Aperiodic pseudorandom number generators based on infinite words

Ľubomíra Balková[1], Michelangelo Bucci[2], Alessandro De Luca[3], Jiří Hladký[1], and Svetlana Puzynina[2,4]

[1] Department of Mathematics, FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Praha 2, Czech Republic
`{lubomira.balkova, hladky.jiri}@gmail.com`

[2] Department of Mathematics, University of Turku, FI-20014 Turku, Finland
`{michelangelo.bucci, svepuz}@utu.fi`

[3] DIETI, Università degli Studi di Napoli Federico II
via Claudio, 21, 80125 Napoli, Italy
`alessandro.deluca@unina.it`

[4] Sobolev Institute of Mathematics, Russia

Abstract. In this paper we study how certain families of aperiodic infinite words can be used to produce aperiodic pseudorandom number generators (PRNGs) with good statistical behavior. We introduce the *well distributed occurrences* (WELLDOC) combinatorial property for infinite words, which guarantees absence of the lattice structure defect in related pseudorandom number generators. An infinite word u on a d-ary alphabet has the WELLDOC property if, for each factor w of u, positive integer m, and vector $\mathbf{v} \in \mathbb{Z}_m^d$, there is an occurrence of w such that the Parikh vector of the prefix of u preceding such occurrence is congruent to $\mathbf{v}$ modulo m. We prove that Sturmian words, and more generally Arnoux-Rauzy words and some morphic images of them, have the WELLDOC property. Using the TestU01 [10] and PractRand [6] statistical tests, we moreover show that not only the lattice structure is absent, but also other important properties of PRNGs are improved when linear congruential generators are combined using infinite words having the WELLDOC property.

Introduction

Pseudorandom number generators aim to produce random numbers using a deterministic process. No wonder they suffer from many defects. The most usual ones – linear congruential generators – are known to produce periodic sequences having a defect called the lattice structure. Guimond et al. [13] proved that when two linear congruential generators are combined using infinite words coding certain classes of quasicrystals or, equivalently, of cut-and-project sets, the resulting sequence is aperiodic and has no lattice structure. For some other related results concerning aperiodic pseudorandom generators we refer to [11, 12].

We have found a combinatorial condition – *well distributed occurrences*, or WELLDOC for short – that also guarantees absence of the lattice structure.

The WELLDOC property for an infinite word u over an alphabet $\mathcal{A}$ means that for any integer m and any factor w of u, the set of Parikh vectors modulo m of prefixes of u preceeding the occurrences of w coincides with $\mathbb{Z}_m^{|\mathcal{A}|}$ (see Definition 2.1). In other words, among Parikh vectors modulo m of such prefixes one has all possible vectors. Besides giving generators without lattice structure, the WELLDOC property is an interesting combinatorial property of infinite words itself. We prove that the WELLDOC property holds for the family of Sturmian words, and more generally for Arnoux-Rauzy words.

Sturmian words constitute a well studied family of infinite aperiodic words. Let u be an infinite word, i. e., an infinite sequence of elements from a finite set called an alphabet. The *(factor) complexity* function counts the number of distinct factors of u of length n. A fundamental result of Morse and Hedlund [16] states that a word u is eventually periodic if and only if for some n its complexity is less than or equal to n. Infinite words of complexity $n+1$ for all n are called *Sturmian words,* and hence they are aperiodic words of the smallest complexity. The most studied Sturmian word is the so-called Fibonacci word

$$0100101001001010010100100101001 0\ldots$$

fixed by the morphism $0 \mapsto 01$ and $1 \mapsto 0$. The first systematic study of Sturmian words was given by Morse and Hedlund in [17]. Such sequences arise naturally in many contexts, and admit various types of characterizations of geometric and combinatorial nature (see, e.g., [14]). For example, each Sturmian word may be realized geometrically by an irrational rotation on the circle. That is, every Sturmian word is obtained by coding the symbolic orbit of a point x on the circle (of circumference one) under a rotation by an irrational angle α, where the circle is partitioned into two intervals of length α and $1-\alpha$. Conversely, each such coding gives rise to a Sturmian word.

Arnoux-Rauzy words were introduced in [1] as natural extensions of Sturmian words to multiliteral alphabets (see Definition 4.4). Despite the fact that they were introduced as generalizations of Sturmian words, Arnoux-Rauzy words display a much more complex behavior. In particular, we have two different proofs of the WELLDOC property for Sturmian words, and only one them can be generalized to Arnoux-Rauzy words. In the sequel we provide both of them.

An infinite word with the WELLDOC property is then used to combine two linear congruential generators and form an infinite aperiodic sequence with good statistical behavior. Using the TestU01 [10] and PractRand [6] statistical tests, we have moreover shown that not only the lattice structure is absent, but also other important properties of PRNGs are improved when linear congruential generators are combined using infinite words having the WELLDOC property.

The paper is organized as follows. In the next section, we give some background on pseudorandom number generation. Next, in Section 2, we give the basic combinatorial definitions needed for our main results, including the WELLDOC property, and we prove that the WELLDOC property of u guarantees absence of the lattice structure of the PRNG based on u. In Sections 3 and 4, we prove that the property holds for Sturmian and Arnoux-Rauzy words, respectively. Finally, in the last section, we present results of empirical tests of PRNGs based on words having the WELLDOC property.

A preliminary version of this paper [2], using the acronym *WDO* instead of WELLDOC, was presented at the WORDS 2013 conference in Turku.

1 Pseudorandom Number Generators and Lattice Structure

For the sake of our discussion, any infinite sequence of integers can be understood as a *pseudorandom number generator (PRNG)*; see also [13]. The most widely used generators – linear congruential generators – are known to suffer from a defect called the lattice structure (they possess it already from dimension 2 as shown in [15]).

Let $Z = (Z_n)_{n\in\mathbb{N}}$ be a PRNG whose output is a finite set $M \subset \mathbb{N}$. We say that Z has the *lattice structure* if there exists $t \in \mathbb{N}$ such that the set

$$\{(Z_i, Z_{i+1}, \dots, Z_{i+t-1}) \bigm| i \in \mathbb{N}\}$$

is covered by a family of parallel equidistant hyperplanes and at the same time, this family does not cover the whole lattice

$$M^t = \{(A_1, A_2, \dots, A_t) \bigm| A_i \in M \text{ for all } i \in \{1, \dots, t\}\}.$$

Recall that a *linear congruential generator* (LCG) $(Z_n)_{n\in\mathbb{N}}$ is given by parameters $a, m, c \in \mathbb{N}$ and defined by the recurrence relation $Z_{n+1} = aZ_n + c \mod m$. Let us mention a famous example of a LCG whose lattice structure is striking. For $t = 3$, the set of triples of RANDU, i.e., $\{(Z_i, Z_{i+1}, Z_{i+2}) \bigm| i \in \mathbb{N}\}$ is covered by only 15 parallel equidistant hyperplanes, see Figure 1.

In the paper of Guimond et al. [13], it is possible to reveal a sufficient condition on absence of the lattice structure.

Proposition 1.1. *Let Z be a PRNG whose output is a finite set $M \subset \mathbb{N}$ containing at least two elements. Assume there exists for any $A, B \in M$ and for any $\ell \in \mathbb{N}$ an ℓ-tuple $(A_1, A_2, \dots, A_\ell)$ such that both $(A_1, A_2, \dots, A_\ell, A)$ and $(A_1, A_2, \dots, A_\ell, B)$ are $(\ell+1)$-tuples of the generator Z. Then Z does not have the lattice structure.*

Remark 1.2. In terms of combinatorics on words – see Section 2 – one can reformulate Proposition 1.1: Let Z be a PRNG whose output is a finite set $M \subset \mathbb{N}$ containing at least two elements. If Z has for any $A, B \in M$ and any length ℓ a right special factor of length ℓ with right extensions A and B, then Z does not have the lattice structure.

Since Proposition 1.1 is formulated for a restricted class of generators in [13] (see Lemma 2.3 ibidem), we will provide its proof. Let us however point out that all ideas of the proof are taken from [13]. We start with a lemma that will be helpful for the proof.

Let us denote $\lambda = \gcd\{A - B \mid A, B \in M\}$.

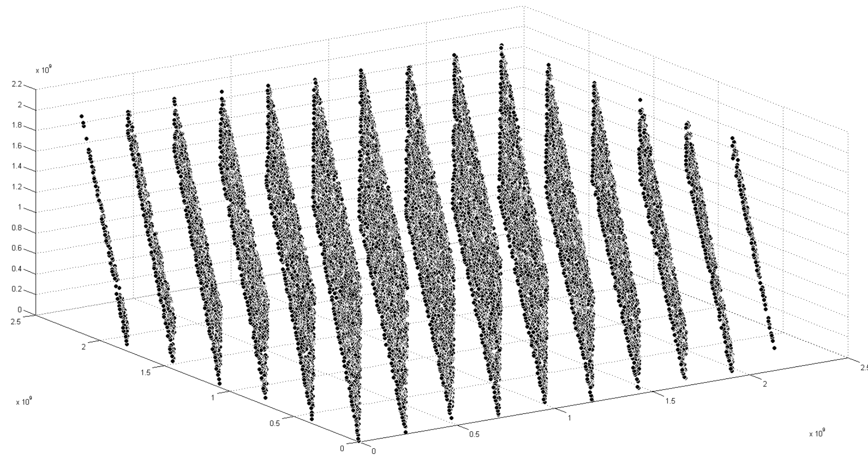

Fig. 1. The triples of RANDU – the LCG with $a = (2^{16} + 3), m = 2^{31}, c = 0$ – are covered by as few as 15 parallel equidistant planes.

Lemma 1.3. *Let Z be a PRNG satisfying all assumptions of Proposition 1.1. Let $\bar{\mathbf{n}}$ be the unit normal vector of a family of parallel equidistant hyperplanes covering all t-tuples of Z. Assume $\bar{\mathbf{e}}_i$ (the i-th vector of the canonical basis of $\mathbb{R}^t$) is not orthogonal to $\bar{\mathbf{n}}$. Then the distance d_i of adjacent hyperplanes in the family along $\bar{\mathbf{e}}_i$ is of the form λ/k for some $k \in \mathbb{N}$.*

Remark 1.4. The distance d_i of adjacent hyperplanes W_0, W_1 along $\bar{\mathbf{e}}_i$ means $|x_i - y_i|$ for any $\bar{\mathbf{x}} \in W_0$ and $\bar{\mathbf{y}} \in W_1$, where the j-th components of $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ satisfy $x_j = y_j$ for all $j \in \{1, \ldots, t\}, j \neq i$. This is a well defined term because the hyperplanes in the family are of the form $W_j \equiv \bar{\mathbf{x}} \cdot \bar{\mathbf{n}} = \alpha + jd,\ j \in \mathbb{Z}$, where d is the distance of adjacent hyperplanes in the family and $\cdot$ denotes the standard scalar product. Thus, without loss of generality, consider the adjacent hyperplanes

$$W_0 \equiv \bar{\mathbf{x}} \cdot \bar{\mathbf{n}} = \alpha \quad \text{and} \quad W_1 \equiv \bar{\mathbf{x}} \cdot \bar{\mathbf{n}} = \alpha + d.$$

Then for any $\bar{\mathbf{x}} \in W_0$ and $\bar{\mathbf{y}} = \bar{\mathbf{x}} + s\bar{\mathbf{e}}_i$ from W_1, we have

$$\begin{aligned}\bar{\mathbf{y}} \cdot \bar{\mathbf{n}} &= \alpha + d = \bar{\mathbf{x}} \cdot \bar{\mathbf{n}} + d,\\ \bar{\mathbf{y}} \cdot \bar{\mathbf{n}} &= \bar{\mathbf{x}} \cdot \bar{\mathbf{n}} + s\bar{\mathbf{e}}_i \cdot \bar{\mathbf{n}} = \bar{\mathbf{x}} \cdot \bar{\mathbf{n}} + sn_i,\end{aligned}$$

where n_i is the i-th component of $\bar{\mathbf{n}}$. Consequently, $d_i = |s| = \left|\frac{d}{n_i}\right|$ and is the same for any choice of $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ different only in their i-th component belonging to adjacent hyperplanes.

Proof (Proof of Lemma 1.3). Let us start with a useful observation. Let $\bar{\mathbf{z}}$ belong to a hyperplane W of the family in question.

1. If $\bar{\mathbf{e}}_j$ is orthogonal to $\bar{\mathbf{n}}$, then we may change the j-th component of $\bar{\mathbf{z}}$ in an arbitrary way and the resulting vector will belong to the same hyperplane, i.e., if $W \equiv \bar{\mathbf{x}} \cdot \bar{\mathbf{n}} = \alpha$, then clearly $(\bar{\mathbf{z}} + \beta\bar{\mathbf{e}}_j) \cdot \bar{\mathbf{n}} = \bar{\mathbf{z}} \cdot \bar{\mathbf{n}} = \alpha$ for any $\beta \in \mathbb{R}$, thus $\bar{\mathbf{z}} + \beta\bar{\mathbf{e}}_j$ belongs to W.
2. If $\bar{\mathbf{e}}_j$ is not orthogonal to $\bar{\mathbf{n}}$ and the distance d_j of adjacent hyperplanes along $\bar{\mathbf{e}}_i$ in the family is of the form λ/k for some $k \in \mathbb{N}$, then $\bar{\mathbf{z}} + r\lambda\bar{\mathbf{e}}_j$ belongs to the family for any $r \in \mathbb{Z}$. This follows from a repeated application of the fact that if $\bar{\mathbf{z}}$ belongs to a hyperplane W, then $\bar{\mathbf{z}} + \frac{\lambda}{k}\bar{\mathbf{e}}_j$ belongs to an adjacent hyperplane of W.

Let us proceed by contradiction, i.e., we assume that there exists $i \in \{1, \ldots, t\}$ such that $\bar{\mathbf{e}}_i$ is not orthogonal to $\bar{\mathbf{n}}$ and the distance along $\bar{\mathbf{e}}_i$ of adjacent hyperplanes of the family in question is not of the form $\lambda/k,\ k \in \mathbb{N}$. Take the largest of such indices and denote it by ℓ. Choose $A, B \in M$ arbitrarily. According to assumptions, there exists an $(\ell-1)$-tuple $(A_1, A_2, \ldots, A_{\ell-1})$ such that both $(A_1, A_2, \ldots, A_{\ell-1}, A)$ and $(A_1, A_2, \ldots, A_{\ell-1}, B)$ are ℓ-tuples of Z. It is therefore possible to find two t-tuples of Z such that the first one is of the form $(A_1, A_2, \ldots, A_{\ell-1}, A, A_{\ell+1}, \ldots, A_t)$ and the second one of the form $(A_1, A_2, \ldots, A_{\ell-1}, B, \hat{A}_{\ell+1}, \ldots, \hat{A}_t)$. These two t-tuples – considered as vectors in $\mathbb{R}^t$ – belong by the assumption of Lemma 1.3 to some hyperplanes in the family. Since all vectors $\bar{\mathbf{e}}_j,\ j \in \{\ell+1, \ldots, t\}$ are either orthogonal to $\bar{\mathbf{n}}$ or the distance of adjacent hyperplanes along $\bar{\mathbf{e}}_j$ is of the form λ/k for some $k \in \mathbb{N}$, we can change the last $t-\ell$ coordinates $\hat{A}_{\ell+1}, \ldots, \hat{A}_t$ of the second vector to arbitrary values from M (we transform them into $A_{\ell+1}, \ldots, A_t$) and it will still belong to a hyperplane in the family. This is a consequence of the observation at the beginning of this proof. Hence, both vectors $(A_1, A_2, \ldots, A_{\ell-1}, A, A_{\ell+1}, \ldots, A_t)$ and $(A_1, A_2, \ldots, A_{\ell-1}, B, A_{\ell+1}, \ldots, A_t)$ belong to some hyperplanes of the family. Their distance along $\bar{\mathbf{e}}_\ell$ equals $|A - B|$, i.e., d_ℓ divides $A - B$. Since A, B have been chosen arbitrarily, it follows that d_ℓ divides λ, i.e., $\lambda = kd_\ell$ for some $k \in \mathbb{N}$, which is a contradiction with the choice of $\bar{\mathbf{e}}_\ell$. □

Proof (Proof of Proposition 1.1). Let $\bar{\mathbf{n}}$ be the unit normal vector of a family of parallel equidistant hyperplanes covering all t-tuples of Z. Suppose without loss of generality that $\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_\ell$ are not orthogonal to $\bar{\mathbf{n}}$ and $\bar{\mathbf{e}}_{\ell+1}, \ldots, \bar{\mathbf{e}}_t$ are orthogonal to $\bar{\mathbf{n}}$. Let $\bar{\mathbf{z}} = (Z_n, Z_{n+1}, \ldots, Z_{n+t-1})$ be a t-tuple of Z, thus $\bar{\mathbf{z}}$ belongs to one of the hyperplanes. Take any vector $\bar{\mathbf{y}} \in M^t$ and let us show that it belongs to a hyperplane in the family.

1. Any vector from M^t which differs from $\bar{\mathbf{z}}$ only in the first ℓ components belongs to a hyperplane of the family. This comes from Lemma 1.3 because when we change for $i \in \{1, \ldots, \ell\}$ the i-th component of $\bar{\mathbf{z}}$ by $d_i = \frac{\lambda}{k}$, then we jump on the adjacent parallel hyperplane. So, any transformation of the i-th component of $\bar{\mathbf{z}}$ into another value from M means a finite number of jumps from one hyperplane onto another. Hence, we may transform $\bar{\mathbf{z}}$ so that it has the first ℓ components equal to $\bar{\mathbf{y}}$ and the obtained vector $\bar{\mathbf{x}}$ belongs to a hyperplane in the family.

2. Any vector from M^t which differs from $\bar{\mathbf{x}}$ only in the last $t-\ell$ components belongs to the same hyperplane as $\bar{\mathbf{x}}$. This comes from the orthogonality $\bar{\mathbf{e}}_i \perp \bar{\mathbf{n}}$ for $i > \ell$ (the argument is the same as in the proof of Lemma 1.3). Since $\bar{\mathbf{y}}$ differs from $\bar{\mathbf{x}}$ only in the last $t-\ell$ components, $\bar{\mathbf{y}}$ belongs to a hyperplane in the family. $\square$

2 Combinatorics on Words and the WELLDOC Property

2.1 Backgrounds on Combinatorics on Words

In the following, $\mathcal{A}$ denotes a finite set of symbols called *letters*; the set $\mathcal{A}$ is therefore called an *alphabet*. A finite string $w = w_1 w_2 \dots w_n$ of letters from $\mathcal{A}$ is said to be a *finite word*, its length is denoted by $|w| = n$ and $|w|_a$ denotes the number of occurrences of $a \in \mathcal{A}$ in w. The empty word, a neutral element for concatenation of finite words, is denoted ε and it is of zero length. The set of all finite words over the alphabet $\mathcal{A}$ is denoted by $\mathcal{A}^*$.

Under an *infinite word* we understand an infinite sequence $u = u_0 u_1 u_2 \dots$ of letters from $\mathcal{A}$. A finite word w is a *factor* of a word v (finite or infinite) if there exist words p and s such that $v = pws$. If $p = \varepsilon$, then w is said to be a *prefix* of v; if $s = \varepsilon$, then w is a *suffix* of v. The set of factors and prefixes of v are denoted by $\mathrm{Fact}(v)$ and $\mathrm{Pref}(v)$, respectively. If $v = ps$ for finite words v, p, s, then we write $p = vs^{-1}$ and $s = p^{-1}v$.

An infinite word u over the alphabet $\mathcal{A}$ is called *eventually periodic* if it is of the form $u = vw^\omega$, where v, w are finite words over $\mathcal{A}$ and ω denotes an infinite repetition. An infinite word is called *aperiodic* if it is not eventually periodic.

For any factor w of an infinite word u, every index i such that w is a prefix of the infinite word $u_i u_{i+1} u_{i+2} \dots$ is called an *occurrence* of w in u. An infinite word u is *recurrent* if each of its factors has infinitely many occurrences in u.

The *factor complexity* of an infinite word u is a map $\mathcal{C}_u : \mathbb{N} \mapsto \mathbb{N}$ defined by $\mathcal{C}_u(n) :=$ the number of factors of length n contained in u. The factor complexity of eventually periodic words is bounded, while the factor complexity of an aperiodic word u satisfies $\mathcal{C}_u(n) \geq n+1$ for all $n \in \mathbb{N}$. A *right extension* of a factor w of u over the alphabet $\mathcal{A}$ is any letter $a \in \mathcal{A}$ such that wa is a factor of u. Of course, any factor of u has at least one right extension. A factor w is called *right special* if w has at least two right extensions. Similarly, one can define a *left extension* and a *left special* factor. A factor is *bispecial* if it is both right and left special. An aperiodic word contains right special factors of any length.

The *Parikh vector* of a finite word w over an alphabet $\{0, 1, \dots, d-1\}$ is defined as $(|w|_0, |w|_1, \dots, |w|_{d-1})$. For a finite or infinite word $u = u_0 u_1 u_2 \dots$, $\mathrm{Pref}_n\, u$ will denote the prefix of length n of u, i.e., $\mathrm{Pref}_n\, u = u_0 u_1 \dots u_{n-1}$.

Some of the examples we consider are morphic words. A *morphism* is a function $\varphi : \mathcal{A}^* \to \mathcal{B}^*$ such that $\varphi(\varepsilon) = \varepsilon$ and $\varphi(wv) = \varphi(w)\varphi(v)$, for all $w, v \in \mathcal{A}^*$. Clearly, a morphism is completely defined by the images of the letters in the domain. A morphism is *prolongable* on $a \in \mathcal{A}$, if $|\varphi(a)| \geq 2$ and a is a prefix of $\varphi(a)$. If φ is prolongable on a, then $\varphi^n(a)$ is a proper prefix of $\varphi^{n+1}(a)$, for all $n \in \mathbb{N}$. Therefore, the sequence $(\varphi^n(a))_{n\geq 0}$ of words defines an infinite word u that is a fixed point of φ. Such a word u is a (pure) *morphic* word.

Let us introduce a combinatorial condition on infinite words that – as we will see later – guarantees no lattice structure for the associated PRNGs.

Definition 2.1 (The WELLDOC property). *We say that an aperiodic infinite word u over the alphabet $\{0,1,\ldots,d-1\}$ has* well distributed occurrences *(or has* the WELLDOC property*) if u satisfies for any $m \in \mathbb{N}$ and any factor w of u the following condition. If $i_0, i_1, \ldots$ denote the occurrences of w in u, then*

$$\left\{\left(|\operatorname{Pref}_{i_j} u|_0, \ldots, |\operatorname{Pref}_{i_j} u|_{d-1}\right) \bmod m \mid j \in \mathbb{N}\right\} = \mathbb{Z}_m^d\,;$$

that is, the Parikh vectors of $\operatorname{Pref}_{i_j} u$ *for $j \in \mathbb{N}$, when reduced modulo m, give the whole set $\mathbb{Z}_m^d$.*

We define the WELLDOC property for aperiodic words since it clearly never holds for periodic ones.

With the above notation, it is easy to see that if a recurrent infinite word u has the WELLDOC property, then for every vector $\mathbf{v} \in \mathbb{Z}_m^d$ there are infinitely many values of j such that the Parikh vector of $\operatorname{Pref}_{i_j} u$ is congruent to $\mathbf{v}$ modulo m.

Example 2.2. The Thue-Morse word

$$u = 0110100110010110100101100110 1001\cdots,$$

which is a fixed point of the morphism $0 \mapsto 01$, $1 \mapsto 10$, does not satisfy the WELLDOC property. Indeed, take $m = 2$ and $w = 00$, then w occurs only in odd positions i_j so that $(|\operatorname{Pref}_{i_j} u|_0 + |\operatorname{Pref}_{i_j} u|_1) = i_j$ is odd. Thus, e.g.,

$$(|\operatorname{Pref}_{i_j} u|_0, |\operatorname{Pref}_{i_j} u|_1) \bmod 2 \neq (0,0),$$

and hence

$$\{(|\operatorname{Pref}_{i_j} u|_0, |\operatorname{Pref}_{i_j} u|_1) \bmod 2 \mid j \in \mathbb{N}\} \neq \mathbb{Z}_2^2.$$

Example 2.3. We say that an infinite word u over an alphabet $\mathcal{A}$, $|\mathcal{A}| = d$, is *universal* if it contains all finite words over $\mathcal{A}$ as its factors. It is easy to see that any universal word satisfies the WELLDOC property. Indeed, for any word $w \in \mathcal{A}^*$ and any m there exists a finite word v such that if $i_0, i_1, \ldots, i_k$ denote the occurrences of w in v, then

$$\left\{\left(|\operatorname{Pref}_{i_j} v|_0, \ldots, |\operatorname{Pref}_{i_j} v|_{d-1}\right) \bmod m \mid j \in \{0,1,\ldots,k\}\right\} = \mathbb{Z}_m^d\,.$$

Since u is universal, v is a factor of u. Denoting by i an occurrence of v in u, one gets that the positions $i + i_j$ are occurrences of w in u. Hence

$$\begin{aligned}&\left\{\left(|\operatorname{Pref}_{i+i_j} u|_0, \ldots, |\operatorname{Pref}_{i+i_j} u|_{d-1}\right) \bmod m \mid j \in \{0,1,\ldots,k\}\right\} = \\ &= (|\operatorname{Pref}_{i} u|_0, \ldots, |\operatorname{Pref}_{i} u|_{d-1}) + \\ &+ \left\{\left(|\operatorname{Pref}_{i_j} v|_0, \ldots, |\operatorname{Pref}_{i_j} v|_{d-1}\right) \bmod m \mid j \in \{0,1,\ldots,k\}\right\} = \mathbb{Z}_m^d\,.\end{aligned}$$

Therefore, u satisfies the WELLDOC property.

2.2 Combination of PRNGs

In order to eliminate the lattice structure, it helps to combine PRNGs in a smart way. Such a method was introduced in [12]. Let $X = (X_n)_{n\in\mathbb{N}}$ and $Y = (Y_n)_{n\in\mathbb{N}}$ be two PRNGs with the same output $M \subset \mathbb{N}$ and the same period $m \in \mathbb{N}$, and let $u = u_0u_1u_2\ldots$ be a binary infinite word over the alphabet $\{0, 1\}$.

The PRNG

$$Z = (Z_n)_{n\in\mathbb{N}} \tag{1}$$

based on u is obtained by the following algorithm:

1. Read step by step the letters of u.
2. When you read 0 for the i-th time, copy the i-th symbol from X to the end of the constructed sequence Z.
3. When you read 1 for the i-th time, copy the i-th symbol from Y to the end of the constructed sequence Z.

Of course, it is possible to generalize this construction: Using infinite words over a multiliteral alphabet, one can combine more than two PRNGs. Remark that following terminology from [3], the sequence Z is obtained as a *shuffle* of the sequences X and Y with the steering word u.

In order to distinguish between generators and infinite words used for their combination, we always denote generators with capital letters $X, Y, Z, \ldots$ and words with lower-case letters u, v, w (the same convention is applied for their outputs: $A, B, \ldots$ for output values of generators (elements of M), $a, b, \ldots$ for letters of words). Finite sequences of successive elements $\bar{\mathbf{x}} = (X_i, X_{i+1}, \ldots, X_{i+t-1})$ of a PRNG X are called t-tuples, or vectors, while in the case of an infinite word u, we call $u_iu_{i+1}\ldots u_{i+t-1}$ a factor of length t.

2.3 The WELLDOC Property and Absence of the Lattice Structure

Guimond et al. in [13] have shown that PRNGs based on infinite words coding a certain class of cut-and-project sets have no lattice structure. In the sequel, we will generalize their result and find larger classes of words guaranteeing no lattice structure for associated generators.

Let us consider binary words in the sequel. However, the proofs would work for multiliteral words as well (and for combination of more generators therefore). They would only be more technical.

Theorem 2.4. *Let Z be the PRNG based on a binary infinite word u with the WELLDOC property. Then Z has no lattice structure.*

Proof. According to Proposition 1.1, it suffices to check that its assumptions are met. Let $A, B \in M$ and $\ell \in \mathbb{N}$. Assume $A = X_i$ and $B = Y_j$, where $X = (X_n)_{n\in\mathbb{N}}$ and $Y = (Y_n)_{n\in\mathbb{N}}$ are the two combined PRNGs with the same output $M \subset \mathbb{N}$ and the same period $m \in \mathbb{N}$. Consider a right special factor w of u of length ℓ, i.e., both words $w0$ and $w1$ are factors of u (such a factor w exists since u is an aperiodic word because of the WELLDOC property). By Definition 2.1, it is possible to find an occurrence i_k of $w0$ in u such that

$$|\operatorname{Pref}_{i_k} u|_0 = i - |w|_0 - 1 \bmod m,$$

$$|\operatorname{Pref}_{i_k} u|_1 = j - |w|_1 - 1 \bmod m.$$

When reading the word $w0$ at the occurrence i_k, the corresponding ℓ-tuple $(A_1, A_2, \ldots, A_\ell)$ of the generator Z consists of symbols

$$X_{(i-|w|_0) \bmod m}, \ldots, X_{(i-1) \bmod m} \text{ and } Y_{(j-|w|_1) \bmod m}, \ldots, Y_{(j-1) \bmod m}.$$

When reading 0 after w, the symbol $X_i = A$ from the first generator follows $(A_1, A_2, \ldots, A_\ell)$.

Again, by Definition 2.1, it is possible to find an occurrence i_s of $w1$ in u such that

$$|\operatorname{Pref}_{i_s} u|_0 = i - |w|_0 - 1 \bmod m,$$

$$|\operatorname{Pref}_{i_s} u|_1 = j - |w|_1 - 1 \bmod m.$$

When reading the word w at the occurrence i_s, the same ℓ-tuple $(A_1, A_2, \ldots, A_\ell)$ of Z as previously occurs. This time, however, $(A_1, A_2, \ldots, A_\ell)$ is followed by B because we read $w1$ and $Y_j = B$. Thus, we have found an ℓ-tuple $(A_1, A_2, \ldots, A_\ell)$ of Z followed in Z by both A and B. $\square$

Remark 2.5. The WELLDOC property is sufficient, but not necessary for absence of the lattice structure. Let us illustrate this fact on an example. Consider a modified Fibonacci word $\hat{u}$ where the letter 2 is inserted after each letter, i.e., $\hat{u} = 0212020212021202\ldots$. As one can easily verify such a word does not have well distributed occurrences. However, let us show the following statement: Let Z be the PRNG combining three generators $X = (X_n)_{n\in\mathbb{N}}, Y = (Y_n)_{n\in\mathbb{N}}$ and $V = (V_n)_{n\in\mathbb{N}}$ with the same output $M \subset \mathbb{N}$ and the same period $m \in \mathbb{N}$ according to the modified Fibonacci word $\hat{u}$. Then Z has no lattice structure.

It suffices to verify assumptions of Proposition 1.1. Let $A, B \in M$ and $\ell \in \mathbb{N}$, ℓ an even number (the proof will be analogous for ℓ odd). Assume $A = X_i$ and $B = Y_j$. Consider w a right special factor of the Fibonacci word u of length $\ell/2$. Since u satisfies the WELLDOC property, it is possible to find an occurrence i_k of $w0$ in u such that

$$|\operatorname{Pref}_{i_k} u|_0 = i - |w|_0 - 1 \bmod m,$$

$$|\operatorname{Pref}_{i_k} u|_1 = j - |w|_1 - 1 \bmod m.$$

Then if we insert the letter 2 after each letter of w, we obtain a right special factor $\hat{w}$ of the modified Fibonacci word $\hat{u}$ of length ℓ. It holds then that

$$\begin{aligned}
&|\operatorname{Pref}_{2i_k} \hat{u}|_0 = i - |w|_0 - 1 \bmod m = i - |\hat{w}|_0 - 1 \bmod m,\\
&|\operatorname{Pref}_{2i_k} \hat{u}|_1 = j - |w|_1 - 1 \bmod m = j - |\hat{w}|_1 - 1 \bmod m,\\
&|\operatorname{Pref}_{2i_k} \hat{u}|_2 = i - |w|_0 - 1 + j - |w|_1 - 1 \bmod m = i + j - |\hat{w}|_2 - 2 \bmod m.
\end{aligned}$$

When reading the word $\hat{w}0$ at the occurrence $2i_k$, the corresponding ℓ-tuple $(A_1, A_2, \ldots, A_\ell)$ of the generator Z is followed by the symbol $X_i = A$ from the first generator.

Again, by the WELLDOC property of u, it is possible to find an occurrence i_s of $w1$ in u such that

$$|\operatorname{Pref}_{i_s} u|_0 = i - |w|_0 - 1 \bmod m,$$

$$|\operatorname{Pref}_{i_s} u|_1 = j - |w|_1 - 1 \bmod m.$$

It holds then that

$$\begin{aligned}
&|\operatorname{Pref}_{2i_s} \hat{u}|_0 = i - |w|_0 - 1 \bmod m = i - |\hat{w}|_0 - 1 \bmod m,\\
&|\operatorname{Pref}_{2i_s} \hat{u}|_1 = j - |w|_1 - 1 \bmod m = j - |\hat{w}|_1 - 1 \bmod m,\\
&|\operatorname{Pref}_{2i_s} \hat{u}|_2 = i - |w|_0 - 1 + j - |w|_1 - 1 \bmod m = i + j - |\hat{w}|_2 - 2 \bmod m.
\end{aligned}$$

When reading the word $\hat{w}$ at the occurrence $2i_s$, the same ℓ-tuple $(A_1, A_2, \ldots, A_\ell)$ of Z as previously occurs. This time, however, $(A_1, A_2, \ldots, A_\ell)$ is followed by B because we read $\hat{w}1$ and $Y_j = B$. Thus, we have found an ℓ-tuple $(A_1, A_2, \ldots, A_\ell)$ of Z followed in Z by both A and B. Therefore Z has no lattice structure.

Remark 2.6. As we can see in the proof of Theorem 2.4, the modulus m considered in the WELLDOC property is set to be equal to the period of the combined generators. Therefore, if we require absence of the lattice structure for a PRNG obtained when combining PRNGs with a fixed period $\hat{m}$, then it is sufficient to use an infinite word u that satisfies the WELLDOC property for the modulus $m = \hat{m}$. This means for instance that the Thue-Morse word is not completely out of the game, but it cannot be used to combine periodic PRNGs with the period being a power of 2.

We have formulated a combinatorial condition – well distributed occurrences – guaranteeing no lattice structure of the associated generator. It is now important to find classes of words satisfying such a condition.

3 Sturmian Words

In this section we show that Sturmian words have well distributed occurrences.

Definition 3.1. *An aperiodic infinite word u is called* Sturmian *if its factor complexity satisfies $\mathcal{C}_u(n) = n + 1$ for all $n \in \mathbb{N}$.*

So, Sturmian words are by definition binary and they have the lowest possible factor complexity among aperiodic infinite words. Sturmian words admit various types of characterizations of geometric and combinatorial nature. One of such characterizations is via irrational rotations on the unit circle. In [17] Hedlund and Morse showed that each Sturmian word may be realized measure-theoretically by an irrational rotation on the circle. That is, every Sturmian word is obtained by coding the symbolic orbit of a point on the circle of circumference one under a rotation R_α by an irrational angle[1] α, $0 < \alpha < 1$, where the circle is partitioned into two complementary intervals, one of length α and the other of length $1-\alpha$. Conversely, each such coding gives rise to a Sturmian word.

Definition 3.2. *The* rotation *by angle α is the mapping R_α from $[0,1)$ (identified with the unit circle) to itself defined by $R_\alpha(x) = \{x+\alpha\}$, where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. Considering a partition of $[0,1)$ into $I_0 = [0, 1-\alpha)$, $I_1 = [1-\alpha, 1)$, define a word*

$$s_{\alpha,\rho}(n) = \begin{cases} 0 & \text{if } R_\alpha^n(\rho) = \{\rho + n\alpha\} \in I_0, \\ 1 & \text{if } R_\alpha^n(\rho) = \{\rho + n\alpha\} \in I_1. \end{cases}$$

One can also define $I_0' = (0, 1-\alpha]$, $I_1' = (1-\alpha, 1]$, the corresponding word is denoted by $s'_{\alpha,\rho}$.

Remark that some but not all Sturmian words are morphic. In fact, it is known that a characteristic Sturmian word (i.e., $\rho = \alpha$) is morphic if and only if the continuous fraction expansion of α is periodic. For more information on Sturmian words we refer to [14, Chapter 2].

Theorem 3.3. *Let u be a Sturmian word on $\{0,1\}$. Then u has the WELLDOC property.*

Proof. In the proof we use the definition of Sturmian word via rotation. The main idea is controlling the number of 1's modulo m by taking circle of length m, and controlling the length taking the rotation by $m\alpha$.

For the proof we will use an equivalent reformulation of the theorem:

Let u be a Sturmian word on $\{0,1\}$, for any natural number m and any factor w of u let us denote $i_0, i_1, \ldots$ the occurrences of w in u. Then

$$\left\{ \left(i_j, |\operatorname{Pref}_{i_j} u|_1 \right) \bmod m \mid j \in \mathbb{N} \right\} = \mathbb{Z}_m^2.$$

That is, we will control the number of 1's and the length instead of the number of 0's.

Since a Sturmian word can be defined via rotations by an irrational angle on a unit circle, without loss of generality we may assume that $u = s_{\alpha,\rho}$ for some $0 < \alpha < 1$, $0 \le \rho < 1$, α irrational (see Definition 3.2). Equivalently, we can consider m copies of the circle connected into one circle of length m with m intervals I_1^i of length α corresponding to 1. The Sturmian word is obtained by rotation by α on this circle of length m (see Fig. 2).

Namely, we define the rotation $R_{\alpha,m}$ as the mapping from $[0,m)$ (identified with the circle of length m) to itself defined by $R_{\alpha,m}(x) = \{x+\alpha\}_m$, where $\{x\}_m = x - \lfloor x/m \rfloor m$ and for $m = 1$ coincides with the fractional part of x.

[1] Measured by arc length (thus equivalent to $2\pi\alpha$ radians).

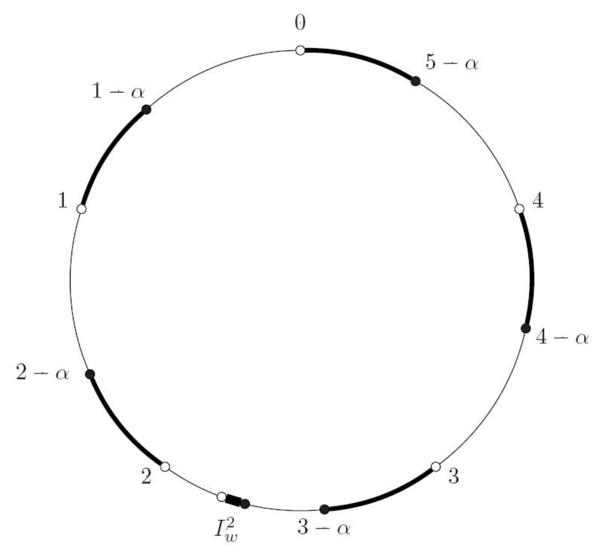

Fig. 2. Illustration to the proof of Theorem 3.3: the example for $m = 5$.

A partition of $[0, m)$ into $2m$ intervals $I_0^i = [i, i+1-\alpha)$, $I_1^i = [i+1-\alpha, i+1)$, $i = 0, \dots, m-1$ defines the Sturmian word $u = s_{\alpha,\rho}$:

$$s_{\alpha,\rho}(n) = \begin{cases} 0 & \text{if } R^n_{\alpha,m}(\rho) = \{\rho + n\alpha\}_m \in I_0^i \text{ for some } \ i = 0, \dots, m-1, \\ 1 & \text{if } R^n_{\alpha,m}(\rho) = \{\rho + n\alpha\}_m \in I_1^i \text{ for some } \ i = 0, \dots, m-1. \end{cases}$$

It is well known that any factor $w = w_0 \cdots w_{k-1}$ of u corresponds to an interval I_w in $[0, 1)$, so that whenever you start rotating from the interval I_w, you obtain w. Namely, $x \in I_w$ if and only if $x \in I_{w_0}, R_\alpha(x) \in I_{w_1}, \dots, R_\alpha^{|w|-1}(x) \in I_{w_{|w|-1}}$.

Similarly, we can define m intervals corresponding to w in $[0, m)$ (circle of length m), so that if $I_w = [x_1, x_2)$, then $I_w^i = [x_1 + i, x_2 + i)$, $i = 0, \dots, m-1$.

Fix a factor w of u, take arbitrary $(j, i) \in \mathbb{Z}_m^2$. Now let us organize (j, i) among the occurrences of w, i.e., find l such that $u_l \dots u_{l+|w|-1} = w$, $l \mod m = j$ and $|\mathrm{Pref}_l u|_1 \mod m = i$:

Consider rotation $R_{m\alpha,m}(x)$ by $m\alpha$ instead of rotation by α, and start m-rotating from $j\alpha + \rho$. Formally, $R_{m\alpha,m}(x) = \{x + m\alpha\}_m$, where, as above, $\{x\}_m = x - [x/m]m$. This rotation will put us to positions $mk + j$, $k \in \mathbb{N}$, in the Sturmian word: for $a \in \{0, 1\}$ one has $s_{\alpha,\rho}(mk + j) = a$ if $R^k_{m\alpha,m}(j\alpha + \rho) = \{j\alpha + \rho + km\alpha\}_m \in I_a^i$ for some $i = 0, \dots, m-1$.

Remark that the points in the orbit of an m-rotation of a point on the m-circle are dense, and hence the rotation comes infinitely often to each interval.

So pick k when $j\alpha + mk\alpha + \rho \in I_w^i \subset [i, i+1)$ (and actually there exist infinitely many such k). Then the length l of the corresponding prefix is equal to $km + j$, and the number of 1's in it is $i + mp$, where p is the number of complete circles you made, i.e., $p = [(j\alpha + mk\alpha + \rho)/m]$. □

4 Arnoux-Rauzy Words

In this section we show that Arnoux-Rauzy words [1], which are natural extensions of Sturmian words to larger alphabets, also satisfy the WELLDOC property. Note that the proof for Sturmian words cannot be generalized to Arnoux-Rauzy words, because it is based on the geometric interpretation of Sturmian words via rotations, while this interpretation does not extend to Arnoux-Rauzy words.

4.1 Basic Definitions

The definitions and results we remind in this subsection are well-known and mostly taken from [1, 9] and generalize the ones given for binary words in [5].

Definition 4.1. *Let $\mathcal{A}$ be a finite alphabet. The* reversal operator *is the operator* $\sim: \mathcal{A}^* \mapsto \mathcal{A}^*$ *defined by recurrence in the following way:*

$$\tilde{\varepsilon} = \varepsilon, \quad \widetilde{va} = a\tilde{v}$$

for all $v \in \mathcal{A}^*$ and $a \in \mathcal{A}$. The fixed points of the reversal operator are called palindromes*.*

Definition 4.2. *Let $v \in \mathcal{A}^*$ be a finite word over the alphabet $\mathcal{A}$. The* right palindromic closure *of v, denoted by $v^{(+)}$, is the shortest palindrome that has v as a prefix. It is readily verified that if p is the longest palindromic suffix of $v = wp$, then $v^{(+)} = wp\tilde{w}$.*

Definition 4.3. *We call the* iterated (right) palindromic closure operator *the operator ψ recurrently defined by the following rules:*

$$\psi(\varepsilon) = \varepsilon, \quad \psi(va) = (\psi(v)a)^{(+)}$$

for all $v \in \mathcal{A}^*$ and $a \in \mathcal{A}$. The definition of ψ may be extended to infinite words u over $\mathcal{A}$ as $\psi(u) = \lim_n \psi(\mathrm{Pref}_n\, u)$, i.e., $\psi(u)$ is the infinite word having $\psi(\mathrm{Pref}_n\, u)$ as its prefix for every $n \in \mathbb{N}$.

Definition 4.4. *Let Δ be an infinite word on the alphabet $\mathcal{A}$ such that every letter occurs infinitely often in Δ. The word $c = \psi(\Delta)$ is then called a* characteristic (or standard) Arnoux-Rauzy word *and Δ is called the* directive sequence *of c. An infinite word u is called an Arnoux-Rauzy word if it has the same set of factors as a (unique) characteristic Arnoux-Rauzy word, which is called the characteristic word of u. The directive sequence of an Arnoux-Rauzy word is the directive sequence of its characteristic word.*

Let us also recall the following well-known characterization (see e.g. [9]):

Theorem 4.5. *Let u be an aperiodic infinite word over the alphabet $\mathcal{A}$. Then u is a standard Arnoux-Rauzy word if and only if the following hold:*

1. $\mathrm{Fact}(u)$ *is closed under reversal (that is, if v is a factor of u so is $\tilde{v}$).*
2. *Every left special factor of u is also a prefix.*
3. *If v is a right special factor of u then va is a factor of u for every $a \in \mathcal{A}$.*

From the preceding theorem, it can be easily verified that the bispecial factors of a standard Arnoux-Rauzy correspond to its palindromic prefixes (including the empty word), and hence to the iterated palindromic closure of the prefixes of its directive sequence. That is, if

$$\varepsilon = b_0, b_1, b_2, \ldots$$

is the sequence, ordered by length, of bispecial factors of the standard Arnoux-Rauzy word u, $\Delta = \Delta_0 \Delta_1 \cdots$ its directive sequence (with $\Delta_i \in \mathcal{A}$ for every i), we have $b_{i+1} = (b_i \Delta_i)^{(+)}$.

A direct consequence of this, together with the preceding definitions, is the following statement, which will be used in the sequel.

Lemma 4.6. *Let u be a characteristic Arnoux-Rauzy word and let Δ and $(b_i)_{i \geq 0}$ be defined as above. If Δ_i does not occur in b_i, then $b_{i+1} = b_i \Delta_i b_i$. Otherwise let $j < i$ be the largest integer such that $\Delta_j = \Delta_i$. Then $b_{i+1} = b_i b_j^{-1} b_i$.*

4.2 Parikh Vectors and Arnoux-Rauzy Factors

Where no confusion arises, given an Arnoux-Rauzy word u, we will denote by

$$\varepsilon = b_0, b_1, \ldots, b_n, \ldots$$

the sequence of bispecial factors of u ordered by length and we will denote for any $i \in \mathbb{N}$, $\bar{\mathbf{b}}_i$ the Parikh vector of b_i.

Remark 4.7. By the pigeonhole principle, it is clear that for every $m \in \mathbb{N}$ there exists an integer $N \in \mathbb{N}$ such that, for every $i \geq N$, the set $\{j > i \mid \bar{\mathbf{b}}_j \equiv_m \bar{\mathbf{b}}_i\}$ is infinite. Where no confusion arises and with a slight abuse of notation, fixed m, we will always denote by N the smallest of such integers.

Lemma 4.8. *Let u be a characteristic Arnoux-Rauzy word and let $m \in \mathbb{N}$. Let*

$$\alpha_1 \bar{\mathbf{b}}_{j_1} + \cdots + \alpha_k \bar{\mathbf{b}}_{j_k} \equiv_m \bar{\mathbf{v}} \in \mathbb{Z}_m^d$$

be a linear combination of Parikh vectors such that $\sum_{i=1}^{k} \alpha_i = 0$, with $j_i \geq N$ and $\alpha_i \in \mathbb{Z}$ for all $i \in \{1, \ldots k\}$. Then, for any $\ell \in \mathbb{N}$, there exists a prefix v of u such that the Parikh vector of v is congruent to $\bar{\mathbf{v}}$ modulo m and vb_ℓ is also a prefix of u.

Proof. Without loss of generality, we can assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$, hence there exists k' such that

$$\alpha_1 \geq \alpha_{k'} \geq 0 \geq \alpha_{k'+1} \geq \alpha_k.$$

We will prove the result by induction on $\beta = \sum_{j=1}^{k'} \alpha_j$. If $\beta = 0$, trivially, we can take $v = \varepsilon$ and the statement is clearly verified. Let us assume the statement true for all $0 \leq \beta < n$ and let us prove it for $\beta = n$. By the remark preceding this lemma, for every ℓ we can choose $i' > j' > \ell$ such that $\bar{\mathbf{b}}_{j_1} \equiv_m \bar{\mathbf{b}}_{i'}$ and $\bar{\mathbf{b}}_{j_k} \equiv_m \bar{\mathbf{b}}_{j'}$. Since every bispecial factor is a prefix and suffix of all the bigger ones, in particular we have that $b_{j'}$ is a suffix of $b_{i'}$, and b_ℓ is a prefix of $b_{j'}$; this implies that $b_{i'} b_{j'}^{-1} b_\ell$ is actually a prefix of $b_{i'}$. By assumption, the Parikh vector of $b_{i'} b_{j'}^{-1}$ is clearly $\bar{\mathbf{b}}_{i'} - \bar{\mathbf{b}}_{j'} \equiv_m \bar{\mathbf{b}}_{j_1} - \bar{\mathbf{b}}_{j_k}$. Since $\alpha_1 \geq 1$ implies $\alpha_k \leq -1$, we have, by induction hypothesis, that there exists a prefix w of u such that the Parikh vector of w is congruent modulo m to

$$(\alpha_1 - 1)\bar{\mathbf{b}}_{j_1} + \cdots + (\alpha_k + 1)\bar{\mathbf{b}}_{j_k}$$

and $wb_{i'}$ is a prefix of u. Hence $wb_{i'} b_{j'}^{-1} b_\ell$ is also a prefix of u and, by simple computation, the Parikh vector of $v = wb_{i'} b_{j'}^{-1}$ is congruent modulo m to $\bar{\mathbf{v}} = \alpha_1 \bar{\mathbf{b}}_{j_1} + \cdots + \alpha_k \bar{\mathbf{b}}_{j_k}$. □

Definition 4.9. *Let $n \in \mathbb{Z}$. We will say that an integer linear combination of integer vectors is a n-*combination *if the sum of all the coefficients equals n.*

Lemma 4.10. *Let u be a characteristic Arnoux-Rauzy word and let $n \in \mathbb{N}$. Every n-combination of Parikh vectors of bispecial factors can be expressed as an n-combination of Parikh vectors of arbitrarily large bispecials. In particular, for every $K, L \in \mathbb{N}$, it is possible to find a finite number of integers $\alpha_1, \ldots, \alpha_k$ such that $\bar{\mathbf{b}}_K = \alpha_1 \bar{\mathbf{b}}_{j_1} + \cdots + \alpha_k \bar{\mathbf{b}}_{j_k}$ with $j_i > L$ for every i and $\alpha_1 + \cdots + \alpha_k = 1$.*

Proof. A direct consequence of Lemma 4.6 is that for every i such that Δ_i appears in b_i, we have $\bar{\mathbf{b}}_{i+1} = 2\bar{\mathbf{b}}_i - \bar{\mathbf{b}}_j$, where $j < i$ is the largest such that $\Delta_j = \Delta_i$. This in turn (since every letter in Δ appears infinitely many times from the definition of Arnoux-Rauzy word) implies that *for every* non-negative integer j, there exists a positive k such that $\bar{\mathbf{b}}_j = 2\bar{\mathbf{b}}_{j+k} - \bar{\mathbf{b}}_{j+k+1}$, that is, we can substitute each Parikh vector of a bispecial with a 1-combination of Parikh vectors of strictly larger bispecials. Simply iterating the process, we obtain the statement. □

In the following we will assume the set $\mathcal{A}$ to be a finite alphabet of cardinality d. For every set $X \subseteq \mathcal{A}^*$ of finite words, we will denote by $\mathrm{PV}(X) \subseteq \mathbb{Z}^d$ the set of Parikh vectors of elements of X and for every $m \in \mathbb{N}$ we will denote by $\mathrm{PV}_m(X) \subseteq \mathbb{Z}_m^d$ the set of elements of $\mathrm{PV}(X)$ reduced modulo m.

For an infinite word u over $\mathcal{A}$, and a factor v of u, let $S_v(u)$ denote the set of all prefixes of u followed by an occurrence of v. In other words,

$$S_v(u) = \{p \in \mathrm{Pref}(u) \mid pv \in \mathrm{Pref}(u)\}.$$

Definition 4.11. *For any set of finite words $X \subseteq \mathcal{A}^*$, we will say that u* has the property $\mathcal{P}_X$ *(or, for short, that u has $\mathcal{P}_X$) if, for every $m \in \mathbb{N}$ and for every $v \in X$ we have that*

$$\mathrm{PV}_m(S_v(u)) = \mathbb{Z}_m^d.$$

That is to say, for every vector $\bar{\mathbf{w}} \in \mathbb{Z}_m^d$ there exists a word $w \in S_v(u)$ such that the Parikh vector of w is congruent to $\bar{\mathbf{w}}$ modulo m.

With this notation, an infinite word u has the WELLDOC property if and only if it has the property $\mathcal{P}_{\mathrm{Fact}(u)}$.

Proposition 4.12. *Let u be a characteristic Arnoux-Rauzy word over the d-letter alphabet $\mathcal{A}$. Then u has the property $\mathcal{P}_{\mathrm{Pref}(u)}$.*

Proof. Let us fix an arbitrary $m \in \mathbb{N}$. We want to show that, for every $v \in \mathrm{Pref}(u)$, $\mathrm{PV}_m(S_v(u)) = \mathbb{Z}_m^d$. Let then $\bar{\mathbf{v}} \in \mathbb{Z}^d$ and ℓ be the smallest number such that v is a prefix of b_ℓ. Let $i_1 < i_2 < \cdots < i_d$ be such that Δ_{i_j} does not appear in b_{i_j}, where Δ is the directive word of u. Without loss of generality, we can rearrange the letters so that each Δ_{i_j} is lexicographically smaller than $\Delta_{i_{j+1}}$. With this assumption if, for every j, we set $\bar{\mathbf{v}}_j = \bar{\mathbf{b}}_{i_j+1}$, i.e., equal to the Parikh vector of b_{i_j+1}, which, by the first part of Lemma 4.6, equals $b_{i_j}\Delta_{i_j}b_{i_j}$, we can find $j-1$ positive integers $\mu_1, \ldots, \mu_{j-1}$ such that $\bar{\mathbf{v}}_j = (\mu_1, \mu_2, \ldots, \mu_{j-1}, 1, 0, \ldots, 0)$. It is easy to show, then, that the set $V = \{\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_d\}$ generates $\mathbb{Z}^d$, hence there exists an integer n such that $\bar{\mathbf{v}}$ can be expressed as an n-combination of elements of V (which are Parikh vectors of bispecial factors of u). Trivially, then, $\bar{\mathbf{v}} = \bar{\mathbf{v}} - n\bar{\mathbf{0}} = \bar{\mathbf{v}} - n\bar{\mathbf{b}}_0$; thus, it is possible to express $\bar{\mathbf{v}}$ as a 0-combination of Parikh vectors of (by the previous Lemma 4.10) arbitrarily large bispecial factors of u. By Lemma 4.8, then there exists a prefix p of u whose Parikh vector $\bar{\mathbf{p}}$ satisfies $\bar{\mathbf{p}} \equiv_m \bar{\mathbf{v}}$ and pb_ℓ is a prefix of u. Since we picked ℓ such that v is a prefix of b_ℓ, we have that $p \in S_v(u)$. From the arbitrariness of v, $\bar{\mathbf{v}}$ and m, we obtain the statement. □

As a corollary of Proposition 4.12, we obtain the main result of this section.

Theorem 4.13. *Let u be an Arnoux-Rauzy word over the d-letter alphabet $\mathcal{A}$. Then u has the property $\mathcal{P}_{\mathrm{Fact}(u)}$, or equivalently, u has the WELLDOC property.*

Proof. Let m be a positive integer and let c be the characteristic word of u. Let v be a factor of u and xvy be the shortest bispecial containing v. By Proposition 4.12, we have that $\mathrm{PV}_m(S_{xv}(c)) = \mathbb{Z}_m^d$ and, since the set is finite, we can find a prefix p of c such that $\mathrm{PV}_m(S_{xv}(p)) = \mathbb{Z}_m^d$. Let w be a prefix of u such that wp is a prefix of u. If $\bar{\mathbf{x}}$ and $\bar{\mathbf{w}}$ are the Parikh vectors of, respectively, x and w, it is easy to see that

$$\bar{\mathbf{w}} + \bar{\mathbf{x}} + \mathrm{PV}(S_{xv}(p)) \subseteq \bar{\mathbf{w}} + \mathrm{PV}(S_v(p)) \subseteq \mathrm{PV}(S_v(u))$$

Since we have chosen p such that $\mathrm{PV}_m(S_{xv}(p)) = \mathbb{Z}_m^d$, we clearly obtain that $\mathrm{PV}_m(S_v(u)) = \mathbb{Z}_m^d$ and hence, by the arbitrariness of v and m, the statement. □

Remark 4.14. Note the following simple method of obtaining words satisfying the WELLDOC property. Take a word u with the WELLDOC property over an alphabet $\{0, 1, \dots, d-1\}$, $d > 2$, apply a morphism $\varphi : d-1 \mapsto 0, i \mapsto i$ for $i = 0, \dots, d-2$, i.e., φ joins two letters into one. It is straightforward that $\varphi(u)$ has the WELLDOC property. So, taking Arnoux-Rauzy words and joining some letters, we obtain other words than Sturmian and Arnoux-Rauzy satisfying the WELLDOC property.

Remark 4.15. No we introduce another class of morphisms preserving the WELLDOC property. Recall that the *adjacency matrix* Φ of a morphism $\varphi : \mathcal{A} \to \mathcal{A}$, with $\mathcal{A} = \{0, 1, \dots, d-1\}$, is defined by $\Phi_{i,j} = |\varphi(j-1)|_{i-1}$ for $1 \le i, j \le d$. By definition, it follows that if $\bar{\mathbf{v}}$ is the Parikh vector of $v \in \mathcal{A}^*$, then $\Phi\bar{\mathbf{v}}$ is the Parikh vector of $\varphi(v)$.

Let us show that if $\det \Phi = \pm 1$ and u has the WELLDOC property, then so does $\varphi(u)$. Indeed, let w be any factor of $\varphi(u)$, and suppose $xwy = \varphi(v)$ for some $v \in \mathrm{Fact}(u)$ and $x, y \in \mathcal{A}^*$. We then have $S_w(\varphi(u)) \supseteq \varphi(S_v(u))x$, so that, writing $\bar{\mathbf{x}}$ for the Parikh vector of x, we have for any $m > 0$

$$\mathrm{PV}_m(S_w(\varphi(u))) \supseteq \Phi \cdot \mathrm{PV}_m(S_v(u)) + \bar{\mathbf{x}} \bmod m\,.$$

Since u has the WELLDOC property, $\mathrm{PV}_m(S_v(u)) = \mathbb{Z}_m^d$. As $\det \Phi = \pm 1$, Φ is invertible (even modulo m), so that $\Phi \cdot \mathbb{Z}_m^d + \bar{\mathbf{x}} \bmod m = \mathbb{Z}_m^d$. Hence $\mathrm{PV}_m(S_w(\varphi(u))) = \mathbb{Z}_m^d$, showing that $\varphi(u)$ has the WELLDOC property by the arbitrariness of w and m.

5 Statistical Tests of PRNGs

In the previous part, we have explained that PRNGs based on infinite words with well distributed occurrences have no lattice structure. In the sequel we show that their results in empirical statistical tests are very good, provided that sufficiently good LCGs are combined.

5.1 Computer Generation of Morphic Words

Any real computer is a finite state machine and hence it can generate only finite prefixes of infinite words. From practical point of view it is important to find algorithms that are efficient both in memory footprint and CPU time. Patera has introduced in [18] an efficient algorithm for generating the Fibonacci word – the prefix of length n is generated in $O(\log(n))$ space and $O(n)$ time. We have generalized this method for any Sturmian and Arnoux-Rauzy word being a fixed point of a morphism φ. The main ingredient is that we consider φ^n instead of φ; we precompute and store in the memory $\varphi^n(a)$ for any $a \in \mathcal{A}$. The runtime to generate 10^{10} letters of the Fibonacci and the Tribonacci word is summarized in Table 1. There are two observations we would like to point out:

1. Using the method from [18] together with our improvement for generation of Sturmian and Arnoux-Rauzy words, the speed of generation of their prefixes is much higher than the speed of generation of LCGs output values. For example, generation of 10^{10} 32-bit values using a LCG modulo 2^{64} takes 14.3 seconds on our machine. Compare it to 0.5 seconds for generation of 10^{10} letters of a fixed point of a morphism with the same hardware. Thus, using a fixed point to combine LCGs causes only a negligible runtime penalty.
2. The speed of generation can be further improved by using a higher initial memory footprint and CPU that can effectively copy such larger chunks of memory (size of L1 data cache is a limiting factor). Thus the new method scales nicely and can benefit form the future CPUs with higher L1 caches. The only requirement is to precompute $\varphi^n(a)$, $a \in \mathcal{A}$, for larger n. Our program does this automatically based on the limit on the initial memory consumption provided by the user.

Word	Fibonacci	Tribonacci
φ morphism rule	115s	107s
φ^n morphism rule	0.41s	0.36s

Table 1. The comparison of time in seconds to generate the first 10^{10} letters of the Fibonacci and the Tribonacci word using the original [18] and the new algorithm. The iteration n in the φ^n rule was chosen so that the length of $\varphi^n(a)$ does not exceed 4096 bytes for any $a \in \mathcal{A}$. The measurement was done on Intel Core i7-3520M CPU running at 2.90GHz.

5.2 Testing PRNGs Based on Sturmian and Arnoux-Rauzy words

We will present results for PRNGs based on:

- the Fibonacci word (as an example of a Sturmian word), i.e., the fixed point of the morphism $0 \mapsto 01, 1 \mapsto 0$,
- the modified Fibonacci word – Fibonacci2 – with the letter 2 inserted after each letter (see Remark 2.5),
- the Tribonacci word (as the simplest example of a ternary Arnoux-Rauzy word), i.e., the fixed point of $0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$.

We have implemented PRNGs for more morphic Sturmian words and ternary Arnoux-Rauzy words. Since the results are similar, we present in the sequel only the above three representatives. Our program generating PRNGs based on morphic words is available online, together with a description at [4].

Remark that we included the modified Fibonacci word that does not have the WELLDOC property, but at the same time it guarantees no lattice structure for the arising generator. The reason for including it is that we would like to illustrate that such a word leads to worse results in testing than the Fibonacci word.

Combining LCGs Instead of combining plain LCGs, we will execute some modifications before their combination. Those modifications turn out to be useful according to the known weaknesses of LCGs.

We have chosen LCGs with the period m in range from $2^{47}-115$ to 2^{64}, but we use only their upper 32 bits as the output since the statistical tests require 32-bit sequences as the input. Their output is thus in all cases $M = \{0, 1, \ldots, 2^{32} - 1\}$. (We do not use directly 2^{32} as modulus since it is known that the k-th bit of a LCG whose modulus is 2^{ℓ} for some $\ell \in \mathbb{N}$ is equal to k.)

We use two batteries of random tests – TestU01 BigCrush and PractRand. They operate differently. The first one includes 160 statistical tests, many of them tailored to the specific classes of PRNGs. It is a reputable test, however its drawback is that it works with a fixed amount of data and discards the least significant bit (for some tests even two bits) of the 32-bit numbers being tested. The second battery consists of three different tests where one is adapted on short range correlations, one reveals long range violations, and the last one is a variation on the classical Gap test. Details can be found in [7] and [8]. Moreover, the PractRand battery applies automatically various filters on the input data. For our purpose the lowbit filter is interesting – it is passing various number of the least significant bits to the statistical tests. As we have already mentioned, the LCGs with $m = 2^{\ell}$ have a much shorter period than the LCG itself. Therefore the lowbit filter is useful to check whether this weakness disappears when LCGs are combined according to an infinite word. The PractRand tests are able to treat very long input sequences, up to a few exabytes. To control the runtime we have limited the length of input sequences to 16TB.

The first column of Table 2 shows the list of tested LCGs. The BigCrush column is showing how many tests of the TestU01 BigCrush battery failed. The PractRand column gives the $\log_2$ of sample datasize in Bytes for which the results of the PractRand tests started to be "very suspicious" (p-values smaller than 10^{-5}). One LCG did not show any failures in the PractRand tests which is denoted as > 16TB. The last column provides time in seconds to generate the first 10^{10} 32-bit sequences of output on Intel i7-3520M CPU running at 2.90GHz.

From Table 2 it can be seen that the LCGs with $m \in \{2^{47} - 115, 2^{63} - 25\}$ have the best statistical properties from the chosen LCGs. At the same time, these LCGs are 20 times slower than the other LCGs used. This is because we cannot avoid the 128-bit integer arithmetic to compute their internal state as well as we cannot avoid explicit modulo operation. As the CPU used does not have the 128-bit integer arithmetic, it has to be implemented in software (in this case via GCC's `__int128` type) which is much slower than the 64-bit arithmetic wired on CPU.

Results in Statistical Tests We will present results for the PRNGs based on the Fibonacci, Fibonacci2 and Tribonacci word using the different combinations of LCGs from Table 2. It includes also the situations where the instances of the same LCG are used. Each instance has its own state. The LCGs were seeded with the value 1. The PRNGs were warmed up by generating 10^9 values before

Generator	Legend	BigCrush	PractRand	Time 10^{10}
$\mathrm{LCG}(2^{47}-115, 71971110957370, 0)$	L47-115	14	40	281
$\mathrm{LCG}(2^{63}-25, 2307085864, 0)$	L63-25	2	>44	277
$\mathrm{LCG}(2^{59}, 13^{13}, 0)$	L59	19	27	14.1
$\mathrm{LCG}(2^{63}, 5^{19}, 1)$	L63	19	33	14.4
$\mathrm{LCG}(2^{64}, 2862933555777941757, 1)$	L64_28	18	35	14.0
$\mathrm{LCG}(2^{64}, 3202034522624059733, 1)$	L64_32	14	34	14.1
$\mathrm{LCG}(2^{64}, 3935559000370003845, 1)$	L64_39	13	33	14.0

Table 2. List of the used LCGs with parameters $\mathrm{LCG}(m, a, c)$. Results in the BigCrush (number of failed tests) and in the PractRand ($\log_2$ of sample size for which the test started to fail) battery of statistical tests. Time in seconds to generate the first 10^{10} 32-bit words of output on Intel i7-3520M CPU running at 2.90GHz.

statistical tests started. Since the relative frequency of the letters in the aperiodic words differ a lot (for example for the Fibonacci word the ratio of zeroes to ones is given by $\tau = \frac{1+\sqrt{5}}{2}$), the warming procedure will guarantee that the state of instances of LCGs will differ even when the same LCGs are used. Even more importantly, the distance between the LCGs is growing as the new output of PRNGs is generated.

Summary of results is in Table 3. The BigCrush column is using the following notation: the first number indicates how many tests from the BigCrush battery have clearly failed and the optional second number in parenthesis denotes how many tests have suspiciously low p-value in the range from 10^{-6} to 10^{-4}. The PractRand column gives the $\log_2$ of sample datasize in Bytes for which the results of the PractRand tests started to be “very suspicious” (p-values smaller than 10^{-5}). The maximum sample data size used was 16TB $\doteq 2^{44}$B. The Time column gives runtime in seconds to generate the first 10^{10} 32-bit words of output on Intel i7-3520M CPU running at 2.90GHz. The source code of the testing programs is in [4].

Table 3: Summary of results of statistical tests for PRNGs based on the Fibonacci, Fibonacci2 and Tribonacci word and different combinations of LCGs from Table 2.

Word	Group	0	1	2	BigCrush	PractRand	Time 10^{10}
Fib	A	L64_28	L64_28		0	41	30.2
		L64_32	L64_28		0(1)	41	29.3
		L64_39	L64_28		0 (2)	41	31
		L64_28	L64_32		0	41	30.2
		L64_32	L64_32		0	41	30.1

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Table 3 – Continued from the previous page

Word	Group	0	1	2	BigCrush	PractRand	Time 10^{10}
		L64_39	L64_32		0	41	30.1
		L64_28	L64_39		0	42	30.2
		L64_32	L64_39		0	40	30.5
		L64_39	L64_39		0	42	30.1
	B	L47-115	L47-115		1(1)	>44	302
		L63-25	L63-25		0(1)	>44	299
		L59	L59		0(1)	34	28.7
		L63	L63		0	40	29.8
	C	L63-25	L59		0	38	198
		L59	L63-25		0(1)	35	134
		L63-25	L64_39		0	>44	199
		L64_39	L63-25		0	41	135
		L59	L64_39		0	35	30.4
		L64_39	L59		0	37	31.3
Fib2	A	L64_28	L64_28	L64_28	0	40	28.4
		L64_39	L64_28	L64_28	0(2)	40	27.9
		L64_39	L64_32	L64_28	0	39	27.5
		L64_28	L64_39	L64_28	0	40	27.3
		L64_32	L64_39	L64_28	0	40	27.5
		L64_39	L64_39	L64_28	0	40	27.4
		L64_39	L64_28	L64_32	0	40	27.3
		L64_28	L64_39	L64_32	0	40	27.9
		L64_28	L64_28	L64_39	0(1)	40	27.4
		L64_32	L64_28	L64_39	0	39	27.7
		L64_39	L64_28	L64_39	0	40	27.3
		L64_28	L64_32	L64_39	0	40	27.3
		L64_28	L64_39	L64_39	0	40	27.3
		L64_39	L64_39	L64_39	0	40	27.4
	B	L47-115	L47-115	L47-115	0(2)	>44	297.0
		L63-25	L63-25	L63-25	0(2)	>44	293.0
		L59	L59	L59	0(1)	32	27.4
		L63	L63	L63	0	38	27.3
	C	L63-25	L59	L64_39	0(1)	39	113.0
		L63-25	L64_39	L59	0	32	113.0
		L59	L63-25	L64_39	0	38	81.1
		L59	L64_39	L63-25	0	39	158.3
		L64_39	L63-25	L59	0	31	81.0
		L64_39	L59	L63-25	0	42	159.0
Trib	A	L64_28	L64_28	L64_28	0(2)	42	27.2
		L64_39	L64_28	L64_28	0	43	27.1
		L64_39	L64_32	L64_28	0(1)	42	28.0
		L64_28	L64_39	L64_28	0(1)	42	28.1

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Table 3 – Continued from the previous page

Word	Group	0	1	2	BigCrush	PractRand	Time 10^{10}
		L64_32	L64_39	L64_28	0	42	27.1
		L64_39	L64_39	L64_28	0(1)	42	27.2
		L64_39	L64_28	L64_32	0	43	27.1
		L64_28	L64_39	L64_32	0(1)	42	27.1
		L64_28	L64_28	L64_39	0	42	28.0
		L64_32	L64_28	L64_39	0	42	27.2
		L64_39	L64_28	L64_39	0(1)	43	27.1
		L64_28	L64_32	L64_39	0	43	27.1
		L64_28	L64_39	L64_39	0(2)	42	27.3
		L64_39	L64_39	L64_39	0	43	27.1
	B	L47-115	L47-115	L47-115	1	>44	299.0
		L63-25	L63-25	L63-25	0(1)	>44	298.0
		L59	L59	L59	0	35	27.2
		L63	L63	L63	0(1)	41	27.2
	C	L63-25	L59	L64_39	0(1)	39	172.0
		L63-25	L64_39	L59	0(1)	41	173.0
		L59	L63-25	L64_39	0	35	106.0
		L59	L64_39	L63-25	0	34	70.5
		L64_39	L63-25	L59	0	41	107.0
		L64_39	L59	L63-25	0(1)	40	74.3

We can make the following observations based on the results in statistical tests:

1. The quality of LCGs has improved substantially when we combined them according to infinite words with the WELLDOC property. This can be seen in the TestU01 BigCrush results. While for LCGs 13 to 19 tests have clearly failed (the only exception is the generator L63-25 with two failures – see Table 2), almost all of the BigCrush tests passed. The worst result was to have one BigCrush test failed for the Tribonacci combination (or two for the Fibonacci combination) of L47-115 generators. The likely reason is that the generator L47-115 has the shortest period of all tested LCGs.
2. The results of the PractRand battery confirm the above findings. For instance, in the case of LCGs with modulo 2^{64}, the test started to find irregularities in the distribution of the least significant bit of tested PRNGs output at around 2TB sample size. Compare it with the sample size of 8GB to 32GB when fast plain LCGs started to fail the test. The PractRand battery applies different filters on the input stream and all failures appeared for `Low1/32` filter where only the least significant bit of the PRNG output is used. It corresponds to a known weakness of LCGs: lower bits of the output have significantly smaller period than the LCG itself. The quality of the PRNGs can be therefore further improved by combining LCGs that do not show flaws for the least significant bits or by using for example just 16 upper bits of the LCGs output.

3. The quality of the PRNG is linked to the quality of the underlying LCG. When looking at the group B in Table 3, we observe that the PractRand results of the arising PRNGs are closely related to the succes of LCGs from Table 2 in the PractRand tests.
4. Another interesting observation is that using the instances of the same LCG (with only sufficiently distinct seeds) produces as good results as combination of different LCGs (multipliers and shifts are different, but the modulus is the same). It is just important to make sure that starting states of the LCGs are far apart enough. Refer to the group A in Table 3.
5. The lower quality LCG will dictate the quality of resulting PRNG. When mixing LCGs with different quality, use better ones as replacement for more frequent letters in the aperiodic word.

   Please refer to the group C in Table 3. For example for the Fibonacci word compare first two rows in the group C - the order of LCGs is merely swapped but the difference in the sample size for which PractRand starts to fail is 8×. This is even more significant for the Tribonacci based generators where the difference between the worst and best PractRand results when reordering the underlying LCGs is given by factor 128×.
6. On the other hand, results from the group A in Table 3 demonstrate that when using generators of similar quality (same modulus, similar deficiencies), the order in which generators are used to substitute the letters of the infinite word does not influence the quality of the resulting generator.
7. We can also see that the modified Fibonacci word (see Remark 2.5) does not produce better results than the Fibonacci word. Clearly, a regular structure of 2's on every other position does not help to produce a better random sequence even if we mix now three LCGs instead of two as in the case of the Fibonacci word.
8. Results for the Tribonacci word are better than for the Fibonacci word. (We have observed this fact for all ternary Arnoux-Rauzy words in comparison to Sturmian words.) It seems therefore that mixing three LCGs is better than using just two LCGs, assuming that an infinite word with the WELLDOC property is used for mixing. We expect naturally that the better chosen LCGs (or even some other modern fast linear PRNGs, e.g. *mt19937* or nonlinear PRNGs based on the AES cipher) we combine according to an infinite word with the WELLDOC property, the better their results in statistical tests will be.
9. We have also tested LCGs with $m = 2^{31} - 1$. It has revealed that if the underlying generators have poor statistical properties, then the PRNG will not be able to mask it. In particular, you cannot expect that PRNGs – despite their infinite aperiodic nature – will fix the short period problem. Once the period of the underlying LCG is exhausted, statistical tests will find irregularities in the output of the PRNG.

In conclusion, summing up the main results from the user point of view:

- Using different instances of the same LCG to form a new generator based on the infinite word with the WELLDOC property will yield a generator with improved statistical properties.
- The introduced method of generation of morphic words is very fast and supports parallel processing.
- The period of underlying generators has to be large enough – much larger than the number of needed values.
- When using different types of the underlying LCGs to form a PRNG, close attention has to be paid to the right order of the combined LCGs. The generator with the worst properties should be used to replace the least frequent letter of the aperiodic word. Moreover, statistical properties of the resulting PRNG will be ruled by the deficiencies of the worst used generator.

6 Open problems and future research

There are many open problems left. Concerning the combinatorial part of our paper, one of the interesting questions there is finding large families of infinite words satisfying the WELLDOC property. For example, which morphic words have the WELLDOC property? Also, it seems to be meaningful to study a weaker WELLDOC property where in Definition 2.1 instead of every $m \in \mathbb{N}$ we consider only a particular m. For instance, one can search for words satisfying such a modified WELLDOC condition for $m = 2$, $m = 2^{\ell}$ etc. Another question to be asked is how to construct words with the WELLDOC property over larger alphabets using words with such a property over smaller alphabets. Regarding statistical tests, it remains to explain why PRNGs based on infinite words with the WELLDOC property succeed in tests and to compare their results with other comparably fast generators.

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