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DIPLOMOVÁ PRÁCE

# Minimální pokrytí diferenční množiny
## (Minimal Covering of the Difference Set)

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Tuto diplomovou práci jsem vypracovala samostatně a uvedla jsem všechnu použitou literaturu.

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# Contents

# Notation

Throughout this work the following notation concerning points and sets is used. The $d$-dimensional Euclidean space is denoted by $\mathbb{R}^d$. Points of $\mathbb{R}^d$ are always denoted by $x, y, z, a, b$, and other lowercase Latin letters and employ lower indices. The coordinate of a vector is denoted by the same letter, with an upper index $1, ..., d$. Thus $x = (x^1, ..., x^d)$. Vector $x^T$ means the transposition of the vector $x$, i.e. $x^T$ is a column vector. Integers are denoted by $i, j, k, l, m, n, s$, with or without lower indices. For real numbers $\lambda, \delta, \varepsilon, \rho, \nu, \alpha$, and other lowercase Greek letters are mostly used. The scalar product of vectors $x, y$ is

$$(x, y) = \sum_{i=1}^{d} x^i y^i.$$

The Gramm matrix of the set of vectors $(x_1, ..., x_d)$ is

$$(x_i, x_j) = \begin{pmatrix} (x_1, x_1) & (x_1, x_2) & ... & (x_1, x_d) \\ (x_2, x_1) & (x_2, x_2) & ... & (x_2, x_d) \\ . & & & \\ . & & & \\ . & & & \\ (x_d, x_1) & (x_d, x_2) & ... & (x_d, x_d) \end{pmatrix}.$$

The length of a vector $x$ is

$$|x| = \sqrt{(x, x)} = \sqrt{(x^1)^2 + ... + (x^d)^2}.$$

Arbitrary sets in $\mathbb{R}^d$ are denoted by capital Latin and Greek letters. The set of points for which some given property $P(x)$ holds is denoted by $\{x | P(x)\}$. $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0$ denotes the set of non-negative integers, i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Furthermore, the following notations are used

$$x + M = \{x + y \mid y \in M\},$$

$$\alpha M = \{\alpha x \mid x \in M\},$$

$$M_1 + M_2 = \{x + y \mid x \in M_1, y \in M_2\},$$

$$M_1 \oplus M_2 = \{z \mid (\exists_1 x \in M_1)(\exists_1 y \in M_2)(z = x + y)\},$$

$$M_1 - M_2 = \{x - y \mid x \in M_1, y \in M_2\},$$

$$M_1 \setminus M_2 = \{x \in M_1 \mid x \notin M_2\}.$$

In general, the set $M + M$ and $2M$ are not identical. The set $M - M$ is also called the *difference set* of $M$. The symbols $\cup, \subset, \supset$ are used to denote the set-theoretical union and inclusion relations, respectively. Further, $\emptyset$ denotes an empty set. The interior of a set $M$ is denoted by $M^\circ$, the closure of $M$ by $\overline{M}$, and the boundary of $M$ by $\partial M$. A set $H$ is called *centrally symmetric* if $H = -H$, i.e. $(\forall x \in H)(-x \in H)$.

# Chapter 1

# Introduction

In this thesis we deal with a specific covering problem, namely covering of the difference set $\Omega - \Omega$ by translated non-rotated open copies of $\Omega$ for a convex compact $\Omega \subset \mathbb{R}^d$. The interest for such a particular study is stimulated by the theory of mathematical quasicrystals.

Quasicrystals take their origin in physics. The logic of the name is often expressed as 'aperiodic crystals'. They were discovered in 1984 [20], and eversince they have been a subject of intensive experimental and theoretical studies. In mathematics, the quasicrystals appeared about a decade before the discovery of quasicrystalline materials in physics under the name Penrose tilings of the plane [18]. Their algebraic theory was then developed by de Bruijn [3]. Much later [15, 16, 12] it was recognized that quasicrystals can be understood as a special case in the general theory of Y. Meyer [13, 14]. Meyer defined a mathematical quasicrystal as a set $\Sigma \subset \mathbb{R}^d$ which fulfils the so-called Delone property (Definition 4.4) and such that

$$\Sigma - \Sigma \subset \Sigma + F \tag{1.1}$$

for a finite set $F$. For such set the name 'Meyer sets' has been adopted. Note that Meyer sets are by (1.1) a natural generalization of lattices, which satisfy $\mathcal{L} - \mathcal{L} \subset \mathcal{L}$. Property (1.1) also implies that the set $\Sigma$ has only finitely many local configurations of a fixed size. (In the case of lattices the configuration is unique.) The number of local configurations relates to the 'complexity' of the quasicrystal model. It is influenced by the cardinality of the finite set $F$ in the Meyer property.

The most commonly used construction of quasicrystal models is the so-called cut-and-project scheme. Even the well known Penrose tiling can be recast in that formalism. Cut-and-project sets (Definition 7.3) $\Sigma = \Sigma(\Omega)$ depend on a compact set $\Omega$, called the acceptance window. The choice of $\Omega$ strongly influences the properties of the cut-and-project set $\Sigma(\Omega)$. In this work especially convex compact sets are treated. Convexity is a reasonable assumption for the acceptance window since, under some additional assumptions, it ensures abundance of scaling symmetries for the quasicrystal model.

Moody [16] has shown that cut-and-project sets are Meyer sets. Thus asking for the finite set $F$ from the Meyer property is a natural question. The aim of this thesis is to investigate the cardinality of the set $F$ for different acceptance windows. This problem can be transformed into investigation of the function $f$ defined by

$$f(\Omega) = \text{ the minimal number of translated copies of } \Omega^\circ \text{ needed for covering of } \Omega - \Omega. \tag{1.2}$$

The value $f(\Omega)$ is called the Meyer number of $\Omega$.

The main result of the thesis is that the function $f$ is bounded on the space of convex compact sets $\Omega$ (Theorem 3.2). We further show that convexity is an essential assumption for such a result since we can construct a sequence of non-convex compact sets $(\Omega_n)_{n\in\mathbb{N}}$ such that $f(\Omega_n)$ tends to infinity with growing $n$ (Chapter 6). In other words, there exists a universal minimal upper bound $K_d$ on the Meyer number $f(\Omega)$ for all convex compacts $\Omega \subset \mathbb{R}^d$, depending only on the dimension $d$, but this is not the case of the family of non-convex $\Omega$.

It is not easy to determine the Meyer number for a given convex compact set $\Omega$. However, we have proved that the function $f$ is upper semicontinuous on the space of convex compact sets $\Omega$ (Theorem 3.3). Such topological study may help in the classification of convex compacts into classes according to their Meyer number.

Further in the thesis we determine the Meyer number $f(\Omega)$ for some planar convex sets and estimate the universal constant $K_d$ in dimensions $d \geq 2$. For this purpose it was useful to compare with different covering problems (and related packing problems), which are studied in discrete geometry. Usually, these problems are very complicated, and only the simplest cases are solved. The best known covering problem asks for the most efficient covering of a bounded shape or of the entire space by balls of equal volume. Usually, coverings of bounded shapes are much more complicated. In dimension $d = 2$, only few results are known about the so-called disk covering problem. However, the most efficient covering of the plane by equal disks has been determined to be the one when disks are placed in points of the hexagonal lattice. The known results about coverings in the plane were not sufficient for determination or, at least, estimation of the constant $K_2$, thus we have used our proper tools for finding the estimate (Section 5.3). We have also determined the Meyer number of some classes of planar convex sets, namely centrally symmetric convex sets (Section 5.2) and regular polygons (Section 5.1).

In dimensions $d \geq 3$, the problems get much more difficult, and although the covering of bounded shapes is probably most efficient in a different way, we use approximation by the known most efficient lattice coverings of the space (Section 4.1).

In Chapter 7 we return to the quasicrystals. We explain in detail the transition from the Meyer property of cut-and-project sets to the problem of minimal covering of the difference set. We construct a particular model of quasicrystals with five-fold symmetry (which corresponds to the physically observed icosahedral symmetries of quasicrystalline materials), and explain how the cardinality of the finite set $F$ from (1.1) influences the number of local configurations in the modeling set.

# Chapter 2

# Topology of spaces of convex sets

Before we start description of spaces of convex sets, which plays an essential role in our main researches, it is useful to sum up basic properties of topological and metric spaces.

## 2.1 Summary of general knowledge of topology

In this part basic topological knowledge is summarized. In the following sections we will refer to points of this summary, or we will even use them without any reference considering them as self-evident facts. For more details see [23].

### 2.1.1 Compactness of topological spaces

**Definition 2.1.** *A topological space $\mathcal{X}$ is called compact if every open covering of $\mathcal{X}$ has a finite open subcovering, i.e.*

$$(\mathcal{X} \subset \cup_{i\in I} S_i,\ S_i \subset \mathcal{X},\ S_i \text{ open}) \Rightarrow (\exists k\ \in \mathbb{N})(\exists j_1, ..., j_k \in I)(\mathcal{X} \subset \cup_{i=1}^{k} S_{j_i}).$$

**Theorem 2.1.** *Let $\mathcal{X}$ be a compact space and let $A$ be a set closed in $\mathcal{X}$. Then $A$ is compact.*

**Theorem 2.2.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. Moreover, let $\mathcal{X}$ be a compact space and let $F$ be a continuous map: $\mathcal{X} \to \mathcal{Y}$. Then $F(\mathcal{X})$ is compact in $\mathcal{Y}$.*

*Proof.* Let $F(\mathcal{X}) \subset \cup_{i\in I} S_i$. Then $(F^{-1}(S_i))_{i\in I}$ is an open covering of $\mathcal{X}$. As $\mathcal{X}$ is compact, there exist indices $i_1, ..., i_k$ so that $(F^{-1}(S_{i_j}))_{j=1}^{k}$ is an open subcovering of $\mathcal{X}$. It implies that $F(\mathcal{X}) \subset \cup_{j=1}^{k} S_{i_j}$. □

**Theorem 2.3.** *Let $\mathcal{X}$ be a compact space and let $f$ be a continuous function: $\mathcal{X} \to \mathbb{R}$. Then $f$ reaches its maximum and minimum on $\mathcal{X}$.*

*Proof.* As $f(\mathcal{X})$ is compact in $\mathbb{R}$, it is closed and bounded. Therefore $\sup f(\mathcal{X}) \in f(\mathcal{X})$ and $\inf f(\mathcal{X}) \in f(\mathcal{X})$. □

### 2.1.2 Compactness of metric spaces

**Definition 2.2.** *Let $\mathcal{X}$ be a topological space. Real function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a metric on $\mathcal{X}$ if $\rho$ satisfies for all $x, y, z \in \mathcal{X}$ the following three properties:*

1. $\rho(x, y) \geq 0$, *moreover* $\rho(x, y) = 0 \Leftrightarrow x = y$,
2. $\rho(x, y) = \rho(y, x)$ *(symmetry)*,
3. $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ *(triangle inequality)*.

**Theorem 2.4.** *Let $A$ be a subset of a metric space $\mathcal{X}$. Then $A$ is closed if and only if every sequence in $A$ has all its limit points in $A$.*

**Theorem 2.5 (Weierstrass).** *Let $\mathcal{X}$ be a metric space. Then $\mathcal{X}$ is compact if and only if every sequence in $\mathcal{X}$ has a convergent subsequence, i.e. a subsequence which convergs to a point of $\mathcal{X}$.*

**Definition 2.3.** *Let $(x_n)_{n=1}^{\infty}$ be a sequence in the metric space $\mathcal{X}$ with the metric $\rho$. $(x_n)_{n=1}^{\infty}$ is called Cauchy sequence if*

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall m, n > n_0)(\rho(x_n, x_m) < \varepsilon).$$

**Definition 2.4.** *A metric space $\mathcal{X}$ is called complete if every Cauchy sequence has its limit in $\mathcal{X}$.*

**Definition 2.5.** *A metric space $\mathcal{X}$ with a metric $\rho$ is called totally bounded if for every $\varepsilon > 0$ there exists a finite $\varepsilon$-net, i.e. for every $\varepsilon > 0$ there exists a finite set $N(\varepsilon)$ such that*

$$(\forall x \in \mathcal{X})(\exists y \in N(\varepsilon))(\rho(x, y) \leq \varepsilon).$$

**Theorem 2.6.** *A metric space $\mathcal{X}$ is compact if and only if $\mathcal{X}$ is complete and totally bounded.*

**Corollary 2.6.1.** *Let $A$ be a subset of linear normed space $\mathbb{R}^d$. $A$ is bounded if and only if $A$ is totally bounded. Therefore $A$ is compact if and only if $A$ is closed and bounded.*

## 2.2 Topology of spaces of convex sets

This section is immediately connected with the main subject which deals with the difference set $\Omega - \Omega$, where $\Omega$ is a convex compact set with non-empty interior in $\mathbb{R}^d$. Therefore it is necessary to get acquainted with topology of spaces of convex sets at first.

**Definition 2.6.** *A set $H$ in $\mathbb{R}^d$ is called convex, if, for any two points $x, y \in H$, it contains all points of the line segment joining $x$ and $y$, i.e.*

$$(\forall x, y \in H)(\forall \lambda \in [0,1])(\lambda x + (1-\lambda)y \in H).$$

Consider $\mathbb{R}^d$ with the Euclidean norm $|\cdot|$.
Firstly, we define topological structures which will be used to describe properties of spaces of convex sets. One defines distance of a set $A \subset \mathbb{R}^d$ from a point $x$ as

$$\rho(x, A) := \inf\{|x-y| \bigm| y \in A\}.$$

An open ball of radius $\varepsilon$ centered at $a$ is defined by

$$B(a, \varepsilon) := \{x \in \mathbb{R}^d \bigm| |x-a| < \varepsilon\}.$$

An $\varepsilon$-neighbourhood of the set $A$ for $\varepsilon > 0$ is the set

$$A_\varepsilon := \cup_{a \in \mathrm{A}} B(a, \varepsilon) = \{x \in \mathbb{R}^d | \rho(x, A) < \varepsilon\}.$$

Clearly, $A_\varepsilon$ is an open set in $\mathbb{R}^d$.
An $(-\varepsilon)$-neighbourhood of a bounded set $A$ is defined by

$$A_{-\varepsilon} := \{x \in A \ \mid \rho(x, \partial A) > \varepsilon\}.$$

Let us show two properties of $(-\varepsilon)$-neighbourhood of a set in $\mathbb{R}^d$.

**Lemma 2.1.** *Let $0 < a < b$. $\Omega_{-b} \subset \Omega_{-a}$ and $\Omega_a \subset \Omega_b$.*

*Proof.* Both inclusions are clear from the corresponding definitions. ∎

**Lemma 2.2.** *Let* $\alpha > 0$. $\Omega_{-\alpha} = \Omega^{\circ}_{-\alpha}$.

*Proof.* $\Omega_{-\alpha} = \{x \in \Omega | \rho(x, \partial\Omega) > \alpha\}$ is an open set. □

**Theorem 2.7.** *Let* $H_1, H_2 \subset \mathbb{R}^d$ *be non-empty closed convex sets which have no common points and at least one of them is compact. Then there exists a hyperplane separating* $H_1, H_2$. *More precisely, there exist* $c \in \mathbb{R}^d$ *and* $\alpha \in \mathbb{R}$ *such that for every* $x \in H_1$ *and for every* $y \in H_2$ *it holds*

$$cx^T > \alpha > cy^T.$$

*Then* $\{z \in \mathbb{R}^d \mid cz^T = \alpha\}$ *is the searched hyperplane.*

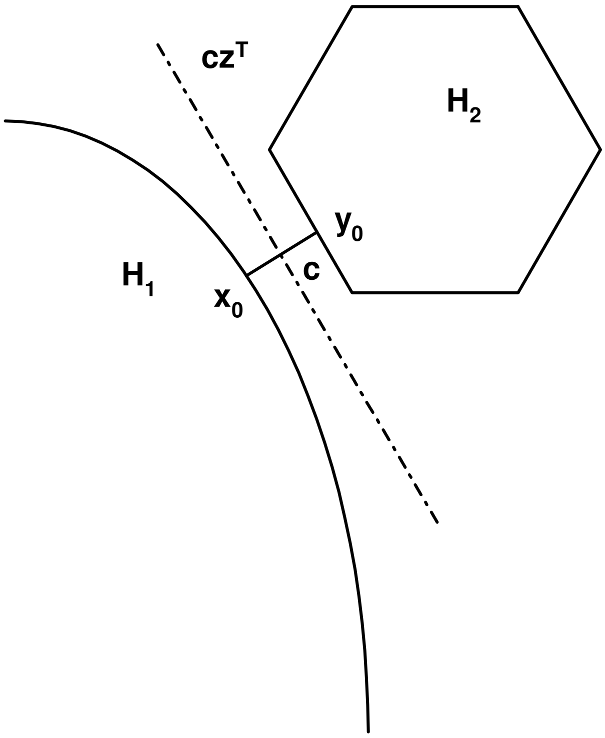



Figure 2.1: Illustaration of the situation in Theorem (2.7).

*Proof.* We roughly describe the main ideas of the proof at first.

1. We show that there exist $x_0 \in H_1$ and $y_0 \in H_2$ such that $\inf\{|x - y| \mid x \in H_1, y \in H_2\} = |x_0 - y_0|$. We denote $c := x_0 - y_0$.

2. Then we take an arbitrary $x \in H_1$ and construct the line segment $\overline{xx_0}$. We choose an arbitrary point of $\overline{xx_0}$ and make profit of the fact that its distance from $y_0$ is greater then $|c|$. We continue analogically for $H_2$ and obtain the searched inequalities

$$(\forall x \in H_1)(\forall y \in H_2)(cx^T > \alpha > cy^T).$$

Let us step up to the precise proof. Let $H_2$ be compact. Define by

$$\nu := \inf\{|x - y| \mid x \in H_1, y \in H_2\}.$$

Using the definition of infimum, we have

$$(\forall n \in \mathbb{N})(\exists x_n \in H_1, y_n \in H_2)(\nu \le |x_n - y_n| < \nu + \tfrac{1}{n}).$$

Both sequences are bounded due to two facts:

1. $H_2$ is compact therefore bounded, which implies that $(\exists K > 0)(\forall n \in \mathbb{N})(|y_n| \le K)$.
2. $(\forall n \in \mathbb{N})(|x_n| \le |x_n - y_n| + |y_n| < \nu + 1 + K)$.

It is possible to choose a Cauchy subsequence from any bounded sequence. As $H_1, H_2$ are closed, any Cauchy sequence in $H_1, H_2$ has its limit in $H_1, H_2$ respectively, i.e. it holds

$$(\exists y_{n_k})(\lim_{k\to\infty} y_{n_k} =: y_0 \in H_2),$$

$$(\exists x_{n_{k_l}})(\lim_{l\to\infty} x_{n_{k_l}} =: x_0 \in H_1).$$

We obtain consequently

$$\nu \le |x_{n_{k_l}} - y_{n_{k_l}}| < \nu + \tfrac{1}{n_{k_l}}.$$

Let $l$ tends to infinity so that we have $\nu \le |x_0 - y_0| \le \nu$. As the vectors $x_0, y_0$ are elements of sets which have no common points, we obtain that $\nu > 0$. Denote by $c := x_0 - y_0$.

Take an arbitrary $x \in H_1$. Due to convexity of $H_1$, it holds

$$(\forall \lambda \in [0,1])(x_0 + \lambda(x - x_0) \in H_1).$$

Note the following considerations

$$|x_0 + \lambda(x - x_0) - y_0| \ge \nu = |x_0 - y_0|,$$

$$|c + \lambda(x - x_0)|^2 \ge |c|^2,$$

$$(\forall \lambda \in (0,1])(\lambda|x - x_0|^2 + 2c(x - x_0)^T \ge 0).$$

Let $\lambda \to 0^+$. Then we have

$$cx^T \ge cx_0^T.$$

Take an arbitrary $y \in H_2$. Due to convexity of $H_2$, it holds

$$(\forall \lambda \in [0,1])(y_0 + \lambda(y - y_0) \in H_2).$$

We use analogical considerations as above.

$$|y_0 + \lambda(y - y_0) - x_0| \ge \nu = |x_0 - y_0|,$$

$$|\lambda(y - y_0) - c|^2 \ge |c|^2,$$

$$(\forall \lambda \in (0,1])(\lambda|y - y_0|^2 \ge 2c(y - y_0)^T).$$

Let $\lambda \to 0^+$. Then we have

$$cy_0^T \ge cy^T.$$

Using the fact $c(x_0 - y_0)^T = cc^T > 0$ and the two previous inequalities, we obtain for every $x \in H_1$ and every $y \in H_2$

$$cx^T \ge cx_0^T > cy_0^T \ge cy^T.$$

It suffices to define $\alpha$ for instance

$$\alpha := \frac{1}{2}(cx_0^T + cy_0^T).$$

Now, we can see that the searched hyperplane is the set $\{z \in \mathbb{R}^d \mid cz^T = \alpha\}$. □

### 2.2.1 Hausdorff metric

Let us define two spaces of compact sets, which will be important in our subsequent considerations. The topology on these spaces is given by the metric Dist, sometimes called the Hausdorff metric.

**Definition 2.7.** *Denote by $\mathcal{N}$ the space of all closed subsets of $\overline{B(0,1)}$ in $\mathbb{R}^d$.*
*Let $\alpha > 0$. Denote by $\mathcal{M} := \{\Omega \in \mathcal{N} \mid \Omega$ convex and $\overline{B(0,\alpha)} \subset \Omega\}$.*

**Definition 2.8.** *Let $A, B$ be compact sets in $\mathbb{R}^d$. We define a real function Dist by*

$$Dist(A,B) := \max\big\{\inf\{\varepsilon > 0 | A \subset B_\varepsilon\}, \inf\{\varepsilon > 0 | B \subset A_\varepsilon\}\big\} = \inf\{\varepsilon > 0 | A \subset B_\varepsilon \wedge B \subset A_\varepsilon\}.$$

**Remark 1.** *The minimal $\varepsilon_1$ such that $B \subset A_{\varepsilon_1}$ and the minimal $\varepsilon_2$ such that $A \subset B_{\varepsilon_2}$ are generally different, as illustrated in Figure 2.2 and 2.3.*

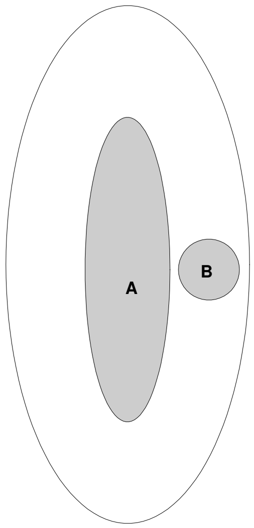



Figure 2.2: Illustration of $A_{\varepsilon_1}$ such that $B \subset A_{\varepsilon_1}$.

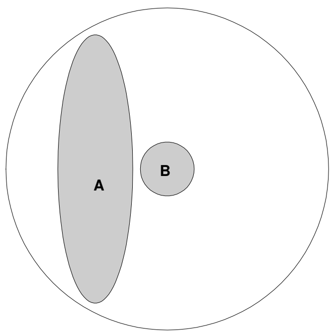



Figure 2.3: Illustration of $B_{\varepsilon_2}$ such that $A \subset B_{\varepsilon_2}$.

**Proposition 2.1.** *Dist is a metric on the space of all compact subsets of $\mathbb{R}^d$. In particular Dist is a metric on spaces $\mathcal{N}$ and $\mathcal{M}$.*

For the proof of Propositon 2.1 we use the following lemma.

**Lemma 2.3.** *Let $B, C$ be compact sets in $\mathbb{R}^d$ and let $\rho, \delta > 0$. Then $C \subset B_\delta$ implies $C_\rho \subset B_{\rho+\delta}$.*

*Proof.* We want to show $(\forall z \in C_\rho)(\exists y \in B)(|z-y| < \delta + \rho)$. We use two facts:

1. $C \subset B_\delta \Rightarrow (\forall x \in C)(\exists y \in B)(|x-y| < \delta)$,
2. $(\forall z \in C_\rho)(\exists x \in C)(|z-x| < \rho)$.

By using previous facts, we have

$$(\forall z \in C_\rho)(\exists y \in B)(|z-y| \le |x-y| + |x-z| < \delta + \rho).$$

Thus $C_\rho \subset B_{\rho+\delta}$. □

*Proof of Proposition 2.1.* To prove the proposition, it is necessary to show the three properties of a metric. For all $A, B, C$ compact in $\mathbb{R}^d$ we have to verify:

1. $Dist(A, B) \ge 0$.
   $Dist(A, B) = 0 \Leftrightarrow A = B$.
2. $Dist(A, B) = Dist(B, A)$.
3. $Dist(A, B) \le Dist(A, C) + Dist(C, B)$.

ad 1. The only implication which does not follow directly from the definition is

$$Dist(A, B) = 0 \Rightarrow A = B.$$

Let us show this by contradiction. Suppose $Dist(A, B) = 0$ and $A \ne B$. Without loss of generality this means

$$(\exists x \in B)(x \notin A).$$

As $A$ is closed, $\rho(x, A) > 0$. Denote $\delta := \rho(x, A) > 0$. $Dist(A, B) = 0$ implies

$$(\forall \varepsilon > 0)(A \subset B_\varepsilon \wedge B \subset A_\varepsilon).$$

Let us denote $\varepsilon := \frac{\delta}{2}$. We have

$$x \in B \subset A_{\frac{\delta}{2}} = \{y \in \mathbb{R}^d \mid \rho(y, A) < \frac{\delta}{2}\}$$

which is contradiction with the fact $\rho(x, A) = \delta$.

ad 2. Symmetry is clear from the definition of $Dist$.

ad 3. Denote $\rho := Dist(A, C)$, $\delta := Dist(C, B)$.

Using the definition of $Dist$, we have

$$A \subset B_{\rho+\delta} \text{ and } B \subset A_{\rho+\delta} \Rightarrow Dist(A, B) \le \rho + \delta = Dist(A, C) + Dist(B, C).$$

Hence, it suffices to verify

$$A \subset B_{\rho+\delta} \text{ and } B \subset A_{\rho+\delta}.$$

Using Lemma 2.3, we have

$$A \subset C_\rho \subset B_{\rho+\delta}.$$

The second inclusion $B \subset A_{\rho+\delta}$ follows analogically. This completes the proof of the triangle inequality.

□

**Observation 2.1.** *Compactness of A,B is necessary in order so that $Dist(A, B)$ is a metric. Otherwise $Dist(A, B) = 0 \not\Rightarrow A = B$, as illustrated in the following example.*

**Example 2.1.** *Let $A := B(0, 1)$, $B := \overline{B(0, 1)}$. Then $Dist(A, B)$ is clearly 0, however, $A \ne B$.*

### 2.2.2 Compactness of the metric space $\mathcal{M}$

As we have already mentioned, the spaces $\mathcal{M}$ and $\mathcal{N}$ are important for our subsequent considerations. Especially the fact that both of them are compact will play an essential role in our following researches.

**Theorem 2.8 ([4]).** *The space $\mathcal{N}$ of all closed subsets of $\overline{B(0,1)}$ is compact.*

Let us introduce some lemmas which will be useful for the proof of Theorem 2.9 which states that also the metric space $\mathcal{M}$ is compact.

**Lemma 2.4.** *Let $(\Omega_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{N}$, i.e. $(\Omega_n)_{n=1}^\infty$ is a sequence of closed subsets of $\overline{B(0,1)}$ in $\mathbb{R}^d$. Denote $\Omega := \lim_{n\to\infty} \Omega_n$. Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence such that $x_n \in \Omega_n$ for all $n \in N$. Then $\lim_{n\to\infty} x_n = x \in \Omega$.*

*Proof.* As $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^d$, there exists $x := \lim_{n\to\infty} x_n \in \mathbb{R}^d$. We want to show that $x \in \Omega$. Since $\mathcal{N}$ is compact, $\Omega$ is closed, and thus $x \in \Omega$ is equivalent with

$$(\forall \varepsilon > 0)(\rho(x, \Omega) \leq \varepsilon).$$

As $\lim_{n\to\infty} \Omega_n = \Omega$, we have

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(x_n \in \Omega_n \subset \Omega_\varepsilon).$$

Using the definition of $\Omega_\varepsilon$, we obtain

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(\rho(x_n, \Omega) < \varepsilon).$$

Note that $\rho(x, \Omega)$ is continuous as a function of $x$. Hence, when we let $n$ tend to infinity, we have

$$(\forall \varepsilon > 0)(\lim_{n\to\infty} \rho(x_n, \Omega) = \rho(x, \Omega) \leq \varepsilon),$$

which was to show. □

**Lemma 2.5.** *Let $(\Omega_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{N}$ and denote $\Omega := \lim_{n\to\infty} \Omega_n$. Then*

$$(\forall x \in \Omega)(\forall n \in \mathbb{N})(\exists x_n \in \Omega_n)(\lim_{n\to\infty} x_n = x).$$

*Proof.* As $\lim_{n\to\infty} \Omega_n = \Omega$, we have

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(\Omega \subset (\Omega_n)_\varepsilon).$$

This implies for $x \in \Omega$ that

$$(\forall m \in \mathbb{N})(\exists n_0)(\forall n > n_0)(\exists x_n^{(m)} \in \Omega_n)(|x - x_n^{(m)}| < \tfrac{1}{m}).$$

As $(x_n^{(m)})$ is a sequence of sequences, we can use for instance diagonal choice. We choose the sequence $(x_n^{(n)})_{n=1}^\infty$ which satisfies

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(|x - x_n^{(n)}| \leq \varepsilon).$$

Thus $x_n := x_n^{(n)} \in \Omega_n$ is the searched sequence such that $\lim_{n\to\infty} x_n = x$. □

**Lemma 2.6.** *Let $(\Omega_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{N}$ and denote $\Omega := \lim_{n\to\infty} \Omega_n$. Then*

$$\lim_{n\to\infty} \rho(x, \Omega_n) = 0 \Leftrightarrow x \in \Omega.$$

*Proof.* One has to prove two implications.
$(\Rightarrow)$ : $\lim_{n\to\infty} \rho(x,\Omega_n) = 0$ implies that

$$(\forall m \in \mathbb{N})(\exists n_0)(\forall n > n_0)(\exists x_n^{(m)} \in \Omega_n)(|x - x_n^{(m)}| < \tfrac{1}{m}).$$

By using the diagonal choice, we obtain a Cauchy sequence $(x_n^{(n)})_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, $x_n^{(n)} \in \Omega_n$. Using Lemma 2.4, we have $x := \lim_{n\to\infty} x_n^{(n)} \in \Omega$.
$(\Leftarrow)$ : As $\lim \Omega_n = \Omega$, we have

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n > n_0)(x \in \Omega \subset (\Omega_n)_\varepsilon).$$

As $(\Omega_n)_\varepsilon = \{x \in \mathbb{R}^d \mid \rho(x,\Omega_n) < \varepsilon\}$, the result is that $\lim_{n\to\infty} \rho(x,\Omega_n) = 0$. □

**Theorem 2.9.** *The space $\mathcal{M}$ of all convex closed sets $\Omega$ in $\mathbb{R}^d$, which satisfy $\overline{B(0,\alpha)} \subset \Omega \subset \overline{B(0,1)}$, is compact.*

*Proof.* It suffices to show that $\mathcal{M}$ is closed in $\mathcal{N}$. Take an arbitrary Cauchy sequence $(\Omega_n)_{n=1}^{\infty}$ in $\mathcal{M}$. $(\Omega_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{N}$, too, and $\mathcal{N}$ is compact and consequently complete. Therefore the limit $\Omega$ is an element of $\mathcal{N}$, i.e. $\Omega$ is a closed subset of $\overline{B(0,1)}$. The only questions left are whether $\overline{B(0,\alpha)} \subset \Omega$ and whether $\Omega$ is convex.

- Let us prove the inclusion $\overline{B(0,\alpha)} \subset \Omega$ by contradiction. Assume $\overline{B(0,\alpha)} \not\subset \Omega$. This means

  $$\big(\exists x \in \overline{B(0,\alpha)}\big)\big(x \notin \Omega\big).$$

  Moreover, we know that for all $n \in \mathbb{N}$ $\quad x \in \overline{B(0,\alpha)} \subset \Omega_n$ and $\Omega = \lim_{n\to\infty} \Omega_n$.
  According to Lemma 2.6, it implies that $x \in \Omega$ which is contradiction with assumption $x \notin \Omega$.

- Convexity: We want to verify

  $$(\forall x, y \in \Omega)(\forall \lambda \in [0,1])(\lambda x + (1-\lambda)y \in \Omega).$$

  Due to Lemma 2.5, there exist sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ such that for all $n \in \mathbb{N}$, $x_n, y_n \in \Omega_n$ and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. As $\Omega_n$ is convex for every $n$, we have

  $$\lambda x + (1-\lambda)y = \lambda \lim_{n\to\infty} x_n + (1-\lambda) \lim_{n\to\infty} y_n = \lim_{n\to\infty} \big(\lambda x_n + (1-\lambda)y_n\big) \in \Omega.$$

□

# Chapter 3

# Covering of the difference set

We have already indicated in the introduction part that the main interest will be devoted to covering of the difference set $\Omega - \Omega$ by translated copies of the interior $\Omega^\circ$, where $\Omega$ is a convex compact set with non-empty interior in $\mathbb{R}^d$. We will focus on estimation of the sufficient number of these copies. The main result is given as Theorem 3.2. It states that there exists a universal constant $K$ such that for all convex compact sets $\Omega$ with non-empty interior in $\mathbb{R}^d$, $K$ translated copies of $\Omega^\circ$ are sufficient to cover $\Omega - \Omega$.

**Definition 3.1.** *Denote by $\kappa_d$ the space of all convex compact sets in $\mathbb{R}^d$ with non-empty interior.*

Let $\Omega$ be a set in $\kappa_d$. We are interested in the set $\Omega - \Omega = \{x - y \mid x, y \in \Omega\}$.

**Theorem 3.1.** *For every set $\Omega$ in $\kappa_d$ there exists a finite set*

$$A = \{a_1, a_2, ..., a_k \mid a_i \in \mathbb{R}^d \ \forall i \in \{1, ..., k\}\}$$

*such that*

$$\Omega - \Omega \subset (\Omega^\circ + a_1) \cup (\Omega^\circ + a_2) \cup .... \cup (\Omega^\circ + a_k).$$

*Proof.* Due to compactness of $\Omega$, the difference set $\Omega - \Omega$ is compact, too (Section 3.1). It follows that for every open covering of $\Omega - \Omega$ there exists a finite open subcovering. Using this property, we have for an arbitrary element $y \in \Omega^\circ$

$$\Omega - \Omega \subset \bigcup_{x \in \Omega - \Omega} (x + (\Omega^\circ - y)) \Rightarrow (\exists k \ \in \mathbb{N})\big(\Omega - \Omega \subset \bigcup_{i=1}^{k} (x_i + (\Omega^\circ - y))\big),$$

where $x_i \in \Omega - \Omega$ for all $i \in \{1, ..., k\}$.
The open subcovering of $\Omega - \Omega$ has the form $\Omega^\circ + A = \bigcup_{i=1}^{k} (x_i + (\Omega^\circ - y))$. Thus one obtains the searched set $A = \{x_i - y \mid i \in \{1, ..., k\}\}$. □

For the purpose to estimate the sufficient number of translated copies of $\Omega^\circ$, which can cover the difference set $\Omega - \Omega$, let us define the function $f$ which to any $\Omega \in \kappa_d$ associates the minimal number of translated copies of $\Omega^\circ$ needed for covering of $\Omega - \Omega$. We explore its properties further on.

**Definition 3.2.** *We define a function $f : \kappa_d \to \mathbb{N}$ by*

$$f(\Omega) := \min\{k \in \mathbb{N} | (\exists \ a_1, ..., a_k \in \mathbb{R}^d)(\Omega - \Omega \subset (a_1 + \Omega^\circ) \cup ... \cup (a_k + \Omega^\circ))\}.$$

*The value $f(\Omega)$ is called the* **Meyer number of** $\Omega$.

**Observation 3.1.** *The function $f$ is well defined. Theorem 3.1 implies that for every $\Omega$ in $\kappa_d$ there exists a positive number $k$ such that $f(\Omega) = k$.*

Let us introduce the most important theorem which answers the question whether the number of translated copies of $\Omega^\circ$, which suffice for covering of the difference set $\Omega - \Omega$, is bounded by a universal constant for all convex compact sets $\Omega$ with non-empty interior in $\mathbb{R}^d$.

**Theorem 3.2.** *The function $f$ is bounded on the metric space $\kappa_d$ with the metric Dist, i.e.*

$$(\exists K > 0)(\forall \Omega \in \kappa_d)(f(\Omega) \leq K).$$

**Definition 3.3.** *The minimal upper bound on the set of all Meyer numbers in $\mathbb{R}^d$ from Theorem 3.2 is denoted by $K_d$.*

**Remark 2.** *As $f$ reaches only a finite number of values in $\kappa_d$, convex sets in $\kappa_d$ are devided into classes according to the value of $f$.*

We will prove Theorem 3.2 in Section 3.4. It is useful to investigate properties of $\Omega - \Omega$ and the function $f$ at first.

## 3.1 Properties of $\Omega - \Omega$

Let us state some basic properties of the difference set $\Omega - \Omega$ for $\Omega \in \kappa_d$.

**Claim 3.1.** *$\Omega - \Omega$ is centrally symmetric.*

*Proof.* For each $z \in \Omega - \Omega$ there exist $x, y \in \Omega$ such that $z = x - y$. Definition of the set $\Omega - \Omega$ implies $y - x = -z \in \Omega - \Omega$. □

**Claim 3.2.** *$\Omega - \Omega$ is independent of the translation of $\Omega$, i.e.*

$$(\forall a \in \mathbb{R}^d)((\Omega + a) - (\Omega + a) = \Omega - \Omega).$$

*Proof.* Take an arbitrary $a \in \mathbb{R}^d$.
($\subseteq$): For each $z \in (\Omega + a) - (\Omega + a)$ there exist $x+a,\ y+a \in \Omega + a$ such that $z = (x+a)-(y+a) = x - y \in \Omega - \Omega$.
($\supseteq$): For each $z \in \Omega - \Omega$ there exist $x, y \in \Omega$ such that $z = x - y = (x + a) - (y + a) \in (\Omega + a) - (\Omega + a)$. □

**Claim 3.3.** *$\Omega - \Omega$ is a convex set.*

*Proof.* One has to show that convexity of $\Omega$ implies convexity of $\Omega - \Omega$:

$$(\forall y, z \in \Omega)(\forall \lambda \in [0, 1])(\lambda y + (1 - \lambda)z \in \Omega) \Rightarrow$$

$$\Rightarrow (\forall x_1, x_2 \in \Omega - \Omega)(\forall \lambda \in [0, 1])(\lambda x_1 + (1 - \lambda)x_2 \in \Omega - \Omega).$$

For each $x_1, x_2 \in \Omega - \Omega$ there exist $y_1, y_2, z_1, z_2 \in \Omega$ such that $x_1 = y_1 - z_1$ and $x_2 = y_2 - z_2$.

$$\lambda x_1 + (1 - \lambda)x_2 = \lambda(y_1 - z_1) + (1 - \lambda)(y_2 - z_2) =$$
$$= \lambda y_1 + (1 - \lambda)y_2 - (\lambda z_1 + (1 - \lambda)z_2) \in \Omega - \Omega.$$

□

**Claim 3.4.** *If $\Omega$ is centrally symmetric then $\Omega - \Omega = 2\Omega$.*

*Proof.* As $\Omega$ is centrally symmetric, we have $\Omega = -\Omega$ and consequently $\Omega - \Omega = \Omega + \Omega$.
(a) Inclusion $2\Omega \subset \Omega + \Omega$ is trivial.
(b) Let us verify $\Omega + \Omega \subset 2\Omega$. Take an arbitrary $z \in \Omega + \Omega$, then $z = x + y$, where $x, y \in \Omega$. Due to convexity of $\Omega$, it holds $\frac{x+y}{2} \in \Omega$. Consequently, we have $z = x+y = \frac{x+y}{2} + \frac{x+y}{2} = 2(\frac{x+y}{2}) \in 2\Omega$. □

**Claim 3.5.** *$\Omega - \Omega$ is compact.*

*Proof.* To prove compactness of $\Omega - \Omega$, one needs to prove that $\Omega - \Omega$ is bounded and closed.
(a) $\Omega - \Omega$ is bounded if there exists a ball of finite radius such that $\Omega - \Omega$ is its subset. As $\Omega$ is bounded, there exists $R > 0$ such that $\Omega \subset \overline{B(0, R)}$. Consequently, one gets

$$\Omega - \Omega \subset \overline{B(0, R)} - \overline{B(0, R)} = \overline{B(0, 2R)}.$$

The last equality holds owing to the fact that $\overline{B(0, R)}$ is centrally symmetric.
(b) $\Omega - \Omega$ is closed if $\Omega - \Omega = \overline{\Omega - \Omega}$. Take an arbitrary $z \in \overline{\Omega - \Omega}$. There exists a sequence $(z_n)_{n=1}^{\infty} \in \Omega - \Omega$ such that $\lim_{n\to\infty} z_n = z$. Moreover $z_n = x_n - y_n$, where $x_n, y_n \in \Omega$. Since $\Omega$ is bounded, every sequence in $\Omega$ has a Cauchy subsequence. We have Cauchy sequences $(x_{k_n})_{n=1}^{\infty}, (y_{l_n})_{n=1}^{\infty}$. It is possible to choose subsequences with the same indices: $(x_{s_n})_{n=1}^{\infty}$, $(y_{s_n})_{n=1}^{\infty}$ for which

$$\lim_{n\to\infty} x_{s_n} = x \in \overline{\Omega} \quad \text{and} \quad \lim_{n\to\infty} y_{s_n} = y \in \overline{\Omega}.$$

The equality $z_{s_n} = x_{s_n} - y_{s_n}$ for every $n \in \mathbb{N}$ implies $z = x - y \in \overline{\Omega} - \overline{\Omega} = \Omega - \Omega$. The last equality results from the fact that $\Omega$ is closed. □

The following property is useful for our considerations further on.

**Claim 3.6.** *Let $\delta > 0$, then $\Omega_\delta - \Omega_\delta \subset (\Omega - \Omega)_{2\delta}$.*

*Proof.* Take an arbitrary $x \in \Omega_\delta - \Omega_\delta$. There exist $x_1, x_2 \in \Omega_\delta$ so that $x = x_1 - x_2$. For all $y \in \Omega - \Omega$ there exist $y_1, y_2 \in \Omega$ so that $y = y_1 - y_2$.

$$|x - y| = |x_1 - x_2 - (y_1 - y_2)| \leq |x_1 - y_1| + |x_2 - y_2| < \delta + \delta = 2\delta.$$

Therefore $x \in (\Omega - \Omega)_{2\delta}$. □

We illustrate the construction of a difference set for $\Omega$ being a line segment and a triangle.

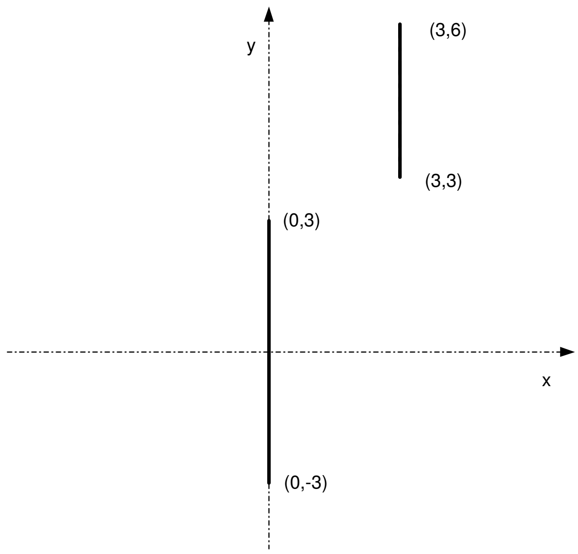



Figure 3.1: $\Omega - \Omega$, where $\Omega$ is the convex hull of points (3,3) and (3,6).

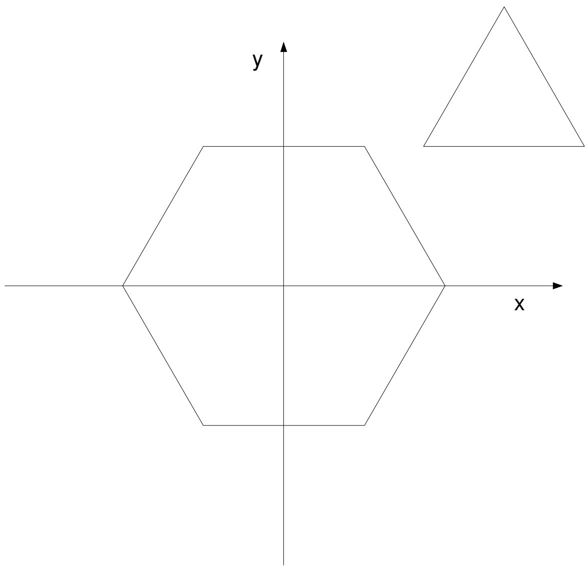



Figure 3.2: $\Omega - \Omega$, where $\Omega$ is a triangle.

## 3.2 Semicontinuity of $f$ on $\mathcal{M}$

We want to prove that the function $f$ is bounded on the space $\kappa_d$. It is useful to show as the first step that the function $f$ is upper semicontinuous on the space $\mathcal{M}$ of all convex compact sets $\Omega$ in $\mathbb{R}^d$ such that $\overline{B(0,\alpha)} \subset \Omega \subset \overline{B(0,1)}$.

Let us introduce lemmas and claims, which will be useful to prove that $f$ satisfies the property of semicontinuity.

Realize that the following lemma is the only moment when we need convexity of the set $\Omega$.

**Lemma 3.1.** *Let $\Omega, \tilde{\Omega} \in \kappa_d$ and let $Dist(\Omega, \tilde{\Omega}) < \delta$. Then $\Omega_{-\delta} \subset \tilde{\Omega} \subset \Omega_\delta$.*

*Proof.* Inclusion $\tilde{\Omega} \subset \Omega_\delta$ is valid for any bounded sets $\Omega$ and $\tilde{\Omega}$ directly from the definition of $\delta$-neighbourhood of a set. We prove $\Omega_{-\delta} \subset \tilde{\Omega}$ by contradiction. Let us suppose that there exists $x \in \Omega_{-\delta}$ such that $x \notin \tilde{\Omega}$. Since $\tilde{\Omega}$ is convex and closed, there exists a hyperplane $H$ such that $\tilde{\Omega}$ is all contained in one of the half-spaces bounded by $H$ and $x$ belongs to the other half-space. We will use the two following statements:

1. $B(x, \delta) \subset \Omega$.

   Suppose the opposite, i.e. there exists $y \in B(x, \delta)$ and $y \notin \Omega$. Since $x \in \Omega$, it holds $|x - y| \geq \rho(x, \partial\Omega) \geq \delta$, which is contradiction with the assumption $y \in B(x, \delta)$.

2. $Dist(\Omega, \tilde{\Omega}) < \delta$.

As $x$ is in another half-plane than $\tilde{\Omega}$, more than half a ball $B(x, \delta)$ lies in the same half-plane as $x$. Hence, there exists $z \in B(x, \delta) \subset \Omega$ such that $\rho(z, \tilde{\Omega}) > \delta$ which is contradiction with the fact $\Omega \subset \tilde{\Omega}_\delta$. □

Take an arbitrary $\Omega$ and denote $k := f(\Omega) \in \mathbb{N}$. It means that

$$\Omega - \Omega \subset (\Omega^\circ + a_1) \cup (\Omega^\circ + a_2) \cup \ldots\ldots \cup (\Omega^\circ + a_k) =: P. \tag{3.1}$$

Denote also

$$\varepsilon := \inf\{\rho(x, \mathbb{R}^d \setminus P) | x \in \Omega - \Omega\}. \tag{3.2}$$

**Lemma 3.2.** $\varepsilon > 0$.

*Proof.* If $\varepsilon = 0$, there exist $(y_n)_{n=1}^\infty \in \mathbb{R}^d \setminus P$ and $(z_n)_{n=1}^\infty \in \Omega - \Omega$ such that $\lim_{n\to\infty} |y_n - z_n| = 0$. As $\Omega - \Omega$ is compact, there exists a Cauchy subsequence $z_{l_n}$ such that

$$\lim_{n\to\infty} z_{l_n} = z \in \Omega - \Omega.$$

Since $\lim_{n\to\infty} |y_{l_n} - z_{l_n}| = 0$ and $\mathbb{R}^d \setminus P$ is closed, it holds

$$\lim_{n\to\infty} y_{l_n} = z \in \mathbb{R}^d \setminus P.$$

Consequently, we have $z \in \Omega - \Omega \subset P$ and $z \in \mathbb{R}^d \setminus P$, which is contradiction. □

Take an arbitrary $x \in \Omega - \Omega$, then there exists at least one index $j \in \{1, ..., k\}$ such that $x \in a_j + \Omega^\circ$.
Let us define

$$v(x) := \max\{\rho(x, \partial(a_j + \Omega^\circ) | x \in (a_j + \Omega^\circ)\ j \in \{1, ..., k\}\}. \tag{3.3}$$

$x$ belongs to an open set and the sets $\partial(a_j + \Omega^\circ)$ are closed, therefore $v(x) > 0$.

**Claim 3.7.** *There exists $\nu > 0$ such that*

$$(\forall x \in \overline{(\Omega - \Omega)_{\frac{1}{2}\varepsilon}})(v(x) \geq \nu > 0).$$

*Proof.* $v(x)$ is a continuous and positive function on the compact set $\overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}}$, therefore $v(x)$ has its minimum on this set. Denoting

$$\nu := \min\{v(x) \mid x \in \overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}}\} \tag{3.4}$$

completes the proof. □

**Claim 3.8.** $\overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}} \subset (a_1+\Omega^\circ_{-\frac{1}{2}\nu}) \cup (a_2+\Omega^\circ_{-\frac{1}{2}\nu}) \cup \dots\dots \cup (a_k+\Omega^\circ_{-\frac{1}{2}\nu})$.

*Proof.* Take an arbitrary $x \in \overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}}$. There exists $s \in \{1, ..., k\}$ such that $v(x) = \rho(x, \partial(a_s+\Omega^\circ)) \geq \nu > \frac{1}{2}\nu$. Therefore $x \in (a_s+\Omega^\circ)_{-\frac{1}{2}\nu} = (a_s+\Omega^\circ_{-\frac{1}{2}\nu})$. □

**Lemma 3.3.** $\overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}} \subset P$.

*Proof.* It is a direct consequence of Claim 3.8. $\overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}} \subset (a_1+\Omega^\circ_{-\frac{1}{2}\nu}) \cup (a_2+\Omega^\circ_{-\frac{1}{2}\nu}) \cup \dots\dots \cup (a_k+\Omega^\circ_{-\frac{1}{2}\nu}) \subset (\Omega^\circ+a_1) \cup (\Omega^\circ+a_2) \cup \dots\dots \cup (\Omega^\circ+a_k) = P$. □

**Theorem 3.3.** *The function $f$ is upper semicontinuous in $\mathcal{M}$. It means*
$(\forall\Omega \in \mathcal{M})\ (\forall\varepsilon > 0)(\exists\delta > 0)(\forall\tilde{\Omega} \in \mathcal{M})\ (Dist(\Omega,\tilde{\Omega}) < \delta)(f(\tilde{\Omega}) \leq f(\Omega)+\varepsilon)$.

*Proof.* As values of $f$ are only positive integer numbers, one has to prove $(\forall\Omega \in \mathcal{M})\ (\exists\delta > 0)(\forall\tilde{\Omega} \in \mathcal{M})\ (Dist(\Omega,\tilde{\Omega}) < \delta)(f(\tilde{\Omega}) \leq f(\Omega))$.

Denote

$$\delta := min\{\tfrac{1}{4}\varepsilon, \tfrac{1}{2}\nu\}. \tag{3.5}$$

Using previous lemmas and claims, we obtain the following inclusions. $Dist(\Omega,\tilde{\Omega}) < \delta$ implies that

$$\tilde{\Omega} - \tilde{\Omega} \subset \overline{\Omega_\delta - \Omega_\delta}.$$

Claim 3.6 says that

$$\overline{\Omega_\delta - \Omega_\delta} \subset \overline{(\Omega-\Omega)_{2\delta}}.$$

Using the definition of $\delta$ (3.5), we have

$$\overline{(\Omega-\Omega)_{2\delta}} \subset \overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}}.$$

Claim 3.8 states

$$\overline{(\Omega-\Omega)_{\frac{1}{2}\varepsilon}} \subset (a_1+\Omega^\circ_{-\frac{1}{2}\nu}) \cup (a_2+\Omega^\circ_{-\frac{1}{2}\nu}) \cup \dots\dots \cup (a_k+\Omega^\circ_{-\frac{1}{2}\nu}).$$

Using the definition of $\delta$,

$$(a_1+\Omega^\circ_{-\frac{1}{2}\nu}) \cup (a_2+\Omega^\circ_{-\frac{1}{2}\nu}) \cup \dots\dots \cup (a_k+\Omega^\circ_{-\frac{1}{2}\nu}) \subset (a_1+\Omega^\circ_{-\delta}) \cup (a_2+\Omega^\circ_{-\delta}) \cup \dots\dots \cup (a_k+\Omega^\circ_{-\delta}).$$

Using Lemma 3.1, we complete the proof

$$(a_1+\Omega^\circ_{-\delta}) \cup (a_2+\Omega^\circ_{-\delta}) \cup \dots\dots \cup (a_k+\Omega^\circ_{-\delta}) \subset (a_1+\tilde{\Omega}^\circ) \cup (a_2+\tilde{\Omega}^\circ) \cup \dots\dots \cup (a_k+\tilde{\Omega}^\circ).$$

Therefore $\tilde{\Omega} - \tilde{\Omega} \subset (a_1+\tilde{\Omega}^\circ) \cup \dots \cup (a_k+\tilde{\Omega}^\circ)$, i.e. $f(\tilde{\Omega}) \leq f(\Omega) = k$. □

### 3.2.1 Property of semicontinuous functions

In the previous part we have shown that the function $f$ is upper semicontinuous in $\mathcal{M}$. Let us find analogy in behavior of continuous and upper semicontinuous functions on compact sets.

**Theorem 3.4.** *The function $f$ reaches its maximum on the space $\mathcal{M}$, i.e.*

$$(\exists K > 0)(\forall \Omega \in \mathcal{M})(f(\Omega) \leq K).$$

*Proof.* Denote $H_\Omega^\delta := \{\tilde{\Omega} \in \mathcal{M} \mid Dist(\Omega, \tilde{\Omega}) < \delta\}$. It is an open set with respect to topology induced by the metric $Dist$. Owing to semicontinuity of $f$, one can associate to any $\Omega \in \mathcal{M}$ an open set $H_\Omega^{\delta_\Omega}$ with the following property

$$(\forall \tilde{\Omega} \in H_\Omega^{\delta_\Omega})(f(\tilde{\Omega}) \leq f(\Omega)).$$

As $\mathcal{M}$ is compact, there exists an open finite subcovering of the open covering

$$\mathcal{M} \subset \cup_{\Omega \in \mathcal{M}} H_\Omega^{\delta_\Omega}.$$

Denote the finite subcovering by

$$\mathcal{M} \subset \cup_{i=1}^n H_{\Omega_i}^{\delta_i}.$$

By the choice of $H_\Omega^{\delta_\Omega}$ we have

$$(\forall \tilde{\Omega} \in \mathcal{M})(\exists\ i \in \{1, 2, ..., n\})(f(\tilde{\Omega}) \leq f(\Omega_i)).$$

It implies that there exists a finite subset $\{\Omega_1, \Omega_2, ..., \Omega_n\} \subset \mathcal{M}$ such that

$$\sup_{\Omega \in \mathcal{M}} f(\Omega) \leq \max_{i \leq n} f(\Omega_i).$$

This proves that $f$ is bounded above. As $f$ has only positive integer values, it reaches its maximum. □

It remains to confirm that the function $f$ reaches its maximum on the space $\kappa_d$, too.

## 3.3 Independence of $f$ of affine transformations of $\Omega$

The next step on the way, which leads to the proof of boundedness of $f$ on the space $\kappa_d$, is to show independence of $f$ of affine transformations of $\Omega$. We will prove it in Proposition 3.2 at the end of this section. Let us remind a well-known property of linear functions at first and let us use it successively in proof of independence of $f$ of linear transformations of $\Omega$.

**Lemma 3.4.** *Any linear map $L$: $\mathbb{R}^d \to \mathbb{R}^d$ is continuous.*

*Proof.* Let $(e_1, ..., e_d)$ be the standard basis of $\mathbb{R}^d$. Take an arbitrary $x \in \mathbb{R}^d$ and denote its coordinates in the standard basis $(\alpha_1, ..., \alpha_d)$. Define $K := \sqrt{\sum_{j=1}^d |Le_j|^2}$. By using the triangle inequality and the Schwarz inequality, we have

$$|Lx| = |L(\sum_{j=1}^d \alpha_j\ e_j)| = |\sum_{j=1}^d \alpha_j\ Le_j| \leq \sum_{j=1}^d |\alpha_j||Le_j| \leq K\sqrt{\sum_{j=1}^d (\alpha_j)^2} = K|x|.$$

It confirms that $L$ is a bounded and therefore continuous map. □

**Proposition 3.1.** *Let $L$: $\mathbb{R}^d \to \mathbb{R}^d$ be a linear bijection. Then $f(L\Omega) = f(\Omega)$ for every $\Omega \in \kappa_d$.*

*Proof.* $L$ is a linear bijection, thus Lemma 3.4 shows that it is a continuous bijection. Consequently, it holds

$$L\Omega = \overline{L\Omega} \quad \text{and} \quad L\Omega^\circ = (L\Omega)^\circ.$$

Let us suppose that $f(\Omega) = k$, i.e.

$$\Omega - \Omega \subset (\Omega^\circ + a_1) \cup (\Omega^\circ + a_2) \cup ... \cup (\Omega^\circ + a_k),$$

where $a_i \in \mathbb{R}^d \quad \forall i \in \{1, ..., k\}$.
Making use of the fact that $f$ is a linear and continuous bijection, we have

$$L\Omega - L\Omega = L(\Omega - \Omega) \subset (L\Omega^\circ + b_1) \cup (L\Omega^\circ + b_2) \cup ... \cup (L\Omega^\circ + b_k) =$$

$$= ((L\Omega)^\circ + b_1) \cup ((L\Omega)^\circ + b_2) \cup ... \cup ((L\Omega)^\circ + b_k),$$

where

$$b_i = La_i \quad \forall i \in \{1, ..., k\}.$$

This confirms that $f(L\Omega) = k = f(\Omega)$ for every compact convex set $\Omega$ in $\mathbb{R}^d$. □

Now, we arrive at the important proposition about independence of $f$ of affine transformations of $\Omega$.

**Proposition 3.2.** *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be an affine bijection. Then $f(A\Omega) = f(\Omega)$ for every $\Omega \in \kappa_d$.*

*Proof.* Let us suppose that $f(\Omega) = k$, i.e.

$$\Omega - \Omega \subset (\Omega^\circ + a_1) \cup (\Omega^\circ + a_2) \cup ... \cup (\Omega^\circ + a_k),$$

where $a_i \in \mathbb{R}^d \quad \forall i \in \{1, ..., k\}$.
We use the following two facts:

1. As $A$ is an affine map, there exist $z \in \mathbb{R}^d$ and a linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ so that for every $x \in \mathbb{R}^d$ we have
$$Ax = z + Lx.$$

2. As $A$ is an affine bijection, $L$ is a linear bijection, and we obtain the following equations for every $\Omega \subset \mathbb{R}^d$
$$A\Omega^\circ = z + L\Omega^\circ = z + (L\Omega)^\circ = (A\Omega)^\circ,$$
$$A\Omega = z + L\Omega = z + \overline{(L\Omega)} = \overline{A\Omega}.$$

Using the previous facts, we have

$$A\Omega - A\Omega = (z + L\Omega) - (z + L\Omega) = L\Omega - L\Omega \subset ((L\Omega)^\circ + La_1) \cup ((L\Omega)^\circ + La_2) \cup ... \cup ((L\Omega)^\circ + La_k) =$$

$$= ((A\Omega)^\circ + c_1) \cup ((A\Omega)^\circ + c_2) \cup ... \cup ((A\Omega)^\circ + c_k),$$

where

$$c_i = La_i - z \quad \forall i \in \{1, ..., k\}.$$

This confirms that $f(A\Omega) = k = f(\Omega)$ for every compact convex set $\Omega$ in $\mathbb{R}^d$. □

## 3.4 Proof of Theorem 3.2

Let us introduce a theorem from [10] which enables to complete the proof of Theorem 3.2, of course by using all of previously shown properties of the function $f$.

**Theorem 3.5 (John).** *Every $\Omega \in \kappa_d$ contains an ellipsoid $E + z$ such that $E + z \subset \Omega \subset dE + z$ ($z$ being the centre of the ellipsoid $E + z$).*

Now, we dispose of enough pieces of information to be able to prove boundedness of $f$ on $\kappa_d$.

*Proof of Theorem 3.2.* For $\alpha := \frac{1}{d}$ we have already proved in Section 3.2 that $f$ is upper semicontinuous on the space $\mathcal{M}$ of all convex closed sets $\Omega$ such that

$$\overline{B(0, \tfrac{1}{d})} \subset \Omega \subset \overline{B(0,1)}$$

and therefore $f$ reaches its maximum $K$ in $\mathcal{M}$. Using this fact, we can easily verify that $f$ is bounded on the space $\kappa_d$ of all convex compact sets $\Omega$ in $\mathbb{R}^d$ with non-empty interior. Theorem 3.5 says that for every set $\Omega$ in $\kappa_d$ there exists an ellipsoid such that $E + z \subset \Omega \subset dE + z$. Let $A$ be an affine map: $\mathbb{R}^d \to \mathbb{R}^d$ such that $A(E + z) = \overline{B(0, \frac{1}{d})}$. It implies

$$\overline{B(0, \tfrac{1}{d})} \subset A(\Omega) \subset \overline{B(0,1)}.$$

Hence, $A\Omega \in \mathcal{M}$ and there exists $K > 0$ such that $f(A\Omega) \leq K$. This fact and the independence of $f$ of affine transformations confirms that

$$(\exists K > 0)(\forall \Omega \in \kappa_d)(f(\Omega) = f(A\Omega) \leq K).$$

□

Let us mention that Theorem 3.5 has an important corollary which enables to estimate the minimal upper bound $K_d$ on the Meyer numbers.

**Corollary 3.5.1.** *Let $\Omega \in \kappa_d$. Then $K_d \leq$ the number of translated copies of the open unit ball $B(0,1)$ needed to cover the closed ball $\overline{B(0, 2d)}$.*

*Proof.* Take an arbitrary $\Omega \in \kappa_d$. Using Theorem 3.5, there is an ellipsoid such that $E + z \subset \Omega \subset dE + z$. There exists an affine transformation $A$ acting followingly

$$A(E + z) = \overline{B(0,1)} \text{ and thus } A(dE + z) = \overline{B(0, d)}.$$

One obtains

$$\overline{B(0,1)} \subset A\Omega \subset \overline{B(0, d)}.$$

The following inclusions hold

$$A\Omega - A\Omega \subset \overline{B(0,d)} - \overline{B(0,d)} = \overline{B(0,2d)} \subset \bigcup_{x \in \overline{B(0,2d)}} (x + B(0,1)).$$

Since $\overline{B(0, 2d)}$ is compact, there exists a finite subcovering $\overline{B(0, 2d)} \subset \bigcup_{i=1}^{k} (a_i + B(0, 1))$. Consequently, we have

$$A\Omega - A\Omega \subset (B(0,1)+a_1) \cup (B(0,1)+a_2) \cup ... \cup (B(0,1)+a_k) \subset (\Omega^\circ + a_1) \cup (\Omega^\circ + a_2) \cup ... \cup (\Omega^\circ + a_k).$$

Thanks to independence of $f$ of affine transformations, the previous inclusions confirm that $f(A\Omega) = f(\Omega) \leq k$ for any $\Omega \in \kappa_d$, thus $K_d \leq k$, where $k$ is the number of translated copies of $B(0, 1)$ needed to cover $\overline{B(0, 2d)}$. □

# Chapter 4

# Lattices, packing, and covering problems

## 4.1 Ball-covering problems

Corollary 3.5.1 transforms the problem to find the minimal upper bound $K_d$ on the Meyer numbers for convex sets $\Omega \subset \mathbb{R}^d$ into the problem to determine the minimal number of open unit balls needed to cover the closed ball with radius $2d$. This problem in spite of being simply formulated seems to be considerably complicated. Let us concentrate on this problem for dimension $d = 2$. Dual formulation of the problem can be found in mathematical literature under the name *disk covering problem.*

**Definition 4.1.** *Given a closed unit disk, find the smallest radius $\rho(n)$ required for $n$ equal closed disks to completely cover the closed unit disk.*

It is evident that the sequence $\rho(n)$ is decreasing.

**Remark 3.** *If for $n_0$ it holds $\rho(n_0) < \frac{1}{4}$ then a disk having radius $2d = 4$ can be covered by $n_0$ disks of radius $< 1$, thus the closed ball $\overline{B(0,4)}$ can be covered by $n_0$ open unit balls. Using the established notation, we have*

$$K_2 \leq n_0 \quad \textit{if} \quad \rho(n_0) < \frac{1}{4}.$$

Values of the sequence $\rho(n)$ are known only for following indices $n$:

$$\rho(1) = 1, \quad \rho(2) = 1, \quad \rho(3) = \frac{1}{2}\sqrt{3}, \quad \rho(4) = \frac{1}{2}\sqrt{2},$$

$$\rho(5) \doteq 0.609, \quad \rho(6) \doteq 0.555, \quad \rho(7) = \frac{1}{2},$$

$$\rho(8) \doteq 0.437, \quad \rho(9) \doteq 0.422, \quad \rho(10) \doteq 0.398.$$

Unfortunately, neither of known values $\rho$ is less than $\frac{1}{4}$. Let us remark that values of $\rho$ for $n = 6, 8, 9, 10$ were obtained by using computer experimentation by Zahn (1962). Even if it is difficult to determine values of $\rho(n)$, it is possible to specify asymptotic behavior of $\rho(n)$. It is given by the following formula

$$\rho(n)^2 \sim \frac{2\pi}{3\sqrt{3}}\frac{1}{n}.$$

The reason why it is easier to describe asymptotic behavior of $\rho(n)$ is the fact that this problem can be transformed into the so-called *covering problem of the plane.* Let us define generally in $\mathbb{R}^d$ the covering problem and the packing problem associated to it.

**Definition 4.2.** *The general ball-packing problem asks for the densest packing of equal balls in d-dimensional Euclidean space where the density* $\theta$ *of a packing is defined by*

$$\theta := \lim_{r\to+\infty} \frac{\textit{number of balls in } B(0,r) \textit{ multiplied by volume of 1 ball}}{\textit{volume of } B(0,r)}.$$

**Definition 4.3.** *The general ball-covering problem asks for the most economical way to cover d-dimensional Euclidean space with equal overlapping balls, i.e. for the arrangement where the thickness* $\Theta$ *of a covering defined by*

$$\Theta := \lim_{r\to+\infty} \frac{\textit{number of balls in } B(0,r) \textit{ multiplied by volume of 1 ball}}{\textit{volume of } B(0,r)}$$

*is the smallest.*

It is well-known [22] that the densest packing of the plane has the density $\theta = \frac{\pi}{12}$ and centers of balls forming the packing are located in vertices of a hexagonal lattice. The most efficient covering has the thickeness $\Theta = \frac{2\pi}{3\sqrt{3}}$ and is reached on the same lattice.

It was already in the 17th century that the ball-covering problems attracted attention. In 1611 Kepler voiced a hypothesis that the most efficient covering of the space is reached by arranging balls to vertices of the so-called *orange lattice*. The Kepler hypothesis was proved only in 1998 by Thomas Hales intensively supported by computer technology. For higher dimensions only estimates of the density of packing and the thickness of covering in $\mathbb{R}^d$ are known. Consider $d+1$ unit balls packed in $d$-dimensional space, whose centers coincide with the vertices of a regular $(d+1)$-simplex having edges of length 2. (A regular $(d+1)$-simplex is a convex hull of $(d+1)$ points, say $x_0, x_1, x_2, ..., x_d$, lying on a sphere, where the vectors $x_1 - x_0, x_2 - x_0, ..., x_d - x_0$ are linearly independent.) Let $\sigma_d$ denote the ratio of the volume of that part of the simplex covered by the balls to the volume of the entire simplex. Rogers has shown that the density of the densest packing of $d$-space with unit balls does not exceed $\sigma_d$. Daniels has given the following asymptotic formula for $\sigma_d$

$$\sigma_d \sim \frac{d}{2^{\frac{d}{2}} e}.$$

On the strength of this result, it has been induced that the thickeness of the most efficient ball-covering in $\mathbb{R}^d$ is greater than $(\frac{2d}{d+1})^{\frac{d}{2}}\sigma_d$. Another interesting result is that one of Bezdek and Hyperberg (1991) who have constructed packings of equal ellipsoids of densities arbitrarily close to 0.753355. This number is greater than the maximal density $\frac{\pi}{3\sqrt{2}} = 0.74048...$ possible for ball-packing.

## 4.2 Delone sets and lattices

In the packing and covering problems we situate centers of balls in positions which are in certain sense 'uniformly distributed' in the space. The set of such positions must not have accumulation points and increasing gaps. Sets with these properties are called *Delone sets.*

**Definition 4.4.** *A set* $\Sigma \subset \mathbb{R}^d$ *is called Delone if it satisfies two conditions:*

1. $\Sigma$ *is uniformely discrete, i.e. there exists* $r > 0$ *such that* $|x - y| > r$ *for any* $x, y \in \Sigma$, $x \neq y$. *This condition assures that* $\Sigma$ *has no accumulation points. The maximal* $r$ *with this property is called the* **minimal distance**. *The* **packing radius** *of* $\Sigma$ *is defined as*

$$r_\Sigma = \frac{1}{2}\sup\{r > 0 \mid |x - y| \geq r, x, y \in \Sigma, x \neq y\}.$$

2. $\Sigma$ *is relatively dense, i.e. there exists* $R > 0$ *such that* $B(x, R) \cap \Sigma \neq \emptyset$ *for any* $x \in \mathbb{R}^d$. *This condition tells that in* $\Sigma$ *there are no increasing gaps. The union of balls with radius* $R$

*centered at points of $\Sigma$ cover the space $\mathbb{R}^d$. The minimal $R$ with this property is called the* **covering radius** *of $\Sigma$*

$$R_\Sigma = \inf\{R > 0 \mid B(x, R) \cap \Sigma \neq \emptyset, x \in \mathbb{R}^d\}.$$

A simple Delone set is a lattice.

**Definition 4.5.** *Let $(x_1, x_2, ..., x_d)$ be an arbitrary system of linearly independent vectors in $\mathbb{R}^d$. The set*

$$\mathcal{L} = \{\sum_{i=1}^{d} a_i x_i \mid a_i \in \mathbb{Z} \quad \forall i \in \{1, 2, ..., d\}\}$$

*is called a* **lattice** *in $\mathbb{R}^d$. The system $(x_1, x_2, ..., x_d)$ is called a* **basis** *of $\mathcal{L}$. The parallelotop consisting of points*

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_d x_d,$$

*where $0 \leq \alpha_i < 1$ for all $i \in \{1, 2, ..., d\}$, is called the* **fundamental region** *of the lattice $\mathcal{L}$.*

A fundamental region is a building block which when periodically repeated fills the entire space without overlaps and in such a way that in each copy of the fundamental region there is just one lattice point. Figure 4.1 shows as an example a part of a 2-dimensional lattice and the fundamental region determined by the basis $x_1, x_2$.

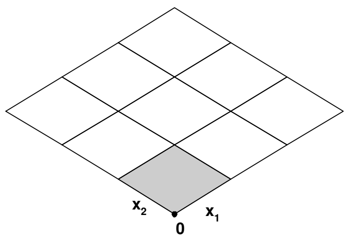



Figure 4.1: Illustration of a 2-dimensional lattice and one of its possible fundamental regions.

**Definition 4.6.** *Let $\mathcal{L}$ be a lattice in $\mathbb{R}^d$ with basis $(x_1, x_2, ..., x_d)$. Denote by $E$ the matrix whose rows are the components of vectors $x_i$. $E$ is said to be a* **generator matrix** *of the lattice $\mathcal{L}$. The symmetric matrix $M = EE^T$, where $E^T$ is the transpose of $E$, is the* **Gram matrix** *for $\mathcal{L}$. We define the* **determinant** *of $\mathcal{L}$ by*

$$\det \mathcal{L} = \det M = \det EE^T = (\det E)^2.$$

Note that the Gram matrix has as its $(i, j)$-th entry the scalar product of $x_i$ and $x_j$, thus its diagonal entries are the squared norms of the basis vectors. There are many different possibilities of choosing a basis and thus a fundamental region for a lattice $\mathcal{L}$. Namely, if $E$ is a generator matrix of $\mathcal{L}$ and $A$ is a $d \times d$ integer matrix of determinant $\pm 1$, then $AE$ is also a generator matrix of $\mathcal{L}$. Conversely, if $E$ and $\tilde{E}$ are two bases of $\mathcal{L}$ then there exists a matrix $A \in \mathbb{Z}^{d \times d}$ of determinant $\pm 1$ such that $\tilde{E} = AE$. It follows that the volume of any fundamental region of the lattice $\mathcal{L}$ is the same, and the square of this volume is equal to the determinant of the lattice. In other words, the lattice determinant is independent of the basis.

Among lattices the so-called *integral lattice* plays an important role.

**Definition 4.7.** *A lattice $\mathcal{L}$ is called integral if $|x|^2 \in \mathbb{Z}$ for every lattice element $x \in \mathcal{L}$.*

Any lattice has an associated **dual lattice** $\mathcal{L}^*$, the set of vectors whose scalar products with the vectors of $\mathcal{L}$ are integers.

**Definition 4.8.** *The dual lattice $\mathcal{L}^*$ of $\mathcal{L}$ is the set of vectors $y \in \mathbb{R}^d$ defined by*

$$y \in \mathcal{L}^* \iff (y, x) \in \mathbb{Z} \text{ for all } x \in \mathcal{L}.$$

If $(x_1, x_2, ..., x_d)$ is a basis of $\mathcal{L}$ then vectors $(x_1^*, x_2^*, ..., x_d^*)$ defined by $(x_i, x_j^*) = \delta_{ij}$ (with $\delta_{ij}$ the Kronecker delta) are linearly independent; they form a basis of $\mathcal{L}^*$. It follows that if $E$ is a generator matrix for $\mathcal{L}$ then $(E^{-1})^T$ is a generator matrix for $\mathcal{L}^*$, and the Gram matrix of $\mathcal{L}^*$ is $M^* = M^{-1}$.

In crystallography, $\mathcal{L}^*$ is usually called the 'reciprocal lattice' because the interplanar spacings of any two neighbouring lattice planes of $\mathcal{L}^*$ are inversely proportional to those of $\mathcal{L}$. Dual lattices play an important role in crystallography, particularly in the interpretation of diffraction patterns.

## 4.3 Voronoi tiles

Lattices are associated with tilings of space by blocks called *unit cells*, whose edges are parallel to the basis vectors of the lattice. The cell is usually chosen to be the fundamental region for some choice of the basis. In this case its volume is equal to $\det \mathcal{L}$. Every lattice can be devided into unit cells in infinitely many ways (one for each choice of basis). Information carried by the unit cell is limited because it cannot always be chosen to show the symmetry of the lattice. There is another polytope having the same volume as a unit cell, that shows the symmetry of the lattice, and moreover, it is independent of the choice of basis. It is the famous *Voronoi tile*, which can be defined for any Delone set.

**Definition 4.9.** *Let $\Sigma$ be a Delone set. The Voronoi tile of a point $x \in \Sigma$ is defined by*

$$V(x) = \{y \in \mathbb{R}^d \mid |x - y| \leq |z - y| \text{ for all } z \in \Sigma\}.$$

Voronoi tiles are convex polytopes in $\mathbb{R}^d$ that cover the entire space $\mathbb{R}^d$ without thick overlaps and without gaps. In such a way, they form a perfect tiling of $\mathbb{R}^d$, called the Voronoi tiling of $\Sigma$. It can be shown that for determining the Voronoi tile $V(x)$ it suffices to study only the local configuration $\Sigma \cap B(x, 2R_\Sigma)$, where $R_\Sigma$ is the covering radius of $\Sigma$. An example of the construction of a Voronoi tile of a point in the plane is given in Figure 4.2. Note in Figure 4.2 that the neighbours whose Voronoi tiles share a vertex $v$ lie on a sphere, centered at $v$, that has no points of $\Sigma$ in its interior. If $\Sigma$ is a lattice then Voronoi tiles of all its points have the same shape.

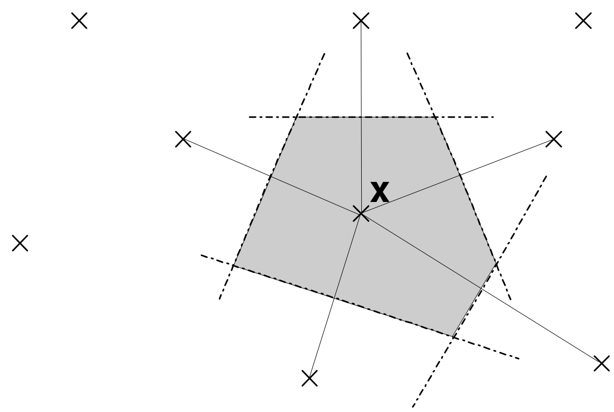



Figure 4.2: Illustration of a Voronoi tile in $\mathbb{R}^2$.

**Proposition 4.1.** *Let $\mathcal{L}$ be a lattice. Then for all $x \in \mathcal{L}$, $V(x) = V(0) + x$.*

Note that for a lattice $\mathcal{L}$ the maximum of norms of points lying on vertices of $V(0)$ is the covering radius of $\mathcal{L}$.

## 4.4 Covering and packing problems on lattices

The general ball-packing and ball-covering problems seem to be very difficult. Therefore mathematicians have directed their attention to a simplier problem of investigation of the maximal density of packing or the minimal thickness of covering only in case when centers of balls are situated in space regularily, i.e. they form a lattice. Now, recalling facts introduced in Section 4.1, one obtains for the density of a lattice packing or the thickeness of a lattice covering

$$\frac{\text{volume of 1 ball}}{\sqrt{\det \mathcal{L}}} = \frac{V_d \rho^d}{\sqrt{\det \mathcal{L}}},$$

where $V_d$ is the volume of a $d$-dimensional unit ball calculated by

$$V_d = \frac{\pi^{\frac{d}{2}}}{(\frac{d}{2})!} \quad \text{for } d \text{ even},$$

$$V_d = \frac{2^d \pi^{\frac{d-1}{2}} (\frac{d-1}{2})!}{d!} \quad \text{for } d \text{ odd},$$

and $\rho$ is the radius of balls forming the packing. Since any lattice is a Delone set $\Sigma$, it is correct to ask for its packing and covering radius (Definition 4.4). It is evident from the definition that balls of radius $R_\Sigma$ (the covering radius) centered at points of $\Sigma$ will form the best covering- they will cover $\mathbb{R}^d$, and no smaller will do. Analogically, balls of radius $r_\Sigma$ centered at points of $\Sigma$ form the best packing- they can be packed in $\mathbb{R}^d$ without overlaps, and no greater can be.

Now, the question is how to find the packing and the covering radius. To find the answer, we will pay attention to Voronoi tiles. We remind that around each point $x$ of a Delone set $\Sigma$ there is its Voronoi tile:

$$V(x) = \{y \in \mathbb{R}^d \mid |x-y| \le |z-y| \text{ for all } z \in \Sigma\}.$$

For a lattice packing with Voronoi tiles being translations of a polytope $V(0)$, the packing radius $r_\mathcal{L}$ is the radius of the largest ball inscribed to $V(0)$, while the covering radius $R_\mathcal{L}$ is the radius of the smallest ball circumscribed to $V(0)$.

Let us list the known results about lattice packings and coverings. Firstly, consider the situation for dimension $d = 2$. For more details see [22].

**Proposition 4.2 (Thue).** *The most efficient lattice packing and lattice covering in $\mathbb{R}^2$ is reached on the hexagonal lattice.*

The corresponding packing density is $\theta = \frac{\pi}{12}$ and covering thickness is $\Theta = \frac{2\pi}{3\sqrt{3}}$. A generator matrix of the hexagonal lattice is

$$E = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

The corresponding Gram matrix is

$$EE^T = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

The hexagonal lattice is illustrated in Figure 4.3. Notice that its Voronoi tiles are hexagons hence its name.

Now, let us consider this problem for dimension $d = 3$.

**Proposition 4.3 (Gauss, 1831).** *The densest lattice packing in $\mathbb{R}^3$ is reached on the face centered cubic lattice.*

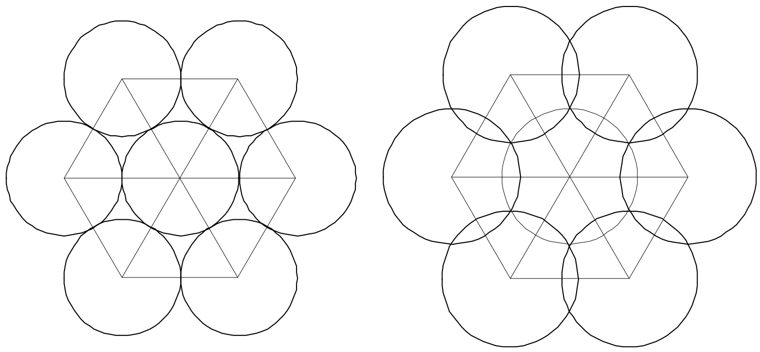


Figure 4.3: Illustration of the densest packing of the plane on the left side and the most efficient covering of the plane on the right side.

Note that arrangement of points to a face-centered cubic lattice (fcc) is found in pyramids of oranges on any fruit stand. A generator matrix of the fcc lattice is

$$E_{fcc} = E = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The corresponding Gram matrix is

$$EE^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

$\det \mathcal{L} = \det EE^T = 4$. The packing radius is $r_{\mathcal{L}} = \frac{1}{\sqrt{2}}$ and the covering radius $R_{\mathcal{L}} = r_{\mathcal{L}}\sqrt{2} = 1$. Consequently, one has the density $\theta = \frac{\pi}{3\sqrt{2}} \doteq 0.7405$ and the thickness $\Theta = \frac{2\pi}{3} \doteq 2.0944$.

**Proposition 4.4 (Bambah, [2]).** *The most efficient lattice covering in 3-dimensional space is the one with balls located in points of the body-centered cubic lattice (or bcc).*

A generator matrix of the bcc lattice is

$$E_{bcc} = E = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The corresponding Gram matrix is

$$EE^T = \frac{1}{4}\begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{pmatrix},$$

$\det \mathcal{L} = \det EE^T = 4$, the packing radius is $r_{\mathcal{L}} = \frac{\sqrt{3}}{4}$ and the covering radius $R_{\mathcal{L}} = r_{\mathcal{L}}\sqrt{\frac{5}{3}} = \frac{\sqrt{5}}{4}$. Consequently, one has the density $\theta = \frac{\pi\sqrt{3}}{8} \doteq 0.6802$ and the thickness $\Theta = \frac{5\pi\sqrt{5}}{24} \doteq 1.4635$. This is at first sight surprising, since as we have recently seen, the densest lattice packing is a different lattice unlike the case $d = 2$, where both are reached on the same hexagonal lattice. We can understand this result if we consider Voronoi tiles of the fcc and bcc lattice. The Voronoi tile for the fcc lattice is a rhombic dodecahedron. On the other hand, the Voronoi tile for the bcc lattice is a truncated octahedron, one of the famous Archimedean polyhedra. Both are illustrated in Figure 4.4.

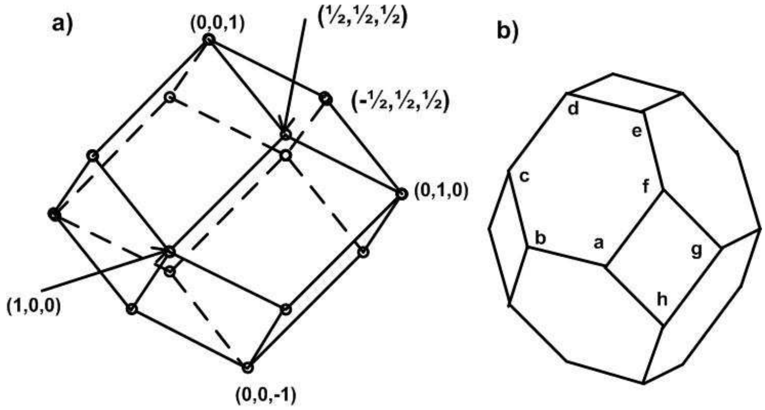



Figure 4.4: **a)** Rhombic dodecahedron (the Voronoi tile for the fcc lattice) centered at origin. There are 6 vertices ($(\pm 1, 0, 0)$ + its permutations), 8 vertices $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. **b)** Truncated octahedron (the Voronoi tile for the bcc lattice) centered at origin. There are 24 vertices ($(\pm 1, \pm\frac{1}{2}, 0)$ + its permutations). For example $a = (0, \frac{1}{2}, 1), b = (0, 1, \frac{1}{2}), c = (\frac{1}{2}, 0, 1), d = (1, \frac{1}{2}, 0), e = (1, 0, \frac{1}{2}), f = (\frac{1}{2}, 0, 1), g = (0, -\frac{1}{2}, 1)), h = (-\frac{1}{2}, 0, 1)$.

Being acquainted with these polytopes, one can easily calculate the packing and covering radius of balls arranged to points of the fcc and bcc lattice. Thus although the fcc lattice is a better packing, the bcc lattice is indeed a better covering.

**Remark 4.** *The lattices fcc and bcc are mutually dual. As it was mentioned a generator matrix of the dual lattice to the fcc lattice can be found as*

$$E^*_{fcc} = (E^{-1}_{fcc})^T = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

*Since a basis of a lattice is not uniquely determined, we can obtain another generator matrix by multiplying of the above matrix by any unimodular integer matrix $A$. If we use*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

*then*

$$AE^*_{fcc} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} = E_{bcc}.$$

Situation for dimension $d \geq 4$ is much more complicated. As we are interested mostly in the covering problems, let us devote our attention to this case. The best known values of thickness of the general ball-covering problems for dimension $3 \leq d \leq 23$ are reached on the lattices which have a basis given by the following generator matrix

$$E = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 \\ \frac{-d}{d+1} & \frac{d}{d+1} & \frac{d}{d+1} & \dots & \frac{d}{d+1} \end{pmatrix}.$$

The corresponding Gram matrix is

$$E = \begin{pmatrix} d & -1 & -1 & \dots & -1 \\ -1 & d & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & d \end{pmatrix}.$$

The covering radius of this lattice is $R_{\mathcal{L}} = \sqrt{\frac{d(d+2)}{12(d+1)}}$. Moreover it is proved that for $3 \leq d \leq 5$ these coverings are the best possible lattice coverings. Let us remark that unlike the case for dimension $d = 3$, for higher dimensions it does not hold any more that the lattice for the best covering is dual to the lattice of the best packing.

# Chapter 5

# Estimates of the Meyer numbers

In the previous part we have proved that there exists a constant $K > 0$ such that the Meyer number of every $\Omega \in \kappa_d$ is bounded above by $K$. (We recall that $\kappa_d$ denotes the space of convex compact subsets of $\mathbb{R}^d$ with non-empty interior.) In this chapter we will estimate the universal minimal upper bound $K_d$ on the Meyer numbers in $\kappa_d$, and we will estimate the Meyer numbers of $\Omega$ on some special subsets of $\kappa_d$.

We will deal with the following cases:

1. $\Omega$ being a regular polygon in $\mathbb{R}^2$,
2. $\Omega$ being a centrally symmetric set in $\kappa_d$ and especially in $\kappa_2$,
3. $\Omega$ being a general set in $\kappa_d$ and especially in $\kappa_2$.
4. $\Omega$ being a general set in $\kappa_3$.

## 5.1 Meyer numbers of regular polygons

The aim is to determine the Meyer numbers of regular polygons in $\mathbb{R}^2$. Due to the fact that $f$ is invariant under affine transformations of $\Omega$, it suffices to consider only regular polygons centered at the origin and having radius 1.

For estimation of the upper bound on $f(\Omega)$ for regular $n$-gons $\Omega$ with $n \geq 7$, it is useful to know that $f(\overline{B(0,1)}) = 8$ (this claim will be proved in Section 5.2) and to estimate the minimal radius $r$ such that 8 copies of the open disk $B(0,r)$ are sufficient to cover the closed disk $\overline{B(0,2)}$.

**Proposition 5.1.** *Let* $r > c = \frac{2}{1+2\cos\frac{2\pi}{7}}$. *Then there exist points* $a_1, a_2, ..., a_8$ *satisfying*

$$\overline{B(0,2)} \subset (B(0,r) + a_1) \cup (B(0,r) + a_2) \cup ... \cup (B(0,r) + a_8).$$

*Proof.* Let us situate disks of radius $c$ followingly: One of them to the origin and 7 of them to vertices of a regular 7-gon of radius $\rho = 2c\cos\frac{\pi}{7}$. Now, let us verify that the two points of intersection of the closed disks $\overline{B(a_1,c)}$ and $\overline{B(a_2,c)}$ (illustrated in Figure 5.1) satisfy that $P_1 \in \partial B(0,c)$ and $P_2 \in \partial B(0,2)$. Then it will be proved that 8 closed disks of radius $c$ cover $\overline{B(0,2)}$ and so do 8 open disks of radius $r > c$.

Let the coordinates (in standard basis) be $a_1 = (\rho, 0), a_2 = (\rho\cos\frac{2\pi}{7}, \rho\sin\frac{2\pi}{7})$. Then any point of intersection $P = (x,y)$ satisfies

$$|P - a_1| = c \quad \text{and} \quad |P - a_2| = c.$$

One obtains equations for coordinates

$$(x-\rho)^2 + y^2 = c^2,$$

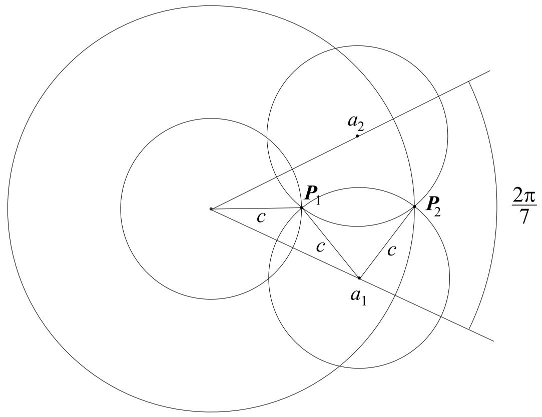



Figure 5.1: Illustration of the situation in the proof of Proposition 5.1.

$$(x - \rho \cos \tfrac{2\pi}{7})^2 + (y - \rho \sin \tfrac{2\pi}{7})^2 = c^2.$$

The solution of these equations is $P_1 = (c \cos \frac{\pi}{7}, c \sin \frac{\pi}{7})$ and $P_2 = (2 \cos \frac{\pi}{7}, 2 \sin \frac{\pi}{7})$ therefore it holds $P_1 \in \partial B(0, c)$ and $P_2 \in \partial B(0, 2)$. □

**Remark 5.** *The approximate value of c is* $c \doteq 0,89$. *Let us remind that we can obtain even a more precise value of c by using results of the disk covering problem (Section 4.1). We dispose of the fact that* $\rho(8) \doteq 0.437$, *i.e. 8 closed disks of radius approximately equal to* 0.437 *suffice to cover the closed disk* $\overline{B(0,1)}$. *Consequently, 8 open disks of radius* $r > 0.874$ *suffice to cover the closed disk* $\overline{B(0,2)}$. *Thus the value c of Proposition 5.1 can be even lowered to* 0.874.

**Claim 5.1.** *Let* $\Omega$ *be a regular* $n$*-gon for* $n \geq 7$. *Then* $f(\Omega) \leq 8$.

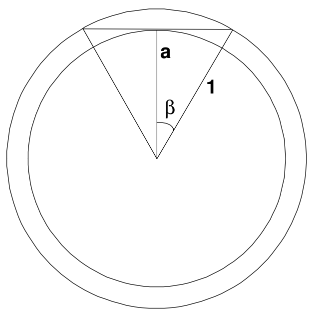



Figure 5.2: Illustration of the situation in the proof of Claim 5.1.

*Proof.* If there exists $\varepsilon > 0$ such that

$$B(0, c + \varepsilon) \subset \Omega^\circ \subset B(0, 1), \tag{5.1}$$

where $c = \frac{2}{1+2\cos\frac{2\pi}{7}}$, then using Proposition 5.1 we obtain

$$\Omega - \Omega \subset \overline{B(0,2)} \subset (a_1 + B(0, c + \varepsilon)) \cup ... \cup (a_8 + B(0, c + \varepsilon)) \subset$$

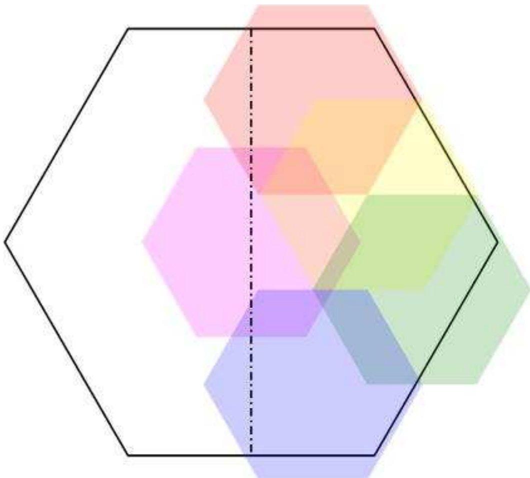
$$\subset (a_1 + \Omega^\circ) \cup ... \cup (a_8 + \Omega^\circ).$$

We want to estimate the central angle $\gamma$ of the $n$-gon $\Omega$ such that the inclusions (5.1) hold. Considering Figure 5.2, we have an implicite equation for $\beta$

$$\cos(\tfrac{\beta}{2}) = c. \tag{5.2}$$

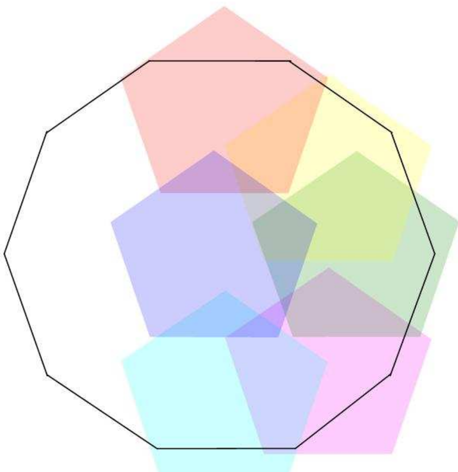
Any regular polygon which has the central angle $\gamma$ smaller than $\beta$ can be covered by 8 translated copies of $B(0, c+\varepsilon)$. Using equation (5.2), we obtain $\frac{2\pi}{6} > \beta > \frac{2\pi}{7}$, therefore $f(\Omega) \leq 8$ for every $n$-gon $\Omega$ with $n \geq 7$. ☐

**Claim 5.2.** *Let $\Omega$ be a regular hexagon. Then $f(\Omega) = 9$.*



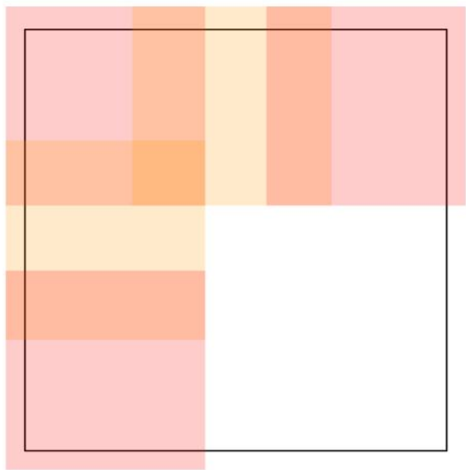
*Proof.* If $\Omega$ is a hexagon having radius of length 1 then $\Omega - \Omega$ is a hexagon having radius of double length. To cover the cicumference of the closed hexagon of radius 2, we need 8 open hexagons of radius 1, and to cover the centre, one more open hexagon of radius 1 is necessary. ☐

**Claim 5.3.** *Let $\Omega$ be a regular pentagon. Then $f(\Omega) = 9$.*



*Proof.* If $\Omega$ is a pentagon having radius of length 1 then $\Omega - \Omega$ is a regular 10-gon having radius of length $2\sin\frac{\pi}{5}$.
The same explanation as by hexagon. $\square$

**Claim 5.4.** *Let $\Omega$ be a square. Then $f(\Omega) = 9$.*



*Proof.* If $\Omega$ is a square having sides of length 1 then $\Omega - \Omega$ is a square of double size. To cover the upper side of the closed square of length 2, we need 3 open squares of half-size. The same for the lower side and the middle side. $\square$

**Claim 5.5.** *Let $\Omega$ be a triangle. Then $f(\Omega) = 12$.*

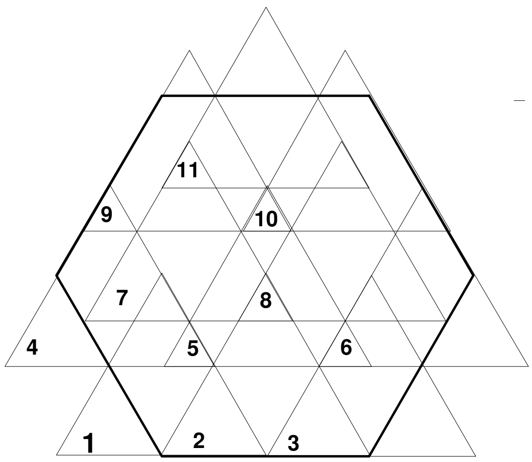



Figure 5.3: Illustration of covering by closed copies.

*Proof.* If $\Omega$ is a triangle with sides of length 1 then $\Omega - \Omega$ is a hexagon with radius of length 1. In Figure 5.3 one can see that 11 copies of a closed triangle suffice to cover the hexagon, and 12 copies are needed to cover it by open triangles as illustrated in Figure 5.4. $\square$

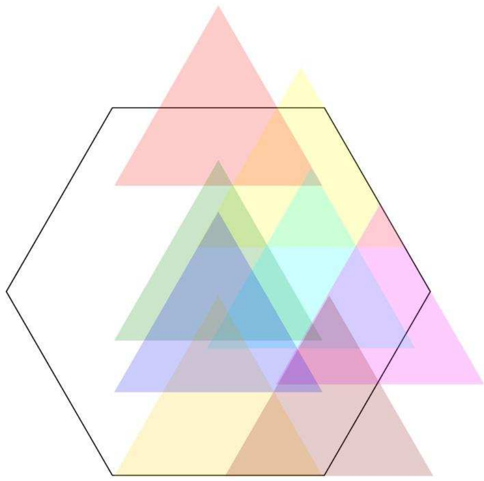


Figure 5.4: Illustration of covering by open copies.

We have thus investigated all the Meyer numbers of regular polygons in the plane.

**Proposition 5.2.** *Let $\Omega$ be a regular $n$-gon in $\mathbb{R}^2$.*
*If $n \geq 7$ then $f(\Omega) = 8$.*
*If $n = 4, 5, 6$ then $f(\Omega) = 9$.*
*If $n = 3$ then $f(\Omega) = 12$.*

## 5.2 Estimate of the universal minimal constant for centrally symmetric sets

The aim of this part is to estimate the upper bound on $f(\Omega)$ for all centrally symmetric sets in $\kappa_d$. Firstly, we consider the general case when $\Omega$ is a centrally symmetric set in $\kappa_d$, and then we limit our considerations to dimension $d = 2$.

Let us introduce an important result of John [10] which concerns centrally symmetric sets in $\kappa_d$.

**Theorem 5.1 (John).** *For every centrally symmetric set $\Omega \in \kappa_d$ there exists a centrally symmetric ellipsoid $E$ such that $E \subset \Omega \subset d^{\frac{1}{2}} E$.*

*Proof.* Let $\Omega$ be a centrally symmetric set in $\kappa_d$ and let $E$ be a centrally symmetric ellipsoid of maximum volume contained in $\Omega$. The existence of such an ellipsoid follows from the compactness of the set of collections $\{a_1, a_2, ..., a_d\}$, where the vectors $a_i$ are mutually orthogonal and the ellipsoid with semi-axes $a_1, ..., a_d$ is contained in $\Omega$. Its volume is a continuous function on a compact space. Hence, it reaches its maximum.
We take an arbitrary point $a$ on the boundary of $\Omega$ and prove that it belongs to $d^{\frac{1}{2}} E$.

We may suppose that $E$ is the ball $|x| < 1$ and that $a$ has the form $(\alpha, 0, ..., 0)$, where $\alpha > 1$. Let us consider the convex hull $\Omega'$, say, of $E$ and the points $\pm a$. Let $\Omega'_2$ be the intersection of $\Omega'$ and the plane $x_3 = x_4 = ... = x_d = 0$ and, for $0 < \theta \leq 1$, let $F_\theta$ denote the linear transformation of that plane given by $x'_1 = \theta x_1, x'_2 = x_2$. Then $\Omega'_2$ is the convex hull of the circular disk $(x_1)^2 + (x_2)^2 \leq 1$ and the points $(\pm\alpha, 0)$, it is bounded by two arcs of this disk and by segments of the four lines $\pm\alpha_{-1} x_1 \pm \beta_{-1} x_2 = 1$, where $\beta = \alpha(\alpha^2 - 1)^{-\frac{1}{2}}$. Thus $F_\theta \Omega'_2$ is bounded by two arcs of the ellipse $(x_1/\theta)^2 + (x_2)^2 \leq 1$ and by segments of the lines $\pm(\theta\alpha)^{-1} x_1 \pm \beta^{-1} x_2 = 1$. It is easy to verify that,

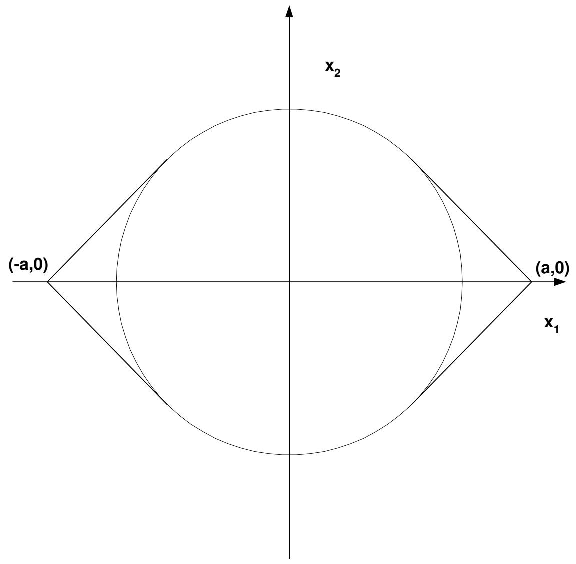



Figure 5.5: Illustration of $\Omega_2'$ from the proof of Theorem 5.1.

for $0 < \theta \leq 1$, $F_\theta \Omega_2'$ contains the disk $(x_1)^2 + (x_2)^2 \leq \rho^2$, where $\rho = \theta\alpha\beta(\theta^2\alpha^2 + \beta^2)^{-\frac{1}{2}}$. So $\Omega_2'$ contains the ellipse $(x_1/\theta)^2 + (x_2)^2 \leq \rho^2$ and so $\Omega'$ contains the ellipsoid $E_\theta$ given by

$$(\theta x_1)^2 + (x_2)^2 + ... + (x_d)^2 \leq \rho^2. \tag{5.3}$$

Since $\Omega'$ is contained in $\Omega$, it is also true that $\Omega$ contains $E_\theta$. By the choice of $E$ this implies that $V(E_\theta) \leq V(E)$, for all $\theta$ with $0 < \theta \leq 1$. We deduce from this fact that $\alpha \leq d^{\frac{1}{2}}$ in the following way. Let $V_d$ denote the volume of the unit ball. Then

$$V(E_\theta) = \theta^{-1}\rho^d V_d = \theta^{d-1}(\alpha^{-2} + \theta^2(1-\alpha^{-2}))^{-\frac{1}{2}d} V_d \tag{5.4}$$

because $\rho^2 = \theta^2(\theta^2\beta^{-2} + \alpha^{-2})^{-1} = \theta^2(\theta^2(1-\alpha^{-2}) + \alpha^{-2})^{-1}$. The expression in the right hand member of (5.4), as a function of $\theta$, tends to zero as $\theta \to 0$ or $\theta \to \infty$ and attains a strong maximum if

$$\frac{d-1}{\theta} - \frac{1}{2}d \cdot \frac{2\theta(1-\alpha^{-2})}{\alpha^{-2} + \theta^2(1-\alpha^{-2})} = 0 \text{ , i.e., if } \frac{d-1}{d} = \frac{\theta^2(\alpha^2-1)}{1+\theta^2(\alpha^2-1)},$$

or also $\theta^2 = (d-1)/(\alpha^2-1)$. However, by the foregoing, the maximum can only be attained for a value $\theta = \theta_0$ with $\theta_0 \geq 1$. So we have $d - 1 \geq \alpha^2 - 1$, and so $\alpha \leq d^{\frac{1}{2}}$.

The last result means that the boundary point $a$ belongs to $d^{\frac{1}{2}}E$. By the arbitrariness of $a$, this proves the theorem. □

From now on, we limit our considerations to dimension $d = 2$. We apply the result of John on centrally symmetric sets in $\kappa_2$.

**Corollary 5.1.1.** *Let $\Omega$ be a centrally symmetric set in $\kappa_2$. Then*

$$f(\Omega) \leq \textit{ the number of copies } B(0,1) \textit{ which are needed for covering of } \overline{B(0, 2\sqrt{2})}.$$

*Proof.* Thanks to compactness of $\overline{B(0, 2\sqrt{2})}$, we can find $(a_1, a_2, ..., a_k) \subset \mathbb{R}^2$ such that $\overline{B(0, 2\sqrt{2})} \subset \bigcup_{i=1}^k (a_i + B(0,1))$. Since $E \subset \Omega \subset \sqrt{2}E$, there exists a linear map $L$ satisfying

$$\overline{B(0,1)} \subset L\Omega \subset \overline{B(0, \sqrt{2})}.$$

Using the previous facts, we have

$$L\Omega - L\Omega \subset \overline{B(0, 2\sqrt{2})} \subset (a_1 + B(0,1)) \cup ... \cup (a_k + B(0,1)) \subset (a_1 + (L\Omega)^\circ) \cup ... \cup (a_k + (L\Omega)^\circ).$$

Since $f$ is independent of linear transformations of $\Omega$, the conclusion is $f(\Omega) = f(L\Omega) \leq k$. $\square$

**Proposition 5.3.** *Let $\Omega$ be a centrally symmetric set in $\kappa_2$. Then $f(\Omega) \leq 16$.*

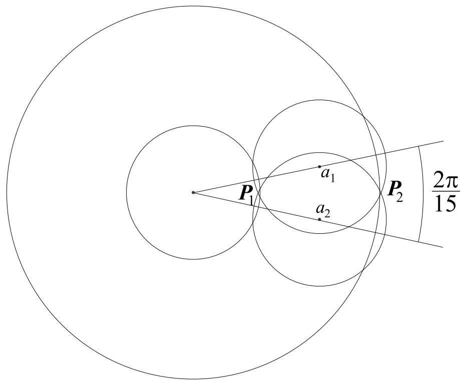



Figure 5.6: 15 translated copies of $B(0,1)$ are sufficient for covering of the circumference of $\overline{B(0, 2\sqrt{2})}$ .

*Proof.* Let $a_1, a_2$ be neighbouring vertices of a regular 15-gon centered at 0 and having radius $r = 1,95$. If we show that one of the points of intersection of $\overline{B(a_1, 1)}$ and $\overline{B(a_2, 1)}$ lies in $B(0,1)$ and the other one out of $\overline{B(0, 2\sqrt{2})}$, then it is clear that 16 open disks of radius 1 suffice to cover $\overline{B(0, 2\sqrt{2})}$. More precisely, 15 disks suffice to cover the circumference of $\overline{B(0, 2\sqrt{2})}$ and one more is needed to cover the middle. Let us determine the coordinates (in standard basis) of the points of intersection, say $P_i = (x_i, y_i)$, where $i = 1, 2$. $P_i \in \partial B(a_1, 1)$ and at one $P_i \in \partial B(a_2, 1)$, where $a_1 = (r, 0)$ and $a_2 = (r \cos \frac{2\pi}{15}, r \sin \frac{2\pi}{15})$, thus we have the following equations for the points of intersection $P_i = (x_i, y_i)$

$$(x_i - r)^2 + y_i^2 = 1, \tag{5.5}$$
$$(x_i - r \cos \tfrac{2\pi}{15})^2 + (y_i - r \sin \tfrac{2\pi}{15})^2 = 1.$$

We obtain

$$y_i = \frac{1 - \cos \frac{2\pi}{15}}{\sin \frac{2\pi}{15}} x_i \tag{5.6}$$

and we substitute $y_i$ in (5.5). Then we have

$$\Big(\big(\tfrac{1-\cos\frac{2\pi}{15}}{\sin\frac{2\pi}{15}}\big)^2 + 1\Big) x_i^2 - 2x_i r + r^2 - 1 = 0$$

with two roots

$$x_1 = \frac{r - \sqrt{r^2 - (r^2-1)\Big(\big(\frac{1-\cos\frac{2\pi}{15}}{\sin\frac{2\pi}{15}}\big)^2 + 1\Big)}}{\Big(\big(\frac{1-\cos\frac{2\pi}{15}}{\sin\frac{2\pi}{15}}\big)^2 + 1\Big)}, \quad x_2 = \frac{r + \sqrt{r^2 - (r^2-1)\Big(\big(\frac{1-\cos\frac{2\pi}{15}}{\sin\frac{2\pi}{15}}\big)^2 + 1\Big)}}{\Big(\big(\frac{1-\cos\frac{2\pi}{15}}{\sin\frac{2\pi}{15}}\big)^2 + 1\Big)}.$$

Now, using (5.6), one can easily calculate the values of $y_1, y_2$ and notice that $P_1 = (x_1, y_1)$ lies in $B(0,1)$ and $P_2 = (x_2, y_2)$ lies out of $\overline{B(0, 2\sqrt{2})}$. The coordinates of points of intersection are

$$P_1 = (x_2, y_2) \doteq (0,9716; 0,2065),\ P_2 = (x_1, y_1) \doteq (2,7289; 0,58).$$

□

**Remark 6.** *Unfortunately, the disk covering problem (Section 4.1) is solved only for $n \leq 10$ and $\rho(10) = 0.398 > \frac{1}{2\sqrt{2}} \doteq 0.354$. If we knew $n \in \mathbb{N}$ for which $\rho(n) < \frac{1}{2\sqrt{2}}$ then it would be possible to cover the disk $\overline{B(0, 2\sqrt{2})}$ by $n$ copies of open unit disks. For this moment, owing to Proposition 5.3, we know that $\rho(16) \leq \frac{1}{2\sqrt{2}}$.*

It is likely that the centrally symmetric set in $\kappa_2$, for which the function $f$ reaches its minimum, is a disk.

**Claim 5.6.** *$f(\overline{B(0,1)}) = 8$, i.e. $\overline{B(0,2)} \subset (a_1 + B(0,1)) \cup ... \cup (a_8 + B(0,1))$.*

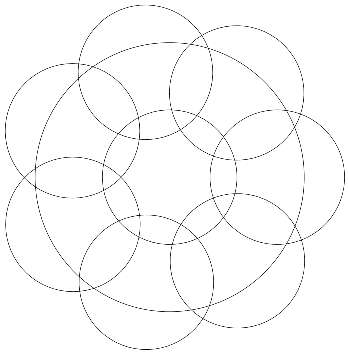


Figure 5.7: 8 open disks suffice.

*Proof.* Considering Figure 5.8, we can see that 6 translated copies of $\overline{B(0,1)}$ are sufficient to cover a hexagon having radius of length 2, therefore 6 translated copies of the disk $\overline{B(0,1)}$ are sufficient to cover the circumference of $\overline{B(0,2)}$, but 6 open copies of $B(0,1)$ are not sufficient. Figure 5.7 illustrates that 7 open translated copies of $B(0,1)$ are sufficient to cover the circumference of $\overline{B(0,2)}$, and one more is needed to cover the interior.

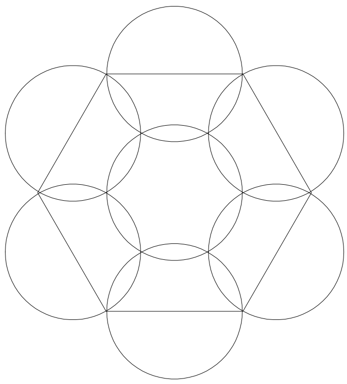


Figure 5.8: 7 open disks are not sufficient.

□

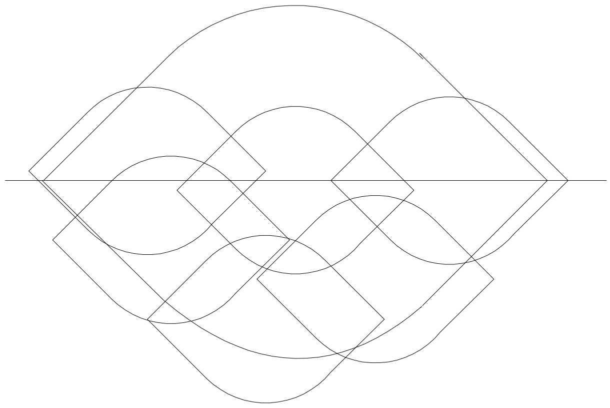


Figure 5.9: Illustration of the convex hull from Claim 5.7.

**Remark 7.** *Recalling the disk covering problem, one knows the value* $\rho(8) = 0.437 \leq \frac{1}{2}$*, which also confirms that 8 open unit disks suffice to cover the closed disk* $\overline{B(0,2)}$.

We conjecture that the Meyer number of any centrally symmetric $\Omega \in \kappa_2$ is bounded by constant $K = 9$. Paying attention to the proof of Theorem 5.1, the role of the most problematic set, which raises the constant, might play the convex hull of $\overline{B(0,1)}$ and the points $(\pm\sqrt{2}, 0)$.

**Claim 5.7.** *Let* $\Omega$ *be the convex hull of* $\overline{B(0,1)}$ *and the points* $(\pm\sqrt{2}, 0)$*. Then* $f(\Omega) = 9$.

*Proof.* Considering Figure 5.9, one can notice that 8 translated copies of the interior of the convex hull of $\overline{B(0,1)}$ and the points $(\pm\sqrt{2}, 0)$ are needed for covering of the circumference of the convex hull of $\overline{B(0,2)}$ and the points $(\pm 2\sqrt{2}, 0)$, and one more is necessary to cover the interior. □

By using the previous result and results for regular polygons, we arrived at the following conjecture.

**Conjecture 1.** *Let* $\Omega$ *be a centrally symmetric set in* $\kappa_2$*. Then* $8 \leq f(\Omega) \leq 9$.

## 5.3 Estimate of the universal minimal constant

The aim of this section is to estimate the minimal upper bound $K_d$ (Definition 3.3). Firstly, we consider general sets in $\kappa_d$ and then we limit our considerations to dimension $d = 2$ and $d = 3$.

**Proposition 5.4.** *Let* $d \geq 2$ *and let* $\Omega$ *be a set in* $\kappa_d$*. Then* $f(\Omega) \leq (2d^2 + 1)^d$.

*Proof.* Using Corollary 3.5.1 of Theorem 3.5, it suffices to prove that the closed ball $\overline{B(0, 2d)}$ can be covered by $k$ translated copies of $B(0,1)$, where $k \leq (2d^2 + 1)^d$. Let us show that

$$\overline{B(0, 2d)} \subset \bigcup_{x \in J} B(x, 1),$$

where $J = \{\frac{1}{d}(x^1, x^2, ..., x^d) \in \mathbb{R}^d \mid x^i = 0, 1, ..., 2d^2,\ i = 1, 2, ..., d\}$. Take an arbitrary $y = (y^1, y^2, ..., y^d) \in \overline{B(0, 2d)}$, then there exists $z = (z^1, z^2, ..., z^d) \in J$ which fulfils

$$|y^1 - z^1|^2 + |y^2 - z^2|^2 + ... + |y^d - z^d|^2 \leq \tfrac{1}{d^2} + \tfrac{1}{d^2} + ... + \tfrac{1}{d^2} = \tfrac{1}{d} < 1,$$

i.e. there exists $z \in J$ such that $y \in B(z, 1)$. One can easily notice that such a covering of $\overline{B(0, 2d)}$ has cardinality $\#J = (2d^2 + 1)^d$. Therefore we obtain

$$(\forall \Omega \in \kappa_d)\big(f(\Omega) \leq (2d^2 + 1)^d\big).$$

□

The estimation from Proposition 5.4 is universal and consequently rough. We will limit our considerations to dimension $d = 2$ and search for a more precise estimation.

### 5.3.1 Estimate of the universal minimal constant in $\mathbb{R}^2$

We restate Corollary 3.5.1 for sets in $\mathbb{R}^2$.

**Corollary 5.1.2.** *Let $\Omega$ be a set in $\kappa_2$. Then*

$$f(\Omega) \leq \text{ the number of translated copies of } B(0,1) \text{ needed for covering of } \overline{B(0,4)}.$$

**Proposition 5.5.** *Let $\Omega$ be a set in $\kappa_2$. Then $f(\Omega) \leq 26$.*

*Proof.* Figure 5.10 shows that it is possible to cover $\overline{B(0,4)}$ by 26 translated copies of the disk $B(0,1)$.

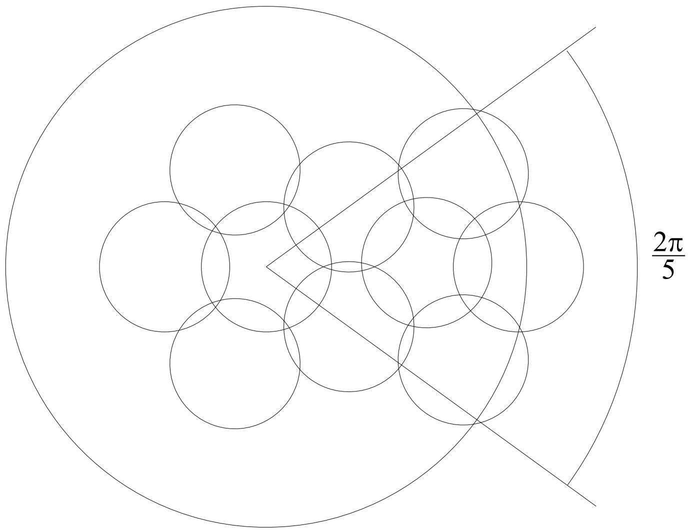



Figure 5.10: Illustration of covering of $\overline{B(0,4)}$.

□

**Remark 8.** *The disk covering problem cannot help us because $\rho(10) \doteq 0.398 > \frac{1}{4}$. But conversely Proposition 5.5 shows that $\rho(26) \leq \frac{1}{4}$.*

We conjecture that this estimate is fairly rough. Considering the case of polygons, the shape which seems to raise the upper bound on $f(\Omega)$ for $\Omega \in \kappa_2$, is a triangle. For $\Omega$ being a triangle, we have $f(\Omega) = 12$. It is likely that the minimal value of the function $f$ for general sets in $\kappa_2$ is 8, reached for instance on $n$-gons, where $n \geq 7$.

Previous considerations lead us to the following conclusion.

**Conjecture 2.** *Let $\Omega$ be a set $\kappa_2$. Then $8 \leq f(\Omega) \leq 12$.*

### 5.3.2 Estimate of the universal minimal constant in $\mathbb{R}^3$

We use the same tool to find an upper bound on the Meyer numbers for convex $\Omega \subset \mathbb{R}^3$ as we did in the case of $\Omega \subset \mathbb{R}^2$. Corollary 3.5.1 claims that for $\Omega \subset \mathbb{R}^3$ the value $f(\Omega)$ is less or equal to the number of translated copies of the open unit ball $B(0,1)$ needed to cover the closed ball $\overline{B(0,6)}$. We estimate this number by arrangement of centres of unit balls to a lattice so that they cover the closed ball $\overline{B(0,6)}$.

**Remark 9.** *For covering of* $\overline{B(0,6)}$ *by open translated copies of* $B(0,1)$ *we use at first the orthogonal lattice* $(\frac{2}{\sqrt{3}}-\varepsilon)\mathbb{Z}^3$ *for* $\varepsilon > 0$*, which will be specified later. If to every lattice point* $(\frac{2}{\sqrt{3}}-\varepsilon)(k,l,m)$ *of norm* $\leq 7$ *we put an open cube of side-length* $\frac{2}{\sqrt{3}}$*, we cover the ball* $\overline{B(0,6)}$*. (See Figure 5.11.)*

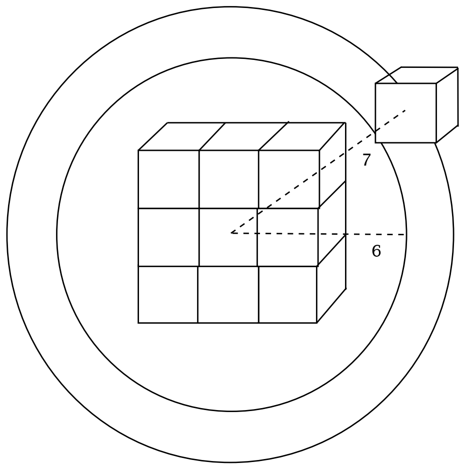



Figure 5.11: Illustration of the arrangement of cubes which cover the closed ball $\overline{B(0,6)}$.

*Replacing cubes by escribed balls, we obtain a covering of* $\overline{B(0,6)}$ *by open copies of* $B(0,1)$*. The number of such cubes/balls is equal to the number of all integer solutions* $(k,l,m)$ *of the equation*

$$(\frac{2}{\sqrt{3}}-\varepsilon)^2k^2 + (\frac{2}{\sqrt{3}}-\varepsilon)^2l^2 + (\frac{2}{\sqrt{3}}-\varepsilon)^2m^2 \leq 49,$$

*which has the same number of integer solutions as*

$$k^2 + l^2 + m^2 \leq [(\frac{2}{\sqrt{3}}-\varepsilon)^{-2}49] = 36$$

*for sufficiently small* $\varepsilon$*. The number of such solutions is 925. Thus* $f(\Omega) \leq 925$ *for any convex compact* $\Omega \subset \mathbb{R}^3$ *with non-empty interior.*

One may naturally asks whether it is possible to lower the estimate by choice of another lattice. The answer is positive. To find the right lattice, we will make use of theory introduced in Section 4.1. We have learned that the best covering uses the bcc lattice. It is likely that if we arrange centres of translated copies of $B(0,1)$ to points of a bcc lattice having a modified size so that they cover the closed ball $\overline{B(0,6)}$, we will obtain a better estimate of the Meyer numbers in $\mathbb{R}^3$.

**Proposition 5.6.** *Let* $\Omega$ *be a convex compact set in* $\mathbb{R}^3$ *with non-empty interior. Then* $f(\Omega) \leq 531$.

*Proof.* Let us consider the bcc lattice of double size, i.e. with the basis

$$\Psi = \{x_1, x_2, x_3\} = \{(2,0,0),(0,2,0),(1,1,1)\}.$$

Such a lattice has the covering radius $R = \frac{\sqrt{5}}{2}$, i.e. closed balls of radius $\frac{\sqrt{5}}{2}$ located at points of this lattice cover the entire space $\mathbb{R}^3$. As we need a covering by open unit balls, we consider instead the lattice with the basis $(\frac{2}{\sqrt{5}}-\varepsilon)\Psi$ for $\varepsilon > 0$, which will be specified later. The number of translated copies of the open unit ball $B(0,1)$ needed to cover the closed ball $\overline{B(0,6)}$ is less or equal to the number of lattice points located in the ball $\overline{B(0,7)}$. This number corresponds to the number of integer solutions $(k,l,m)$ of the equation

$$(\frac{2}{\sqrt{5}}-\varepsilon)|kx_1 + lx_2 + mx_3| \leq 7,$$

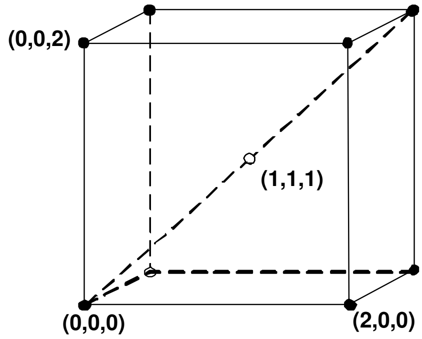



Figure 5.12: Illustration of the bcc lattice of double size.

which has the same number of integer solutions as

$$(2k+m)^2 + (2l+m)^2 + m^2 \leq [(\tfrac{2}{\sqrt{5}} - \varepsilon)^{-2}49] = 61$$

for sufficiently small $\varepsilon$. The number of such solutions is 531. Thus $f(\Omega) \leq 531$ for any convex compact $\Omega \subset \mathbb{R}^3$ with non-empty interior. □

**Remark 10.** *Considering arrangement of ball centres to points of a lattice, we conjecture that the smallest number of lattice points located in $\overline{B(0,7)}$, such that translated copies of $B(0,1)$ centered at lattice points cover $\overline{B(0,6)}$, is reached for the bcc lattice having a modified size. To illustrate this conjecture, let us estimate the number for the fcc lattice. Let us consider the fcc lattice with the basis*

$$\Psi = \{x_1, x_2, x_3\} = \{(1,0,1)), (0,1,1), (1,1,0)\}.$$

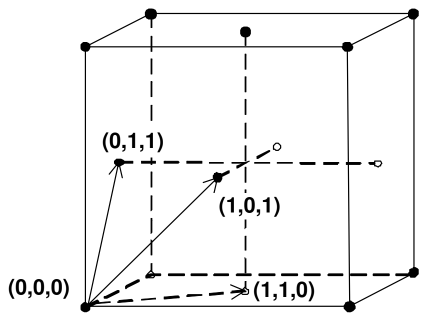



Figure 5.13: Illustration of the fcc lattice.

*Such a lattice has the covering radius $R = 1$, i.e. closed balls of radius 1 located at points of this lattice cover the entire space $\mathbb{R}^3$. The number of points of the lattice with the basis $(1-\varepsilon)\Psi$, where $\varepsilon > 0$ will be specified later, located in the ball $\overline{B(0,7)}$ corresponds to the number of integer solutions $(k,l,m)$ of the equation*

$$(1-\varepsilon)|kx_1 + lx_2 + mx_3| \leq 7,$$

*which has the same number of integer solutions as*

$$(k+m)^2 + (l+m)^2 + (k+l)^2 \leq [(1-\varepsilon)^{-2}49] = 49$$

*for sufficiently small $\varepsilon$. The number of such solutions is 683, i.e. it is greater than 531 reached for the bcc lattice.*

# Chapter 6

# Unboundedness of the Meyer numbers for non-convex sets

So far we have examined values of the function $f$ on the space of convex compact sets in $\mathbb{R}^d$ with non-empty interior. There appears a natural question: Is convexity of the set $\Omega$ necessary for boundedness of the function $f$? The answer is positive. We will prove this statement by construction of a sequence of non-convex compact sets $(\Omega_n)_{n=1}^{\infty}$ with the property $\overline{\Omega_n^\circ} = \overline{\Omega_n} \neq \emptyset$, and we will show that

$$\lim_{n\to\infty} f(\Omega_n) = +\infty.$$

For this purpose let us introduce the notion of star-shaped sets in $\mathbb{R}^d$.

**Definition 6.1.** *Let $\Omega$ be a set in $\mathbb{R}^d$. Then $\Omega$ is called star-shaped if it holds*

$$(\exists x \in \Omega)(\forall y \in \Omega)(\overline{xy} \in \Omega),$$

*where $\overline{xy}$ is a line segment connecting $x, y$.*

**Remark 11.** *Apparently, star-shaped sets are the nearest generalisation of convex sets. In spite of this fact, the function $f$ is not bounded on the space of star-shaped sets as follows from the proof of the following theorem.*

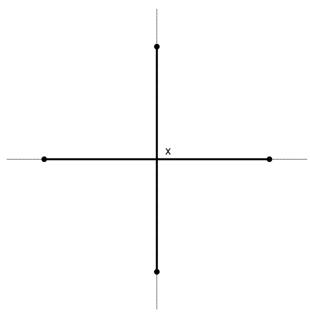



Figure 6.1: Illustration of a star-shaped set.

**Theorem 6.1.** *The function $f$ is not bounded on the space of compact sets in $\mathbb{R}^d$ with the property $\overline{\Omega^\circ} = \overline{\Omega} \neq \emptyset$.*

*Proof.* We divide the proof into three parts. We will show unboundedness of the function $f$ on the space of compact sets in $\mathbb{R}^2$ by construction of a sequence of star-shaped sets $\Omega_n$ fulfilling the property $\overline{\Omega^\circ} = \overline{\Omega} \neq \emptyset$ and such that $\lim_{n\to\infty} f(\Omega_n) = +\infty$. Then we will prove unboundedness of the function $f$ on the space of compact sets in $\mathbb{R}^d$ by construction of a sequence of prisms having a star-shaped set as their base. Finally, we will deal with the case in $\mathbb{R}^1$, where any star-shaped set is convex. Therefore we will use another tool to prove unboundedness of the function $f$ on the space of compact sets in $\mathbb{R}$.

1. Let dimension $d = 2$. We define a sequence of star-shaped compact sets $(\Omega_n)_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $\overline{\Omega_n^\circ} = \overline{\Omega_n} \neq \emptyset$ in the following way. For every $n \in \mathbb{N}$, $\Omega_n$ is the union of five convex hulls (coordinates of points are written in the standard basis of $\mathbb{R}^2$):

$$H_1 \text{ is the convex hull of points } (\tfrac{1}{n}, \tfrac{1}{n}), (\tfrac{-1}{n}, \tfrac{1}{n}), (\tfrac{-1}{n}, \tfrac{-1}{n}), (\tfrac{1}{n}, \tfrac{-1}{n}),$$

i.e. $H_1$ is a square with side-length $\frac{2}{n}$ centered at the origin.

$$H_2 \text{ is the convex hull of points } (\tfrac{1}{n}, \tfrac{1}{n}), (0, 1), (\tfrac{-1}{n}, \tfrac{1}{n}).$$

$$H_3 \text{ is the convex hull of points } (\tfrac{-1}{n}, \tfrac{1}{n}), (-1, 0), (\tfrac{-1}{n}, \tfrac{-1}{n}).$$

$$H_4 \text{ is the convex hull of points } (\tfrac{-1}{n}, \tfrac{-1}{n}), (0, -1), (\tfrac{1}{n}, \tfrac{-1}{n}).$$

$$H_5 \text{ is the convex hull of points } (\tfrac{1}{n}, \tfrac{-1}{n}), (1, 0), (\tfrac{1}{n}, \tfrac{1}{n}).$$

$$\Omega_n := \bigcup_{i=1}^{5} H_i. \tag{6.1}$$

The set $\Omega_n$ is illustrated in Figure 6.2 bellow.

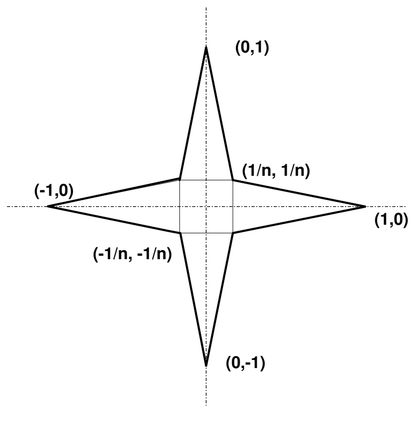



Figure 6.2: Illustration of $\Omega_n$ for $n = 6$.

We can calculate the volume of $\Omega_n$:

$$\text{vol}\, \Omega_n = 4\tfrac{1}{n}(1 - \tfrac{1}{n}) + 4\tfrac{1}{n^2} = \tfrac{4}{n}.$$

It is not difficult to construct the difference set $\Omega_n - \Omega_n$ for the above $\Omega_n$, see Figure 6.3. Note that $\Omega_n - \Omega_n$ contains the square of side-length 2 centered at the origin. Let us explain

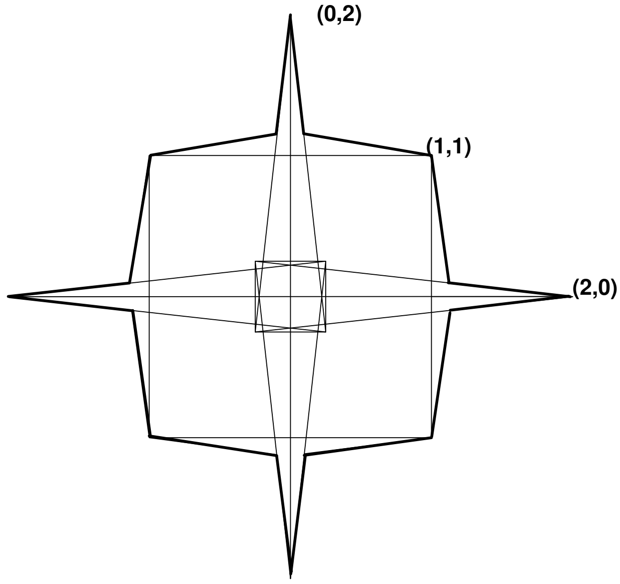



Figure 6.3: Illustration of $\Omega_n - \Omega_n$ for $n = 6$.

why this is true for every $n \in \mathbb{N}$. It is due to the fact that for every point $(x_1, x_2)$ of the square, i.e.

$$-1 \leq x_1 \leq 1,$$
$$-1 \leq x_2 \leq 1,$$

it is possible to write

$$(x_1, x_2) = (x_1, 0) - (0, -x_2),$$

where $(x_1, 0), (0, -x_2)$ are points of line segments, which are parts of $\Omega_n$ for every $n \in \mathbb{N}$.

For the volume of $\Omega_n - \Omega_n$ we have

$$\text{vol}(\Omega_n - \Omega_n) \geq 4.$$

One can realize that with the growing $n$ the volume of $\Omega_n$ tends to zero while the volume of $\Omega_n - \Omega_n$ remains greater than 4. The fact that

$$\lim_{n\to\infty} f(\Omega_n) \geq \lim_{n\to\infty} \frac{\text{vol } (\Omega_n - \Omega_n)}{\text{vol } \Omega_n} \geq \lim_{n\to\infty} \frac{4}{\frac{4}{n}} = \lim_{n\to\infty} n = +\infty$$

proves unboundedness of $f$ on the space of compact sets in $\mathbb{R}^2$.

2. We will use analogical considerations for the proof that the function $f$ is not bounded on the space of general compact sets in $\mathbb{R}^d$ with $d > 2$. Let us define a sequence of compact sets $(\Omega_n)_{n=1}^{\infty} \subset \mathbb{R}^d$ satisfying the condition $\overline{\Omega_n^\circ} = \overline{\Omega_n} \neq \emptyset$ by

$$\Omega_n := \{(x_1, x_2, ..., x_d) \in R^d \bigm| (x_1, x_2) \in \bigcup_{i=1}^{5} H_i, x_j \in [0, 1] \text{ for } j = 3, ..., d\},$$

where $\bigcup_{i=1}^{5} H_i$ is the union of convex hulls from (6.1).

In $\mathbb{R}^3$, $\Omega_n$ is a prism having the star-shaped set from Figure 6.2 as its base.

Considering this sequence, we can calculate the volume of $\Omega_n$, we have vol $\Omega_n = \frac{4}{n}$.

Let us explain that $\Omega_n - \Omega_n$ contains a $d$-dimensional cube having side-length 2 and centered at the origin for every $n \in \mathbb{N}$. Take an arbitrary element of that $d$-dimensional cube, i.e.

$$(x_1, x_2, ..., x_d),$$

where $-1 \leq x_i \leq 1$ for every $i \in \{1, 2, ..., d\}$. Denote by $y := (x_1, 0, y_3, ...y_d)$, where $y_i = x_i$ for $x_i > 0$, otherwise $y_i = 0$ and denote by $z := (0, -x_2, -z_3, ... - z_d)$, where $z_j = x_j$ for $x_j < 0$, otherwise $z_j = 0$. The element of the cube can be written using the above notation

$$(x_1, x_2, ..., x_d) = y - z,$$

where $y \in \Omega_n$ and $z \in \Omega_n$. Therefore $\Omega_n - \Omega_n$ contains a $d$-dimensional cube of side-length 2 and $\text{vol}(\Omega_n - \Omega_n) \geq 2^d$.

Using the knowledge of volumes, we have

$$\lim_{n\to\infty} f(\Omega_n) \geq \lim_{n\to\infty} \frac{\text{vol } (\Omega_n - \Omega_n)}{\text{vol } \Omega_n} \geq \lim_{n\to\infty} \frac{2^d}{\frac{4}{n}} = +\infty,$$

which proves that the function $f$ is unbounded on the space of general compact sets in $\mathbb{R}^d$ with $d \geq 2$.

3. The last question left is unboundedness of the function $f$ on the space of general compact sets in $\mathbb{R}^1$. In 1-dimensional case, any compact set is star-shaped if and only if it is convex. We cannot use analogical constructions as in the previous cases to prove unboundedness of $f$ because the function $f$ is bounded on the space of compact star-shaped sets in $\mathbb{R}$ with non-empty interior. We will show unboundedness of $f$ for general compact sets in $\mathbb{R}$ by the following construction. Let us define a sequence $(\Omega_n)_{n=1}^{\infty}$ of compact sets in $\mathbb{R}$ fulfilling the condition $\overline{\Omega_n^\circ} = \overline{\Omega_n} \neq \emptyset$. The sequence $(\Omega_n)_{n=1}^{\infty}$ arises from the interval $[0, 1]$. $\Omega_1$ is the interval $[0, 1]$. For $n \geq 2$, $\Omega_n$ is the union of the following intervals:

$$[0, \tfrac{1}{n^2}], [\tfrac{1}{n} - \tfrac{1}{n^2}, \tfrac{1}{n} + \tfrac{1}{n^2}], [\tfrac{2}{n} - \tfrac{1}{n^2}, \tfrac{2}{n} + \tfrac{1}{n^2}], ...., [\tfrac{n-2}{n} - \tfrac{1}{n^2}, \tfrac{n-2}{n} - \tfrac{1}{n^2}] \text{ and } [\tfrac{n-1}{n}, 1],$$

i.e. $\Omega_n$ contains one interval of the length $\frac{1}{n^2}$, $n-2$ intervals of the length $\frac{2}{n^2}$ and one interval of the length $\frac{1}{n}$.

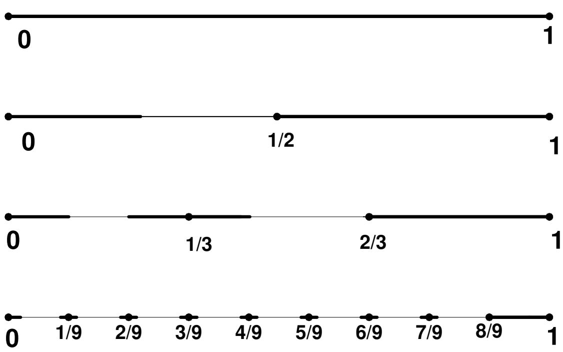



Figure 6.4: Construction of $(\Omega_n)_{n=1}^{\infty}$ in 1-dimensional case.

Considering the construction in Figure 6.4, we have

$$\text{vol } \Omega_n = \tfrac{1}{n^2} + (n-2)\tfrac{2}{n^2} + \tfrac{1}{n}.$$

It is easy to prove that $\Omega_n - \Omega_n = [-1, 1]$ for every $n \in \mathbb{N}$. Mind the following explanation. $\Omega_n$ contains the points $0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1$, and the interval $[\frac{n-1}{n}, 1]$. Take an arbitrary $k \in \{0, 1, 2, ..., n\}$, then the difference set

$$[\tfrac{n-1}{n}, 1] - \tfrac{k}{n} = [\tfrac{k}{n} - 1, \tfrac{k-n+1}{n}] \cup [\tfrac{n-1-k}{n}, 1 - \tfrac{k}{n}]$$

is subset of $\Omega_n - \Omega_n$ for every $k \in \{0, 1, 2, ..., n\}$. Therefore the difference set $\Omega_n - \Omega_n = \cup_{k=0}^{n}([\frac{n-1}{n}, 1] - \frac{k}{n}) = [-1, 1]$ and its volume is 2. Considering the volumes, we obtain

$$\lim_{n\to\infty} f(\Omega_n) \geq \lim_{n\to\infty} \frac{\text{vol } (\Omega_n - \Omega_n)}{\text{vol } \Omega_n} = \lim_{n\to\infty} \frac{2}{\frac{1}{n^2} + (n-2)\frac{2}{n^2} + \frac{1}{n}} = +\infty,$$

which proves that the function $f$ is not bounded on the space of general compact sets in $\mathbb{R}$.

□

We arrive at the conclusion that convexity of the set $\Omega \subset \mathbb{R}^d$ is essential in order to obtain a universal minimal upper bound $K_d$ on the Meyer numbers in $\mathbb{R}^d$.

# Chapter 7

# Quasicrystals

For more information about this topic consult [19].

## 7.1 Quasicrystals as a class of crystals

Shechtman with his collegues [20] produced in 1984 an alloy of aluminium and manganese by a rapid cooling technique. Its diffraction images showed icosahedral symmetry, long believed to be impossible for matter in the crystalline state. Icosahedral symmetry does not coincide with translational periodicity of a lattice. Therefore the examined matter could not have been considered as a crystal in a traditional conception, since persuasion of translational periodicity of space arrangement of atoms belonged to the essential axioms of crystallography. However, the sharp bright diffraction pattern ratified long-range orientational order of atoms. Shechtman's discovery deeply shattered with crystallography of that time, and many fundamental and fascinating questions arised:

1. What are the suitable mathematical models for crystalline structure if we no longer define a crystal to be a solid with a periodic atomic pattern?
2. What kind of geometric properties must a point set have so that its diffraction patterns could show sharp bright spots?
3. How can such point sets be generated?

Let us explain why the icosahedral symmetry containing five-fold rotational axis is incompatible with a periodic pattern. Let us study under which isometries a periodic pattern can be invariant. Recall that every isometry is given by an orthogonal matrix $A$. (Note that a matrix of an isometry which preserves a periodic pattern must have integer entries. We denote the group of all orthogonal matrices in $\mathbb{Z}^{d\times d}$ by $\mathcal{O}(d,\mathbb{Z})$.) The smallest $k$ such that $A^k = I_d$ is called the order of rotation.

**Proposition 7.1 (the crystallographic restriction).** *If a periodic pattern in $\mathbb{R}^d$, $d = 2$ or $3$, is invariant under a rotation of order $k$ then $k \in \{2, 3, 4, 6\}$.*

*Proof.* (a) in $\mathbb{R}^2$
Let us assume that there exists a plane pattern that is both periodic and invariant under $k$-fold rotation. Let $x$ be a rotation center. Since the pattern is periodic, it has a countable infinity of such centers (one in each fundamental region) and there is a minimal distance, say $d$, between them. Assume that $y$ is another such center and the distance $|x - y| = d$. A counter-clockwise rotation through $\frac{2\pi}{k}$ radians about $x$ carries $y$ to another rotation center $y'$, and we must have $|y - y'| \geq d$. This is possible only if $k \leq 6$ as illustrated in Figure 7.1. If $k = 5$ then $|y - y'| \geq d$, but another problem arises. Since $y$ is also a rotation center, clockwise rotation about $y$ must carry $x$ to another center $x'$. But $|x - x'| < d$. See Figure 7.1.
(b) in $\mathbb{R}^3$

The same argument is valid in 3-dimensional space because every rotation is a rotation of a plane about an axis orthogonal to it. □

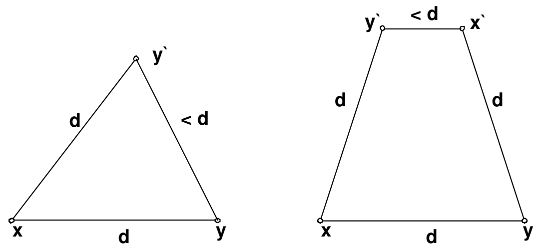



Figure 7.1: (a) The angle of rotation cannot be less than $\frac{\pi}{3}$. (b) The angle of rotation cannot be $\frac{2\pi}{5}$.

As Shechtman discovered a material with icosahedral symmetry, i.e. it contained a five-fold rotational axis, its atoms could not form a periodic pattern. The discovery shattered the fundamental law of the crystallographic restriction, not by showing it to be logically false but by showing that periodicity is not necessarily a consequence of long-range order, if by 'long-range order' we mean whatever order necessary for a crystal to produce a diffraction pattern with sharp bright spots. It came to light that we may not know what 'longe-range order' means, nor what a 'crystal' is, nor how 'symmetry' should be defined. As a consequence, new definitions of these fundamental notions were needed. Let us acquaint you with a brief historical review which will lead into formulation of new definitions.

In the nineteenth century, discrete point sets generated by 'rigid' motions were called *regular systems of points*. Under 'rigid' motions only translations were understood at first. The simplest regular system of points is a lattice (Definition 4.5).

Jordan generalized the definition of a regular system of points to any point set whose points were invariant under translations or rotations. Fedorov extended the concept of 3-dimensional regular systems to point sets whose symmetries included not only the motions studied by Jordan but also reflections, glide reflections, and rotary reflections, and the 230 generating groups - now known as the crystallographic groups of $\mathbb{R}^3$ - had been enumerated.

The Russian mathematician B. N. Delone together with Aleksandrov, and Padurov clarified and extended Fedorov's work and developed a coherent body of $d$-dimensional mathematical crystallography, incorporating the work of Minkowski, Voronoi, and others. According to Delone, the study of crystallography should begin with very general point sets, constrained by two simple, physically reasonable properties, 'discreteness' and 'homogeneity'. No regularity or other symmetry conditions are imposed. Discreteness is an important condition, it expresses the fact that centers of atoms in any gas, liquid, or solid, cannot come arbitrarily close together. The homogeneity condition models the fact that atoms in these phases tend to distribute themselves more or less uniformly throughout available space. Discrete, homogeneous point sets can serve as models for a broad range of structures, from highly amorphous to highly symmetric. Discreteness and homogeneity are formulated as 'uniform descreteness' and 'relative density' in the Delone property (see Definition 4.4 in Section 4.2).

Local configurations in Delone sets can be characterized in various ways. Usually we consider 'circular' configurations centered at points of $\Sigma$:

**Definition 7.1.** *Let $r > 0$. The $r$-star at $x \in \Sigma$ is the finite point set $B(x, r) \cap \Sigma$.*

Two sets $M_1, M_2 \subset \mathbb{R}^d$ are congruent if there exist $z_1, z_2 \in \mathbb{R}^d$ and an isometry $A \in \mathcal{O}(d, \mathbb{Z})$ such that

$$A(M_1 - z_1) = M_2 - z_2.$$

Bieberbach [5] proved a fundamental statement, Schoenflies [21] had established it for the special case $d = 3$. It confirms the essential role of lattices in crystallography.

**Theorem 7.1 (Bieberbach).** *Let $\Sigma$ be a Delone set in $\mathbb{R}^d$ such that the $r$-stars of all of its points are congruent for every $r > 0$. Then $\Sigma$ is a union of a finite number of congruent lattices.*

It is evident that Delone sets which have only one $r$-star (up to congruence) for every $r > 0$ are periodic patterns and therefore cannot have icosahedral symmetry. Thus a generalized definition of a crystal must allow more $r$-stars. From the physical point of view, it is however reasonable to keep the following two requirements on the Delone set $\Sigma$ modeling a crystal:

- $\Sigma$ has *finite local complexity*, i.e. for every $r > 0$ there is only a finite number of $r$-stars up to congruence in $\Sigma$.
- $\Sigma$ is repetitive, i.e. for all $r > 0$, every $r$-star appears infinitely many times in $\Sigma$.

## 7.2 Meyer sets and cut-and-project sets

A large class of Delone sets with finite local complexity has been proposed by Meyer [15]. A Delone set $\Sigma \subset \mathbb{R}^d$ is said to have the Meyer property if there exists a finite set $F \subset \mathbb{R}^d$ satisfying $\Sigma - \Sigma \subset \Sigma + F$.

**Proposition 7.2.** *If $\Sigma$ is a Meyer set then $\Sigma$ has finite local complexity.*

*Proof.* In order to show that a Meyer set has for all $r > 0$ only finitely many $r$-stars up to congruence, it suffices to show that shifting $r$-stars of all point of $\Sigma$ to the origin, one obtains a finite set. We have

$$\big(B(x,r) \cap \Sigma\big) - x = B(0,r) \cap (\Sigma - x) \subset B(0,r) \cap (\Sigma - \Sigma) \subset$$
$$\subset B(0,r) \cap (\Sigma + F).$$

The number of different shapes of $r$-stars is thus bounded by the number of configurations which can be formed by elements in the set $B(0,r) \cap (\Sigma + F)$, which is finite. □

Note from the proof that the cardinality of the set $B(0,r) \cap (\Sigma + F)$ determining the number of local configurations in $\Sigma$ depends on the cardinality of the set $F$.

A rich class of Meyer sets can be obtained by the so-called *cut-and-project method* [9]. Roughly speaking, one projects points of a higher dimensional lattice to a lower dimensional subspace and then chooses projections which have their projection to the complementary subspace in a given bounded region, one obtains a *cut-and-project set*.

Let us step up to a correct definition. Let $V_1, V_2$ be subspaces of $\mathbb{R}^{c+d}$ such that $V_1 \oplus V_2 = \mathbb{R}^{c+d}$. Let $\pi_1 : \mathbb{R}^{c+d} \to V_1$ be the projector onto $V_1$ along $V_2$ and $\pi_2 : \mathbb{R}^{c+d} \to V_2$ the projector onto $V_2$ along $V_1$. Keeping this notation, we have the following definitions.

**Definition 7.2.** *The cut-and-project scheme is a triplet $(V_1, V_2, L)$, where $\mathcal{L}$ is a lattice in $\mathbb{R}^{c+d}$, which fulfils:*

1. *restriction $\pi_1$ to $\mathcal{L}$ is an injection,*
2. *$\pi_2(\mathcal{L})$ is dense in $V_2$.*

**Definition 7.3.** *Let $(V_1, V_2, \mathcal{L})$ be a cut-and-project scheme and let $\Omega \subset V_2$ such that $\Omega$ is bounded, $\Omega^\circ \neq \emptyset$, and $\overline{\Omega^\circ} = \overline{\Omega}$. Then the set*

$$\Sigma(\Omega) := \{\pi_1(x) \mid x \in \mathcal{L}, \pi_2(x) \in \Omega\}$$

*is called a cut-and-project set with the* **acceptance window** *$\Omega$. The space $V_1$ is usually called* **inner space** *and $V_2$* **physical space**.

Cut-and-project sets are aperiodic Delone sets, and under the additional condition that the boundary of the acceptance window has an empty intersection with $\pi_2(\mathcal{L})$, the set $\Sigma(\Omega)$ is repetitive. It is also not difficult to see that a cut-and-project set satisfies the Meyer property

$$\Sigma(\Omega) - \Sigma(\Omega) \subset \Sigma(\Omega) + F. \tag{7.1}$$

As we have said, to estimate the number of local configurations in $\Sigma(\Omega)$, knowledge of the cardinality of $F$ is important. It is obvious from the definition of $\Sigma(\Omega)$ that the finite set $F$ in the Meyer property (7.1) of $\Sigma(\Omega)$ satisfies $F \subset \pi_1(\mathcal{L})$. For the finite set $G := \pi_2(\pi_1^{-1}(F))$ we have

$$\Omega - \Omega \subset \Omega + G\,. \tag{7.2}$$

Thus the Meyer property of $\Sigma(\Omega)$ implies relation (7.2) for its acceptance window which corresponds to covering of the difference set $\Omega - \Omega$ by translated copies of $\Omega$. The finite set $G$ determines the translation vectors.

The converse is however not that simple. Having a finite set $G \subset V_2$ which satisfies (7.2), it is not always possible to find $F$ of the same cardinality so that (7.1) holds. This comes from the fact that $G$ may not be subset of $\pi_2(\mathcal{L})$. However, this inconvenience can be avoided if instead of (7.2) we study covering of the difference set $\Omega - \Omega$ by copies of the interior $\Omega^\circ$,

$$\Omega - \Omega \subset \Omega^\circ + G\,. \tag{7.3}$$

Having such $G$ and due to the fact that $\pi_2(\mathcal{L})$ is dense in $V_2$, we can clearly find a set $\tilde{G} \subset \pi_2(\mathcal{L})$ of the same cardinality as $G$ and satisfying (7.2). Therefore we may set $F = \pi_1(\pi_2^{-1}(\tilde{G}))$ to obtain (7.1) with $|F| = |G|$. The above considerations justify the study of the minimal covering of the difference set which was in focus of this thesis.

## 7.3 Cut-and-project sets with five-fold symmetry

We have not discussed any symmetries yet, however, the aim of crystallographers is to find a mathematical model of materials which have been observed experimentally, i.e. those with icosahedral symmetry. We have got acquinted with the crystallographic restriction (Proposition 7.1) which does not allow such symmetry for periodic patterns in $\mathbb{R}^3$. There can be stated an analogue of the crystallographic restriction for lattices in $\mathbb{R}^d$. Let $d(k)$ be the least value of $d$ for which an element of order $k$ appears in $\mathcal{O}(d,\mathbb{Z})$. Recall Euler's function, defined for positive integers: $\phi(k)$ is the number of integers $n$ less than $k$ and such that gcd(k,n)=1, where gcd is the greatest common devisor. Now define

$$\Phi(k) = \phi(k) \quad \text{if} \quad k = p^\alpha, \text{where } p \text{ is a prime and } \alpha \in \mathbb{N}\ ,$$

$$\Phi(k) = \Phi(k_1) + \Phi(k_2) \quad \text{if } k = k_1 k_2 \text{ and } \gcd(k_1, k_2)\text{=1}.$$

**Theorem 7.2 ([8]).** $d(k) = \Phi(k)$.

**Corollary 7.2.1.** *Rotations of order 5 first appear in $\mathbb{R}^4$ and the icosahedral symmetry with its rotations of order 2,3, and 5 becomes possible for a lattice only when $d \geq 6$.*

*Proof.* (a) Using Theorem 7.2, we have $d(5) = \Phi(5) = \phi(5) = 4$, therefore rotations of order 5 become possible in $\mathbb{R}^4$. (b) For the proof of the fact that the icosahedral symmetry firstly appears in $\mathbb{R}^6$ see [7]. □

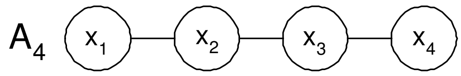
Although one cannot have a periodic pattern in $\mathbb{R}^3$ with icosahedral symmetry, we can use Corollary 7.2.1 to find an aperiodic Delone set with finite local complexity which reveals this symmetry. Cut-and-project scheme allows us to use the symmetry of higher dimensional lattices in lower dimensional spaces. For simplicity we show construction of a cut-and-project set $\Sigma(\Omega)$ in $\mathbb{R}^2$ with five-fold symmetry. However, similar considerations can be undertaken to obtain cut-and-project set $\Sigma(\Omega) \subset \mathbb{R}^3$ with icosahedral symmetry.

### 7.3.1 Construction of a cut-and-project set with five-fold symmetry

The first step is to find a lattice with five-fold rotational symmetry. The generalized crystallographic restriction gives us to understand that this is firstly possible in $\mathbb{R}^4$. Let $\mathcal{L} = \left\{\sum_{i=1}^{4} a_i x_i \;\middle|\; a_i \in \mathbb{Z}\ \forall i \in \{1,2,3,4\}\right\}$ be a lattice generated by vectors with the following Gram matrix

$$M = (x_i, x_j) = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

The maximal group of isometries under which this lattice is invariant is $A_4$ generated by four reflections $r_1, r_2, r_3, r_4$ in $\mathbb{R}^4$, which can be described by the so-called Coxeter graph.




This graph represents the bilateral position of mirrors for reflections $r_i, r_j$. If $i$ and $j$ are not connected by an edge then the mirrors are mutually orthogonal. If the vertices are connected by an edge then the angle between the mirrors is $\frac{\pi}{3}$.

Accordingly, reflections $r_1, r_2, r_3, r_4$ are defined in the following way

$$(\forall x \in \mathbb{R}^4)\left(r_i(x) = x - \frac{2(x, x_i)}{(x_i, x_i)} x_i = x - 2(x, x_i) x_i\right).$$

As the angle between the vectors $x_i, x_j$ for $|i-j| = 1$ is $\frac{2\pi}{3}$, we obtain

$$r_i(x_i) = -x_i,$$

$$r_i(x_{i\pm1}) = x_i + x_{i\pm1},$$

$$r_i(x_j) = x_j \text{ for } |j-i| > 1.$$

Let us show that by a composition of these reflections we obtain the searched rotation of order 5.

**Claim 7.1.** *The map $R = r_1 r_3 r_2 r_4 : \mathbb{R}^4 \to \mathbb{R}^4$ is an isometry of order 5.*

*Proof.* $R$ is a composition of isometries, i.e. it is an isometry, too, and moreover $R(\mathcal{L}) = \mathcal{L}$ because $\mathcal{L}$ is invariant under $r_i$ for all $i \in \{1,2,3,4\}$. Now, let us verify that $(\forall x \in \mathbb{R}^4)(R^5 x = x)$. As $R$ is linear, it suffices to prove that

$$(\forall i \in \{1,2,3,4\})(R^5 x_i = x_i).$$

We show this property for the vector $x_1$, one can continue analogically for the remaining vectors $x_2, x_3, x_4$.

$R(x_1) = r_1 r_3 r_2 (r_4(x_1)) = r_1 r_3 (r_2(x_1)) = r_1(r_3(x_2 + x_1)) = r_1(x_3 + x_2 + x_1) = x_3 + x_1 + x_2 - x_1 =$
$= x_3 + x_2.$

$R^2(x_1) = R(x_3 + x_2) = r_1 r_3 r_2 (r_4(x_3 + x_2)) = r_1 r_3 (r_2(x_4 + x_3 + x_2)) = r_1(r_3(x_4 + x_2 + x_3 - x_2)) =$
$= r_1(r_3(x_4 + x_3)) = r_1(x_4 + x_3 - x_3) = r_1(x_4) = x_4.$

$R^3(x_1) = R(x_4) = r_1 r_3 r_2 (r_4(x_4)) = r_1 r_3 (r_2(-x_4)) = r_1(r_3(-x_4)) = r_1(-x_3 - x_4) = -x_3 - x_4.$

$R^4(x_1) = R(-x_3 - x_4) = r_1r_3r_2(r_4(-x_3 - x_4)) = r_1r_3(r_2(-x_4 - x_3 + x_4)) = r_1r_3(r_2(-x_3)) =$
$= r_1(r_3(-x_2 - x_3)) = r_1(-x_3 - x_2 + x_3) = r_1(-x_2) = -x_1 - x_2.$

$R^5(x_1) = R(-x_1-x_2) = r_1r_3r_2(r_4(-x_1-x_2)) = r_1r_3(r_2(-x_1-x_2)) = r_1(r_3(-x_2-x_1+x_2)) =$
$= r_1(r_3(-x_1)) = r_1(-x_1) = x_1.$

□

As it was said at the beginning, we want to find such a projection $\pi_1$ of $\mathbb{R}^4$ on the 2-dimensional physical space that keeps the five-fold symmetry of $\mathcal{L}$. We have calculated images of the vector $x_1$ by compositions of the map $R$.

$$R: \; x_1 \;\rightarrow\; x_2 + x_3 \;\rightarrow\; x_4 \;\rightarrow\; -x_3 - x_4 \;\rightarrow\; -x_1 - x_2 \;\rightarrow\; x_1,$$

therefore we can formulate our demand on the five-fold symmetry of the projection $\pi_1(\mathcal{L})$ in the following way. The projections of the vectors above should form vertices of a regular pentagon in $\mathbb{R}^2$. Let $u := \pi_1(x_1)$ and $v := \pi_1(x_4)$. We have two possibilities how to choose the angle between $u$ and $v$. Either $\frac{4\pi}{5}$ or $\frac{2\pi}{5}$. Let us choose the first possibility, i.e. the angle between vectors $u$ and $v$ is $\frac{4\pi}{5}$.

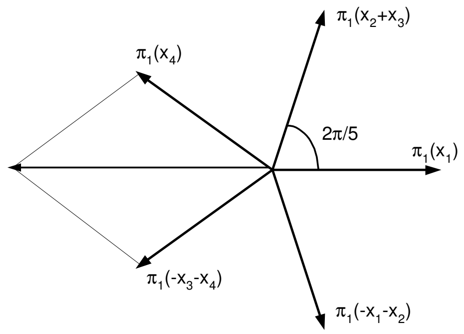



Figure 7.2: Illustration of the projection $\pi_1$.

Considering Figure 7.2, one can determine the projections $\pi_1(x_2)$ and $\pi_1(x_3)$. We have namely

$$\pi_1(x_2) = -\big(\pi_1(x_1) + \pi_1(-x_1 - x_2)\big) = \tau\pi_1(x_4) = \tau v,$$

$$\pi_1(x_3) = -\big(\pi_1(x_4) + \pi_1(-x_3 - x_4)\big) = \tau\pi_1(x_1) = \tau u,$$

$$\text{where } \tau = 2\cos(\tfrac{\pi}{5}).$$

It is useful to enumerate the value of $\tau$. For this purpose let us define a linear map on $\pi_1(\mathcal{L})$

$$R_2 := \pi_1(r_2r_4)\pi_1^{-1}.$$

We consider two facts:

1. $R_2(\tau u) = \pi_1(r_2r_4)\pi_1^{-1}(\tau u) = \pi_1(r_2r_4)(x_3) = \pi_1(x_2 + x_3 + x_4) = \tau v + \tau u + v.$
2. Linearity of $R_2$ implies
$$R_2(\tau u) = \tau R_2(u) = \tau u + \tau^2 v.$$

We obtain

$$\tau v + v + \tau u = \tau u + \tau^2 v.$$

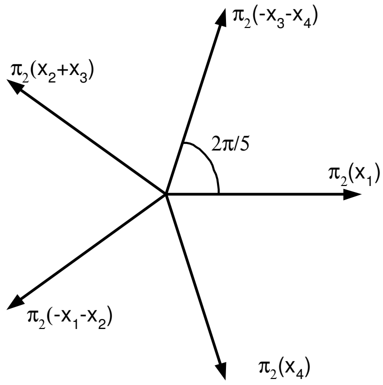



Figure 7.3: Illustration of the projection $\pi_2$.

We can notice that $\tau$ must fulfil the equation $\tau^2 = \tau + 1$ and $\tau = 2\cos(\frac{\pi}{5})$, which is a positive number. As a result we have that $\tau = \frac{1+\sqrt{5}}{2} \doteq 1,618$ (the famous 'golden cut'). Let us denote by $V := \pi_1(\mathcal{L})$. It is possible to describe $V$ as

$$V = \{(a + b\tau)u + (c + d\tau)v \mid a, b, c, d \in \mathbb{Z}\}.$$

Now, we use the second possibility of the angle choice, and we define $u^* := \pi_2(x_1)$ and $v^* := \pi_2(x_4)$, where the angle between $u^*$ and $v^*$ is $\frac{2\pi}{5}$. Considering Figure 7.3, one can determine the projections $\pi_2(x_2), \pi_2(x_3)$.

$$\pi_2(x_2) = \pi_2(x_2+x_3)+\pi_2(-x_3-x_4)+\pi_2(x_4) = -\tau\pi_2(x_4)+\pi_2(x_4) = (1-\tau)\pi_2(x_4) = (1-\tau)v^* = \tau' v^*,$$

$$\pi_2(x_3) = \pi_2(x_2 + x_3) + \pi_2(-x_1 - x_2) + \pi_2(x_1) = -\tau\pi_2(x_1) + \pi_2(x_1) = (1 - \tau)\pi_2(x_1) = \tau' u^*,$$

$$\text{where } \tau' = \frac{1 - \sqrt{5}}{2} \doteq -0,618.$$

Let us denote by $V^* := \pi_2(\mathcal{L})$. It is possible to describe $V^*$ as

$$V^* = \{(a + b\tau')u^* + (c + d\tau')v^* \mid a, b, c, d \in \mathbb{Z}\}.$$

We have obtained the cut-and-project scheme $(V, V^*, \mathcal{L})$.

$$\mathcal{L} = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 + \mathbb{Z}x_4 \subset \mathbb{R}^4,$$

$$V = \pi_1(\mathcal{L}) = (\mathbb{Z} + \mathbb{Z}\tau)u + (\mathbb{Z} + \mathbb{Z}\tau)v,$$

$$V^* = \pi_2(\mathcal{L}) = (\mathbb{Z} + \mathbb{Z}\tau')u^* + (\mathbb{Z} + \mathbb{Z}\tau')v^*.$$

Since $\tau' = 1 - \tau$ and $u^* = u$, $v^* = -u - \tau v$, the sets $V$ and $V^*$ coincide. It is possible to define a bijection $*$ on $V = V^*$ by

$$\big((a + b\tau)u + (c + d\tau)v\big)^* = (a + b\tau')u^* + (c + d\tau')v^*. \tag{7.4}$$

In order to state some properties of the map $*$, let us summarize some facts from the theory of numbers.

**Remark 12.** *$\mathbb{Z}[\tau] = \mathbb{Z} + \mathbb{Z}\tau$ is the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{5})$. The Galois automorphism of this field defines an automorphism on the ring $\mathbb{Z}[\tau]$ by*

$$\alpha = a + b\tau \in \mathbb{Z}[\tau] \quad \to \quad \alpha' = a + b\tau' = a + b - b\tau \in \mathbb{Z}[\tau].$$

*The morphism property of the map implies*

*1.* $(\forall \alpha, \beta \in \mathbb{Z}[\tau])\big((\alpha\beta)' = \alpha'\beta'\big)$,

*2.* $(\forall \alpha, \beta \in \mathbb{Z}[\tau])\big((\alpha+\beta)' = \alpha' + \beta'\big)$.

*The norm $N$ on the field $\mathbb{Q}(\sqrt{5})$ is given by*

$$N(\alpha) = \alpha\alpha' = (a+b\tau)(a+b\tau') = a^2 + ab - b^2 \in \mathbb{Q},$$

*for $\alpha \in \mathbb{Q}(\sqrt{5})$, where we have used $\tau + \tau' = 1$, $\tau\tau' = -1$. The norm satisfies $N(\alpha) \in \mathbb{Z}$ for $\alpha \in \mathbb{Z}[\tau]$ and $N(\alpha) = 0 \Leftrightarrow \alpha = 0$. Note that $N(\tau) = \tau\tau' = -1$ and thus $\tau$ is a unit in $\mathbb{Z}[\tau]$. It follows that $\tau\mathbb{Z}[\tau] = \mathbb{Z}[\tau]$. Due to the uniform distribution of $n\theta \mod 1$ in $[0,1)$ for $\theta$ irrational* [24]*, the ring $\mathbb{Z}[\tau]$ is dense on the real line.*

**Claim 7.2.** *The Galois automorphism is an everywhere discontinuous map.*

*Proof.* We want to verify that

$$(\forall \alpha \in \mathbb{Z}[\tau])(\exists \varepsilon > 0)(\forall \delta > 0)(\exists \beta \in \mathbb{Z}[\tau])(0 < |\alpha - \beta| < \delta \wedge |\tilde{\alpha} - \tilde{\beta}| \geq \varepsilon).$$

Take an arbitrary $\alpha$. Since $\mathbb{Z}[\tau]$ is dense in $\mathbb{R}$, for every $\delta > 0$ there exists $\beta \in \mathbb{Z}[\tau]$ such that $0 < |\alpha - \beta| < \delta$. Denote $\alpha - \beta := a + b\tau \neq 0$, then $\alpha' - \beta' = a + b\tau'$. We have

$$0 \neq N(\alpha - \beta) = (\alpha - \beta)(\alpha' - \beta') \in \mathbb{Z}.$$

We arrive at the conclusion that $|(\alpha - \beta)(\tilde{\alpha} - \tilde{\beta})| \geq 1$, which says

$$|\tilde{\alpha} - \tilde{\beta}| \geq \frac{1}{|\alpha - \beta|} > \frac{1}{\delta}.$$

It confirms that the Galois map is everywhere discontinuous. □

Let us now state some properties of the star map defined in (7.4).

**Claim 7.3.** *1.* $(\forall x, y \in V)\big((x+y)^* = x^* + y^*\big)$,

*2.* $(\forall x \in V)(\forall \alpha \in \mathbb{Z}[\tau])\big((\alpha x)^* = \alpha' x^*\big)$,

*3.* $*$ *is an everywhere discontinuous map.*

*Proof.* 1. and 2. follow trivially using the fact that $'$ is an automorphism on $\mathbb{Z}[\tau]$. Property 3. is an obvious consequence of the discontinuity of the Galois map. Let us prove the discontinuity of the star map in 0 at first, i.e.

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in V)(0 < |x| < \delta \wedge |x^*| \geq \varepsilon).$$

Since $\mathbb{Z}[\tau]$ is dense in $\mathbb{R}$, we can find for every $\delta > 0$ an $\alpha \in \mathbb{Z}[\tau]$ so that $x = \alpha u$ and $0 < |x| = |\alpha||u| < \delta$.

At the same time

$$|x^*| = |\alpha' u^*| \geq \frac{1}{|\alpha|}|u^*| > \frac{1}{\delta}|u^*|,$$

where we have again used $|N(\alpha)| = |\alpha\alpha'| \geq 1$ for $\alpha \neq 0$. It confirms that the star map is discontinuous in 0. It is easy to generalise this result. If we take an arbitrary vector $x \in V$ then for every $\delta > 0$ there exists a vector $y \in V$ such that $|x - y| < \delta$. We denote $z := x - y$, and so we transform the problem of the discontinuity everywhere again to the discontinuity in 0. □

At this moment we can finally define the cut-and-project set for an acceptance window $\Omega$ by

$$\Sigma(\Omega) = \{(a+b\tau)u + (c+d\tau)v \mid a,b,c,d \in \mathbb{Z}, (a+b\tau')u^* + (c+d\tau')v^* \in \Omega\} = \{x \in V \mid x^* \in \Omega\},$$

where $\Omega$ is a convex compact set in $\mathbb{R}^2$ with non-empty interior.

Let us remind the construction of $V$, we demanded $V$ to have five-fold symmetry. Now, we can realize that it has even ten-fold symmetry. How does the symmetry of $V$ influences the symmetry of the cut-and-project set $\Sigma(\Omega)$? The answer can be found in the following theorem [17].

**Theorem 7.3.** *Let $\Omega$ be a convex compact set in $\mathbb{R}^2$ with non-empty interior. By the above defined projections $\pi_1, \pi_2$, $\Omega$ has ten-fold symmetry if and only if $\Sigma(\Omega)$ has ten-fold symmetry.*

### 7.3.2 Estimate of the number of different Voronoi tiles

Let us show a possible application of investigated Meyer numbers. For simplicity we consider an acceptance window without ten-fold symmetry. Let $I$ be an interval in $\mathbb{R}$ and let $\Omega$ be a convex compact set in $\mathbb{R}^2$ with non-empty interior such that

$$\Omega = Iu^* + Iv^* = \{au^* + bv^* \mid a, b \in I\}.$$

Our aim is to estimate the number of different tiles in the perfect Voronoi tiling in $\Sigma(\Omega)$. We recall that the number of $r$-stars (up to congruence) is bounded by the number of subsets of $(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0, r)$. Since a Voronoi tile centered at $x$ is influenced only by points in $\Sigma(\Omega) \cap B(x, 2R_{\Sigma(\Omega)})$, we obtain that the number of possibly different Voronoi tiles is less or equal to the number of subsets of $(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0, 2R_{\Sigma(\Omega)})$. Let us make use of the Meyer property

$$\#(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0, 2R_{\Sigma(\Omega)}) \leq \#(\Sigma(\Omega) + F) \cap B(0, 2R_{\Sigma(\Omega)}) \leq \#F \frac{\text{vol } B(0, 2R_{\Sigma(\Omega)})}{\text{vol } B(0, r_{\Sigma(\Omega)})}.$$

One can note that we need to know three constants to be able to use this estimate:

1. the cardinality of the set $F$,
2. the covering radius of $R_{\Sigma(\Omega)}$,
3. the packing radius of $r_{\Sigma(\Omega)}$.

The cardinality of $F$ corresponds to the value of the function $f$ defined by

$$f(\Omega) = \text{the minimal number of translated copies of } \Omega^\circ \text{ needed for covering of } \overline{\Omega - \Omega}.$$

We recall Section 5.1 about polygons, where we have proved that for $\Omega$ being regular quadrangles, $\#F \leq 9$.

To determine the covering radius $R_{\Sigma(\Omega)}$ and the packing radius $r_{\Sigma(\Omega)}$, we need to introduce some helpful facts. Let us describe $\Sigma(\Omega)$ in the case when $\Omega$ is a rhombus.

$$\Sigma(\Omega) = \{a_1u + a_2v \mid a_1, a_2 \in \mathbb{Z}[\tau], (a_1u + a_2v)^* \in \Omega\} =$$

$$= \{a_1u + a_2v \mid a_1, a_2 \in \mathbb{Z}[\tau], a_1' \in I, a_2' \in I\} =$$

$$= \{a_1 \in \mathbb{Z}[\tau] \mid a_1' \in I\}u + \{a_2 \in \mathbb{Z}[\tau] \mid a_2' \in I\}v =$$

$$= \Sigma(I)u + \Sigma(I)v.$$

Such $\Sigma(\Omega)$ is sometimes called a quasilattice, which corresponds to Figure 7.4. Note that such $\Sigma(\Omega)$ is a cartesian product of two one-dimensional cut-and-project sets $\Sigma(I)$.

The following claim asserts that without loss of generality it suffices to deal with intervals of length $1 \leq |I| < \tau$.

**Claim 7.4.** *For the previously defined cut-and-project set $\Sigma(\Omega)$ it holds $\tau\Sigma(\Omega) = \Sigma(\tau'\Omega)$. Moreover, for every $k \in \mathbb{Z}$*

$$\tau^k \Sigma(\Omega) = \Sigma(\tau'^k \Omega).$$

*Proof.* We shall use $\tau\mathbb{Z}[\tau] = \mathbb{Z}[\tau]$.

$$\tau\Sigma(\Omega) = \{\tau\gamma u + \tau\delta v \mid \gamma, \delta \in \mathbb{Z}[\tau], \gamma' u^* + \delta' v^* \in \Omega\} =$$

$$\{\alpha u + \beta v \mid \alpha, \beta \in \mathbb{Z}[\tau], \alpha' u^* + \beta' v^* \in \tau'\Omega\}.$$

The second part of the claim follows easily using mathematical induction. ∎

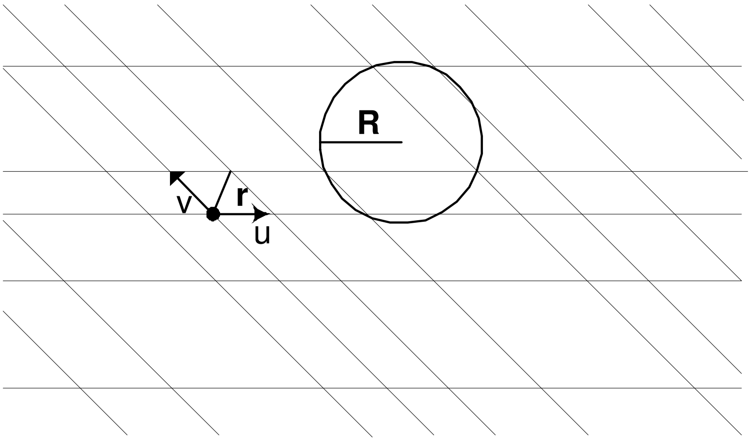



Figure 7.4: Illustration of the quasilattice $\Sigma(\Omega)$.

We apply the previous claim to obtain

$$\tau\Sigma(I) = \Sigma(\tau' I),$$

which confirms that it is enough to consider intervals of length $1 \le |I| < \tau$ because if $\tau^k \le |J| < \tau^{k+1}$ then we have $\Sigma(J) = \frac{1}{\tau^k}\Sigma(I)$, where $I$ is the interval with the corresponding length $1 \le |I| < \tau$.

The following theorem describes the structure of one-dimensional cut-and-project sets $\Sigma(I)$, namely, it determines the distances between neighbouring points. The theorem will be useful further on for determining the values of $R_{\Sigma(\Omega)}$ and $r_{\Sigma(\Omega)}$.

**Theorem 7.4.** *Let $1 \le d < \tau$. If we arrange the elements of the set $\Sigma[c, c+d) = \{a + b\tau \mid a, b \in \mathbb{Z}, c \le a + b\tau' < c + d\}$ in such a way that they form an increasing sequence $(x_n)_{-\infty}^{\infty}$, i.e.*

$$\Sigma[c, c+d) = \{x_n \mid n \in \mathbb{Z}\},$$

*then it holds*

$$x_{n+1} - x_n \in \{1, \tau, \tau^2\}.$$

*Moreover, if $d = 1$ then*

$$x_{n+1} - x_n \in \{\tau, \tau^2\}.$$

*Proof.* By definition, the point $a + b\tau$ belongs to $\Sigma[c, c+d)$ if and only if

$$c \le a + b\tau' = a - \frac{b}{\tau} < c + d,$$

which implies

$$c + \frac{b}{\tau} \le a < c + d + \frac{b}{\tau}. \tag{7.5}$$

As $1 \le d < \tau < 2$, there exist only one or two integers $a$ for every $b \in \mathbb{Z}$, namely $a = \lceil c + \frac{b}{\tau} \rceil$ and eventually $a = \lceil c + \frac{b}{\tau} \rceil + 1$, i.e.

$$x_b = a + b\tau = \lceil c + \frac{b}{\tau} \rceil + b\tau,$$

eventually

$$x_b = \lceil c + \frac{b}{\tau} \rceil + 1 + b\tau.$$

We divide the proof into two steps.

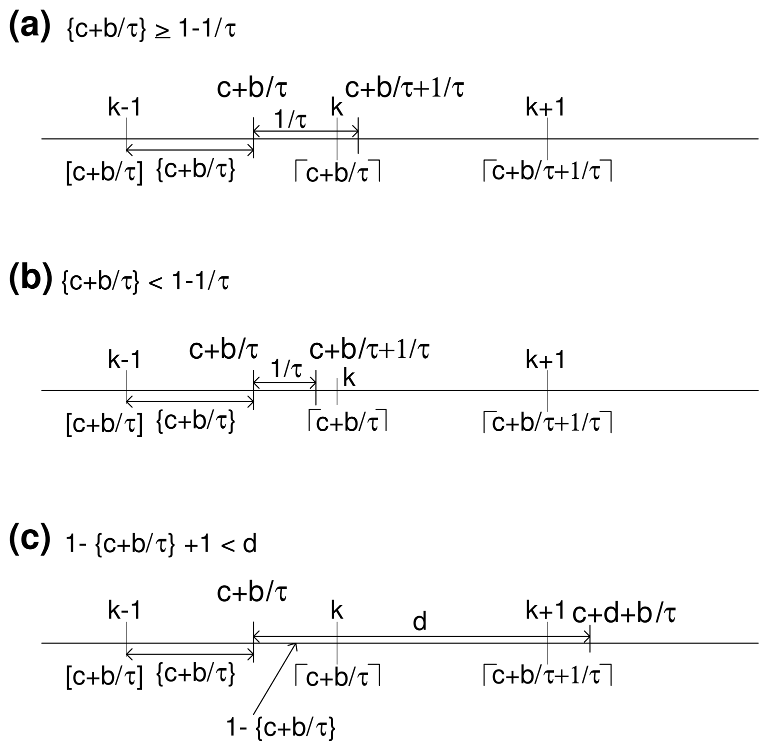



Figure 7.5: Illustration of situations in the proof of Theorem 7.4.

1. Let $d = 1$, then for every $b \in \mathbb{Z}$ there exists precisely one integer $a$ satisfying (7.5). We obtain
$$x_{b+1} - x_b = \lceil c + \tfrac{b+1}{\tau} \rceil + (b+1)\tau - \left( \lceil c + \tfrac{b}{\tau} \rceil + b\tau \right) = \tau + \lceil c + \tfrac{b+1}{\tau} \rceil - \lceil c + \tfrac{b}{\tau} \rceil .$$
Due to the fact that $0 < \frac{1}{\tau} < 1$ there are two cases which can happen. Either there exists an integer $k$ such that $c + \frac{b}{\tau} < k < c + \frac{b}{\tau} + \frac{1}{\tau}$, then $x_{b+1} - x_b = \tau + 1 = \tau^2$. This case happens if $\{c + \frac{b}{\tau}\} \geq 1 - \frac{1}{\tau}$. See Figure 7.5 (a). Otherwise $\{c + \frac{b}{\tau}\} < 1 - \frac{1}{\tau}$, then $\{c + \frac{b}{\tau}\} + \frac{1}{\tau} < 1$, which implies
$$\lceil c + \tfrac{b+1}{\tau} \rceil - \lceil c + \tfrac{b}{\tau} \rceil = 0.$$
The gap between $x_{b+1}$ and $x_b$ has length $\tau$, i.e. $x_{b+1} - x_b = \tau$. See Figure 7.5 (b).

2. Let $\tau > d > 1$. So far we have investigated that the gaps in the sequence $\{x_b = \lceil c + \frac{b}{\tau} \rceil + b\tau \mid b \in \mathbb{Z}\}$ reaches two lengths $\tau$ and $\tau^2$.

   Where can we find the remaining points of $\Sigma[c, c+d)$? Firstly, we state for which $b \in \mathbb{Z}$
$$c + \tfrac{b}{\tau} \leq \lceil c + \tfrac{b}{\tau} \rceil + 1 < c + d + \tfrac{b}{\tau}.$$
Considering Figure 7.4 (c), we have $\{c + \frac{b}{\tau}\} > 1 - (d-1) = 2 - d$. If $2 - d > 1 - \frac{1}{\tau}$ then we find the remaining point $\lceil c + \frac{b}{\tau} \rceil + 1 + b\tau$ in the gap of length $\tau^2$. It is equivalent with the inequality
$$\tau = 1 + \tfrac{1}{\tau} > d,$$
which is fulfiled for $d < \tau$. If there are two integers $a$ for one integer $b \in \mathbb{Z}$ then their distance is 1, and the point $x_b + 1$ divides the gap $\tau^2 = \tau + 1$ into two gaps of lengths 1 and $\tau$. The conclusion sounds that there are three lengths between increasingly arranged points of $\Sigma[c, c+d)$ for $d > 1$, namely $1, \tau, \tau^2$.

□

Now, we can meet the previous promise and determine the covering radius and the packing radius in $\Sigma(\Omega)$. It suffices to consider Figure 7.4, and one can note that the covering radius is radius of a circle circumscribed to a rhombus of side-length $\tau^2$. The value of the covering radius is $R_{\Sigma(\Omega)} = \frac{\tau^2}{\sqrt{\tau+2}}$. The packing radius can be calculated easily. It is one half of the length of a diagonal of the rombus having side-length 1. It is that one of two diagonals which halves the angle $\frac{4\pi}{5}$. Using for instance the cosinus theorem, we obtain

$$r_{\Sigma(\Omega)} = \frac{1}{2}\sqrt{2 - 2\cos(\tfrac{\pi}{5})} = \frac{1}{2}\sqrt{2-\tau} = \frac{1}{2}\sqrt{\tfrac{1}{\tau^2}} = \frac{1}{2\tau}.$$

The searched estimate on the number of different Voronoi tiles in $\Sigma(\Omega)$, where $\Omega$ is a rhombus, is the number of subsets of $(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0, 2R_{\Sigma(\Omega)})$, where

$$\#(\Sigma(\Omega) - \Sigma(\Omega)) \cap B(0, 2R_{\Sigma(\Omega)}) \leq \#F \frac{\text{vol } B(0, 2R_{\Sigma(\Omega)})}{\text{vol } B(0, r_{\Sigma(\Omega)})} =$$

$$= 9\frac{\pi 4 {R_{\Sigma(\Omega)}}^2}{\pi {r_{\Sigma(\Omega)}}^2} = 36\frac{\tau^4}{\tau+2}.\frac{1}{\frac{1}{4\tau^2}} = 144\frac{\tau^6}{\tau+2} \doteq 714.$$

# Chapter 8

# Conclusion

This thesis essentially contributes to the solution of the problem of minimal covering of the difference set $\Omega - \Omega$ by translated copies of $\Omega^\circ$ for a convex compact set $\Omega \subset \mathbb{R}^d$. Let us reformulate this problem using notation established in this work. The main interest of this thesis has been devoted to investigation of properties and values of the function $f$ defined by

$$f(\Omega) = \text{ the minimal number of translated copies of } \Omega^\circ \text{ needed for covering of } \Omega - \Omega.$$

$f(\Omega)$ is called the Meyer number of $\Omega$. The significant result states that, for any dimension $d$, the function $f$ is bounded above. To obtain this result, it was important to investigate topological properties of the domain of the function $f$, i.e. of the space $\kappa_d$ of all convex compact subsets of $\mathbb{R}^d$ with non-empty interior. The space $\kappa_d$ equipped with the well-known Hausdorff metric forms a metric space. We have shown that the space $\mathcal{M}$ of all convex compact sets $\Omega$, satisfying $\overline{B(0,\frac{1}{d})} \subset \Omega \subset \overline{B(0,1)}$, is a compact subspace of $\kappa_d$, and $f$ is upper semicontinuous on the space $\mathcal{M}$. Consequently, $f$ reaches its maximum on the space $\mathcal{M}$. To extend this result to the whole space $\kappa_d$, we made profit of John's theorem.

On top of it, a corollary of John's theorem provides us with a tool for estimation of the minimal upper bound $K_d$ on the set of all Meyer numbers. It says that $K_d$ is less or equal to the number of translated copies of the open unit ball needed to cover the closed ball of radius $2d$. This is in general a difficult problem, therefore we have limited our considerations to lower dimensions.

For dimension $d = 2$, dual formulation of the problem can be found in mathematical literature under the name *disk covering problem*. Given a closed unit disk, one search for the smallest radius $\rho(n)$ required for $n$ equal closed disks to completely cover the closed unit disk. Unfortunately, the disk covering problem is solved only for $n \leq 10$ and consequently is not utilizable for estimation of $K_2$. On the contrary, we contribute to the solution of the disk covering problem by the following results: $\rho(16) \leq \frac{1}{2\sqrt{2}}$ and $\rho(26) \leq \frac{1}{4}$. To estimate $K_2$, we used geometrical considerations and have shown

$$K_2 \leq 26.$$

We can refine this result if we limit our considerations to centrally symmetric convex compact sets $\Omega \subset \mathbb{R}^2$, for which we have derived

$$f(\Omega) \leq 16.$$

It is, however, apparent that these bounds are not reached. In order to find better estimates, we have determined the value of the function $f$ for some special types of convex sets in $\mathbb{R}^2$, namely an ellipse, for which $f(\Omega) = 8$, and regular polygons. These results lead us to conjecture that

$$8 \leq f(\Omega) \leq 12$$

for any convex compact set $\Omega \subset \mathbb{R}^2$. For centrally symmetric convex compact sets the conjecture is even more interesting, namely that $f(\Omega) \in \{8, 9\}$.

For dimension $d = 3$, the situation gets even more complicated. We made use of results known about the famous ball-packing and ball-covering problem. Namely, we situated the centres of open unit balls to points of the body-centered cubic lattice, which provides the most efficient covering of the space, so that they covered the closed ball of radius 6, and by estimation of this number we obtained $K_3 \leq 531$.

Further on, we answered a natural question: Is convexity of the set $\Omega$ essential for boundedness of the function $f$? The answer is positive. We considered star-shaped sets, which can be viewed as the nearest generalisation of convex sets. In spite of this fact, we were able to construct a sequence of star-shaped sets $(\Omega_n)_{n=1}^{\infty}$ such that $f(\Omega_n)$ tends to infinity with growing $n$.

In the end, we described the connection of the problem of minimal covering and cut-and-project sets, a class of mathematical models for quasicrystals fulfilling the so-called Meyer property, which ensures the finite local complexity. We applied our theory on a concrete example, we constructed a cut-and-project set with five-fold symmetry and estimated the number of Voronoi tiles for the acceptance window being a rhombus.

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