All about Infinite Words Associated with Quadratic Non-simple Parry Numbers

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Abstract. Studying of factor complexity, palindromic complexity, and return words of infinite aperiodic words is an interesting combinatorial problem. Moreover, investigation of infinite words associated with β-integers $\mathbb{Z}_\beta$, for β being a Pisot number, can be interpreted as investigation of one-dimensional quasicrystals. In this paper, new results concerning the above combinatorial characteristics for quadratic non-simple Parry number β will be presented. This is the only case among (non-simple) Parry numbers worth of studying palindromes since for non-quadratic cases, there is only a finite number of palindromes in the associated infinite word u_β. We have investigated factor and palindromic complexity, return words, and arithmetics using methods which can be applied for any infinite aperiodic words being fixed points of a substitution.

Abstrakt. Studium komplexity, palindromické komplexity a "return" slov v nekonečných aperiodických slovech je zajímavý kombinatorický problém. Zkoumání nekonečných slov přidružených β-celým číslům lze navíc interpretovat jako zkoumání jednodimenzionálních kvazikrystalů. V tomto článku představíme nové výsledky týkající se výše zmiňovaných charakteristik a také aritmetiky. Námi zkoumaný případ je jediným případem mezi parryovskými čísly, kdy přidružené nekonečné slovo obsahuje nekonečně mnoho palindromů a je tedy zajímavé z hlediska vyšetřování palindromické komplexity. Použité metody se dají aplikovat na celou řadu slov, která jsou pevnými body substitucí.

1 Introduction

Some kinds of infinite aperiodic words can serve as models for one dimensional quasicrystals, i.e., materials with long-range orientational order and sharp diffraction images of non-crystallographic symmetry. We will focus on infinite words u_β associated with β-integers $\mathbb{Z}_\beta \subset \mathbb{R}$. It has been shown that for β being a Pisot number ($\beta > 1$ being an algebraic integer such that all its Galois conjugates have modulus strictly less than one), $\mathbb{Z}_\beta$ is a uniformly discrete and relatively dense set (in one word, it is a Delone set [9]) fulfilling $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ for a finite set F (the Meyer property [10]). Since self-similar Delone sets fulfilling the Meyer property are suitable models for quasicrystalline structure, β-integers for β being a Pisot number serve as models for one dimensional quasicrystals.

2 Preliminaries

First, let us introduce our "language" which will be used throughout this paper. An *alphabet* $\mathcal{A}$ is a finite set of symbols called *letters*. A concatenation of letters is a *word*.

The set $\mathcal{A}^*$ of all finite words (including the empty word ε) provided with the operation of concatenation is a free monoid. We will deal also with right-sided infinite words $u = u_0u_1u_2\ldots$. A finite word w is called a *factor* of the word u (finite or infinite) if there exist a finite word $w^{(1)}$ and a word $w^{(2)}$ (finite or infinite) such that $u = w^{(1)}ww^{(2)}$. The word w is a *prefix* of u if $w^{(1)} = \varepsilon$. A concatenation of k letters a (or words a) will be denoted by a^k, a concatenation of infinitely many letters a (or words a) by a^ω. An infinite word u is said to be *eventually periodic* if there exist words v, w such that $u = vw^\omega$. A word which is not eventually periodic is called *aperiodic*. An infinite word u is *uniformly recurrent* if for any $n \in \mathbb{N}$ there exists an $R(n) \in \mathbb{N}$ such that any factor of u of length $R(n)$ contains all factors of length n. The *language* on u is the set of all factors of a word u. A mapping φ on the free monoid $\mathcal{A}^*$ is called a morphism if $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in \mathcal{A}^*$. Obviously, for determining any morphism it suffices to give $\varphi(a)$ for all $a \in \mathcal{A}$. The action of a morphism can be naturally extended on right-sided infinite words by the prescription

$$\varphi(u_0u_1u_2\ldots) := \varphi(u_0)\varphi(u_1)\varphi(u_2)\ldots$$

A non-erasing morphism φ, for which there exists a letter $a \in \mathcal{A}$ such that $\varphi(a) = aw$ for some non-empty word $w \in \mathcal{A}^*$, is called a substitution. An infinite word u such that $\varphi(u) = u$ is called a fixed point of the substitution φ. Obviously, every substitution has at least one fixed point, namely

$$\lim_{n\to\infty} \varphi^n(a).$$

A substitution φ is primitive if there exists an integer exponent k such that for each pair of letters $a, b \in \mathcal{A}$, the letter a appears in the word $\varphi^k(b)$. Queffélec [12] showed that any fixed point of a primitive substitution is a uniformly recurrent infinite word.

3 Beta-expansions and beta-integers

Let $\beta > 1$ be a real number and let x be a positive real number. Any convergent series of the form:

$$x = \sum_{i=-\infty}^{k} x_i\beta^i,$$

where $x_i \in \mathbb{N}$, is called a *β-representation* of x. As well as it is usual for the decimal system, we will denote the β-representation of x by

$$x_kx_{k-1}\ldots\ x_0 \bullet x_{-1}\ldots \quad ,$$

if $k \geq 0$, otherwise

$$0 \bullet \underbrace{0\ldots\ldots\ldots\ldots 0}_{(-1-k)-times} x_kx_{k-1}\ldots \quad .$$

If a β-representation ends with infinitely many zeros, it is said to be finite and the ending zeros are omitted. A representation of x can be obtained by the following greedy algorithm: There exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k := \lfloor \frac{x}{\beta^k} \rfloor$ and $r_k := \{\frac{x}{\beta^k}\}$, where $\lfloor . \rfloor$ denotes the lower integer part and $\{.\}$ denotes the fractional part. For $i < k$, put $x_i := \lfloor \beta r_{i+1} \rfloor$ and $r_i := \{\beta r_{i+1}\}$. The representation obtained by the greedy algorithm is called *β-expansion* of x and denoted $\langle x\rangle_\beta$. If $x = \sum_{i=-\infty}^{k} x_i\beta^i$ is the β-expansion of a nonnegative number x, then $\sum_{i=-\infty}^{-1} x_i\beta^i$ is called the β-fractional (or simply fractional) part of x. Let us introduce some important notions connected with β-expansions:

- The set of nonnegative numbers with vanishing fractional part are called nonnegative β-integers, formally

$$\mathbb{Z}_\beta^+ := \{x \geq 0 \mid \langle x\rangle_\beta = x_k x_{k-1} \dots x_0\bullet\}.$$

- The set of β-integers is then defined by

$$\mathbb{Z}_\beta := -\mathbb{Z}_\beta^+ \cup \mathbb{Z}_\beta^+.$$

- All the real numbers with a finite β-expansion of $|x|$ form the set $Fin(\beta)$, formally

$$Fin(\beta) := \bigcup_{n\in\mathbb{N}} \frac{1}{\beta^n}\mathbb{Z}_\beta.$$

- For any $x \in Fin(\beta)$, we denote by $fp_\beta(x)$ the length of its fractional part, i.e.,

$$fp_\beta(x) = \min\{l \in \mathbb{N} \mid \beta^l x \in \mathbb{Z}_\beta\}.$$

The sets $\mathbb{Z}_\beta$ and $Fin(\beta)$ are generally not closed under addition and multiplication. The following notion is important for studying of lengths of the fractional parts which may appear as a result of addition and multiplication.

- $L_\oplus(\beta) := \min\{L \in \mathbb{N} \mid x, y \in \mathbb{Z}_\beta, x + y \in Fin(\beta) \Longrightarrow fp_\beta(x + y) \leq L\}.$

- $L_\otimes(\beta) := \min\{L \in \mathbb{N} \mid x, y \in \mathbb{Z}_\beta, xy \in Fin(\beta) \Longrightarrow fp_\beta(xy) \leq L\}.$

If such $L \in \mathbb{N}$ does not exist, we set $L_\oplus(\beta) := \infty$ or $L_\otimes(\beta) := \infty$.

The Rényi expansion of unity simplifies the description of elements of $\mathbb{Z}_\beta$ and $Fin(\beta)$. For its definition, we introduce the transformation $T_\beta(x) := \{\beta x\}$ for $x \in [0, 1]$. The *Rényi expansion of unity* in the base β is defined as

$$d_\beta(1) = t_1 t_2 t_3 \dots, \quad \text{where} \quad t_i := \lfloor \beta T_\beta^{i-1}(1) \rfloor.$$

One can show that every real number β can be characterized by its Rényi expansion of unity (see [11]). Numbers with a finite Rényi expansion of unity are called *simple Parry numbers*. Numbers with an eventually periodic Rényi expansion of unity are called *(non-simple) Parry numbers*.

4 Infinite words associated with beta-integers

In [14], it is shown that the distances occurring between neighbors of $\mathbb{Z}_\beta$ form the set $\{\Delta_k \mid k \in \mathbb{N}\}$, where

$$\Delta_k := \sum_{i=1}^{\infty} \frac{t_{i+k}}{\beta^i} \text{ for } k \in \mathbb{N}. \tag{1}$$

It is evident that the set $\{\Delta_k \mid k \in \mathbb{N}\}$ is finite if and only if $d_\beta(1)$ is eventually periodic.

Every Pisot number, i.e., a real algebraic integer greater than 1, all of whose conjugates are of modulus strictly less than 1, is a Parry number. On the other hand, every quadratic Parry number is Pisot. This explains that if we deal with quadratic Parry numbers β, the set $\mathbb{Z}_\beta$ models one-dimensional quasicrystals.

From now on, we will restrict our considerations to quadratic Parry numbers. The Rényi expansion of unity for a simple quadratic Parry number β is equal to $d_\beta(1) = pq$, where $p \geq q$, in other words, β is the positive root of the polynomial $x^2 - px - q$. Whereas the Rényi expansion of unity for a non-simple quadratic Parry number β is equal to $d_\beta(1) = pq^\omega$, where $p > q \geq 1$, and β is the greater root of the polynomial $x^2 - (p+1)x + p - q$. Drawn on the real line, there are only two distances between neighboring points of $\mathbb{Z}_\beta$. The longer distance is always $\Delta_0 = 1$, the smaller one is Δ_1.

If we assign letters 0, $1B$ to the two types of distances Δ_0 and Δ_1, respectively, and write down the order of distances in $\mathbb{Z}_\beta^+$ on the real line, we naturally obtain an infinite word; we will denote this word by u_β. Since $\beta\mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+$, it can be shown easily that the word u_β is a fixed point of a certain substitution φ (see e.g. [6]); in particular, for the non-simple quadratic Pisot number β, the generating substitution is

$$\varphi(0) = 0^p1, \quad \varphi(1) = 0^q1. \tag{2}$$

5 Combinatorial and arithmetical properties of infinite words

To understand the physical properties of quasicrystals, it is useful to investigate the combinatorial and arithmetical properties of the infinite aperiodic words u_β modeling quasicrystals.

- The number of local configurations of atoms in quasicrystalline materials is described by *factor complexity*. It is a function associating to every integer n the number of different factors of length n contained in u_β.
- The local symmetry of the material corresponds to *palindromic complexity*. It is a function associating to every integer n the number of different palindromes of length n contained in u_β, where *palindrome* is a word which stays the same when read backwards.
- Another interesting characterization of richness of motives appearing in quasicrystals is given by the notion of *return words*. Let w be a factor of u_β. Take an arbitrary occurrence of w in u_β. You obtain a return word of w if you read letters successively,

beginning at the first letter of w and ending with the letter preceding the very next occurrence of w.

There is one more important reason why to deal with β-integers. It is sometimes useful in computer science to consider addition or multiplication in β-arithmetics. $\mathbb{Z}_\beta$ is generally not closed under addition and multiplication, thus it is useful to study the fractional parts that may appear as results of these operations and to estimate their lengths, i.e., to find the values of $L_\oplus(\beta)$ and $L_\otimes(\beta)$.

6 Summary of known results

Let us remind that all the characteristics we consider here, i.e., factor complexity, palindromic complexity, return words, and arithmetics, have been already investigated for quadratic Parry units.

- Factor complexity of the infinite word u_β associated with β being a quadratic Parry unit is equal to $n+1$ [4], i.e., u_β in this case is a Sturmian word. Let us mention that all the eventually periodic words have complexity less or equal to n for some n, hence Sturmian words are the simplest aperiodic words.

- It has been shown in [5] that there are exactly 2 palindromes of any odd length and one palindrome of any even length, hence palindromic complexity P reaches the following values
$$P(2n) = P(0) = 1 \quad \text{and} \quad P(2n+1) = P(1) = 2 \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

- According to [15] it holds that an infinite word on a binary alphabet is Sturmian if and only if for any factor w, there exist two return words of w.

- Results for arithmetics of quadratic Parry units have been found in [4]. For the case of β having the Rényi expansion of unity $d_\beta(1) = p1$, the exact values of $L_\oplus(\beta)$ and $L_\otimes(\beta)$ are $L_\oplus(\beta) = L_\otimes(\beta) = 2$, while for β having the Rényi expansion of unity $d_\beta(1) = p(p-1)^\omega$, it holds $L_\oplus(\beta) = L_\otimes(\beta) = 1$.

A lot of work has been done also for simple Parry numbers. The exact formula for factor complexity of u_β for β being a simple Parry number with the Rényi expansion of unity $d_\beta(1) = t_1t_2...t_m$, where $t_1 = t_2 = ... = t_{m-1}$ or $t_1 > \max\{t_2, ..., t_{m-1}\}$ has been derived in [7]. It is useful to consider palindromic complexity only for the case of $t_1 = t_2 = ... = t_{m-1}$. Otherwise, the language of u_β is not closed under reversal and, hence, contains only a finite number of palindromes. The exact formula for palindromic complexity of u_β associated with a simple Parry number with the Rényi expansion of unity $d_\beta(1) = t_1t_2...t_m$, where $t_1 = t_2 = ... = t_{m-1}$, has been found in [1].

In [8], one can find very precise estimates on $L_\oplus(\beta)$ for β being a quadratic simple Parry number and also some rough estimates on $L_\oplus(\beta)$ for β being a non-simple quadratic Parry number.

7 New results

We will present our results concerning all the characteristics of the infinite word u_β associated with beta-integers for β being a quadratic non-simple non-unit Parry number, i.e., β having the Rényi expansion of unity $d_\beta(1) = pq^\omega$, where $p-1 > q \geq 1$, in other words, β being the larger root of the polynomial $x^2 - (p+1)x + (p-q)$. In this case, u_β is the fixed point of the substitution $\varphi(0) = 0^p1,\ \varphi(1) = 0^q1$. We have investigated factor and palindromic complexity [2], return words, and arithmetics [3]. We have found the exact values of factor complexity C, which implies that $C(n+1) - C(n) \in \{1,2\}$ for all $n \in \mathbb{N}$. We have derived the exact values of palindromic complexity P, which confirms that $P(n) \in \{1,2,3,4\}$ for all $n \in \mathbb{N}$. We have shown that for any factor w of u_β, there exist either 2 or 3 return words of w. In the arithmetics, the upper bound on the number $L_\oplus(\beta)$ reached in [8] has been improved. We have shown that $\left\lfloor \frac{p-1}{q} \right\rfloor \leq L_\oplus(\beta) \leq \left\lceil \frac{p}{q} \right\rceil$. Let us describe in more details factor complexity of u_β and the cardinality of the set of return words of u_β.

7.1 Factor complexity

For proofs and precise statements of this section see [2]. To describe complexity of infinite uniformly recurrent words, one can limit his considerations to description of left special factors. Let us remind that a factor w of u_β is left special if both $0w$ and $1w$ are factors of u_β.

Observation 7.1. *Let us denote by M_n the set of all left special factors of length n of an infinite uniformly recurrent word over a two-letter alphabet. Then the first difference of complexity satisfies*

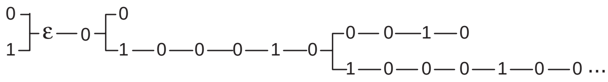
$$\Delta C(n) = C(n+1) - C(n) = \#M_n.$$

To describe all the left special factors of u_β, let us distinguish more types of them.

Definition 7.1. *Let u_β be the infinite word associated with $d_\beta(1) = pq^\omega,\ p-1 > q \geq 1$.*

- *A left special factor $w \in \mathcal{L}(u_\beta)$ is called maximal if neither $w0$ nor $w1$ are left special.*
- *An infinite word v is called an infinite left special factor of u_β if each prefix of v is a left special factor of u_β.*
- *A factor w of u_β is called total bispecial if both $w0$ and $w1$ are left special factors of u_β.*

Example 7.1. *Let us illustrate a few of left special factors of $u_\beta = 00010001000101000100010001 01\ldots$ being the fixed point of the substitution $\varphi(0) = 0001, \varphi(1) = 01$ by construction of the head of a tree containing left special factors. Beginning from the empty word to the right, one can read all left special factors of length $n \in \{1,2,\ldots,14\}$. There are two maximal left special factors 00, 01000100010 and two total bispecial factors 0, 0100010 having length* < 14.



Obviously, every left special factor is a prefix of a maximal or an infinite left special factor. Let us describe all the maximal and infinite left special factors:

- All maximal left special factors have the form:

$$U^{(1)} = 0^{p-1}, \tag{4}$$
$$U^{(n)} = T(U^{(n-1)}) = 0^q 1 \varphi(U^{(n-1)}) 0^q \quad \text{for } n \geq 2.$$

- All total bispecial factors have the form:

$$V^{(1)} = 0^q, \tag{5}$$
$$V^{(n)} = T(V^{(n-1)}) = 0^q 1 \varphi(V^{(n-1)}) 0^q.$$

  Moreover, $V^{(n-1)}$ is a prefix of $V^{(n)}$ and $V^{(n)}$ is a prefix of $U^{(n)}$ for all $n \in \mathbb{N}$.

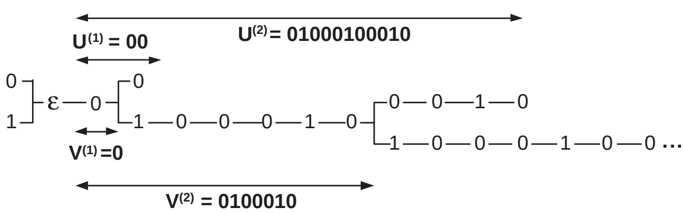
- There exists one infinite left special factor of the form $\lim_{n\to\infty} V^{(n)}$.

For n such that
$$|V^{(k)}| < n \leq |U^{(k)}| \quad \text{for some } k \in \mathbb{N},$$
there exist two left special factors of length n. The lengths $|V^{(k)}|, |U^{(k)}|$ play an essential role for determining of complexity. One can easily obtain the recursive formulae of $|V^{(k)}|, |U^{(k)}|$ using their definition (4) and (5).

Combining all the obtained results, we can determine complexity.

Theorem 7.1. *Let u_β be the fixed point of the substitution $\varphi(0) = 0^p1,\ \varphi(1) = 0^q1,\ p-1 > q \geq 1$. Then for all $n \in \mathbb{N}$*

$$\triangle C(n) = C(n+1) - C(n) = \begin{cases} 2 & |V^{(k)}| < n \leq |U^{(k)}| \quad \textit{for some } k \in \mathbb{N}, \\ 1 & \textit{otherwise.} \end{cases}$$



Obrázek 1: Illustration of the tree of left special factors for u_β being the fixed point of the substitution $\varphi(0) = 0001, \varphi(1) = 01$. We can see total bispecial factors $V^{(k)}$ and maximal left special factors $U^{(k)}$ for $k = 1, 2$.

7.2 Return words

Let us start with an exact definition of a return word. Let w be a factor of an infinite word $u = u_0u_1\dots$ (with $u_j \in \mathcal{A}$), the length $|w| = \ell$. An integer j is an *occurrence* of w in u if $u_ju_{j+1}\dots u_{j+\ell-1} = w$. Let j, k, $j < k$, be successive occurrences of w. Then $u_ju_{j+1}\dots u_{k-1}$ is a *return word* of w. The set of all return words of w is denoted by $M(w)$, i.e.,

$$M(w) = \{u_ju_{j+1}\dots u_{k-1} \mid j, k \text{ being successive occurrences of } w\}.$$

It is not difficult to see that the set of return words of w is finite for any factor w if u is a uniformly recurrent word. In our case, u_β is a fixed point of a primitive substitution, consequently, u_β is uniformly recurrent.

Example 7.2. *Let* $u_\beta = 0010010100100010100101\dots$ *be the fixed point of the substitution* $\varphi(0) = 001, \varphi(1) = 01$. *Let us show examples of return words:*

$$M(0) = \{0, 01\},$$

$$M(00) = \{001, 00101\},$$
$$M(001) = \{001, 00101\},$$
$$M(0010) = \{001, 00101\}.$$

In order to study return words $M(w)$ of factors w of an infinite word u, it is possible to limit our considerations to bispecial factors. Namely, if a factor w is not right special, i.e., if it has a unique right extension $a \in \mathcal{A}$, then the sets of occurrences of w and wa coincide, and

$$M(w) = M(wa).$$

If a factor w has a unique left extension $b \in \mathcal{A}$, then $j \geq 1$ is an occurrence of w in the infinite word u if and only if $j - 1$ is an occurrence of bw. This statement does not hold for $j = 0$. Nevertheless, if u is a uniformly recurrent infinite word, then the set $M(w)$ of return words of w stays the same no matter whether we include the return word corresponding to the prefix w of u or not. Consequently, we have

$$M(bw) = bM(w)b^{-1} = \{bvb^{-1} \mid v \in M(w)\},$$

where bvb^{-1} means that the word v is prolonged to the left by the letter b and it is shortened from the right by erasing the letter b (which is always the suffix of v for $v \in M(w)$).

For an aperiodic uniformly recurrent infinite word u, each factor w can be extended to the left and to the right to a bispecial factor. To describe the cardinality of $M(w)$, it suffices therefore to consider bispecial factors w.

Observation 7.2. *Let w be a bispecial factor of u_β containing at least one 1. Then there exists a bispecial factor v such that $w = \varphi(v)0^q$ and $\#M(w) = \#M(v)$.*

Using Observation 7.2, it suffices to consider bispecial factors of u_β that do not contain 1 to obtain all possible cardinalities of the sets of return words.

Theorem 7.2. *Let w be a factor of u_β. Then $2 \le \#M(w) \le 3$.*

Důkaz. Let us describe return words of bispecial factors that do not contain 1, i.e., that are equal to $0^r, r \le p-1$.

1. Let $r \le q$. Then all the return words of 0^r are the following ones:

$$0 \text{ since there is the block } 0^p \in L(u_\beta),$$

$$0^r1 \text{ since there occurs the block } 0^p10^p \in L(u_\beta).$$

2. Let $q < r \le p-1$. Then all the return words are the following ones:

$$0 \text{ since there is the block } 0^p \in L(u_\beta),$$

$$0^r1 \text{ since there occurs the block } 0^p10^p \in L(u_\beta),$$

$$0^r10^q1 \text{ since there is the block } 0^p10^q10^p \in L(u_\beta).$$

It is apparent that there are no other return words of 0^r. □

Observation 7.3. *From the proof of Theorem 7.2, we can notice that in the case $q = p-1$, $\#M(w) = 2$ for all the prefixes of u_β. This confirms that u_β is Sturmian in this case.*

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