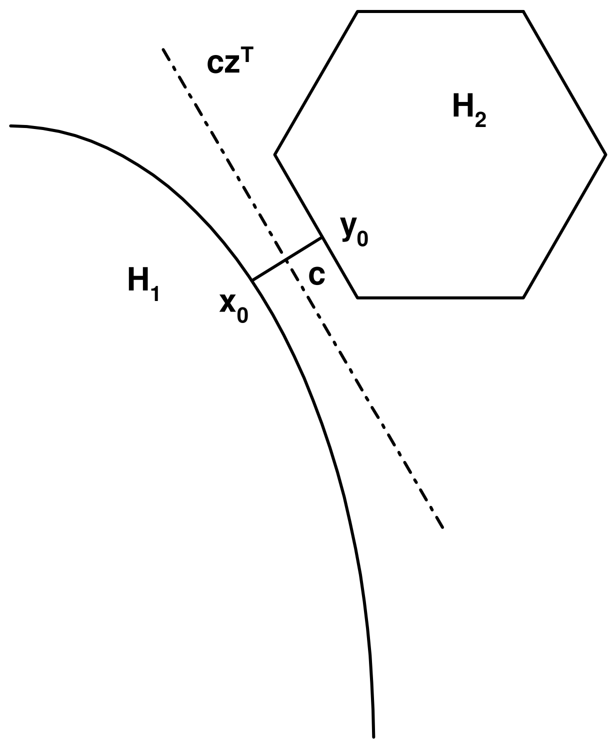


Figure 2.1: Illustaration of the situation in theorem (2.7).

Proof. We roughly describe the main ideas of the proof at first.

1. We show that there exist $x_0 \in H_1$ and $y_0 \in H_2$ such that $\inf\{|x - y| \mid x \in H_1, y \in H_2\} = |x_0 - y_0|$. We denote $c := x_0 - y_0$.

2. Then we take arbitrary $x \in H_1$ and construct the line segment $\overline{xx_0}$. We choose arbitrary point of $\overline{xx_0}$ and we make profit of the fact that its distance from y_0 is greater then $|c|$. We continue analogically for H_2 and we obtain the searched inequalities

$$(\forall x \in H_1)(\forall y \in H_2)(cx^T > \alpha > cy^T).$$

Let us step up to the precise proof. Let H_2 be compact. Define by

$$\nu := \inf\{|x - y| \mid x \in H_1, y \in H_2\}.$$

Using the definition of infimum we have

$$(\forall n \in \mathbb{N})(\exists x_n \in H_1, y_n \in H_2)(\nu \leq |x_n - y_n| < \nu + \tfrac{1}{n}).$$

Both sequences are bounded due to two facts

1. H_2 is compact therefore bounded, which implies that $(\exists K > 0)(\forall n \in \mathbb{N})(|y_n| \leq K)$.
2. $(\forall n \in \mathbb{N})(|x_n| \leq |x_n - y_n| + |y_n| < \nu + 1 + K)$.

It is possible to choose a Cauchy subsequence from any bounded sequence. As H_1, H_2 respectively are closed, any Cauchy sequence in H_1, H_2 respectively has its limit in H_1, H_2 respectively, i.e. it holds

$$(\exists y_{n_k})(\lim_{k\to\infty} y_{n_k} =: y_0 \in H_2).$$

$$(\exists x_{n_{k_l}})(\lim_{l\to\infty} x_{n_{k_l}} =: x_0 \in H_1).$$

We obtain consequently

$$\nu \leq |x_{n_{k_l}} - y_{n_{k_l}}| < \nu + \tfrac{1}{n_{k_l}}.$$

Let l tends to infinity so that we have $\nu \leq |x_0 - y_0| \leq \nu$. As the vectors x_0, y_0 are elements of sets, which have no common points, we obtain that $\nu > 0$. Denote by $c := x_0 - y_0$.

Take arbitrary $x \in H_1$. Due to convexity of H_1 it holds

$$(\forall \lambda \in [0, 1])(x_0 + \lambda(x - x_0) \in H_1).$$

Look at the following considerations

$$|x_0 + \lambda(x - x_0) - y_0| \geq \nu = |x_0 - y_0|,$$

$$|c + \lambda(x - x_0)|^2 \geq |c|^2,$$

$$(\forall \lambda \in (0, 1])(\lambda |x - x_0|^2 + 2c(x - x_0)^T \geq 0).$$

Let $\lambda \to 0^+$. Then we have

$$cx^T \geq cx_0^T.$$

Take arbitrary $y \in H_2$. Due to convexity of H_2 it holds

$$(\forall \lambda \in [0, 1])(y_0 + \lambda(y - y_0) \in H_2).$$

We use analogical considerations as above.

$$|y_0 + \lambda(y - y_0) - x_0| \geq \nu = |x_0 - y_0|,$$

$$|\lambda(y - y_0) - c|^2 \geq |c|^2,$$

$$(\forall \lambda \in (0, 1])(\lambda |y - y_0|^2 \geq 2c(y - y_0)^T).$$

Let $\lambda \to 0^+$. Then we have

$$cy_0^T \geq cy^T.$$

Using the fact $c(x_0 - y_0)^T = cc^T > 0$ and the two previous inequalities we obtain for every $x \in H_1$ and every $y \in H_2$

$$cx^T \geq cx_0^T > cy_0^T \geq cy^T.$$

It suffices to define α for instance

$$\alpha := \frac{1}{2}(cx_0^T + cy_0^T).$$

Now, we can see that the searched hyperplane is the set $\{z \in \mathbb{R}^d \mid cz^T = \alpha\}$. $\square$

2.2.1 Hausdorff metric

Let us define two spaces of compact sets, which will be important in our subsequent considerations. The topology on these spaces is given by the metric Dist, sometimes called the Hausdorff metric.

Definition 2.7. *Denote by $\mathcal{N}$ the space of all closed subsets of $\overline{B(0,1)}$ in $\mathbb{R}^d$.*
Let $\alpha > 0$. Denote by $\mathcal{M} := \{\Omega \in \mathcal{N} \mid \Omega$ convex and $\overline{B(0,\alpha)} \subset \Omega\}$.

Definition 2.8. *Let A,B be compact sets in $\mathbb{R}^d$. We define a real function Dist by*

$$Dist(A,B) := \max\big\{\inf\{\varepsilon > 0 | A \subset B_\varepsilon\}, \inf\{\varepsilon > 0 | B \subset A_\varepsilon\}\big\} = \inf\{\varepsilon > 0 | A \subset B_\varepsilon \wedge B \subset A_\varepsilon\}.$$

Remark 1. *The minimal ε_1 such that $B \subset A_{\varepsilon_1}$ and the minimal ε_2 such that $A \subset B_{\varepsilon_2}$ are generally different.*

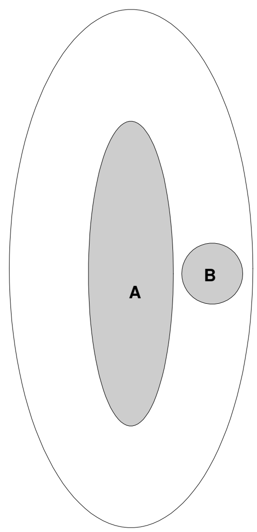


Figure 2.2: Illustration of A_{ε_1} such that $B \subset A_{\varepsilon_1}$.

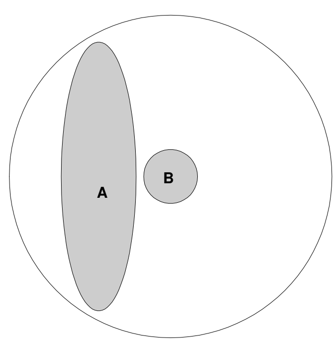


Figure 2.3: Illustration of B_{ε_2} such that $A \subset B_{\varepsilon_2}$.

Proposition 2.1. *Dist is a metric on the space of all compact subsets of $\mathbb{R}^d$. In particular Dist is a metric on spaces $\mathcal{N}$ and $\mathcal{M}$.*

For the proof of Propositon 2.1 we use the following lemma.

Lemma 2.3. *Let B, C be compact sets in $\mathbb{R}^d$ and let $\rho, \delta > 0$. Then $C \subset B_\delta$ implies $C_\rho \subset B_{\rho+\delta}$.*

Proof. We want to show $(\forall z \in C_\rho)(\exists y \in B)(|z - y| < \delta + \rho)$. We use two facts

1. $C \subset B_\delta \Rightarrow (\forall x \in C)(\exists y \in B)(|x - y| < \delta)$,
2. $(\forall z \in C_\rho)(\exists x \in C)(|z - x| < \rho)$.

By using previous facts we have

$$(\forall z \ \in C_\rho)(\exists y \in B)(|z - y| \leq |x - y| + |x - z| < \delta + \rho).$$

Thus $C_\rho \subset B_{\rho+\delta}$. □

Proof of Proposition 2.1. To prove the proposition it is necessary to show the three properties of a metric. For all A, B, C compact in $\mathbb{R}^d$ we have to verify that

1. $Dist(A, B) \geq 0$.
   $Dist(A, B) = 0 \Leftrightarrow A = B$.
2. $Dist(A, B) = Dist(B, A)$.
3. $Dist(A, B) \leq Dist(A, C) + Dist(C, B)$.

ad 1. The only implication which does not follow directly from the definition is

$$Dist(A, B) = 0 \Rightarrow A = B.$$

Let us show this by contradiction. Suppose $Dist(A, B) = 0$ and $A \neq B$. Without loss of generality this means

$$(\exists x \in B)(x \notin A).$$

As A is closed $\rho(x, A) > 0$. Denote $\delta := \rho(x, A) > 0$. $Dist(A, B) = 0$ implies

$$(\forall \varepsilon > 0)(A \subset B_\varepsilon \wedge B \subset A_\varepsilon).$$

Let us denote $\varepsilon := \frac{\delta}{2}$. We have

$$x \in B \subset A_{\frac{\delta}{2}} = \{y \in \mathbb{R}^d \mid \rho(y, A) < \frac{\delta}{2}\},$$

which is contradiction with the fact $\rho(x, A) = \delta$.

ad 2. Symmetry is clear from the definition of $Dist$.
ad 3. Denote $\rho := Dist(A, C)$, $\delta := Dist(C, B)$.

Using the definition of Dist we have

$$A \ \subset B_{\rho+\delta} \ \text{ and } B \subset A_{\rho+\delta} \ \Rightarrow \ Dist(A, B) \leq \rho + \delta = Dist(A, C) + Dist(B, C).$$

Hence, it suffices to verify

$$A \ \subset B_{\rho+\delta} \text{ and } B \subset A_{\rho+\delta}.$$

Using Lemma 2.3 we have

$$A \ \subset C_\rho \subset B_{\rho+\delta}.$$

The second inclusion $B \subset A_{\rho+\delta}$ follows analogically. This completes the proof of the triangle inequality.

□

Observation 2.1. *Compactness of A,B is necessary in order that $Dist(A, B)$ be a metric. Otherwise $Dist(A, B) = 0 \not\Rightarrow A = B$, as illustrated on the following example.*

Example 2.1. *Let $A := B(0, 1)$, $B := \overline{B(0, 1)}$. Then $Dist(A, B)$ is clearly 0, however, $A \neq B$.*

2.2.2 Compactness of the metric space $\mathcal{M}$

As we have already mentioned, the spaces $\mathcal{M}$ and $\mathcal{N}$ are important for our subsequent considerations. Especially the fact, that both of them are compact, will play an essential role in the following researches.

Theorem 2.8. *The space $\mathcal{N}$ of all closed subsets of $\overline{B(0,1)}$ is compact.*

For the proof see [1].

Let us introduce some lemmas which will be useful for the proof of Theorem 2.9, which states that also the metric space $\mathcal{M}$ is compact.

Lemma 2.4. *Let $(\Omega_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{N}$, i.e. $(\Omega_n)_{n=1}^\infty$ is a sequence of closed subsets of $\overline{B(0,1)}$ in $\mathbb{R}^d$. Denote $\Omega := \lim_{n\to\infty} \Omega_n$. Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence such that $x_n \in \Omega_n$ for all $n \in N$. Then $\lim_{n\to\infty} x_n = x \in \Omega$.*

Proof. As (x_n) is a Cauchy sequence in $\mathbb{R}^d$ there exists $x := \lim_{n\to\infty} x_n \in \mathbb{R}^d$. We want to show that $x \in \Omega$. Since $\mathcal{N}$ is compact, Ω is closed, and thus $x \in \Omega$ is equivalent with

$$(\forall \varepsilon > 0)(\rho(x, \Omega) \le \varepsilon).$$

As $\lim_{n\to\infty} \Omega_n = \Omega$ we have

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(x_n \in \Omega_n \subset \Omega_\varepsilon).$$

Using the definition of Ω_ε we obtain

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(\rho(x_n, \Omega) < \varepsilon).$$

Note that $\rho(x, \Omega)$ is continuous as a function of x. Hence, when we let n tend to infinity, we have

$$(\forall \varepsilon > 0)(\lim_{n\to\infty} \rho(x_n, \Omega) = \rho(x, \Omega) \le \varepsilon),$$

which was to show. □

Lemma 2.5. *Let $(\Omega_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{N}$ and denote $\Omega := \lim_{n\to\infty} \Omega_n$. Then*

$$(\forall x \in \Omega)(\forall n \in \mathbb{N})(\exists x_n \in \Omega_n)(\lim_{n\to\infty} x_n = x).$$

Proof. As $\lim_{n\to\infty} \Omega_n = \Omega$ we have

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(\Omega \subset (\Omega_n)_\varepsilon).$$

This implies for $x \in \Omega$ that

$$(\forall m \in \mathbb{N})(\exists n_0)(\forall n > n_0)(\exists x_n^{(m)} \in \Omega_n)(|x - x_n^{(m)}| < \tfrac{1}{m}).$$

As $(x_n^{(m)})$ is a sequence of sequences we can use for instance diagonal choice. We choose the sequence $(x_n^{(n)})_{n=1}^\infty$ which satisfies

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(|x - x_n^{(n)}| \le \varepsilon).$$

Thus $x_n := x_n^{(n)} \in \Omega_n$ is the searched sequence such that $\lim_{n\to\infty} x_n = x$. □

Lemma 2.6. *Let $(\Omega_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{N}$ and denote $\Omega := \lim_{n\to\infty} \Omega_n$. Then*

$$\lim_{n\to\infty} \rho(x, \Omega_n) = 0 \Leftrightarrow x \in \Omega.$$

Proof. One has to prove two implications.
$(\Rightarrow)$: $\lim_{n\to\infty} \rho(x, \Omega_n) = 0$ implies that

$$(\forall m \in \mathbb{N})(\exists n_0)(\forall n > n_0)(\exists x_n^{(m)} \in \Omega_n)(|x - x_n^{(m)}| < \tfrac{1}{m}).$$

By using the diagonal choice we obtain a Cauchy sequence $(x_n^{(n)})_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, $x_n^{(n)} \in \Omega_n$. Using Lemma 2.4 we have $x := \lim_{n\to\infty} x_n^{(n)} \in \Omega$.
$(\Leftarrow)$: As $\lim \Omega_n = \Omega$ we have

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(x \in \Omega \subset (\Omega_n)_\varepsilon).$$

As $(\Omega_n)_\varepsilon = \{x \in \mathbb{R}^d \mid \rho(x, \Omega_n) < \varepsilon\}$ the result is that $\lim_{n\to\infty} \rho(x, \Omega_n) = 0$. □

Theorem 2.9. *The space $\mathcal{M}$ of all convex closed sets Ω in $\mathbb{R}^d$, which satisfy $\overline{B(0,\alpha)} \subset \Omega \subset \overline{B(0,1)}$, is compact.*

Proof. It is enough to show that $\mathcal{M}$ is closed in $\mathcal{N}$. Take an arbitrary Cauchy sequence $(\Omega_n)_{n=1}^{\infty}$ in $\mathcal{M}$. $(\Omega_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{N}$, as well, and $\mathcal{N}$ is compact and consequently complete. Therefore the limit Ω is an element of $\mathcal{N}$, i.e. Ω is a closed subset of $\overline{B(0,1)}$. The only questions left are whether $\overline{B(0,\alpha)} \subset \Omega$ and whether Ω is convex.

- Let us prove the inclusion $\overline{B(0,\alpha)} \subset \Omega$ by contradiction. Assume $\overline{B(0,\alpha)} \not\subset \Omega$. This means

  $$\big(\exists x \in \overline{B(0,\alpha)}\big)\big(x \notin \Omega\big).$$

  Moreover we know $(\forall n \in \mathbb{N})(x \in \overline{B(0,\alpha)} \subset \Omega_n)$ and $\Omega = \lim_{n\to\infty} \Omega_n$.
  According to Lemma 2.6, it implies that $x \in \Omega$ which is contradiction with assumption $x \notin \Omega$.

- Convexity: We want to verify

  $$(\forall x, y \in \Omega)(\forall \lambda \in [0,1])(\lambda x + (1-\lambda)y \in \Omega).$$

  Due to Lemma 2.5, there exist sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ such that for all $n \in \mathbb{N}$, $x_n, y_n \in \Omega_n$ and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. As Ω_n is convex for every n we have

  $$\lambda x + (1-\lambda)y = \lambda \lim_{n\to\infty} x_n + (1-\lambda) \lim_{n\to\infty} y_n = \lim_{n\to\infty} \big(\lambda x_n + (1-\lambda)y_n\big) \in \Omega.$$

□

Chapter 3

Covering of the difference set $\Omega - \Omega$ by Ω

We have already indicated in the introduction part, that the main interest will be devoted to covering of the difference set $\Omega - \Omega$ by translated copies of the interior Ω°, where Ω is a compact convex set with non-empty interior in $\mathbb{R}^d$. We will focus on estimation of the sufficient number of these copies. The main result is given as Theorem 3.2. It states that there exists a universal constant K, such that for all convex compact sets Ω with non-empty interior, K translated copies of Ω° are sufficient to cover $\Omega - \Omega$.

We recall the introduction part of the work, where the following statements were introduced. The number of different types of Voronoi tiles in $\Sigma(\Omega)$ can be estimated if we know the minimal distance $r_{\Sigma(\Omega)}$, the covering radius $R_{\Sigma(\Omega)}$ and the cardinality of the set F from the Meyer property

$$\Sigma(\Omega) - \Sigma(\Omega) \subset \Sigma(\Omega) + F.$$

If we estimate the mentioned constant K we have an estimation on the cardinality of F as well.

Definition 3.1. *Denote by κ the space of all convex compact sets in $\mathbb{R}^d$ with non-empty interior.*

Let Ω be a set in κ. We are interested in the set $\Omega - \Omega = \{x - y \mid x, y \in \Omega\}$.

Theorem 3.1. *For every set Ω in κ there exists a finite set*

$$A = \{a_1, a_2, ..., a_k \mid (a_i \in \mathbb{R}^d)(\forall i \in \{1, ..., k\})\}$$

such that

$$\overline{\Omega - \Omega} \subset (\Omega^\circ + a_1) \cup (\Omega^\circ + a_2) \cup .... \cup (\Omega^\circ + a_k).$$

Proof. As $\overline{\Omega - \Omega}$ is compact, i.e. for every open covering of $\overline{\Omega - \Omega}$ there exists a finite open subcovering. Using this property we have for an arbitrary element y of Ω

$$\overline{\Omega - \Omega} \subset \cup_{x \in \overline{\Omega - \Omega}}(x + (\Omega^\circ - y)) \Rightarrow (\exists k \ \in \mathbb{N})(\overline{\Omega - \Omega} \subset \cup_{i=1}^k (x_i + (\Omega^\circ - y))),$$

where $(\forall i \in \{1, ..., k\})(x_i \in \overline{\Omega - \Omega})$.
The open subcovering of $\overline{\Omega - \Omega}$ has the form $\Omega^\circ + A = \cup_{i=1}^k (x_i + (\Omega^\circ - y))$. Therefore the searched set $A = \{x_i - y \mid i \in \{1, ..., k\}\}$. ∎

For the purpose to estimate the sufficient number of translated copies of Ω°, which can cover the difference set $\Omega - \Omega$, let us define the function f which to any $\Omega \in \kappa$ associates the minimal number of translated copies of Ω° needed for covering of $\overline{\Omega - \Omega}$. We explore its properties further on.

Definition 3.2. *We define a function $f : \kappa \to \mathbb{N}$ by*

$$f(\Omega) := \min\{k \in \mathbb{N} | (\exists a_1, ..., a_k \in \mathbb{R}^d)(\overline{\Omega - \Omega} \subset (a_1 + \Omega^\circ) \cup ... \cup (a_k + \Omega^\circ))\}.$$

Observation 3.1. *The function f is well defined. Theorem 3.1 implies that for every Ω in κ there exists a positive number k such that $f(\Omega) = k$.*

Let us introduce the most important theorem, which answers the question, whether the number of translated copies of Ω°, which suffice for covering the difference set $\Omega - \Omega$, is bounded by a universal constant for all convex compact sets Ω with non-empty interior in $\mathbb{R}^d$.

Theorem 3.2. *The function f is bounded on the metric space κ with the metric Dist, i.e.*

$$(\exists K > 0)(\forall \Omega \in \kappa)(f(\Omega) \leq K).$$

Remark 2. *As f reaches only a finite number of values in κ, convex sets in κ are devided into a few classes according to the value of f.*

We will prove this theorem in section 3.4. It is useful to investigate the properties of $\Omega - \Omega$ and the function f at first.

3.1 Properties of $\Omega - \Omega$

Let us state some basic properties of the difference set $\Omega - \Omega$.

Claim 3.1. *$\Omega - \Omega$ is o-symmetric.*

Proof. For each $z \in \Omega - \Omega$ there exist $x, y \in \Omega$ such that $z = x - y$. Definition of the set $\Omega - \Omega$ implies $y - x = -z \in \Omega - \Omega$. □

Claim 3.2. *$\Omega - \Omega$ is independent on the translation of Ω, i.e.*

$$(\forall a \in \mathbb{R}^d)((\Omega + a) - (\Omega + a) = \Omega - \Omega).$$

Proof. Take an arbitrary a $\in \mathbb{R}^d$.

($\subseteq$): For each $z \in (\Omega + a) - (\Omega + a)$ there exist $x + a$, $y + a \in \Omega + a$ such that $z = (x + a) - (y + a) = x - y \in \Omega - \Omega$.

($\supseteq$): For each $z \in \Omega - \Omega$ there exist $x, y \in \Omega$ such that $z = x - y = (x + a) - (y + a) \in (\Omega + a) - (\Omega + a)$. □

Claim 3.3. *If Ω is a convex set in $\mathbb{R}^d$, then $\Omega - \Omega$ is a convex set in $\mathbb{R}^d$.*

Proof. To prove the claim one has to show

$$(\forall y, z \in \Omega)(\forall \lambda \in [0, 1])(\lambda y + (1 - \lambda)z \in \Omega) \Rightarrow$$
$$\Rightarrow (\forall x_1, x_2 \in \Omega - \Omega)(\forall \lambda \in [0, 1])(\lambda x_1 + (1 - \lambda)x_2 \in \Omega - \Omega).$$

For each $x_1, x_2 \in \Omega - \Omega$ there exist $y_1, y_2, z_1, z_2 \in \Omega$ such that $x_1 = y_1 - z_1$ and $x_2 = y_2 - z_2$.

$$\lambda x_1 + (1 - \lambda)x_2 = \lambda(y_1 - z_1) + (1 - \lambda)(y_2 - z_2) =$$
$$= \lambda y_1 + (1 - \lambda)y_2 - (\lambda z_1 + (1 - \lambda)z_2) \in \Omega - \Omega.$$

□

The following property is useful for our considerations further on.

Claim 3.4. *Let $\delta > 0$, then $\Omega_\delta - \Omega_\delta \subset (\Omega - \Omega)_{2\delta}$.*

Proof. Take an arbitrary $x \in \Omega_\delta - \Omega_\delta$. There exist $x_1, x_2 \in \Omega_\delta$ so that $x = x_1 - x_2$. For all $y \in \Omega - \Omega$ there exist $y_1, y_2 \in \Omega$ so that $y = y_1 - y_2$.

$$|x - y| = |x_1 - x_2 - (y_1 - y_2)| \leq |x_1 - y_1| + |x_2 - y_2| < \delta + \delta = 2\delta.$$

Therefore $x \in (\Omega - \Omega)_{2\delta}$. □

We illustrate the construction of the difference set on the example of a line segment and a triangle.

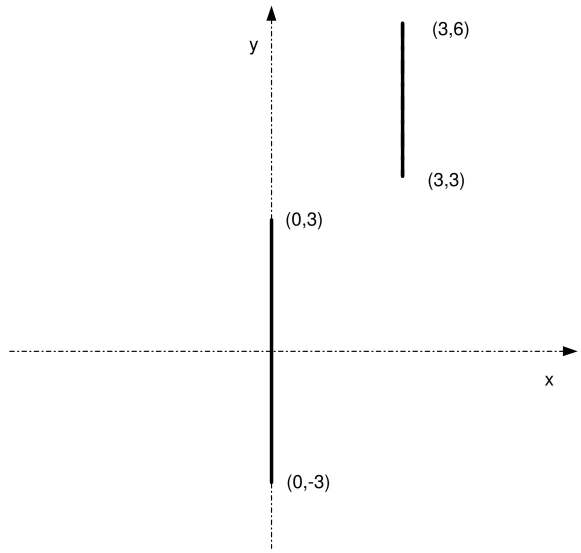


Figure 3.1: $\Omega - \Omega$, where Ω is the convex hull of points (3,3) and (3,6).

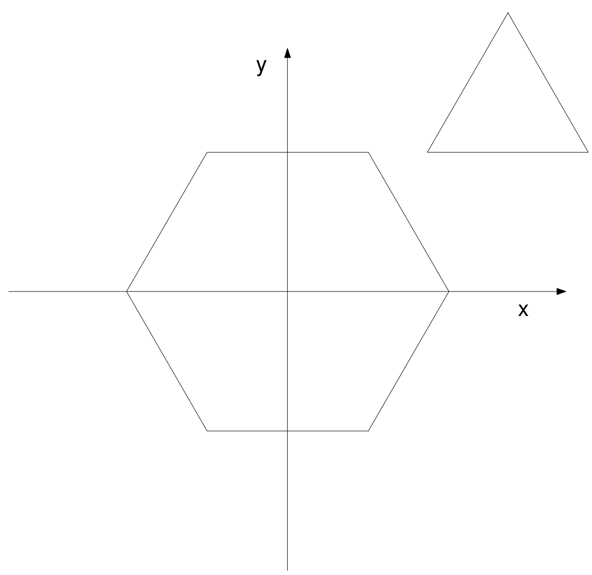


Figure 3.2: $\Omega - \Omega$, where Ω is a triangle.

3.2 Semicontinuity of f on $\mathcal{M}$

We want to prove that the function f is bounded on the space κ. It is useful to show as the first step that the function f is upper semicontinuous on the space $\mathcal{M}$ of all convex compact sets Ω in $\mathbb{R}^d$ such that $\overline{B(0,\alpha)} \subset \Omega \subset \overline{B(0,1)}$.

Let us introduce lemmas and claims, which will be useful to prove that f satisfies the property of semicontinuity.

Realize that the following lemma is the only moment when we need convexity of the set Ω.

Lemma 3.1. *Let $\Omega, \tilde{\Omega}$ be convex compact sets in $\mathbb{R}^d, Dist(\Omega, \tilde{\Omega}) < \delta$. Then $\Omega_{-\delta} \subset \tilde{\Omega} \subset \Omega_\delta$.*

Proof. The inclusion $\tilde{\Omega} \subset \Omega_\delta$ is valid for any bounded sets Ω and $\tilde{\Omega}$ directly from the definition of δ-neighbourhood of a set. We prove $\Omega_{-\delta} \subset \tilde{\Omega}$ by contradiction. Let us suppose: $(\exists x \in \Omega_{-\delta})(x \notin \tilde{\Omega})$. Since $\tilde{\Omega}$ is convex and closed there exists a hyperplane H, such that $\tilde{\Omega}$ is all contained in one of the half-spaces bounded by H, and that x belongs to the other half-space. We will use the two following statements

1. $B(x, \delta) \subset \Omega$.

   Suppose the opposite, i.e. there exists $y \in B(x, \delta)$ and $y \notin \Omega$. Since $x \in \Omega$ it holds $|x - y| \geq \rho(x, \partial\Omega) \geq \delta$, which is contradiction with the assumption $y \in B(x, \delta)$.

2. $Dist(\Omega, \tilde{\Omega}) < \delta$.

As x is in another half-plane than $\tilde{\Omega}$, more than half a ball $B(x, \delta)$ lies in the same half-plane as x. Hence, there exists $z \in B(x, \delta) \subset \Omega$ such that $\rho(z, \tilde{\Omega}) > \delta$, which is contradiction with the fact $\Omega \subset \tilde{\Omega}_\delta$. □

Take an arbitrary Ω and denote $k := f(\Omega) \in \mathbb{N}$. It means that

$$\overline{\Omega - \Omega} \subset (\Omega^\circ + a_1) \cup (\Omega^\circ + a_2) \cup ..... \cup (\Omega^\circ + a_k) =: P. \tag{3.1}$$

Denote also

$$\varepsilon := \inf\{\rho(x, \mathbb{R}^d \setminus P) | x \in \overline{\Omega - \Omega}\}. \tag{3.2}$$

Lemma 3.2. $\varepsilon > 0$.

Proof. If $\varepsilon = 0$, there exists $(y_n)_{n=1}^\infty \in \mathbb{R}^d \setminus P$ and $(z_n)_{n=1}^\infty \in \overline{\Omega - \Omega}$ such that $\lim_{n\to\infty} |y_n - z_n| = 0$. As $\overline{\Omega - \Omega}$ is compact, there exists a Cauchy subsequence z_{l_n} such that

$$\lim_{n\to\infty} z_{l_n} = z \in \overline{\Omega - \Omega}.$$

Since $\lim_{n\to\infty} |y_{l_n} - z_{l_n}| = 0$ and $\mathbb{R}^d \setminus P$ is closed, it holds

$$\lim_{n\to\infty} y_{l_n} = z \in \mathbb{R}^d \setminus P.$$

So that $z \in \overline{\Omega - \Omega} \subset P$ and $z \in \mathbb{R}^d \setminus P$, which is contradiction. □

Take an arbitrary $x \in \overline{\Omega - \Omega}$, then there exists at least one index $j \in \{1, ..., k\}$ such that $x \in a_j + \Omega^\circ$.
Let us define

$$v(x) := \max\{\rho(x, \partial(a_j + \Omega^\circ)) | (x \in a_j + \Omega^\circ)(j \in \{1, ..., k\})\}. \tag{3.3}$$

x belongs to an open set and sets $\partial(a_j + \Omega^\circ)$ are closed, therefore $v(x) > 0$.

Claim 3.5. *There exists $\nu > 0$ such that*

$$(\forall x \in \overline{(\Omega - \Omega)_{\frac{1}{2}\varepsilon}})(v(x) \geq \nu > 0).$$

Proof. $v(x)$ is a continuous and positive function in the compact set $\overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}}$, therefore v(x) has its minimum in this set. Denoting

$$\nu := \min\{v(x) \mid x \in \overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}}\} \tag{3.4}$$

completes the proof. □

Claim 3.6. $\overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}} \subset (a_1+\Omega_{-\frac{1}{2}\nu}) \cup (a_2+\Omega_{-\frac{1}{2}\nu}) \cup ..... \cup (a_k+\Omega_{-\frac{1}{2}\nu})$.

Proof. Take an arbitrary $x \in \overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}}$. There exists such $s \in \{1,...,k\}$ that $v(x) = \rho(x, \partial(a_s+\Omega^\circ)) \geq \nu > \frac{1}{2}\nu$. Therefore $x \in (a_s+\Omega^\circ)_{-\frac{1}{2}\nu}$. □

Lemma 3.3. $\overline{(\overline{(\Omega-\Omega)}_{\frac{1}{2}\varepsilon})} \subset P$.

Proof. Using the knowledge $\overline{\overline{A}} = \overline{A}$, one has to prove $\overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}} \subset P$.

Using Claim (3.5) and Claim (3.6) we have

$$(\forall x \in \overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}})(\exists s \ \in \{1,...,k\})(x \in (a_s+\Omega^\circ)_{-\frac{1}{2}\nu} \subset (a_s+\Omega^\circ) \subset P).$$

□

Theorem 3.3. *The function f is upper semicontinuous in $\mathcal{M}$. That means*
$(\forall \Omega \in \mathcal{M})\ (\forall \varepsilon > 0)(\exists \delta > 0)(\forall \tilde{\Omega} \in \mathcal{M})\ (Dist(\Omega,\tilde{\Omega}) < \delta)(f(\tilde{\Omega}) \leq f(\Omega)+\varepsilon)$.

Proof. As values of f are only natural numbers, one has to prove
$(\forall \Omega \in \mathcal{M})\ (\exists \delta > 0)(\forall \tilde{\Omega} \in \mathcal{M})\ (Dist(\Omega,\tilde{\Omega}) < \delta)(f(\tilde{\Omega}) \leq f(\Omega))$.

Put

$$\delta := min\{\tfrac{1}{4}\varepsilon, \tfrac{1}{2}\nu\}. \tag{3.5}$$

Using previous lemmas and claims we obtain the following inclusions.
$Dist(\Omega,\tilde{\Omega}) < \delta$ implies that

$$\overline{\tilde{\Omega}-\tilde{\Omega}} \subset \overline{\Omega_\delta-\Omega_\delta}.$$

Lemma 3.4 says that

$$\overline{\Omega_\delta-\Omega_\delta} \subset \overline{(\Omega-\Omega)_{2\delta}}.$$

Using the definition of δ (3.5) we have

$$\overline{(\Omega-\Omega)_{2\delta}} \subset \overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}}.$$

Claim 3.6 states

$$\overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}} \subset (a_1+\Omega_{-\frac{1}{2}\nu}) \cup (a_2+\Omega_{-\frac{1}{2}\nu}) \cup ..... \cup (a_k+\Omega_{-\frac{1}{2}\nu}).$$

Lemma 2.1 tells that

$$(a_1+\Omega_{-\frac{1}{2}\nu}) \cup (a_2+\Omega_{-\frac{1}{2}\nu}) \cup ..... \cup (a_k+\Omega_{-\frac{1}{2}\nu}) \subset (a_1+\Omega^\circ_{-\delta}) \cup (a_2+\Omega^\circ_{-\delta}) \cup ..... \cup (a_k+\Omega^\circ_{-\delta}).$$

Using Lemma 3.1 we complete the proof

$$(a_1+\Omega^\circ_{-\delta}) \cup (a_2+\Omega^\circ_{-\delta}) \cup ..... \cup (a_k+\Omega^\circ_{-\delta}) \subset (a_1+\tilde{\Omega}^\circ) \cup (a_2+\tilde{\Omega}^\circ) \cup ..... \cup (a_k+\tilde{\Omega}^\circ).$$

Therefore $\overline{\tilde{\Omega}-\tilde{\Omega}} \subset (a_1+\tilde{\Omega}^\circ) \cup ... \cup (a_k+\tilde{\Omega}^\circ)$, i.e. $f(\tilde{\Omega}) \leq f(\Omega) = k$. □

3.2.1 Properties of semicontinuous functions

In the previous part we have shown that the function f is upper semicontinuous in $\mathcal{M}$. Let us find analogy in behavior of continuous and upper semicontinuous functions on compact sets.

Theorem 3.4. *Let g be an upper semicontinuous function in $\mathcal{X}$ and $\mathcal{X}$ is compact. g reaches its maximum K in $\mathcal{X}$.*

To prove Theorem 3.4 we use the following lemma.

Lemma 3.4. $(\forall \alpha \in \mathbb{R})\{\Omega \in \mathcal{X} \mid g(\Omega) < \alpha\}$ *is an open set.*

Proof. Denote $\mathcal{X}_\alpha := \{\Omega \in \mathcal{X} \mid g(\Omega) < \alpha\}$. One has to prove

$$(\forall \Omega \in \mathcal{X}_\alpha)(\exists \delta > 0)(\forall \tilde{\Omega} \in \mathcal{X})(Dist(\tilde{\Omega}, \Omega) < \delta)(\tilde{\Omega} \in \mathcal{X}_\alpha).$$

It follows directly from the definition of semicontinuous functions

$$(\forall \Omega \in \mathcal{X}_\alpha)(\exists \delta > 0)(\forall \tilde{\Omega} \in \mathcal{X})(Dist(\tilde{\Omega}, \Omega) < \delta)(f(\tilde{\Omega}) \leq f(\Omega) < \alpha).$$

□

Proof of Theorem 3.4. Denote $H_\Omega^\delta := \{\tilde{\Omega} \in \mathcal{X} \mid Dist(\Omega, \tilde{\Omega}) < \delta\}$. Using Lemma 3.4 we have

$$(\forall \Omega \in \mathcal{X})(\exists H_\Omega^{\delta_\Omega})(\forall \tilde{\Omega} \in H_\Omega^{\delta_\Omega})(g(\tilde{\Omega}) < g(\Omega) + 1).$$

As $\mathcal{X}$ is compact there exists a finite subcovering for the covering

$$\mathcal{X} \subset \cup_{\Omega \in \mathcal{X}} H_\Omega^{\delta_\Omega}.$$

Denote the finite subcovering by

$$\mathcal{X} \subset \cup_{i=1}^n H_{\Omega_i}^{\delta_i}.$$

So that we have

$$(\forall i \in \hat{n})(\forall \Omega_i \in \mathcal{X})(\exists H_{\Omega_i}^{\delta_i})(\forall \tilde{\Omega} \in H_{\Omega_i}^{\delta_i})(g(\tilde{\Omega}) < g(\Omega_i) + 1).$$

It implies that there exists a finite subset $\{\Omega_1, \Omega_2, ..., \Omega_n\} \subset \mathcal{X}$ such that

$$\sup_{\Omega \in \mathcal{X}} g(\Omega) \leq \max_{i \leq n} g(\Omega_i) + 1.$$

This proves that g is bounded above. Denote $K := \sup_{\Omega \in \mathcal{X}} g(\Omega)$. For all $n \in \mathbb{N}$ take $\Omega_n \in \mathcal{X}$ with $g(\Omega_n) \geq (K - 1/n)$. By the compactness of $\mathcal{X}$, Ω_n has a cluster point Ω. There exists a subsequence (Ω_{k_n}) of (Ω_n) such that $lim_{n\to\infty}\Omega_{k_n} = \Omega$. Thus we have

$$(\forall n \in \mathbb{N})(g(\Omega_{k_n}) \geq (K - 1/k_n))$$

which implies $g(\Omega) \geq K$. Hence, g reaches its maximum K in $\mathcal{X}$. □

We will apply theorem (3.4) on the function f which is semicontinuous on the compact space $\mathcal{M}$. We arrive at the following corollary.

Corollary 3.4.1. *The function f reaches its maximum on the space $\mathcal{M}$, i.e.*

$$(\exists K > 0)(\forall \Omega \in \mathcal{M})(f(\Omega) \leq K).$$

It remains to confirm that the function f reaches its maximum on the space κ, too.

3.3 Independence of f on affine transformations of Ω

The next step on the way which leads to the proof of boundedness of f on the space κ is to show independence of f on affine transformations of Ω. Independence of f on affine transformations of Ω is proved in Proposition 3.2 at the end of this part. Let us show two useful lemmas at first.

Lemma 3.5. $\overline{\Omega - \Omega} = \overline{\Omega} - \overline{\Omega}$.

Proof. $(\supset) : (\forall x, y \in \overline{\Omega})(\exists (x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \in \Omega)(\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y)$. Thus we have

$$x - y = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n - y_n).$$

As $x_n - y_n \in \Omega - \Omega$ for every $n \in \mathbb{N}$, it holds $\lim_{n\to\infty}(x_n - y_n) \in \overline{\Omega - \Omega}$.
$(\subset) :$ Take an arbitrary $z \in \overline{\Omega - \Omega}$. There exists a sequence $(z_n)_{n=1}^{\infty} \in \Omega - \Omega$ such that $\lim_{n\to\infty} z_n = z$. Moreover $z_n = x_n - y_n$, where $x_n, y_n \in \Omega$. Since Ω is bounded, every sequence in Ω has a Cauchy subsequence. We have Cauchy sequences $(x_{k_n})_{n=1}^{\infty}, (y_{l_n})_{n=1}^{\infty}$. It is possible to choose subsequences with the same indices: $(x_{s_n})_{n=1}^{\infty}$, $(y_{s_n})_{n=1}^{\infty}$ for which

$$\lim_{n\to\infty} x_{s_n} = x \in \overline{\Omega} \ \text{ and } \lim_{n\to\infty} y_{s_n} = y \in \overline{\Omega}.$$

The equality $z_{s_n} = x_{s_n} - y_{s_n}$ for every $n \in \mathbb{N}$ implies $z = x - y \in \overline{\Omega} - \overline{\Omega}$. □

Lemma 3.6. *A linear bijection: $\mathbb{R}^d \to \mathbb{R}^d$ is a continuous mapping.*

Proof. Let $(e_1, ..., e_d)$ be an orthonormal base of $\mathbb{R}^d$. As L is an onto and one-to-one mapping $(Le_1, ..., Le_d)$ is a base of $\mathbb{R}^d$, as well. Take an arbitrary $x \in \mathbb{R}^d$. There exist $(\alpha_1, ..., \alpha_d)$, where $(\forall j \in \{1, ..., d\})$ $(\alpha_j \in \mathbb{R})$ so that

$$x = \sum_{j=1}^{d} \alpha_j \ e_j.$$

Define $K := \max_{j\in\{1,...,d\}} |Le_j|$.

$$|Lx|^2 = |L(\sum_{j=1}^{d} \alpha_j \ e_j)|^2 = |\sum_{j=1}^{d} \alpha_j \ Le_j|^2 \le \sum_{j=1}^{d} (\alpha_j)^2 \ (Le_j)^2 \le K^2 \sum_{j=1}^{d} (\alpha_j)^2 = K^2 |x|^2.$$

It implies that L is a bounded and therefore continuous mapping

$$(\exists K > 0)(\forall x \in \mathbb{R}^d)(|Lx| \le K \ |x|).$$

□

Before we come to the promised proposition about independence of f on affine transformations, we show independence of f on linear transformations.

Proposition 3.1. *Let L: $\mathbb{R}^d \to \mathbb{R}^d$ be a linear bijection (an onto and one-to-one mapping). Then $f(L\Omega) = f(\Omega)$ for every Ω convex compact in $\mathbb{R}^d$.*

Proof. Using Lemma 3.5 we have

$$(\forall \ \Omega \text{ closed})(\Omega - \Omega = \overline{\Omega - \Omega}).$$

We suppose

$$\Omega - \Omega \subset (\Omega^\circ + a_1) \cup (\Omega^\circ + a_2) \cup ... \cup (\Omega^\circ + a_k),$$

where $a_i \in \mathbb{R}^d \quad \forall i \in \{1, ..., k\}$.
We have to verify that there exist $b_1, b_2, ..., b_k \in \mathbb{R}^d$ such that

$$\overline{L\Omega - L\Omega} \subset ((L\Omega)^\circ + b_1) \cup ((L\Omega)^\circ + b_2) \cup ... \cup ((L\Omega)^\circ + b_k).$$

Take arbitrary $x, y \in \Omega$. As L is linear, we have $Lx - Ly = L(x - y)$.
Since $x - y \in \Omega - \Omega$, there exists $z \in \Omega^\circ$ and $i \in \hat{k}$ such that $x - y = z + a_i$.
It implies that $L\Omega - L\Omega \subset (L\Omega^\circ + b_1) \cup (L\Omega^\circ + b_2) \cup ... \cup (L\Omega^\circ + b_k)$, where

$$b_i = La_i.$$

As L is regular (L is a bijection and L is linear and therefore by using Lemma 3.6 L and L^{-1} are continuous), it holds for every $\Omega \subset \mathbb{R}^d$

$$L\Omega^\circ = (L\Omega)^\circ,$$
$$L\overline{\Omega} = \overline{(L\Omega)}.$$

Using the previous facts we have

$$\overline{L\Omega - L\Omega} = L\Omega - L\Omega \subset (L\Omega^\circ + b_1) \cup (L\Omega^\circ + b_2) \cup ... \cup (L\Omega^\circ + b_k) = ((L\Omega)^\circ + b_1) \cup ((L\Omega)^\circ + b_2) \cup ... \cup ((L\Omega)^\circ + b_k),$$

where

$$b_i \in \mathbb{R}^d \quad \forall i \in \{1, ..., k\}.$$

This confirms the statement of the proposition $f(L(\Omega)) = f(\Omega)$ for every compact convex set Ω in $\mathbb{R}^d$. □

Now, we arrive at the important proposition about independence of f on affine transformations of Ω.

Proposition 3.2. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a bijective affine map. Then $f(A\Omega) = f(\Omega)$ for every convex compact set $\Omega \subset \mathbb{R}^d$ with non-empty interior.*

Proof. We know that for every closed set Ω it holds $\Omega - \Omega = \overline{\Omega - \Omega}$. We suppose that $f(\Omega) = k$, i.e. there exist $a_1, a_2, ..., a_k \in \mathbb{R}^d$ such that

$$\Omega - \Omega \subset (\Omega^\circ + a_1) \cup (\Omega^\circ + a_2) \cup ... \cup (\Omega^\circ + a_k).$$

We have to verify that there exist $b_1, b_2, ..., b_k \in \mathbb{R}^d$ such that

$$\overline{A\Omega - A\Omega} \subset ((A\Omega)^\circ + b_1) \cup ((A\Omega)^\circ + b_2) \cup ... \cup ((A\Omega)^\circ + b_k).$$

We use the following two facts:

1. As A is an affine map, there exist $z \in \mathbb{R}^d$ and a linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ so that for every $x \in \mathbb{R}^d$ we have
$$Ax = z + Lx.$$
2. As A is an affine bijection, L is a linear bijection and we obtain following equations for every $\Omega \subset \mathbb{R}^d$
$$A\Omega^\circ = z + L\Omega^\circ = z + (L\Omega)^\circ = (A\Omega)^\circ,$$
$$A\overline{\Omega} = z + L\overline{\Omega} = z + \overline{(L\Omega)} = \overline{A\Omega}.$$

Using the first knowledge we obtain for every $x, y \in \mathbb{R}^d$

$$Ax - Ay = (z + Lx) - (z + Ly) = Lx - Ly = L(x - y) = A(x - y) - z,$$

i.e. it holds for every $\Omega \subset \mathbb{R}^d$

$$A\Omega - A\Omega = A(\Omega - \Omega) - z.$$

Using the previous facts we have

$$\overline{A\Omega - A\Omega} = A\Omega - A\Omega = A(\Omega - \Omega) - z \subset (A\Omega^\circ + Aa_1) \cup (A\Omega^\circ + Aa_2) \cup ... \cup (A\Omega^\circ + Aa_k) - z =$$
$$= \big(((A\Omega)^\circ + z + La_1) \cup ((A\Omega)^\circ + z + La_2) \cup ... \cup ((A\Omega)^\circ + z + La_k)\big) - z = ((A\Omega)^\circ + b_1) \cup ((A\Omega)^\circ + b_2) \cup ... \cup ((A\Omega)^\circ + b_k),$$

where $b_i = La_i$ for every $i \in \{1, 2, ..., k\}$. This confirms the statement of the proposition $f(A(\Omega)) = f(\Omega)$ for every compact convex set Ω in $\mathbb{R}^d$. □

3.4 Proof of Theorem 3.2

Let us introduce a theorem from [3] which enables to complete the proof of Theorem 3.2, of course by using all of previously shown properties of the function f.

Theorem 3.5. *Every closed bounded convex set with non-empty interior Ω in $\mathbb{R}^d$ contains an ellipsoid $E+z$ such that $E+z \subset \Omega \subset dE+z$ (z being the centre of the ellipsoid $E+z$).*

Let us recall the statement of Theorem 3.2 in order to refresh what we actually want to prove. The function f is bounded on the metric space κ with the metric $Dist$.

Proof of Theorem 3.2. For $\alpha := \frac{1}{d}$ we have already proved in section 3.2 that f is upper semicontinuous on the space $\mathcal{M}$ of all convex closed sets Ω such that

$$\overline{B(0,\tfrac{1}{d})} \subset \Omega \subset \overline{B(0,1)}$$

and therefore f reaches its maximum K in $\mathcal{M}$. Using these facts we can easily verify that f is bounded in the space κ of all convex compact sets Ω in $\mathbb{R}^d$ with non-empty interior. Theorem 3.5 says that for every set Ω in κ there exists an ellipsoid such that $E+z \subset \Omega \subset dE+z$. Let A be such an affine mapping: $\mathbb{R}^d \to \mathbb{R}^d$ that $A(E+z) = \overline{B(0,\frac{1}{d})}$. It implies

$$\overline{B(0,\tfrac{1}{d})} \subset A(\Omega) \subset \overline{B(0,1)}.$$

Hence, $A(\Omega) \in \mathcal{M}$ and we know that there exists a positive constant K such that for all subsets $\tilde{\Omega}$ of $\mathcal{M}$, $f(\tilde{\Omega}) \leq K$. This fact and the independence of f on affine transformations,which was shown in section 3.3 and which says $f(\Omega) = f(A(\Omega))$, confirms that

$$(\exists K > 0)(\forall \Omega \in \kappa)(f(\Omega) = f(A\Omega) \leq K),$$

which completes proof of Theorem 3.2. $\square$

Chapter 4

Estimate of the universal constant

Let us recall again the introduction part of the work. We know that the convex compact set Ω with non-empty interior is the acceptance window of the cut-and-project set $\Sigma(\Omega)$. In the following part we will estimate cardinality of the set F in the Meyer property $\Sigma(\Omega) - \Sigma(\Omega) \subset \Sigma(\Omega) + F$ for variable acceptance windows Ω. We will deal with the following cases:

1. Ω being a regular polygon in $\mathbb{R}^2$,
2. Ω being an o-symmetric set in $\mathbb{R}^d$ and especially in $\mathbb{R}^2$,
3. Ω being a general convex compact set in $\mathbb{R}^d$ and especially in $\mathbb{R}^2$.

4.1 Estimate of values of f for polygons

In the previous section we have defined the function f and we have shown that f is bounded on the space κ of all convex compact sets with non-empty interior in $\mathbb{R}^d$. In this part the aim is to estimate the value of the function f for regular polygons, which responds to estimation of the upper bound on the cardinality of F.

Due to that fact that f is invariant under affine transformations of Ω it suffices to consider only regular polygons centered at the origin and having radius 1.

For estimation of the upper bound on $f(\Omega)$ for regular n-gons Ω with $n \geq 7$ it is useful to know that $f(\overline{B(0,1)}) = 8$ (this claim will be proved in the section 4.2) and to determine the minimal radius r, such that 8 copies of the open ball $B(0,r)$ are sufficient to cover the closed ball $\overline{B(0,2)}$.

Proposition 4.1. *Let $r > 0$ such that there exist points $a_1, a_2, ..., a_8$ satisfying*

$$\overline{B(0,2)} \subset (B(0,r) + a_1) \cup (B(0,r) + a_2) \cup ... \cup (B(0,r) + a_8).$$

Then $r > a = \frac{2}{1+2\cos\frac{2\pi}{7}}$.

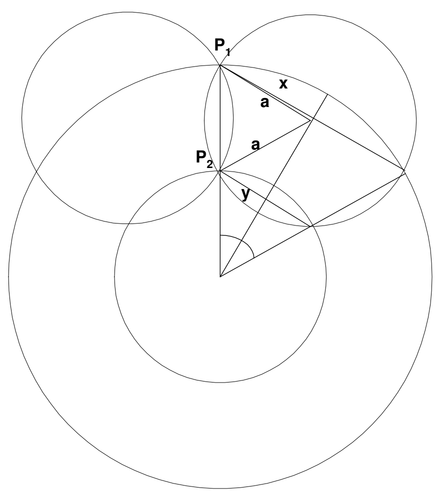


Figure 4.1: Covering of $\overline{B(0,2)}$ with 8 translated balls $\overline{B(0,a)}$.

Proof. Geometrical considerations show that the lower bound on the radius r from the proposition is the value a, for which circumferences of three translated balls $B(0,a)$ meet at points P_1, P_2.

$$\sqrt{2^2 - x^2} = \sqrt{a^2 - x^2} + 2a - 2(a - \sqrt{a^2 - y^2})$$
$$\sqrt{4 - x^2} = \sqrt{a^2 - x^2} + 2\sqrt{a^2 - y^2} \tag{4.1}$$

Using the cosine theorem we have

$$(2y)^2 = a^2 + a^2 - 2a^2\cos(\alpha)$$
$$(2x)^2 = 2^2 + 2^2 - 2 \cdot 2^2 \cos(\alpha)$$
$$x^2 = 2(1 - \cos(\alpha)) \tag{4.2}$$

$$y^2 = \frac{a^2}{2}(1 - \cos(\alpha)) \tag{4.3}$$

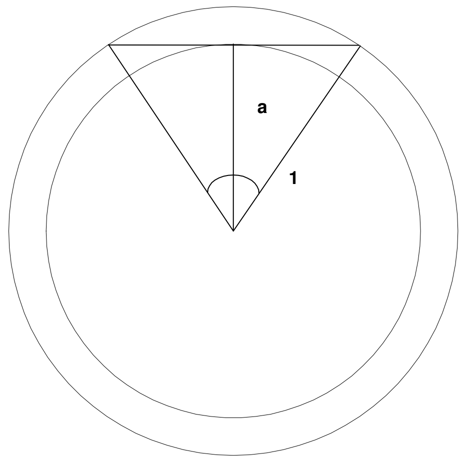
Substituing x and y in equation (4.1) by expressions from (4.2) and (4.3) we have

$$\sqrt{1+\cos(\alpha)}\sqrt{2}(1-a) = \sqrt{a^2 - 2 + 2\cos(\alpha)}.$$

The searched radius $a = \frac{2}{1+2\cos(\alpha)}$, where $\alpha := \frac{2\pi}{7}$, what was to show. $\square$

Remark 3. *The approximate value of a is $a \doteq 0,89$.*

Claim 4.1. *Let Ω be a regular n-gon for $n \geq 7$. Then $f(\Omega) \leq 8$.*



Proof. If there exists $\varepsilon > 0$ such that

$$B(0, a+\varepsilon) \subset \Omega^\circ \subset B(0,1), \tag{4.4}$$

then using Proposition 4.1 we obtain

$$\Omega - \Omega \subset \overline{B(0,2)} \subset (a_1 + B(0, a+\varepsilon)) \cup ... \cup (a_8 + B(0, a+\varepsilon)) \subset$$

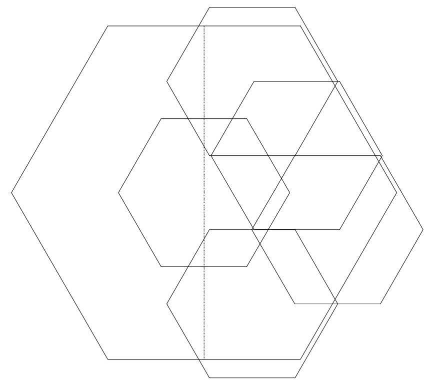
$$(a_1 + \Omega^\circ) \cup ... \cup (a_8 + \Omega^\circ).$$

We want to estimate the central angle γ of the n-gon Ω such that the inclusions(4.4) hold. Considering the figure above we have an implicite equation for β

$$\cos(\tfrac{\beta}{2}) = a. \tag{4.5}$$

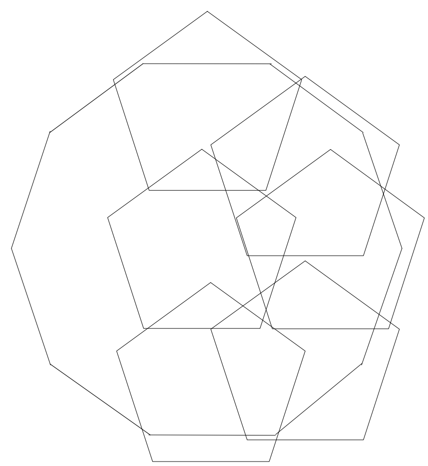
Any regular polygon, which has the central angle γ smaller than β, can be covered by 8 translated copies of $B(0, a+\varepsilon)$. Using equation (4.5) we obtain $\frac{2\pi}{6} > \beta > \frac{2\pi}{7}$, therefore $f(\Omega) \leq 8$ for every n-gon Ω with $n \geq 7$. $\square$

Claim 4.2. *Let Ω be a regular hexagon. Then $f(\Omega) = 9$.*



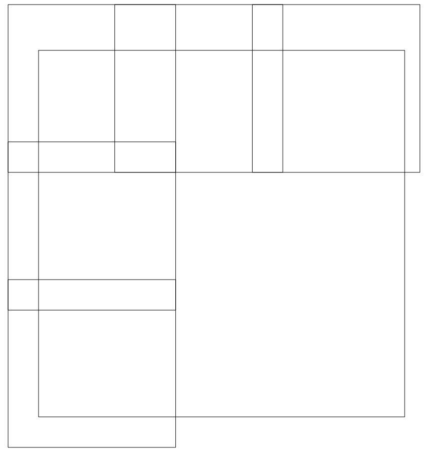
Proof. If Ω is a hexagon having the radius of the length 1, then $\Omega - \Omega$ is a hexagon having radius of double length. To cover the partimeter of the closed hexagon of radius 2, we need 8 open hexagons of radius 1, and to cover the centre one more open hexagon of radius 1 is necessary. ☐

Claim 4.3. *Let Ω be a regular pentagon. Then $f(\Omega) = 9$.*



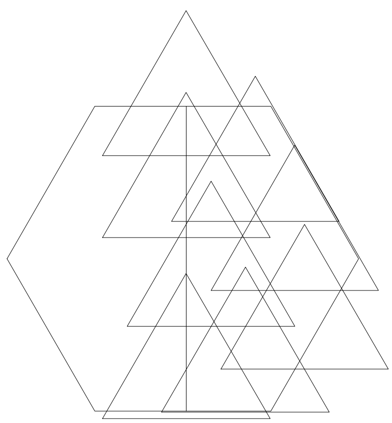
Proof. If Ω is a pentagon having the radius of the length 1, then $\Omega - \Omega$ is a regular 10-gon having the radius of the length $1 + \frac{\sqrt{3}}{2}$.
The same explanation as by hexagon. ☐

Claim 4.4. *Let Ω be a square. Then $f(\Omega) = 9$.*



Proof. If Ω is a square having sides of the length 1, then $\Omega - \Omega$ is a square of double size. To cover the upper side of the closed square of the length 2 we need 3 open squares of half-size. The same for the lower side and the middle side. $\square$

Claim 4.5. *Let Ω be a triangle. Then $f(\Omega) \leq 13$.*



Proof. If Ω is a triangle with sides of the length 1, then $\Omega - \Omega$ is a hexagon with a radius of the length 1. 4 open triangles are needed for covering two left-handed sides of the closed hexagon. 4 more triangles are needed to cover half a hexagon. The most advantageous way is to put 3 of them on the perpendicular axis. Therefore $3 + 2 \times 5$ triangles are sufficient for covering the hexagon. $\square$

We have thus shown the following proposition.

Proposition 4.2. *Let Ω be a regular n-gon.*
If $n \geq 7$, then $f(\Omega) = 8$.
If $n = 4, 5, 6$, then $f(\Omega) = 9$.
If $n = 3$, then $f(\Omega) \leq 13$.

4.2 Estimate of the universal constant for o-symmetric sets

The aim of this part is to estimate the upper bound on $f(\Omega)$ for all o-symmetric convex sets Ω with non-empty interior. Firstly, we consider the general case when the acceptance window Ω is an o-symmetric set in $\mathbb{R}^d$ and then we limit our considerations to dimension $d = 2$.

Definition 4.1. *Let us define the constant K as the minimal universal upper bound on the set $\{f(\Omega) \mid \Omega$ convex, compact, o-symmetric , with non-empty interior$\}$, i.e.*

$$K := \min\{L > 0 \mid (\forall\ \Omega \in \kappa, \Omega \textit{ o-symmetric})(f(\Omega) \le L)\}.$$

Let us introduce a proposition which describes property of $\Omega - \Omega$ for o-symmetric sets. We start with a lemma which is useful to prove the mentioned proposition.

Lemma 4.1. *Let H be a convex set in $\mathbb{R}^d$. Then $H + H = 2H$.*

Proof. As H is convex $(\forall x, y \in H)(\frac{1}{2}(x+y) \in H)$.Thus $x+y \in 2H$. Conversely, every point $x \in H$ can be written as $\frac{1}{2}(x+x)$. Hence $2x = x+x \in H+H$. So we have the formula $H+H = 2H$. $\square$

Proposition 4.3. *Let Ω be a convex o-symmetric set with non-empty interior in $\mathbb{R}^d$.*
Then $\Omega - \Omega = 2\Omega$.

Proof. As Ω is convex, $\Omega + \Omega = 2\Omega$.
As Ω is o-symmetric, $\Omega = -\Omega$.
It implies $\Omega - \Omega = 2\Omega$. $\square$

Now, we deal with the result of John [3], which enables to estimate the searched constant K.

Theorem 4.1 (John)**.** *For every compact o-symmetric set Ω with non-empty interior in $\mathbb{R}^d$ there exists an o-symmetric ellipsoid E such that $E \subset \Omega \subset d^{\frac{1}{2}}E$.*

Proof. Let Ω be a closed bounded o-symmetric convex set with non-empty interior and let E be an o-symmetric ellipsoid of maximum volume contained in Ω. The existence of such an ellipsoid follows from the compactness of the set of collections $\{a_1, a_2, ..., a_d\}$, where the a_i are mutually orthogonal and the ellipsoid with semi-axes $a_1, ..., a_d$ is contained in Ω. Its volume is a continuous function in a compact space. Hence, it reaches its maximum.
We take an arbitrary point a on the boundary of Ω and prove that it belongs to $d^{\frac{1}{2}}E$.

We may suppose that E is the sphere $|x| < 1$ and that a has the form $(\alpha, 0, ..., 0)$, where $\alpha > 1$. Let us consider the convex hull Ω', say, of E and the points $\pm a$. Let Ω'_2 be the intersection of Ω' and the plane $x_3 = x_4 = ... = x_d = 0$ and, for $0 < \theta \le 1$, let F_θ denote the linear transformation of that plane given by $x'_1 = \theta x_1, x'_2 = x_2$. Then Ω'_2 is the convex hull of the circular disc $(x_1)^2 + (x_2)^2 \le 1$ and the points $(\pm\alpha, 0)$, it is bounded by two arcs of this disc and by segments of the four lines $\pm\alpha^{-1}x_1 \pm \beta^{-1}x_2 = 1$, where $\beta = \alpha(\alpha^2 - 1)^{-\frac{1}{2}}$. Thus $F_\theta\Omega'_2$ is bounded by two arcs of the ellipse $(x_1/\theta)^2 + (x_2)^2 \le 1$ and by segments of the lines $\pm(\theta\alpha)^{-1}x_1 \pm \beta^{-1}x_2 = 1$. It is easily verified that, for $0 < \theta \le 1$, $F_\theta\acute{\Omega}_2$ contains the disc $(x_1)^2 + (x_2)^2 \le \rho^2$, where $\rho = \theta\alpha\beta(\theta^2\alpha^2 + \beta^2)^{-\frac{1}{2}}$. So Ω'_2 contains the ellipse $(x_1/\theta)^2 + (x_2)^2 \le \rho^2$ and so Ω' contains the ellipsoid E_θ given by

$$E_\theta : (\theta x_1)^2 + (x_2)^2 + ... + (x_d)^2 \le \rho^2. \tag{4.6}$$

Since Ω' is contained in Ω, it is also true that Ω contains E_θ. By the choice of E this implies that $V(E_\theta) \le V(E)$, for all θ with $0 < \theta \le 1$. We deduce from this fact that $\alpha \le d^{\frac{1}{2}}$, in the following way.

Let κ_d denote the volume of the unit sphere. Then

$$V(E_\theta) = \theta^{-1}\rho^d\kappa_d = \theta^{d-1}(\alpha^{-2} + \theta^2(1 - \alpha^{-2}))^{-\frac{1}{2}d}\kappa_d, \tag{4.7}$$

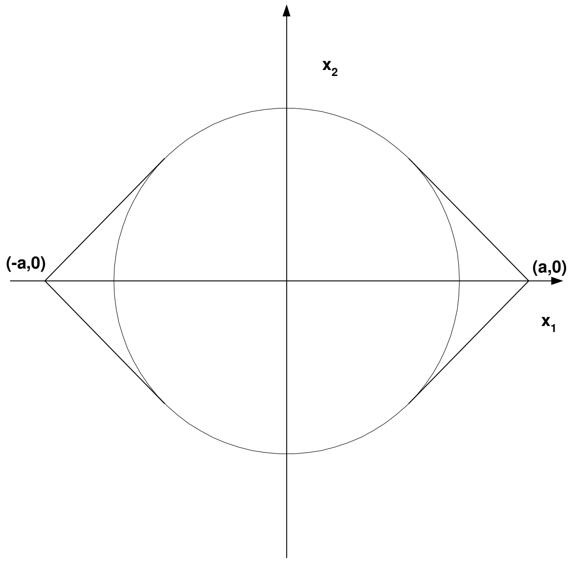


Figure 4.2: Illustration of Ω_2'.

because $\rho^2 = \theta^2(\theta^2\beta^{-2} + \alpha^{-2})^{-1} = \theta^2(\theta^2(1-\alpha^{-2}) + \alpha^{-2})^{-1}$. The expression in the right hand member of (4.7), as a function of θ, tends to zero as $\theta \to 0$ or $\theta \to \infty$ and attains a strong maximum if

$$\frac{d-1}{\theta} - \frac{1}{2}d \cdot \frac{2\theta(1-\alpha^{-2})}{\alpha^{-2} + \theta^2(1-\alpha^{-2})} = 0, \text{ i.e., if } \frac{d-1}{d} = \frac{\theta^2(\alpha^2-1)}{1+\theta^2(\alpha^2-1)},$$

or also $\theta^2 = (d-1)/(\alpha^2-1)$. However, by the foregoing, the maximum can only be attained for a value $\theta = \theta_0$ with $\theta_0 \geq 1$. So we have $d - 1 \geq \alpha^2 - 1$, and so $\alpha \leq d^{\frac{1}{2}}$.

The last result means that the boundary point a belongs to $d^{\frac{1}{2}}E$. By the arbitrariness of a, this proves the theorem. □

From now on, we limit our considerations to dimension $d = 2$. We apply the result of John on o-symmetric compact sets in $\mathbb{R}^2$.

Corollary 4.1.1. *Let Ω be a convex compact o-symmetric set with non-empty interior in $\mathbb{R}^2$. Then*

$$f(\Omega) \leq \text{ the number of copies } B(0,1) \text{ which are needed for covering } \overline{B(0, 2\sqrt{2})}.$$

Proof. Since $E \subset \Omega \subset \sqrt{2}E$ there exists a linear mapping L such that

$$\overline{B(0,1)} \subset L(\Omega) \subset \overline{B(0,\sqrt{2})}.$$

As Ω is o-symmetric, $\Omega - \Omega = 2\Omega$. We have

$L(\Omega) - L(\Omega) = 2L(\Omega) \subset \overline{B(0, 2\sqrt{2})} \subset (a_1 + B(0,1)) \cup ... \cup (a_k + B(0,1)) \subset$
$\subset (a_1 + L(\Omega)^\circ) \cup ... \cup (a_k + L(\Omega)^\circ)$.
Therefore $f(\Omega) = f(L(\Omega)) = K \leq k$. □

Proposition 4.4. *Let Ω be an o-symmetric convex compact set in $\mathbb{R}^2$. Then $f(\Omega) \leq 16$.*

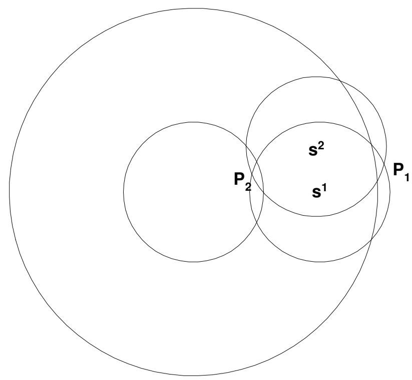


Figure 4.3: For covering the circumference of $\overline{B(0, 2\sqrt{2})}$ 15 translated copies of $B(0,1)$ are sufficient.

Proof. Let s^1, s^2 be neighbouring vertices of a regular 15-gon centered at 0 and having the radius $r = 1,95$. If we show that one of the points of intersection of $B(s^1, 1)$ and $B(s^2, 1)$ lies in $B(0,1)$ and the other one out of $\overline{B(0, 2\sqrt{2})}$, then it is clear that 16 balls of the radius 1 suffice to cover $\overline{B(0, 2\sqrt{2})}$. More precisely, 15 balls suffice to cover the circumference of $\overline{B(0, 2\sqrt{2})}$ and one more is needed to cover the middle. Let us determine the coordinates (in standard basis) of the points of intersection, say $P_i = (x_i, y_i)$, where $i = 1, 2$. $P_i \in \partial B(s^1, 1)$ and at one $P_i \in \partial B(s^2, 1)$, where $s^1 = (r, 0)$ and $s^2 = (r \cos \frac{2\pi}{15}, r \sin \frac{2\pi}{15})$, i.e. we have the following equations for the points of intersection $P_i = (x_i, y_i)$

$$(x_i - r)^2 + y_i^2 = 1, \tag{4.8}$$
$$(x_i - r\cos \tfrac{2\pi}{15})^2 + (y_i - r \sin \tfrac{2\pi}{15})^2 = 1.$$

We obtain

$$y_i = \frac{1 - \cos \frac{2\pi}{15}}{\sin \frac{2\pi}{15}} x_i \tag{4.9}$$

and we substitute y_i in (4.8). Then we have

$$\Big(\big(\tfrac{1-\cos \frac{2\pi}{15}}{\sin \frac{2\pi}{15}}\big)^2 + 1\Big)x_i^2 - 2x_i r + r^2 - 1 = 0$$

with two roots

$$x_1 = \frac{r + \sqrt{r^2 - (r^2-1)\Big(\big(\frac{1-\cos \frac{2\pi}{15}}{\sin \frac{2\pi}{15}}\big)^2 + 1\Big)}}{\Big(\big(\frac{1-\cos \frac{2\pi}{15}}{\sin \frac{2\pi}{15}}\big)^2 + 1\Big)}, \quad x_2 = \frac{r - \sqrt{r^2 - (r^2-1)\Big(\big(\frac{1-\cos \frac{2\pi}{15}}{\sin \frac{2\pi}{15}}\big)^2 + 1\Big)}}{\Big(\big(\frac{1-\cos \frac{2\pi}{15}}{\sin \frac{2\pi}{15}}\big)^2 + 1\Big)}.$$

Now, using (4.9) one can easily calculate the values of y_1, y_2 and notice that $P_1 = (x_1, y_1)$ lies out of $\overline{B(0, 2\sqrt{2})}$ and $P_2 = (x_2, y_2)$ lies in $B(0,1)$. The coordinates of points of intersection are

$$P_1 = (x_1, y_1) \doteq (2,7289; 0,58), \;\; P_2 = (x_2, y_2) \doteq (0,9716; 0,2065).$$

$\square$

It is likely that the o-symmetric convex compact set with non-empty interior, for which the function f reaches its minimum, is a ball.

Claim 4.6. *$f(\overline{B(0,1)}) = 8$, i.e. $\overline{B(0,2)} \subset (a_1 + B(0,1)) \cup ... \cup (a_8 + B(0,1))$.*

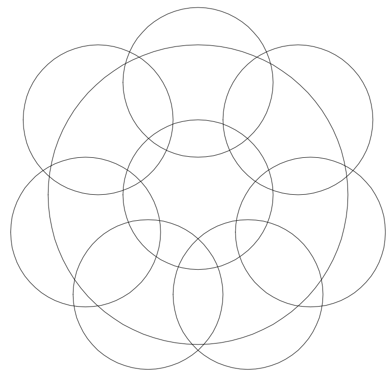


Figure 4.4: 8 open balls suffice.

Proof. Looking at the picture below, we can see that 6 translated copies of $\overline{B(0,1)}$ are sufficient to cover a hexagon having the radius of the length 2, therefore 6 translated copies of the ball $\overline{B(0,1)}$ are sufficient to cover the circumference of $\overline{B(0,2)}$, but 6 open copies of $B(0,1)$ are not sufficient. The picture above illustrates that 7 open translated copies of $B(0,1)$ are sufficient to cover the circumference of $\overline{B(0,2)}$ and one more is needed to cover the interior. $\square$

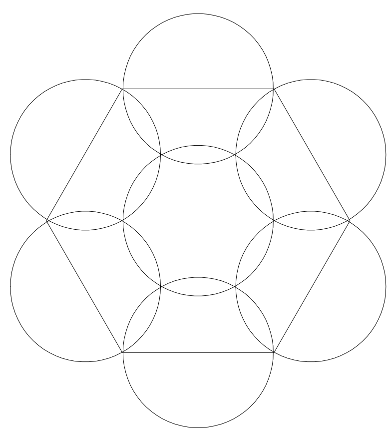
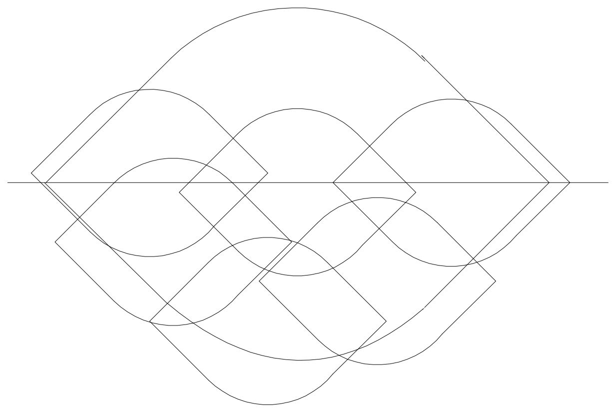


Figure 4.5: 7 open balls are not sufficient.



We conjecture that the function f on the space of all o-symmetric convex compact sets with non-empty interior in $\mathbb{R}^2$ is bounded by the constant $K = 9$. Paying attention to the proof of John's theorem the role of the most problematic set, which raises the constant, could play the convex hull of $\overline{B(0,1)}$ and the points $(\pm\sqrt{2}, 0)$.

Claim 4.7. *Let Ω be the convex hull of $\overline{B(0,1)}$ and the points $(\pm\sqrt{2}, 0)$. Then $f(\Omega) = 9$.*

Proof. Considering the figure above one can notice that 8 translated copies of the convex hull of $\overline{B(0,1)}$ and the points $(\pm\sqrt{2}, 0)$ are needed for covering the circumference of the convex hull of $\overline{B(0,2)}$ and the points $(\pm 2\sqrt{2}, 0)$ and one more is necessary to cover the interior. $\square$

By using the previous result and results for regular o-symmetric polygons we arrived at the following conjecture.

Conjecture 1. *Let Ω be an o-symmetric convex compact set with non-empty interior in $\mathbb{R}^2$. Then*

$$8 \leq f(\Omega) \leq 9.$$

4.3 Estimate of the universal constant

The aim of this section is to estimate the minimum of upper bounds on $f(\Omega)$ for all convex compact sets Ω with non-empty interior in $\mathbb{R}^d$. Firstly, we consider a general acceptance window Ω in $\mathbb{R}^d$ and then we limit our considerations to dimension $d = 2$. In order to find at least a rough estimate let us recall Theorem 3.5 which was mentioned in Chapter 3.

Theorem 4.2. *Every convex compact set Ω with non-empty interior in $\mathbb{R}^d$ contains an ellipsoid $E + z$ such that $E + z \subset \Omega \subset dE + z$ (z being the centre of the ellipsoid $E + z$).*

Considering the above theorem and the fact that f is invariant under affine transformations we arrive at the following proposition.

Proposition 4.5. *Let $d \geq 2$ and let Ω be a convex compact set in $\mathbb{R}^d$ with non-empty interior. Then $f(\Omega) \leq (2d^2 + 1)^d$.*

Proof. Since $z + E \subset \Omega \subset z + dE$ there exists an affine mapping A such that

$$\overline{B(0,1)} \subset A(\Omega) \subset \overline{B(0,d)}.$$

Denote $k :=$ the number of copies $B(0,1)$ which are needed for covering $\overline{B(0,2d)}$.
Then we have
$A(\Omega) - A(\Omega) \subset \overline{B(0,d)} - \overline{B(0,d)} = \overline{B(0,2d)} \subset (a_1 + B(0,1)) \cup ... \cup (a_k + B(0,1)) \subset$
$\subset (a_1 + A(\Omega)^\circ) \cup ... \cup (a_k + A(\Omega)^\circ)$.
Therefore $f(\Omega) = f(A(\Omega)) \leq k$.

Now, we prove $k \leq (2d^2 + 1)^d$. Let us show that

$$\overline{B(0,2d)} \subset \bigcup_{x \in J} B(x,1),$$

where $J = \{\frac{1}{d}(x^1, x^2, ..., x^d) \in \mathbb{R}^d \mid x^i = 0, 1, ..., 2d^2,\ i = 1, 2, ..., d\}$. One can easily notice that $\#J = (2d^2 + 1)^d$. If we confirm that $\overline{B(0,2d)} \subset \bigcup_{x \in J} B(x,1)$, we will have proved that

$$(\forall \Omega \in \kappa)\big(f(\Omega) \leq (2d^2 + 1)^d\big).$$

Take arbitrary $y = (y^1, y^2, ..., y^d) \in \overline{B(0,2d)}$, then there exists $z = (z^1, z^2, ..., z^d) \in J$ which fulfils

$$|y^1 - z^1|^2 + |y^2 - z^2|^2 + ... + |y^d - z^d|^2 \leq \tfrac{1}{d^2} + \tfrac{1}{d^2} + ... + \tfrac{1}{d^2} = \tfrac{1}{d} < 1,$$

i.e. there exists $z \in J$ such that $y \in B(z,1)$. □

The estimation from Proposition 4.5 is universal and consequently rough. We will limit our considerations to dimension $d = 2$ and search for a more precise estimation.

We apply theorem (4.2) on sets in $\mathbb{R}^2$.

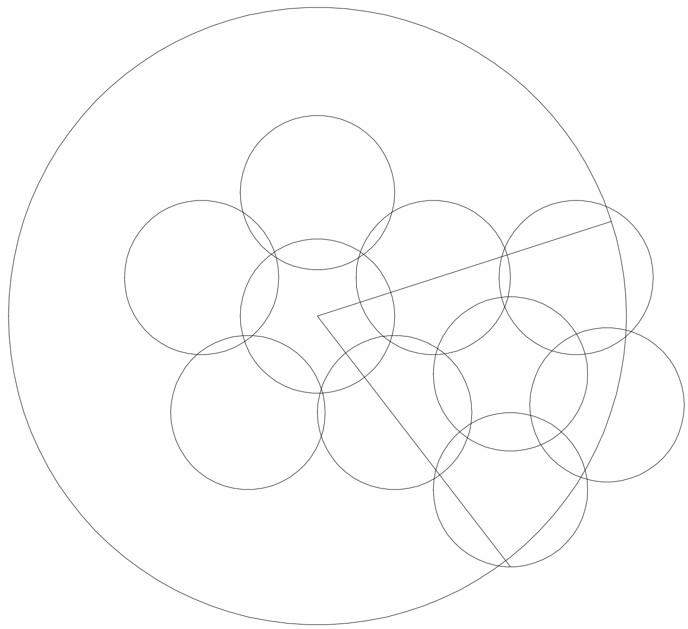
Corollary 4.2.1. *Let Ω be a convex compact set with non-empty interior in $\mathbb{R}^2$. Then*

$$f(\Omega) \leq \text{ the number of copies } B(0,1) \text{ which are needed for covering } \overline{B(0,4)}.$$

Proof. Analogy of the first part of the proof of Proposition 4.5. □

Theorem 4.3. *Let Ω be a convex compact set with non-empty interior in $\mathbb{R}^2$. Then $f(\Omega) \leq 26$.*

Proof. The following figure shows that it is possible to cover $\overline{B(0,4)}$ by 26 translated copies of the ball $B(0,1)$.
Hence, $f(\Omega) \leq 26$ for every convex compact set Ω with non-empty interior in $\mathbb{R}^2$.



□

We conjecture that this estimate is fairly rough. Considering the case of polygons, the shape, which seems to raise the upper bound on $f(\Omega)$ for all nonsymmetric convex sets Ω, is a triangle. For Ω being a triangle we have $f(\Omega) \leq 13$.
It is likely that the minimal value of the function f for general convex sets is reached on n-gons, where $n \geq 7$ and n odd. In this case $f(\Omega) = 8$.

Previous considerations lead us to the following conclusion.

Conjecture 2. *Let Ω be a convex compact set with non-empty interior in $\mathbb{R}^2$. Then $8 \leq f(\Omega) \leq 13$.*

Chapter 5

Unboundedness of the function f on the space of general compact sets

So far we have examined values of the function f on the space of convex compact sets in $\mathbb{R}^d$ with non-empty interior.

We stand in front of a natural question: Is convexity of the set Ω necessary for boundedness of the function f? The answer is positive. We will prove this statement by construction of a sequence of non-convex compact sets $(\Omega_n)_{n=1}^{\infty}$ with the property $\overline{\Omega_n^{\circ}} = \overline{\Omega_n} \neq \emptyset$ and we will show that

$$\lim_{n\to\infty} f(\Omega_n) = +\infty.$$

To this purpose let us introduce the notion of star-shaped sets in $\mathbb{R}^d$.

Definition 5.1. *Let Ω be a set in $\mathbb{R}^d$. Then Ω is called star-shaped if it holds*

$$(\exists x \in \Omega)(\forall y \in \Omega)(\overline{xy} \in \Omega),$$

where $\overline{xy}$ is a line segment connecting x, y.

Remark 4. *Apparently, star-shaped sets are the nearest generalisation of convex sets. In spite of this fact the function f is not bounded on the space of star-shaped sets as follows from the proof of the following theorem.*

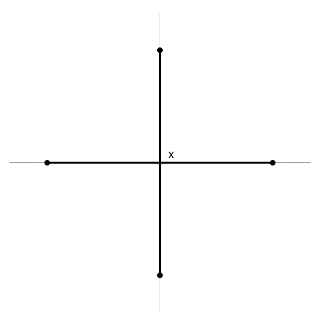


Figure 5.1: Illustration of a star-shaped set.

Theorem 5.1. *The function f is not bounded on the space of compact sets in $\mathbb{R}^d$ with the property $\overline{\Omega^\circ} = \overline{\Omega} \neq \emptyset$.*

Proof. We divide the proof into three parts. We will show unboundedness of the function f on the space of compact sets in $\mathbb{R}^2$ by construction of a sequence of star-shaped sets Ω_n fulfilling the property $\overline{\Omega^\circ} = \overline{\Omega} \neq \emptyset$ and such that $\lim_{n\to\infty} f(\Omega_n) = +\infty$. Then we will prove unboundedness of the function f on the space of compact sets in $\mathbb{R}^d$ by construction of a sequence of prisms having a star-shaped set as their base. Finally, we will deal with the case in $\mathbb{R}^1$, where any star-shaped set is convex. Therefore we will use some other idea to prove unboundedness of the function f on the space of compact sets in $\mathbb{R}$.

1. Let $d = 2$ be the dimension of $\mathbb{R}^d$. We define a sequence of star-shaped compact sets $(\Omega_n)_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $\overline{\Omega_n^\circ} = \overline{\Omega_n} \neq \emptyset$ in the following way. For every $n \in \mathbb{N}$, Ω_n is the union of five convex hulls (coordinates of points are written in the standard basis of $\mathbb{R}^2$):

$$H_1 \text{ is the convex hull of points } (\tfrac{1}{n}, \tfrac{1}{n}), (\tfrac{-1}{n}, \tfrac{1}{n}), (\tfrac{-1}{n}, \tfrac{-1}{n}), (\tfrac{1}{n}, \tfrac{-1}{n}),$$

i.e. H_1 is a square with side-length $\frac{2}{n}$ centered at the origin.

$$H_2 \text{ is the convex hull of points } (\tfrac{1}{n}, \tfrac{1}{n}), (0,1), (\tfrac{-1}{n}, \tfrac{1}{n}).$$

$$H_3 \text{ is the convex hull of points } (\tfrac{-1}{n}, \tfrac{1}{n}), (-1,0), (\tfrac{-1}{n}, \tfrac{-1}{n}).$$

$$H_4 \text{ is the convex hull of points } (\tfrac{-1}{n}, \tfrac{-1}{n}), (0,-1), (\tfrac{1}{n}, \tfrac{-1}{n}).$$

$$H_5 \text{ is the convex hull of points } (\tfrac{1}{n}, \tfrac{-1}{n}), (1,0), (\tfrac{1}{n}, \tfrac{1}{n}).$$

$$\Omega_n := \bigcup_{i=1}^{5} H_i. \tag{5.1}$$

The set Ω_n is illustrated on the image bellow.

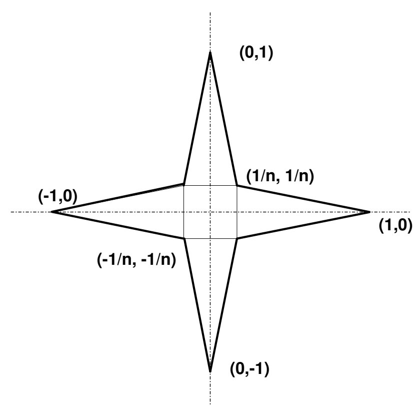


Figure 5.2: Illustration of Ω_n for $n = 6$.

Considering the picture above we can calculate the volume of Ω_n:

$$\text{vol } \Omega_n = 4\tfrac{1}{n}(1 - \tfrac{1}{n}) + 4\tfrac{1}{n^2} = \tfrac{4}{n}.$$

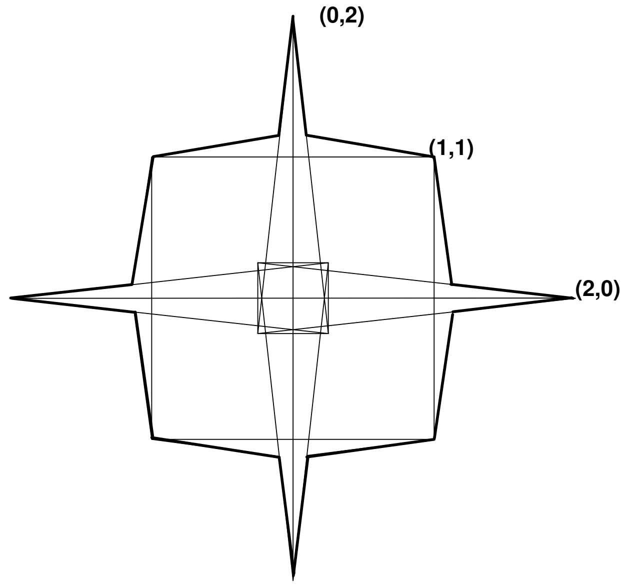


Figure 5.3: Illustration of $\Omega_n - \Omega_n$ for $n = 6$.

It is not difficult to construct the difference set $\Omega_n - \Omega_n$ for the above Ω_n, see figure 5.3. Note that $\Omega_n - \Omega_n$ contains the square of side-length 2 centered at the origin. Let us explain why this is true for every $n \in \mathbb{N}$. It is due to the fact that for every point (x_1, x_2) of the square, i.e.

$$-1 \leq x_1 \leq 1,$$
$$-1 \leq x_2 \leq 1,$$

it is possible to write

$$(x_1, x_2) = (x_1, 0) - (0, -x_2),$$

where $(x_1, 0), (0, -x_2)$ are points of line segments, which are parts of Ω_n for every $n \in \mathbb{N}$.

For the volume of $\Omega_n - \Omega_n$ we have

$$\mathrm{vol}(\Omega_n - \Omega_n) \geq 4.$$

One can realize that with the growing n the volume of Ω_n tends to zero while the volume of $\Omega_n - \Omega_n$ remains greater than 4. The fact that

$$\lim_{n\to\infty} f(\Omega_n) \geq \lim_{n\to\infty} \frac{\mathrm{vol}\ (\Omega_n - \Omega_n)}{\mathrm{vol}\ \Omega_n} \geq \lim_{n\to\infty} \frac{4}{\frac{4}{n}} = \lim_{n\to\infty} n = +\infty$$

proves unboundedness of f on the space of compact sets in $\mathbb{R}^2$.

2. We will use analogical considerations for the proof that the function f is not bounded on the space of general compact sets in $\mathbb{R}^d$ with $d > 2$. Let us define a sequence of compact sets $(\Omega_n)_{n=1}^{\infty} \subset \mathbb{R}^d$ satisfying the condition $\overline{\Omega_n^{\circ}} = \overline{\Omega_n} \neq \emptyset$ by

$$\Omega_n := \{(x_1, x_2, ..., x_d) \in R^d \mid (x_1, x_2) \in \bigcup_{i=1}^{5} H_i, x_j \in [0,1] \text{ for } j = 3, ..., d\},$$

where $\bigcup_{i=1}^{5} H_i$ is the union of convex hulls from (5.1).

In $\mathbb{R}^3$ Ω_n is a prism having the star-shaped set from the figure 5.2 as its base.

Considering this sequence we can calculate the volume of Ω_n, we have vol $\Omega_n = \frac{4}{n}$.

Let us explain that $\Omega_n - \Omega_n$ contains a d-dimensional cube having side-length 2 and centered at the origin for every $n \in \mathbb{N}$. Take arbitrary element of that d-dimensional cube, i.e.

$$(x_1, x_2, ..., x_d),$$

where $-1 \leq x_i \leq 1$ for every $i \in \{1, 2, ..., d\}$. Denote by $y := (x_1, 0, y_3, ...y_d)$, where $y_i = x_i$ for $x_i > 0$, otherwise $y_i = 0$ and denote by $z := (0, -x_2, -z_3, ... - z_d)$, where $z_j = x_j$ for $x_j < 0$, otherwise $z_j = 0$. The element of the cube can be written using the above notation

$$(x_1, x_2, ..., x_d) = y - z,$$

where $y \in \Omega_n$ and $z \in \Omega_n$. Therefore $\Omega_n - \Omega_n$ contains a d-dimensional cube of the side-length 2 and $\text{vol}(\Omega_n - \Omega_n) \geq 2^d$.

Using the knowledge of volumes we have

$$\lim_{n\to\infty} f(\Omega_n) \geq \lim_{n\to\infty} \frac{\text{vol }(\Omega_n - \Omega_n)}{\text{vol }\Omega_n} \geq \lim_{n\to\infty} \frac{2^d}{\frac{4}{n}} = +\infty,$$

which proves that the function f is unbounded on the space of general compact sets in $\mathbb{R}^d$ with $d \geq 2$.

3. The last question left is unboundedness of the function f on the space of general compact sets in $\mathbb{R}^1$. In 1-dimensional case any compact set is star-shaped if and only if it is convex. We cannot use analogical constructions as in the previous cases to prove unboundedness of f because the function f is bounded on the space of compact star-shaped sets in $\mathbb{R}$ with non-empty interior. We will show unboundedness of f for general compact sets in $\mathbb{R}$ by the following construction. Let us define a sequence $(\Omega_n)_{n=1}^{\infty}$ of compact sets in $\mathbb{R}$ fulfilling the condition $\overline{\Omega_n^\circ} = \overline{\Omega_n} \neq \emptyset$. The sequence $(\Omega_n)_{n=1}^{\infty}$ arises from the interval $[0, 1]$. Ω_1 is the interval $[0, 1]$. For $n \geq 2$, Ω_n is the union of the following intervals:

$$[0, \tfrac{1}{n^2}], [\tfrac{1}{n} - \tfrac{1}{n^2}, \tfrac{1}{n} + \tfrac{1}{n^2}], [\tfrac{2}{n} - \tfrac{1}{n^2}, \tfrac{2}{n} + \tfrac{1}{n^2}], ..., [\tfrac{n-2}{n} - \tfrac{1}{n^2}, \tfrac{n-2}{n} - \tfrac{1}{n^2}] \text{ and } [\tfrac{n-1}{n}, 1],$$

i.e. Ω_n contains one interval of the length $\frac{1}{n^2}$, $n-2$ intervals of the length $\frac{2}{n^2}$ and one interval of the length $\frac{1}{n}$.

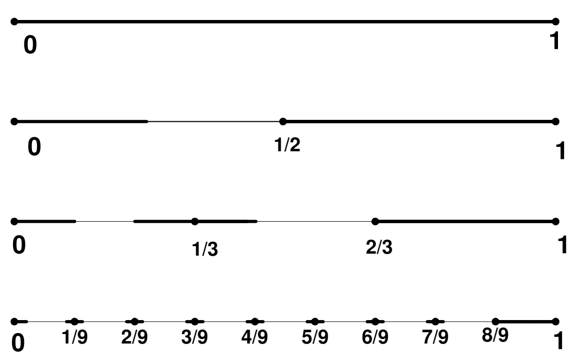


Figure 5.4: Construction of $(\Omega_n)_{n=1}^{\infty}$ in 1-dimensional case.

Considering the construction and the picture above we have

$$\text{vol }\Omega_n = \tfrac{1}{n^2} + (n-2)\tfrac{2}{n^2} + \tfrac{1}{n}.$$

It is easy to prove that $\Omega_n - \Omega_n = [-1,1]$ for every $n \in \mathbb{N}$. Mind the following explanation. Ω_n contains the points $0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1$ and the interval $[\frac{n-1}{n}, 1]$. Take an arbitrary $k \in \{0,1,2,...,n\}$, then the difference set

$$[\tfrac{n-1}{n}, 1] - \tfrac{k}{n} = [\tfrac{k}{n} - 1, \tfrac{k-n+1}{n}] \cup [\tfrac{n-1-k}{n}, 1 - \tfrac{k}{n}]$$

is subset of $\Omega_n - \Omega_n$ for every $k \in \{0,1,2,...,n\}$. Therefore the difference set $\Omega_n - \Omega_n = \cup_{k=0}^{n}([\frac{n-1}{n}, 1] - \frac{k}{n}) = [-1,1]$ and its volume is 2. Considering the volumes we obtain

$$\lim_{n\to\infty} f(\Omega_n) \geq \lim_{n\to\infty} \frac{\text{vol } (\Omega_n - \Omega_n)}{\text{vol } \Omega_n} = \lim_{n\to\infty} \frac{2}{\frac{1}{n^2} + (n-2)\frac{2}{n^2} + \frac{1}{n}} = +\infty,$$

which proves that the function f is not bounded on the space of general compact sets in $\mathbb{R}$.

□

We arrive at the conclusion that convexity of the acceptance window is essential for obtaining a universal upper bound on cardinality of the finite set F in the Meyer property $\Sigma(\Omega) - \Sigma(\Omega) \subset \Sigma(\Omega) + F$. Let us remind that knowledge of cardinality of F is important for obtaining an estimate on the number of different shapes of Voronoi tiles in $\Sigma(\Omega)$.

Chapter 6

A concrete example

Throughout this work we were searching for estimates on the upper bound of the function f on spaces of various compact convex sets Ω with non-empty interior. Let us apply our estimates to a concrete example. We will construct a cut-and-project set with the acceptance window Ω being a rhombus and on the basis of our previous investigations, we will estimate the number of different Voronoi tiles in this cut-and-project set $\Sigma(\Omega)$.

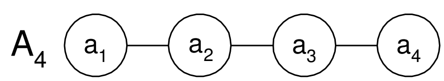
Let us describe, how to construct cut-and-project sets with fivefold rotation symmetry. The first step is to define a lattice in $\mathbb{R}^{c+d}$. Dimension $c+d=4$ is the lowest dimension, where a lattice L with fivefold rotation symmetry can be found. We will work in this dimension in order to make the construction as imaginable as possible.

Let $L = \{\sum_{i=1}^{4} \alpha_i a_i \mid (\forall i \in \{1,2,3,4\})(\alpha_i \in \mathbb{Z})\}$ be a lattice generated by four linearly independent vectors a_1, a_2, a_3, a_4 with the following Gram matrix

$$(a_i, a_j) = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

where $(.,.)$ denotes the scalar product in $\mathbb{R}^d$.

This lattice corresponds to the group A_4 generated by four reflections r_1, r_2, r_3, r_4 in $\mathbb{R}^4$, which can be described by the so-called Coxeter graph.



This graph represents the bilateral position of mirrors for reflections r_i, r_j. If i and j are not connected by an edge, then the mirrors are mutually orthogonal. If the vertices are connected by an edge, then the angle between the mirrors is $\frac{\pi}{3}$.

Accordingly, reflections r_1, r_2, r_3, r_4 are defined in the following way

$$(\forall x \in \mathbb{R}^4)\left(r_i(x) = x - \frac{2(x,a_i)}{(a_i,a_i)}a_i = x - 2(x,a_i)a_i\right).$$

As the angle between the vectors a_i, a_j for $|i-j|=1$ is $\frac{2\pi}{3}$ we obtain

$$r_i(a_i) = -a_i,$$

$$r_i(a_{i\pm1}) = a_i + a_{i\pm1},$$

$$r_i(a_j) = a_j \text{ for } |j-i| > 1.$$

Let us show that by composition of the reflections we obtain the searched rotation of order 5, which confirms that the lattice L has fivefold symmetry.

Claim 6.1. *The mapping $R = r_1r_3r_2r_4 : \mathbb{R}^4 \to \mathbb{R}^4$ is an isometry of order 5, it means that for every $x \in \mathbb{R}^4$ we have $|Rx| = |x|$ and $R^5x = x$.*

Proof. Let us divide the proof into two parts.

1. First let us prove that $(\forall x \in \mathbb{R}^4)(|Rx| = |x|)$.

   $$(\forall x \in \mathbb{R}^4)\left(|r_ix|^2 = \left(x - \tfrac{2(x,a_i)}{(a_i,a_i)}a_i\right)\left(x - \tfrac{2(x,a_i)}{(a_i,a_i)}a_i\right)^T = |x|^2 - \tfrac{4(x,a_i)^2}{(a_i,a_i)} + \tfrac{4(x,a_i)^2}{(a_i,a_i)^2}(a_i,a_i) = |x|^2\right).$$

   We have shown that r_i is an isometry in $\mathbb{R}^4$ for every $i \in \{1,2,3,4\}$. As R is a composition of isometries, it is an isometry in $\mathbb{R}^4$, too.

2. Let us verify that $(\forall x \in \mathbb{R}^4)(R^5x = x)$. We know that r_i is a linear mapping for every $i \in \{1,2,3,4\}$. As R is a composition of linear mappings, it is a linear mapping, too. As a consequence it suffices to prove that

   $$(\forall i \in \{1,2,3,4\})(R^5a_i = a_i).$$

   We show this property for the vector a_1, one can continue analogically for the remaining vectors a_2, a_3, a_4.

   $$R(a_1) = r_1r_3r_2(r_4(a_1)) = r_1r_3(r_2(a_1)) = r_1(r_3(a_2+a_1)) = r_1(a_3+a_2+a_1) = a_3+a_1+a_2-a_1 = \\ = a_3 + a_2.$$

   $$R^2(a_1) = R(a_3+a_2) = r_1r_3r_2(r_4(a_3+a_2)) = r_1r_3(r_2(a_4+a_3+a_2)) = r_1(r_3(a_4+a_2+a_3-a_2)) = \\ = r_1(r_3(a_4+a_3)) = r_1(a_4+a_3-a_3) = r_1(a_4) = a_4.$$

   $$R^3(a_1) = R(a_4) = r_1r_3r_2(r_4(a_4)) = r_1r_3(r_2(-a_4)) = r_1(r_3(-a_4)) = r_1(-a_3-a_4) = -a_3-a_4.$$

   $$R^4(a_1) = R(-a_3-a_4) = r_1r_3r_2(r_4(-a_3-a_4)) = r_1r_3(r_2(-a_4-a_3+a_4)) = r_1r_3(r_2(-a_3)) = \\ = r_1(r_3(-a_2-a_3)) = r_1(-a_3-a_2+a_3) = r_1(-a_2) = -a_1-a_2.$$

   $$R^5(a_1) = R(-a_1-a_2) = r_1r_3r_2(r_4(-a_1-a_2)) = r_1r_3(r_2(-a_1-a_2)) = r_1(r_3(-a_2-a_1+a_2)) = \\ = r_1(r_3(-a_1)) = r_1(-a_1) = a_1.$$

□

Remark 5. *Notice that $R(L) = L$, therefore L has fivefold symmetry.*

As it was said at the beginning we want to find such a projection π_1 of $\mathbb{R}^4$ on the 2-dimensional physical space that keeps the fivefold symmetry of L. We have calculated images of the vector a_1 by compositions of the mapping R.

$$R: \quad a_1 \quad \to \quad a_2 + a_3 \quad \to \quad a_4 \quad \to \quad -a_3 - a_4 \quad \to \quad -a_1 - a_2 \quad \to \quad a_1,$$

therefore we can formulate our demand on the fivefold symmetry of the projection $\pi_1(L)$ in the following way. The projections of the vectors above should form vertices of a regular pentagon in $\mathbb{R}^2$. Let $u := \pi_1(a_1)$ and $v := \pi_1(a_4)$. We have two possibilities, how to choose the angle between u and v. Either $\frac{4\pi}{5}$ or $\frac{2\pi}{5}$. Let us choose the first possibility, i.e. the angle between vectors u and v is $\frac{4\pi}{5}$.

Considering the figure 6.1 one can determine the projections $\pi_1(a_2)$ and $\pi_1(a_3)$. We have namely

$$\pi_1(a_2) = -\big(\pi_1(a_1) + \pi_1(-a_1 - a_2)\big) = \tau\pi_1(a_4) = \tau v,$$

$$\pi_1(a_3) = -\big(\pi_1(a_4) + \pi_1(-a_3 - a_4)\big) = \tau\pi_1(a_1) = \tau u,$$

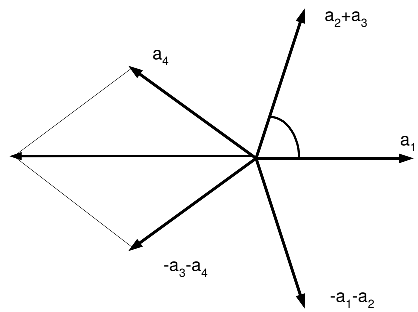


Figure 6.1: Illustration of the projection π_1.

where $\tau = 2\cos(\frac{\pi}{5})$.

It is useful to enumerate the value of τ. To this purpose let us define a linear mapping on $\pi_1(L)$

$$R_2 := \pi_1(r_2r_4)\pi_1^{-1}.$$

Considering two facts

1. $R_2(\tau u) = \pi_1(r_2r_4)\pi_1^{-1}(\tau u) = \pi_1(r_2r_4)(a_3) = \pi_1(a_2 + a_3 + a_4)) = \tau v + \tau u + v.$
2. Linearity of R_2 implies
$$R_2(\tau u) = \tau R_2(u) = \tau u + \tau^2 v.$$

We obtain
$$\tau v + v + \tau u = \tau u + \tau^2 v.$$
We can notice that τ must fulfil the equation $\tau^2 = \tau + 1$ and $\tau = 2\cos(\frac{\pi}{5})$, which is a positive number. As a result we have that $\tau = \frac{1+\sqrt{5}}{2} \doteq 1,618$.

Let us denote by $M := \pi_1(L)$. It is possible to describe M as

$$M = \{(a + b\tau)u + (c + d\tau)v \mid a, b, c, d \in \mathbb{Z}\}.$$

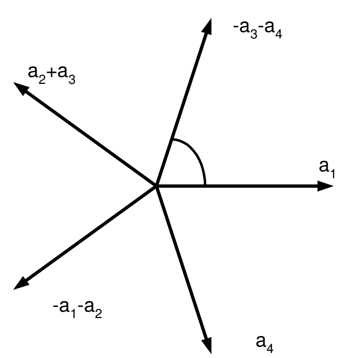


Figure 6.2: Illustration of the projection π_2.

Now, we use the second possibility of the angle choice and we define $u^* := \pi_2(a_1)$ and $v^* := \pi_2(a_4)$, where the angle between u^* and v^* is $\frac{2\pi}{5}$. From the figure 6.2, one can determine the projections $\pi_2(a_2), \pi_2(a_3)$.

$$\pi_2(a_2) = \pi_2(a_2+a_3)+\pi_2(-a_3-a_4)+\pi_2(a_4) = -\tau\pi_2(a_4)+\pi_2(a_4) = (1-\tau)\pi_2(a_4) = (1-\tau)v^* = \tau' v^*,$$

$$\pi_2(a_3) = \pi_2(a_2 + a_3) + \pi_2(-a_1 - a_2) + \pi_2(a_1) = -\tau\pi_2(a_1) + \pi_2(a_1) = (1 - \tau)\pi_2(a_1) = \tau' u^*,$$

$$\text{where } \tau' = \frac{1-\sqrt{5}}{2} \doteq -0,618.$$

Let us denote by $M^* := \pi_2(L)$. It is possible to describe M^* as

$$M^* = \{(a + b\tau')u^* + (c + d\tau')v^* \mid a, b, c, d \in \mathbb{Z}\}.$$

We have obtained the cut-and-project scheme (M, M^*, L).

$$L = \mathbb{Z}a_1 + \mathbb{Z}a_2 + \mathbb{Z}a_3 + \mathbb{Z}a_4 \subset \mathbb{R}^4,$$

$$M = \pi_1(L) = (\mathbb{Z} + \mathbb{Z}\tau)u + (\mathbb{Z} + \mathbb{Z}\tau)v,$$

$$M^* = \pi_2(L) = (\mathbb{Z} + \mathbb{Z}\tau')u^* + (\mathbb{Z} + \mathbb{Z}\tau')v^*.$$

Since $\tau' = 1 - \tau$ and $u^* = u$, $v^* = -u - \tau v$, the sets M and M^* coincide. It is possible to define a bijection $*$ on $M = M^*$ by

$$\big((a + b\tau)u + (c + d\tau)v\big)^* = (a + b\tau')u^* + (c + d\tau')v^*. \tag{6.1}$$

In order to state some properties of the map $*$, let us recall that the set

$$\mathbb{Z}[\tau] = \mathbb{Z} + \mathbb{Z}\tau$$

is the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{5})$. The Galois automorphism of this field defines an automorphism on the ring $\mathbb{Z}[\tau]$ by

$$\alpha = a + b\tau \in \mathbb{Z}[\tau] \quad \to \quad \alpha' = a + b\tau' = a + b - b\tau \in \mathbb{Z}[\tau].$$

The morphism property of the map implies

1. $(\forall\alpha, \beta \in \mathbb{Z}[\tau])(\alpha\beta)' = \alpha'\beta'$,
2. $(\forall\alpha, \beta \in \mathbb{Z}[\tau])(\alpha + \beta)' = \alpha' + \beta'$.

The norm N on the field $\mathbb{Q}(\sqrt{5})$ is given by

$$N(\alpha) = \alpha\alpha' = (a + b\tau)(a + b\tau') = a^2 + ab - b^2 \in \mathbb{Q},$$

for $\alpha \in \mathbb{Q}(\sqrt{5})$, where we have used $\tau + \tau' = 1$, $\tau\tau' = -1$. The norm satisfies $N(\alpha) \in \mathbb{Z}$ for $\alpha \in \mathbb{Z}[\tau]$ and $N(\alpha) = 0 \Leftrightarrow \alpha = 0$. Note that $N(\tau) = \tau\tau' = -1$ and thus τ is a unit in $\mathbb{Z}[\tau]$. It follows that $\tau\mathbb{Z}[\tau] = \mathbb{Z}[\tau]$. Due to the uniform distribution of $n\theta \mod 1$ in $[0, 1)$ for θ irrational [10], the ring $\mathbb{Z}[\tau]$ is dense on the real line.

Claim 6.2. *The Galois automorphism is an everywhere discontinuous map.*

Proof. We want to verify that

$$(\forall\alpha \in \mathbb{Z}[\tau])(\exists\varepsilon > 0)(\forall\delta > 0)(\exists\beta \in \mathbb{Z}[\tau])(0 < |\alpha - \beta| < \delta \wedge |\tilde{\alpha} - \tilde{\beta}| \geq \varepsilon).$$

Take an arbitrary α. Since $\mathbb{Z}[\tau]$ is dense in $\mathbb{R}$, for every $\delta > 0$ there exists $\beta \in \mathbb{Z}[\tau]$ such that $0 < |\alpha - \beta| < \delta$. Denote $\alpha - \beta := a + b\tau \neq 0$, then $\alpha' - \beta' = a + b\tau'$. We have

$$0 \neq N(\alpha - \beta) = (\alpha - \beta)(\alpha' - \beta') \in \mathbb{Z}.$$

We arrive at the conclusion that $|(\alpha - \beta)(\tilde{\alpha} - \tilde{\beta})| \geq 1$, which says

$$|\tilde{\alpha} - \tilde{\beta}| \geq \frac{1}{|\alpha - \beta|} > \frac{1}{\delta}.$$

It confirms that the Galois mapping is everywhere discontinuous. ∎

Let us now state the properties of the star map defined in (6.1).

Claim 6.3. *1.* $(\forall x, y \in M)((x+y)^* = x^* + y^*)$,

2. $(\forall x \in M)(\forall \alpha \in \mathbb{Z}[\tau])(\alpha x)^* = \alpha' x^*$,

3. $*$ *is an everywhere discontinuous map.*

Proof. 1. and 2. follow trivially using the fact that $'$ is an automorphism on $\mathbb{Z}[\tau]$. Property 3. is an obvious consequence of the discontinuity of the Galois mapping. Let us prove the discontinuity of the star-map in 0 at first, i.e.

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in M)(0 < |x| < \delta \wedge |x^*| \geq \varepsilon).$$

Since $\mathbb{Z}[\tau]$ is dense in $\mathbb{R}$, we can find for every $\delta > 0$ an $\alpha \in \mathbb{Z}[\tau]$ so that $x = \alpha u$ and $0 < |x| = |\alpha||u| < \delta$.

At the same time

$$|x^*| = |\alpha' u^*| \geq \frac{1}{|\alpha|}|u^*| > \frac{1}{\delta}|u^*|,$$

where we have again used $|N(\alpha)| = |\alpha\alpha'| \geq 1$ for $\alpha \neq 0$. It confirms that the star-mapping is discontinuous in 0. It is easy to generalise this result. If we take an arbitrary vector $x \in M$, then for every $\delta > 0$ there exists a vector $y \in M$ such that $|x - y| < \delta$. We denote $z := x - y$ and so we transform the problem of the discontinuity everywhere again to the discontinuity in 0. □

At this moment we can finally define the cut-and-project set for an acceptance window Ω by

$$\Sigma(\Omega) = \{(a + b\tau)u + (c + d\tau)v \mid a, b, c, d \in \mathbb{Z}, (a + b\tau')u^* + (c + d\tau')v^* \in \Omega\} = \{x \in M \mid x^* \in \Omega\},$$

where Ω is a convex compact set in $\mathbb{R}^2$ with non-empty interior.

Let us remind the construction of M, we demanded M to have fivefold symmetry. Now, we can realize that it has even tenfold symmetry. How does the symmetry of M influences the symmetry of the cut-and-project set $\Sigma(\Omega)$? The answer can be found in the following theorem [7].

Theorem 6.1. *Let Ω be a convex compact set in $\mathbb{R}^2$ with non-empty interior. By the above defined projections π_1, π_2, Ω has tenfold symmetry if and only if $\Sigma(\Omega)$ has tenfold symmetry.*

At this moment, we can finally step up to the concrete application of the described theory. For simplicity we consider an acceptance window without tenfold symmetry. Let I be an interval in $\mathbb{R}$ and let Ω be a convex compact set in $\mathbb{R}^2$ with non-empty interior such that

$$\Omega = Iu^* + Iv^* = \{au^* + bv^* \mid a, b \in I\}.$$

Our aim is to estimate the number of different cells in the perfect Voronoi tiling in $\Sigma(\Omega)$, which is bounded by the number of subsets of $(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0, 2R_{\Sigma(\Omega)})$. We recall the estimation from the introduction part of this work

$$\#(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0, 2R_{\Sigma(\Omega)}) \leq \#(\Sigma(\Omega) + F) \cap B(0, 2R_{\Sigma(\Omega)}) \leq \#F \frac{\text{vol } B(0, 2R_{\Sigma(\Omega)})}{\text{vol } B(0, \frac{1}{2}r_\Sigma)}.$$

One can notice that we need to know three constants to be able to use this estimate. The first one is the cardinality of the set F, which comes from the Meyer property

$$\Sigma(\Omega) - \Sigma(\Omega) \subset \Sigma(\Omega) + F.$$

The cardinality of the set F corresponds to the value of the function f defined by

$$f(\Omega) = \text{the minimal number of translated copies of } \Omega^\circ \text{ needed for covering of } \overline{\Omega - \Omega}.$$

We recall the section about polygons, where we proved that for Ω being regular quadrangles, $\#F \leq 9$. The second constant is the covering radius $R_{\Sigma(\Omega)}$. The third constant is the minimal distance $r_{\Sigma(\Omega)}$. We will determine these constants here.

Let us describe $\Sigma(\Omega)$ in the case when Ω is a rhombus.

$$\Sigma(\Omega) = \{x_1 u + x_2 v \mid x_1, x_2 \in \mathbb{Z}[\tau], (x_1 u + x_2 v)^* \in \Omega\} =$$

$$= \{x_1 u + x_2 v \mid x_1, x_2 \in \mathbb{Z}[\tau], x_1' \in I, x_2' \in I\} =$$
$$= \{x_1 \in \mathbb{Z}[\tau] \mid x_1' \in I\}u + \{x_2 \in \mathbb{Z}[\tau] \mid x_2' \in I\} =$$
$$= \Sigma(I)u + \Sigma(I)v.$$

Such $\Sigma(\Omega)$ is sometimes called a quasilattice, which corresponds to the the illustration bellow. Note that such $\Sigma(\Omega)$ is a cartesian product of two one-dimensional cut-and-project sets $\Sigma(I)$.

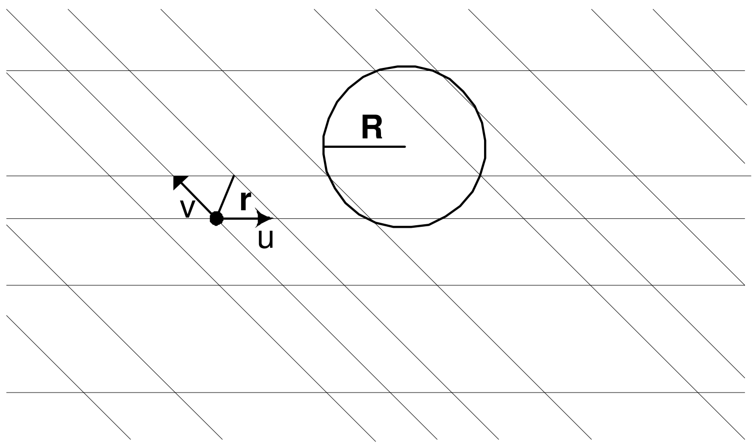


Figure 6.3: Illustration of the quasilattice $\Sigma(\Omega)$.

The following claim asserts that without loss of generality it suffices to deal with intervals of length $1 \leq |I| < \tau$.

Claim 6.4. *For the previously defined cut-and-project set* $\Sigma(\Omega)$ *it holds* $\tau\Sigma(\Omega) = \Sigma(\tau'\Omega)$. *Moreover, for every* $k \in \mathbb{Z}$

$$\tau^k \Sigma(\Omega) = \Sigma({\tau'}^k \Omega).$$

Proof. We shall use $\tau\mathbb{Z}[\tau] = \mathbb{Z}[\tau]$.

$$\tau\Sigma(\Omega) = \{\tau\gamma u + \tau\delta v \mid \gamma, \delta \in \mathbb{Z}[\tau], \gamma' u^* + \delta' v^* \in \Omega\} =$$

$$\{\alpha u + \beta v \mid \alpha, \beta \in \mathbb{Z}[\tau], \alpha' u^* + \beta' v^* \in \tau'\Omega\}.$$

The second part of the claim follows easily using mathematical induction. $\square$

We apply the previous claim to obtain

$$\tau\Sigma(I) = \Sigma(\tau' I),$$

which confirms that it is enough to consider intervals of the length $1 \leq |I| < \tau$, because if $\tau^k \leq |J| < \tau^{k+1}$ we have, $\Sigma(J) = \tau^k \Sigma(I)$, where I is the interval with the corresponding length $1 \leq |I| < \tau$.

The following theorem describes the structure of one-dimensional cut-and-project sets $\Sigma(I)$, namely, it determines the distances between neighbouring points. The theorem will be useful further on for determining the values of $R_{\Sigma(\Omega)}$ and $r_{\Sigma(\Omega)}$.

Theorem 6.2. *Let $1 \le d < \tau$. If we arrange the elements of the set $\Sigma[c, c+d) = \{a + b\tau \mid a, b \in \mathbb{Z}, c \le a + b\tau' < c + d\}$ in such a way that they form an increasing sequence $(x_n)_{-\infty}^{\infty}$, i.e.*

$$\Sigma[c, c+d) = \{x_n \mid n \in \mathbb{Z}\},$$

then it holds

$$x_{n+1} - x_n \in \{1, \tau, \tau^2\}.$$

Moreover, if $d = 1$, then

$$x_{n+1} - x_n \in \{\tau, \tau^2\}.$$

Proof. By definition, the point $a + b\tau$ belongs to $\Sigma[c, c+d)$ if and only if

$$c \le a + b\tau' = a - \frac{b}{\tau} < c + d,$$

which implies

$$c + \frac{b}{\tau} \le a < c + d + \frac{b}{\tau}. \tag{6.2}$$

As $1 \le d < \tau < 2$, there exist only one or two integers a for every $b \in \mathbb{Z}$, namely $a = \lceil c + \frac{b}{\tau} \rceil$ and eventually $a = \lceil c + \frac{b}{\tau} \rceil + 1$, i.e.

$$x_b = a + b\tau = \lceil c + \frac{b}{\tau} \rceil + b\tau,$$

eventually

$$x_b = \lceil c + \frac{b}{\tau} \rceil + 1 + b\tau.$$

We divide the proof into two steps.

1. Let $d = 1$, then for every $b \in \mathbb{Z}$ there exists precisely one integer a satisfying (6.2). We obtain

   $$x_{b+1} - x_b = \lceil c + \tfrac{b+1}{\tau} \rceil + (b+1)\tau - \left(\lceil c + \tfrac{b}{\tau} \rceil + b\tau\right) = \tau + \lceil c + \tfrac{b+1}{\tau} \rceil - \lceil c + \tfrac{b}{\tau} \rceil.$$

   Due to the fact that $0 < \frac{1}{\tau} < 1$ there are two cases which can happen. Either there exists an integer k such that $c + \frac{b}{\tau} < k < c + \frac{b}{\tau} + \frac{1}{\tau}$, then $x_{b+1} - x_b = \tau + 1 = \tau^2$. This case happens if $\{c + \frac{b}{\tau}\} > 1 - \frac{1}{\tau}$. Otherwise $\{c + \frac{b}{\tau}\} < 1 - \frac{1}{\tau}$, then $\{c + \frac{b}{\tau}\} + \frac{1}{\tau} < 1$, which implies

   $$\lceil c + \tfrac{b+1}{\tau} \rceil - \lceil c + \tfrac{b}{\tau} \rceil = 0.$$

   The gap between x_{b+1} and x_b has the length τ, i.e. $x_{b+1} - x_b = \tau$.

2. Let $\tau > d > 1$. So far we have investigated that the gaps in the sequence $\{x_b = \lceil c + \frac{b}{\tau} \rceil + b\tau \mid b \in \mathbb{Z}\}$ reaches two lengths τ and τ^2.

   Where can we find the remaining points of $\Sigma[c, c+d)$? Firstly we state for which $b \in \mathbb{Z}$

   $$c + \tfrac{b}{\tau} \le \lceil c + \tfrac{b}{\tau} \rceil + 1 < c + d + \tfrac{b}{\tau}.$$

   Considering the figure we have $\{c + \frac{b}{\tau}\} > 1 - (d-1) = 2 - d$. If $2 - d > 1 - \frac{1}{\tau}$, then we find the remaining point $\lceil c + \frac{b}{\tau} \rceil + 1 + b\tau$ in the gap of the length τ^2. It is equivalent with the inequality

   $$\tau = 1 + \tfrac{1}{\tau} > d,$$

   which is fulfiled for $d < \tau$. If there are two integers a for one integer $b \in \mathbb{Z}$, then their distance is 1 and the point $x_b + 1$ divides the gap $\tau^2 = \tau + 1$ into two gaps of the lengths 1 and τ. The conclusion sounds that there are three lengths between increasingly arranged points of $\Sigma[c, c+d)$ for $d > 1$, namely $1, \tau, \tau^2$.

$\square$

Now, we can meet the previous promise and determine the covering radius and the minimal distance in $\Sigma(\Omega)$. It suffices to consider the figure 6.3 and one can realize that the covering radius is radius of a circle located largely in a rhombus of side-length τ^2. The value of the covering radius is $R_{\Sigma(\Omega)} = \frac{\tau^2}{\sqrt{\tau+2}}$. The minimal distance can be calculated easily. It is the length of a diagonal of the rombus having side-length 1. It is that one of two diagonals which halves the angle $\frac{4\pi}{5}$. Using for instance the cosinus theorem we obtain

$$r_{\Sigma(\Omega)} = \sqrt{2 - 2\cos(\tfrac{\pi}{5})} = \sqrt{2-\tau} = \sqrt{\tfrac{1}{\tau^2}} = \tfrac{1}{\tau}.$$

The searched estimate on the number of different Voronoi tiles in $\Sigma(\Omega)$, where Ω is a rhombus, is the number of subsets of $(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0, 2R_{\Sigma(\Omega)})$, where

$$\#(\Sigma(\Omega)-\Sigma(\Omega))\cap B(0,2R_{\Sigma(\Omega)}) \leq \#F\frac{\text{vol } B(0,2R_{\Sigma(\Omega)})}{\text{vol } B(0,\frac{1}{2}r_{\Sigma(\Omega)})} = 9\frac{\pi 4 {R_{\Sigma(\Omega)}}^2}{\pi\frac{1}{4}{r_{\Sigma(\Omega)}}^2} = 9.16.\frac{\tau^4}{\tau+2}.\frac{1}{\frac{1}{\tau^2}} = 144\frac{\tau^6}{\tau+2} \doteq 714.$$

Chapter 7

Conclusion

As it was promised in the Introduction, the main interest of the work was devoted to investigation of properties and values of the function f defined by

$$f(\Omega) = \text{the minimal number of translated copies of } \Omega^\circ \text{ needed for covering of } \Omega - \Omega,$$

where Ω is a convex compact subset of $\mathbb{R}^d$ with non-empty interior.

The first step of investigation of the function f was inquiry of the topologic properties of domain of the function f, i.e. of the space κ of all convex compact subsets of $\mathbb{R}^d$ with non-empty interior. The space κ equipped with the well-known Hausdorff metric forms a metric space. We have shown that the space $\mathcal{M}$ of all convex compact sets Ω satisfying $\overline{B(0, \frac{1}{d})} \subset \Omega \subset \overline{B(0,1)}$ is a compact subspace of κ, which was essential for our study of the function f.

Then we have proven two important properties of the function f:

1. f is invariant under affine transformations of Ω,

2. f is an upper semicontinuous function on the space $\mathcal{M}$.

We use both of the above properties in the proof of the main theorem (Theorem 3.2), which says that the function f is bounded on the space κ by a universal constant K which depends only on dimension d.

The method for showing boundedness of f on κ provides us with a tool for determining the universal upper bound on the function f on κ and consequently on the cardinality of the finite set F fulfilling

$$\Sigma(\Omega) - \Sigma(\Omega) \subset \Sigma(\Omega) + F$$

for any cut-and-project set with convex compact $\Omega \subset \mathbb{R}^d$. For this, one needs to find minimal covering of the closed ball $\overline{B(0, 2d)} \subset \mathbb{R}^d$ by unit open balls, which is in general a difficult problem. In Section 4.3 we provide an estimate valid in any dimension, but this universal estimate is naturally very rough. We have focused on dimension $d = 2$ and shown that

$$f(\Omega) \leq 26$$

for any convex compact set $\Omega \subset \mathbb{R}^2$. We can refine the result if we limit our considerations to centrally symmetric convex compact sets $\Omega \subset \mathbb{R}^2$, for which we derive

$$f(\Omega) \leq 16.$$

It is however apparent that these bounds are not reached. In order to find better estimates, we have determined the value of the function f for some special types of convex sets in $\mathbb{R}^2$, namely an ellipse, for which $f(\Omega) = 8$, and regular polygons (see Proposition 4.2). These results lead us to conjecture that

$$8 \leq f(\Omega) \leq 13$$

for any convex compact set $\Omega \subset \mathbb{R}^2$. For centrally symmetric convex compact sets the conjecture is even more interesting, namely that $f(\Omega) \in \{8, 9\}$.

Further on, we answered a natural question: Is convexity of the set Ω essential for boundedness of the function f? The answer is positive. We considered star-shaped sets, which can be viewed as the nearest generalisation of convex sets. In spite of that fact we were able to construct a sequence of star-shaped sets $(\Omega_n)_{n=1}^{\infty}$ such that $f(\Omega_n)$ tends to infinity with growing n.

In the end, we applied our theory on a concrete example, we estimated the number of Voronoi tiles in a cut-and-project set with the acceptance window being a rhombus. Let us mention that the present work opens the door for further investigations. Already in dimension 2 the question about behaviour of the function f is far from beeing completely understood. Proving the mentioned conjectures and stating similar results in dimensions $d \geq 3$ is desirable.

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