Asymptotic behavior of beta-integers

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Abstract

Beta-integers ("β-integers") are those numbers which are the counterparts of integers when real numbers are expressed in irrational basis $\beta > 1$. In quasicrystalline studies β-integers supersede the "crystallographic" ordinary integers. When the number β is a Parry number, the corresponding β-integers realize only a finite number of distances between consecutive elements and somewhat appear like ordinary integers, mainly in an asymptotic sense. In this letter we make precise this asymptotic behavior by proving four theorems concerning Parry β-integers.

1   Introduction: β-integers versus integers in quasicrystals

Aperiodicity of quasicrystals [1] implies the absence of space-group symmetry to which integers are inherent [2]. On the other hand, experimentally observed quasicrystals show self-similarity (*e.g.* in their diffraction pattern). The involved self-similarity factors are quadratic Pisot-Vijayaraghavan (PV) units. We recall that an algebraic integer $\beta > 1$ is called a PV number if all its Galois conjugates have modulus less than one. Mostly observed factors are the *golden mean* $\tau = \frac{1+\sqrt{5}}{2}$ and its square $\tau^2 = \frac{3+\sqrt{5}}{2}$, associated to decagonal and icosahedral quasicrystals respectively. The other ones are $\delta = 1 + \sqrt{2}$ (octagonal symmetry) and $\theta = 2 + \sqrt{3}$ (dodecagonal symmetry). Each of these numbers, here generically denoted by β, determines a discrete set of real numbers, $\mathbb{Z}_\beta = \{b_n \mid n \in \mathbb{Z}\}$ [3]. The set of β-integers b_n aims to play the role that integers play in crystallography. In the same spirit, β-lattices Γ are based on β-integers, like lattices are based on integers: $\Gamma = \sum_{i=1}^{d} \mathbb{Z}_\beta \mathbf{e}_i$, with $(\mathbf{e}_i)$ a base of $\mathbb{R}^d$. These point sets in $\mathbb{R}^d$ are eligible frames in which one could think of the properties of quasiperiodic point-sets and tilings, thus generalizing the notion of lattice

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in periodic cases [4]. As a matter of fact, it has become like a paradigm that geometrical supports of quasi-crystalline structures should be Delaunay sets obtained through "Cut-and-Projection" from higher-dimensional lattices. Now, it can be proved [5] that most of such Cut and Project sets are subsets of suitably rescaled β-lattices.

β-integers and their β-lattice extensions have interesting arithmetic and diffractive properties which generalize to some extent the additive, multiplicative and diffractive properties of integers [4, 6, 7]. Throughout these features, a question arises to which the content of this paper will partially answer. In view of the "quasi"-periodic distribution of the b_n's on the real line, it is natural to investigate the extent to which they differ from ordinary integers, according to the nature of β. If β is endowed with specific algebraic properties, one will see that the corresponding set of β-integers is a mathematically "well-controlled" perturbation of $\mathbb{Z}$ in an asymptotic sense

$$b_n \underset{n \to \infty}{\approx} c_\beta (n + \alpha(n)) \tag{1}$$

where $n \mapsto \alpha(n)$ is a bounded sequence and the scaling coefficient c_β is in $\mathbb{Q}(\beta)$. As a matter of fact, for a large family of numbers β, namely the *Parry* family, β-lattices asymptotically appear as lattices. Parry numbers give rise to β-integers which realize only a finite number of distances between consecutive elements and so appear as the most comparable to ordinary integers.

We give in the present letter a set of rigorous results concerning the asymptotic behavior of β-integers when β is a generic Parry number. More precisely, we will illustrate the similarity between sets $\mathbb{N}$ and $\mathbb{Z}_\beta^+ = \{b_n \mid n \in \mathbb{N}\}$ for β being a Parry number by proving two properties:

1. We will show that $c_\beta = \lim_{n \to \infty} \dfrac{b_n}{n}$ exists and we will provide a simple formula for c_β.

2. For β being moreover a PV number with mutually distinct roots of its Parry polynomial, we will prove that $(b_n - c_\beta n)_{n \in \mathbb{N}}$ is a bounded sequence.

Let us mention that both of the previous asymptotic characteristics are known for β being a quadratic unit. The following proposition providing explicit formulae for β-integers for a quadratic unit β comes from [7].

Proposition 1.1. *If β is a simple quadratic Parry unit, then*

$$\mathbb{Z}_\beta^+ = \left\{ b_n = c_\beta n + \frac{1}{\beta} \frac{1-\beta}{1+\beta} + \frac{\beta-1}{\beta} \left\{ \frac{n+1}{1+\beta} \right\}, \ n \in \mathbb{N} \right\}, \quad \text{where } c_\beta = \frac{1+\beta^2}{\beta(1+\beta)}.$$

If β is a non-simple quadratic Parry unit, then

$$\mathbb{Z}_\beta^+ = \left\{ b_n = c_\beta n + \frac{1}{\beta} \left\{ \frac{n}{\beta} \right\}, \ n \in \mathbb{N} \right\}, \quad \text{where } c_\beta = 1 - \frac{1}{\beta^2},$$

where $\{x\}$ designates the fractional part of a nonnegative real number x.

In Section 2, we provide a brief but self-contained description of the concept of β-integers, with emphasis on β-integers realizing only a finite number of distances between neighbors, i.e., β-integers associated with Parry numbers β. The distances are then coded by letters, and, thanks to the self-similarity of $\mathbb{Z}_\beta$, the obtained infinite words u_β are known to be fixed points of substitutions. As the associated substitution matrices are primitive, the Perron-Frobenius theorem together with some further applications from matrix theory will lead to the results in Section 3 on the asymptotic behavior of β-integers for Parry numbers β.

2 All we need to know on β-integers

2.1 β-representation and β-expansion

Let $\beta > 1$ be a real number and let x be a non-negative real number. Any convergent series of the form $x = \sum_{i=-\infty}^{k} x_i \beta^i \overset{\text{not}}{=} x_k x_{k-1} \cdots x_0 \bullet x_{-1} \cdots$, where $x_i \in \mathbb{N} \overset{\text{def}}{=} \{0, 1, 2, \dots\}$ and $x_k \neq 0$, is called a β-*representation* of x. Let β be a positive integer, then if we admit only $\{0, 1, \dots, \beta - 1\}$ as the set of coefficients and if we avoid the suffix $(\beta - 1)^\omega$, where ω signifies an infinite repetition, there exists a unique β-representation for every x, called the *standard* β-representation. For $\beta = 10$ (resp. $\beta = 2$), it is the usual *decimal* (resp. *binary*) representation.

Even if β is not an integer, every positive number x has at least one β-representation. This representation can be obtained by the following *greedy algorithm:*

1. Find $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$ and put $x_k := \left\lfloor \dfrac{x}{\beta^k} \right\rfloor$ and $r_k := \left\{ \dfrac{x}{\beta^k} \right\}$, where $\lfloor x \rfloor = x - \{x\}$ denotes the lower integer part of x.

2. For $i < k$, put $x_i := \lfloor \beta r_{i+1} \rfloor$ and $r_i := \{\beta r_{i+1}\}$.

This representation is called the β-*expansion* of x and the coefficients of the β-expansion clearly satisfy: $x_k \in \{1, \dots, \lfloor \beta \rfloor\}$ and $x_i \in \{0, \dots, \lfloor \beta \rfloor\}$ for all $i < k$. We use the notation $\langle x \rangle_\beta$ for the β-expansion of x. For β being an integer, the β-expansion coincides with the standard β-representation.

The greedy algorithm implies that among β-representations, the β-expansion is the largest according to the radix order. Actually the latter corresponds to the ordering of real numbers: for all $x, y \in [0, +\infty)$, the inequality $x < y$ holds if and only if $\langle x \rangle_\beta$ is smaller than $\langle y \rangle_\beta$ according to the radix order.

Example 2.1. *Let $\beta = \tau = \frac{1+\sqrt{5}}{2}$. The golden mean τ is the larger root of the polynomial $x^2 - x - 1$. Applying the greedy algorithm, we get for instance the following τ-expansions:*
$\langle \frac{\sqrt{5}-1}{2} \rangle_\tau = 0 \bullet 1$, $\langle \frac{3+\sqrt{5}}{2} \rangle_\tau = 100\bullet$, $\langle \frac{5+3\sqrt{5}}{10} \rangle_\tau = 1 \bullet (0001)^\omega \cdots$.

2.2 Rényi expansion of unity

The β-expansion of numbers from the interval $[0,1)$ can be obtained through the map $T_\beta : [0,1] \rightarrow [0,1)$ defined by

$$T_\beta(x) = \{\beta x\}. \tag{2}$$

It is easy to verify that for every $x \in [0,1)$, it holds $\langle x \rangle_\beta = 0 \bullet x_{-1}x_{-2}\cdots$ if and only if $x_{-i} = \lfloor \beta T_\beta^{-(i+1)}(x) \rfloor$. By extending Formula (2) to $x = 1$, one gets the so-called Rényi expansion of unity [8] in base β:

$$d_\beta(1) = t_1 t_2 t_3 \cdots, \quad \text{where} \quad t_i := \lfloor \beta T_\beta^{i-1}(1) \rfloor. \tag{3}$$

Every number $\beta > 1$ is characterized by its Rényi expansion of unity. Note that $t_1 = \lfloor \beta \rfloor \geq 1$. Parry in [9] has moreover shown that the Rényi expansion of unity enables us to decide whether a given β-representation of x is its β-expansion or not. For this purpose, we define the *infinite Rényi expansion of unity* (it is the largest infinite β-representation of 1 with respect to the radix order)

$$d_\beta^*(1) = \begin{cases} d_\beta(1) & \text{if} \quad d_\beta(1) \quad \text{is infinite,} \\ (t_1 t_2 \cdots t_{m-1}(t_m - 1))^\omega & \text{if} \quad d_\beta(1) = t_1 \cdots t_m \quad \text{with} \quad t_m \neq 0. \end{cases} \tag{4}$$

Proposition 2.2 (Parry condition). *Let $d_\beta^*(1)$ be the infinite Rényi expansion of unity in base β. Let $\sum_{i=-\infty}^k x_i \beta^i$ be a β-representation of a non-negative number x. Then $\sum_{i=-\infty}^k x_i \beta^i$ is the β-expansion of x if and only if*

$$x_i x_{i-1} \cdots \prec d_\beta^*(1) \quad \text{for all} \quad i \leq k, \tag{5}$$

where $\prec$ means smaller in the lexicographical order.

Example 2.3. *For $\beta = \tau = \frac{1+\sqrt{5}}{2}$, the Rényi expansion of unity is $d_\tau(1) = 11$. Then, $d_\tau^*(1) = (10)^\omega$, and, according to the Parry condition, any sequence of coefficients in $\{0,1\}$ which does not end with $(10)^\omega$ and which does not contain the block 11 is the τ-expansion of a non-negative real number.*

2.3 Parry numbers

This paper is concerned by real numbers $\beta > 1$ having an eventually periodic Rényi expansion of unity $d_\beta(1)$, i.e., $d_\beta(1) = t_1 \ldots t_m(t_{m+1} \ldots t_{m+p})^\omega$ for some $m, p \in \mathbb{N}$, called *Parry numbers*. For every Parry number β, it is easy to recover, from the eventual periodicity of $d_\beta(1)$, a monic polynomial with integer coefficients having β as a root, i.e., β is an algebraic integer. However, this so-called *Parry polynomial* is not necessarily the minimal polynomial of β. It was proven in [10] that every Pisot number is a Parry number.

2.4 Definition and properties of β-integers

Nonnegative numbers x with vanishing β-fractional part are called *nonnegative β-integers*, formally, $\mathbb{Z}_\beta^+ := \{x \geq 0 \mid \langle x \rangle_\beta = x_k x_{k-1} \cdots x_0 \bullet\}$. The set of *$\beta$-integers* is then defined by $\mathbb{Z}_\beta :=$

$\left(-\mathbb{Z}_\beta^+\right) \cup \mathbb{Z}_\beta^+$. As already mentioned, the radix order on β-expansions corresponds to the natural order of non-negative real numbers. Consequently, there exists a strictly increasing sequence $(b_n)_{n=0}^\infty$ such that

$$b_0 = 0 \quad \text{and} \quad \{b_n \mid n \in \mathbb{N}\} = \mathbb{Z}_\beta^+. \tag{6}$$

When β is an integer, $\mathbb{Z}_\beta = \mathbb{Z}$ and so the distance between the neighboring elements of $\mathbb{Z}_\beta$ is always 1. The situation changes dramatically if $\beta \notin \mathbb{N}$. In this case, the number of different distances between the neighboring elements of $\mathbb{Z}_\beta$ is at least 2 and the set $\mathbb{Z}_\beta$ keeps only partially the resemblance to $\mathbb{Z}$:

1. $\mathbb{Z}_\beta$ has no accumulation points.

2. $\mathbb{Z}_\beta$ is relatively dense, i.e., the distances between consecutive elements of $\mathbb{Z}_\beta$ are bounded.

3. $\mathbb{Z}_\beta$ is self-similar, thus $\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta$.

4. $\mathbb{Z}_\beta$ is not invariant under translation.

5. $\mathbb{Z}_\beta$ forms a Meyer set if β is a Pisot number, i.e., $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ for a finite set $F \subset \mathbb{R}$ (proved in [3]).

Thurston [11] has shown that distances occurring between neighbors of $\mathbb{Z}_\beta$ form the set $\{\Delta_k \mid k \in \mathbb{N}\}$, where

$$\Delta_k := \sum_{i=1}^\infty \frac{t_{i+k}}{\beta^i} \quad \text{for } k \in \mathbb{N}. \tag{7}$$

It is evident that the set $\{\Delta_k \mid k \in \mathbb{N}\}$ is finite if and only if $d_\beta(1)$ is eventually periodic.

2.5 Infinite words and substitutions associated with β-integers

If the number of distances between neighbors in $\mathbb{Z}_\beta$ is finite, one can associate with every distance a letter. Thus, we obtain an infinite word u_β coding $\mathbb{Z}_\beta^+$ as illustrated in Figure 1. In order to define
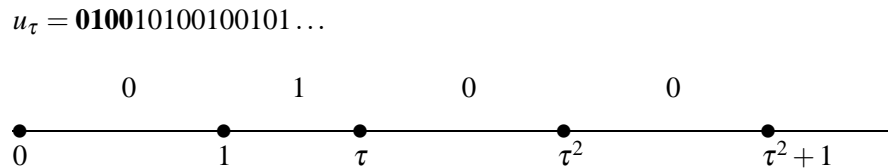
$u_\tau = \mathbf{0100}10100100101\dots$

Figure 1: First elements of $\mathbb{Z}_\tau^+$ (non-negative τ-integers) and of the associated infinite word u_τ.

u_β properly, let us introduce some definitions from combinatorics on words. An *alphabet* $\mathscr{A}$ is a finite set of symbols, called *letters*. A concatenation of letters is a *word*. The set $\mathscr{A}^*$ of all finite words (including the empty word ε) provided with the operation of concatenation is a free monoid. The length of a word $w = w_0 w_1 w_2 \cdots w_{n-1}$ is denoted by $|w| = n$ and $|w|_a$ denotes the number of

letters $a \in \mathscr{A}$ in w. We will deal also with infinite words $u = u_0 u_1 u_2 \cdots \in \mathscr{A}^{\mathbb{N}}$. A finite word w is called a *factor* of the word u (finite or infinite) if there exist a finite word $w^{(1)}$ and a word $w^{(2)}$ (finite or infinite) such that $u = w^{(1)} w w^{(2)}$. The word w is a *prefix* of u if $w^{(1)} = \varepsilon$. A *substitution* over $\mathscr{A}^*$ is a morphism $\varphi : \mathscr{A}^* \to \mathscr{A}^*$ such that there exists a letter $a \in \mathscr{A}$ and a non-empty word $w \in \mathscr{A}^*$ satisfying $\varphi(a) = aw$ and φ is *non-erasing*, i.e., $\varphi(b) \neq \varepsilon$ for all $b \in \mathscr{A}$. Since any morphism satisfies $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in \mathscr{A}^*$, the substitution is uniquely determined by images of letters. Instead of classical $\varphi(a) = w$, we sometimes write $a \to w$. A substitution can be naturally extended to infinite words $u \in \mathscr{A}^{\mathbb{N}}$ by the prescription $\varphi(u) = \varphi(u_0)\varphi(u_1)\varphi(u_2)\ldots$ An infinite word u is said to be a *fixed point* of the substitution φ if it fulfills $u = \varphi(u)$. It is obvious that a substitution φ has at least one fixed point, namely $\lim_{n\to\infty} \varphi^n(a)$. Morphisms on $\mathscr{A}^*$ equipped with the operation of composition form a monoid. Substitutions do not form a monoid as the identity is not a substitution. Nevertheless, if φ is a substitution, φ^n is also a substitution.

The following prescription associates with every substitution φ defined over the alphabet $\mathscr{A} = \{a_1, a_2, \ldots, a_d\}$ a non-negative integer $d \times d$ matrix, called the *substitution matrix M_φ*

$$\left(M_\varphi\right)_{ij} = |\varphi(a_i)|_{a_j}, \quad i, j \in \{1, \ldots, d\}. \tag{8}$$

As an immediate consequence of the definition, it holds for any word w that

$$(|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_d}) M_\varphi = (|\varphi(w)|_{a_1}, |\varphi(w)|_{a_2}, \ldots, |\varphi(w)|_{a_d}). \tag{9}$$

The substitution matrix of the composition of substitutions φ, ψ obeys the formula $M_{\varphi \circ \psi} = M_\psi M_\varphi$.

Example 2.4. *The Fibonacci substitution φ defined on $\{0, 1\}$ by $\varphi(0) = 01$, $\varphi(1) = 0$, has the following substitution matrix*

$$M_\varphi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us finally recall a notion from matrix theory which proves useful in the study of substitutions. A matrix M is called *primitive* if there exists $k \in \mathbb{N}$ such that all entries of M^k are positive. There exists a powerful theorem treating primitive matrices.

Theorem 2.5 (Perron-Frobenius). *Let M be a $d \times d$ primitive matrix. Then:*

1. *the matrix M has a positive eigenvalue λ which is strictly greater than the modulus of any other eigenvalue,*

2. *the eigenvalue λ is algebraically simple,*

3. *to this eigenvalue corresponds a positive eigenvector (i.e., with positive entries only), while no other eigenvalue has a positive eigenvector.*

The eigenvalue λ from the above theorem is called the *Perron-Frobenius eigenvalue* of M. If a primitive matrix M is the substitution matrix of a substitution φ over $\mathscr{A} = \{a_1, \ldots, a_d\}$, according

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to the result of Quefféléc [12], the left eigenvector $(\rho_1, \rho_2, \ldots, \rho_d)$ of λ normalized by $\sum_{i=1}^{d} \rho_i = 1$ is equal to the vector of letter frequencies in any fixed point u of φ, i.e.,

$$(\rho_1, \rho_2, \ldots, \rho_d) = (\rho(a_1), \rho(a_2), \ldots, \rho(a_d)), \tag{10}$$

where $\rho(a_i) = \lim_{n \to \infty} \frac{|u^{(n)}|_{a_i}}{n}$ and $u^{(n)}$ denotes the prefix of u of length n.

2.6 Parry numbers and the infinite words u_β

From the formula for distances (7), we know that the number of distances in $\mathbb{Z}_\beta$ is finite if and only if the Rényi expansion of unity $d_\beta(1)$ is eventually periodic, i.e., if β is a Parry number.

- If $d_\beta(1)$ is finite, i.e., $d_\beta(1) = t_1 t_2 \ldots t_m$, $t_m \neq 0$, β is said to be a *simple Parry number*, and the set of distances is $\{\Delta_0, \Delta_1, \ldots, \Delta_{m-1}\}$, where all of the listed elements are mutually distinct.

- If $d_\beta(1)$ is eventually periodic, but not finite, β is a *non-simple Parry number*. Choose $p, m \in \mathbb{N}$ to be minimal such that $d_\beta(1) = t_1 t_2 \ldots t_m (t_{m+1} \ldots t_{m+p})^\omega$, then the set of all mutually distinct distances is $\{\Delta_0, \Delta_1, \ldots, \Delta_{m+p-1}\}$.

Let us precisely define the infinite word $u_\beta = u_0 u_1 u_2 \ldots$ associated with $\mathbb{Z}_\beta^+$ for a Parry number β. Let $\{\Delta_0, \ldots, \Delta_{d-1}\}$ be the set of distances between neighboring β-integers and let $(b_n)_{n=0}^\infty$ be as defined in (6), then

$$u_n \stackrel{\text{def}}{=} i \quad \text{if} \quad b_{n+1} - b_n = \Delta_i. \tag{11}$$

2.7 Canonical substitutions for Parry numbers

Fabre in [13] has associated with Parry numbers canonical substitutions in the following way.

Let β be a simple Parry number, i.e., $d_\beta(1) = t_1 t_2 \ldots t_m$, for $m \in \mathbb{N}$. Then the corresponding canonical substitution φ is defined over the alphabet $\{0, 1, \ldots, m-1\}$ by

$$\varphi(0) = 0^{t_1} 1, \ \varphi(1) = 0^{t_2} 2, \ \ldots, \ \varphi(m-2) = 0^{t_{m-1}}(m-1), \ \varphi(m-1) = 0^{t_m}. \tag{12}$$

Similarly, let β be a non-simple Parry number, i.e., $d_\beta(1) = t_1 t_2 \ldots t_m (t_{m+1} \ldots t_{m+p})^\omega$. The associated canonical substitution φ is defined over the alphabet $\{0, 1, \ldots, m+p-1\}$ by

$$\varphi(0) = 0^{t_1} 1, \ \varphi(1) = 0^{t_2} 2, \ \ldots, \ \varphi(m-1) = 0^{t_m} m, \ \ldots$$
$$\ldots, \varphi(m+p-2) = 0^{t_{m+p-1}}(m+p-1), \ \varphi(m+p-1) = 0^{t_{m+p}} m. \tag{13}$$

Each of these substitutions has a unique fixed point $\lim_{n \to \infty} \varphi^n(0)$. Moreover, this fixed point turns out to be equal to u_β. It is readily seen that in both cases, the substitution matrix M_φ is primitive. The characteristic polynomial of the substitution matrix M_φ coincides with the Parry polynomial of the number β. Hence, the Perron-Frobenius theorem implies that the Parry number β is a simple root of its Parry polynomial $p(x)$ and therefore $p'(\beta) \neq 0$.

3 Asymptotic behavior of β-integers for Parry numbers

In order to derive some information about asymptotic properties of β-integers, let us recall the essential relation given in [13] between a β-integer b_n and its coding by a prefix of the associated infinite word u_β.

Proposition 3.1. *Let u_β be the infinite word associated with a Parry number β and let φ be the associated substitution of β, then for every $b_n \in \mathbb{Z}_\beta^+$ it holds that $\langle b_n \rangle_\beta = a_{k-1} \ldots a_1 a_0 \bullet$ if and only if $\varphi^{k-1}(0^{a_{k-1}}) \ldots \varphi(0^{a_1}) 0^{a_0}$ is a prefix of u_β of length n.*

Since every prefix of u_β codes a β-integer b_n, Proposition 3.1 provides us with the following corollary.

Corollary 3.2. *Let w be a prefix of u_β, then there exist $k \in \mathbb{N}$ and $a_0, a_1, \ldots, a_{k-1} \in \mathbb{N}$ such that*

$$w = \varphi^{k-1}(0^{a_{k-1}}) \ldots \varphi(0^{a_1}) 0^{a_0},$$

where $a_{k-1} \ldots a_1 a_0 \bullet$ is the β-expansion of a β-integer.

Let us denote $U_i := |\varphi^i(0)|$, then $(U_i)_{i=0}^\infty$ is a canonical numeration system associated with the Parry number β (defined and called β- *numeration system* in [10] and studied in [13]). For details on numeration systems consult [14]. Proposition 3.1 implies that the greedy representation of an integer n in this system is given by

$$n = \sum_{i=0}^{k-1} a_i U_i \quad \text{if} \quad b_n = \sum_{i=0}^{k-1} a_i \beta^i.$$

Applying (8), the sequence $(U_i)_{i=0}^\infty$ may be expressed employing the substitution matrix M of φ in the following way

$$U_i = (1, 0, \ldots, 0) M^i (1, 1, \ldots, 1)^T, \tag{14}$$

where T is for matrix transposition. Let β be a simple Parry number, then the Rényi expansion of unity is of the form $d_\beta(1) = t_1 t_2 \ldots t_m$ and β is the largest root of the Parry polynomial $p(x) = x^m - (t_1 x^{m-1} + t_2 x^{m-2} + \cdots + t_{m-1} x + t_m)$. Let us recall that $p(x)$ may be reducible.

Our first aim is to provide a simple formula for constant c_β such that $b_n \underset{n \to \infty}{\approx} c_\beta n$. For any root γ of $p(x)$, it is easy to verify that $(\gamma^{m-1}, \gamma^{m-2}, \ldots, \gamma, 1)$ is a left eigenvector of the substitution matrix M associated with γ. On the other hand, according to (10), the unique left eigenvector $(\rho_0, \rho_1, \ldots, \rho_{m-1})$ of M with $\sum_{i=0}^{m-1} \rho_i = 1$ is such that ρ_i is the frequency of letter i in u_β. Combining the two previous facts, we obtain for frequencies the following formula

$$\rho_i = \frac{\beta^{m-1-i}}{\sum_{i=0}^{m-1} \beta^i}. \tag{15}$$

Let $(\Delta_0, \Delta_1, \ldots, \Delta_{m-1})^T$ be the right eigenvector of M associated with β such that $\Delta_0 = 1$, then it is easy to verify that Δ_i is the distance between consecutive β-integers which is coded by letter i in the infinite word u_β (see the formula (7) for distances). For our purposes, the following easily derivable formula for distances will be useful

$$\Delta_i = \beta^i - \sum_{j=1}^{i} t_j \beta^{i-j}, \quad i \in \{0, 1, \ldots, m-1\}. \tag{16}$$

Theorem 3.3. *Let $p(x)$ be the Parry polynomial of a simple Parry number β. Then*

$$c_\beta := \lim_{n \to \infty} \frac{b_n}{n} = \frac{\beta - 1}{\beta^m - 1} p'(\beta).$$

Proof. Let us denote by u the prefix of u_β of length n, then

$$b_n = |u|_0 \Delta_0 + |u|_1 \Delta_1 + \cdots + |u|_{m-1} \Delta_{m-1}.$$

Since frequencies of letters exist, $\lim_{n \to \infty} \frac{b_n}{n}$ exists and obeys the following formula

$$\lim_{n \to \infty} \frac{b_n}{n} = \rho_0 \Delta_0 + \rho_1 \Delta_1 + \cdots + \rho_{m-1} \Delta_{m-1}.$$

Applying (15) and (16), we obtain

$$
\begin{aligned}
\lim_{n \to \infty} \frac{b_n}{n} &= \frac{1}{\sum_{i=0}^{m-1} \beta^i} \left(\sum_{i=0}^{m-1} \beta^{m-1-i} \left(\beta^i - \sum_{j=1}^{i} t_j \beta^{i-j} \right) \right) \\
&= \frac{1}{\sum_{i=0}^{m-1} \beta^i} \left(m\beta^{m-1} - \sum_{j=1}^{m-1} t_j \sum_{i=j}^{m-1} \beta^{m-1-j} \right) \\
&= \frac{1}{\sum_{i=0}^{m-1} \beta^i} \left(m\beta^{m-1} - \sum_{j=1}^{m-1} t_j (m-j) \beta^{m-1-j} \right) \\
&= \frac{p'(\beta)}{\sum_{i=0}^{m-1} \beta^i} = \frac{\beta-1}{\beta^m-1} p'(\beta).
\end{aligned}
$$

$\square$

Corollary 3.4. *Let $\beta = \beta_1, \beta_2, \ldots, \beta_m$ be mutually distinct roots of the Parry polynomial $p(x)$ of a simple Parry number β. Then*

$$\lim_{n \to \infty} \frac{b_n}{n} = \frac{\beta - 1}{\beta^m - 1} \prod_{i=2}^{m} (\beta - \beta_i).$$

Proof. $p(x) = \prod_{i=1}^{m} (x - \beta_i)$, $p'(x) = \sum_{k=1}^{m} \prod_{i=1, i \neq k}^{m} (x - \beta_i)$, thus $p'(\beta) = \prod_{i=2}^{m} (\beta - \beta_i)$. $\square$

Remark 3.5. *If $p(x)$ is an irreducible polynomial, then β is an algebraic integer of order m and $\beta_2, \ldots, \beta_m$ are algebraic conjugates of β, and hence mutually distinct.*

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We now study the asymptotic behavior of $(b_n - c_\beta n)_{n\in\mathbb{N}}$. We know already that the limit $\lim_{n\to\infty} \frac{b_n}{n}$ exists. Hence it is enough to consider the limit of the subsequence (U_n)

$$\lim_{n\to\infty} \frac{b_n}{n} = \lim_{n\to\infty} \frac{b_{U_n}}{U_n} = \lim_{n\to\infty} \frac{\beta^n}{U_n}.$$

Under the assumption that all roots of $p(x)$ are mutually different, we will find a useful expression for U_n. Since M is diagonalizable, there exists an invertible matrix P such that PMP^{-1} is diagonal with $\left(PMP^{-1}\right)_{ii} = \beta_i$, $i \in \{1,\dots,m\}$. Using (14), we may write

$$U_n = (1,\, 0,\dots,0)P^{-1}\left(PMP^{-1}\right)^n P\,(1,1,\dots,1)^T. \tag{17}$$

It follows from the Perron-Frobenius theorem that $\beta > |\beta_i|$, hence, the formula (17) leads to the following expression

$$\frac{1}{c_\beta} = \lim_{n\to\infty} \frac{U_n}{\beta^n} = (1,\, 0,\dots,0)P^{-1}P_{e_1}P\,(1,1,\dots,1)^T, \tag{18}$$

where P_{e_1} is the orthogonal projection on the line given by $e_1 = (1,0,\dots,0)^T$, i.e., $(P_{e_1})_{11} = 1$, $(P_{e_1})_{ij} = 0$ otherwise. We now examine the difference $b_n - c_\beta n$. Let $\langle b_n\rangle_\beta = a_{k-1}\dots a_0\bullet$, thus $b_n = \sum_{i=0}^{k-1} a_i\beta^i$ and $n = \sum_{i=0}^{k-1} a_i U_i$. Employing (17) and (18), we obtain

$$\frac{1}{c_\beta}b_n - n = \sum_{i=0}^{k-1} a_i(1,0,\dots,0)P^{-1}\left(\beta_1^i P_{e_1} - \left(PMP^{-1}\right)^i\right)P\,(1,1,\dots,1)^T =$$

$$= (1,0,\dots,0)P^{-1}ZP\,(1,1,\dots,1)^T, \tag{19}$$

where Z is a diagonal matrix with $Z_{11} = 0$, $Z_{jj} = -z_j$ for $j \in \{2,\dots,m\}$, and $z_j = \sum_{i=0}^{k-1} a_i\beta_j^i$. Since the coefficients of β-expansion satisfy $a_i \in \{0,\dots,\lfloor\beta\rfloor\}$ and since for PV numbers β, it holds $|\beta_j| < 1$ for $j = 2,3,\dots,m$, we have

$$|z_j| \leq \sum_{i=0}^{k-1} |a_i||\beta_j^i| \leq \frac{\beta}{1-|\beta_j|}. \tag{20}$$

Remark 3.6. *Suppose that the Parry polynomial $p(x)$ of a Parry number β is reducible, say $p(x) = q(x)\cdot r(x)$, where $q(x)$ is the minimal polynomial of β, and $r(x)$ is a polynomial of degree at least 1. Then the product of the roots of $r(x)$ is an integer and therefore either all roots of $r(x)$ lie on the unit circle or at least one among the roots of $r(x)$ is in modulus larger than 1. It implies that the set of z_j is bounded for all j if and only if β is a Pisot number and its Parry polynomial is the minimal polynomial of β.*

According to Remark 3.6 and as P does not depend on n, we have shown the following theorem.

Theorem 3.7. *Let β be a simple Parry number. If β is moreover a Pisot number and the Parry polynomial of β is its minimal polynomial, then $(b_n - c_\beta n)_{n \in \mathbb{N}}$ is a bounded sequence.*

Now P is a matrix of the Vandermonde's type given by

$$P_{ij} = \beta_i^{m-j}, \quad \text{then} \quad P(1,1,\ldots,1)^T = \left(\tfrac{\beta^m - 1}{\beta - 1}, \tfrac{\beta_2^m - 1}{\beta_2 - 1}, \ldots, \tfrac{\beta_m^m - 1}{\beta_m - 1} \right)^T. \tag{21}$$

In order to have for all $n \in \mathbb{N}$ an explicit formula for $\frac{1}{c_\beta} b_n - n$, it remains to determine $(1, 0, \ldots, 0)P^{-1}$, i.e., the first row of P^{-1}. Since $P^{-1} = \frac{1}{\det P} P^{adj}$, where $(P^{adj})_{1j} = \det P(j,1)$ and $P(j,1)$ is obtained from P by deleting the j-th row and the 1-st column, applying Vandermonde's result yields

$$(P^{-1})_{1j} = \frac{\prod_{i<k,\ i,k \neq j}(\beta_i - \beta_k)}{\prod_{i<k}(\beta_i - \beta_k)} = \frac{(-1)^{j-1}}{\prod_{k \neq j}(\beta_j - \beta_k)} = \frac{(-1)^{j-1}}{p'(\beta_j)}. \tag{22}$$

Notice that since $p(x)$ does not have multiple roots, $p'(\beta_j) \neq 0$.

It follows from formulae (19), (21), and (22) that

$$\frac{1}{c_\beta} b_n - n = \sum_{j=2}^{m} \frac{(-1)^j z_j}{p'(\beta_j)} \frac{1 - \beta_j^m}{1 - \beta_j}.$$

In consequence, using the estimate (20), we may deduce an upper bound on $|b_n - c_\beta n|$

$$|b_n - c_\beta n| \leq 2 c_\beta \beta \sum_{j=2}^{m} \frac{1}{(1 - |\beta_j|)^2} \frac{1}{|p'(\beta_j)|}. \tag{23}$$

Example 3.8. *Let us illustrate the previous results on the case of the simplest simple Parry number - the golden mean $\beta = \tau = \frac{1+\sqrt{5}}{2}$. Rényi expansion of unity is $d_\tau(1) = 11$ and $p(x) = x^2 - x - 1$. The substitution matrix for the Fibonacci substitution has been given in Example 2.4. Consequently, $(U_n)_{n \in \mathbb{N}}$ satisfies $U_n = f_n$ for all $n \in \mathbb{N}$, where $(f_n)_{n \in \mathbb{N}}$ is the Fibonacci sequence given by*

$$f_{n+1} = f_n + f_{n-1}, \ f_0 = 1, \ f_1 = 2.$$

Applying Theorem 3.3, we get

$$c_\tau = \frac{\tau - 1}{\tau^2 - 1} p'(\tau) = \frac{p'(\tau)}{\tau + 1} = \frac{2\tau - 1}{\tau + 1} = \frac{\tau^2 + 1}{\tau^3},$$

which is in correspondence with Proposition 1.1.

Let us denote the second root of $p(x)$ (the Galois conjugate of τ) by τ', $\tau' = \frac{1-\sqrt{5}}{2} = \frac{-1}{\tau}$. If $\langle b_n \rangle_\beta = a_{k-1} \ldots a_1 a_0 \bullet$, then $a_i \in \{0,1\}$ and

$$b_n - c_\tau n = c_\tau \left(\tfrac{1}{2\tau - 1}, \tfrac{1}{2\tau' - 1} \right) \begin{pmatrix} 0 & 0 \\ 0 & -z_2 \end{pmatrix} \begin{pmatrix} 1 + \tau \\ 1 + \tau' \end{pmatrix} = \frac{1 - \tau}{\tau(\tau + 1)} \sum_{i=0}^{k-1} a_i (\tau')^i.$$

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Since $|\tau'| < 1$, the sequence $|b_n - c_\tau n|_{n \in \mathbb{N}}$ is bounded and we may easily determine an upper bound (taking into account that $\tau' < 0$)

$$\frac{1-\tau}{\tau^3} \leq \frac{\tau-1}{\tau(\tau+1)} \sum_{i=1}^{\infty} (\tau')^{2i-1} \leq c_\tau n - b_n \leq \frac{\tau-1}{\tau(\tau+1)} \sum_{i=0}^{\infty} (\tau')^{2i} = \frac{1}{\tau^3},$$

thus, comparing the upper and lower bound, we deduce

$$|b_n - c_\tau n| \leq \frac{1}{\tau^3},$$

which is again in correspondence with Proposition 1.1, where we have replaced the fractional part $\left\{ \frac{n+1}{1+\beta} \right\}$ with 1 in order to get an upper bound on $|b_n - c_\tau n|$.

3.1 Non-simple Parry numbers β

Let β be a non-simple Parry number, then the Rényi expansion of unity is of the form $d_\beta(1) = t_1 t_2 \ldots t_m (t_{m+1} \ldots t_{m+p})^\omega$ with m, p chosen to be minimal and β is the largest root of the Parry polynomial $p(x) = (x^p - 1)\left(x^m - t_1 x^{m-1} - \cdots - t_m\right) - t_{m+1} x^{p-1} - \cdots - t_{m+p-1} x - t_{m+p}$. Let us recall that $p(x)$ may be reducible.

Similarly to the simple case, our first goal is to derive a simple formula for constant c_β such that $b_n \underset{n \to \infty}{\approx} c_\beta n$. For any root γ of the Parry polynomial $p(x)$,

$$(\underbrace{\gamma^{m-1}(\gamma^p - 1),\ \gamma^{m-2}(\gamma^p - 1), \ldots, (\gamma^p - 1)}_{m \text{ components}}, \underbrace{\gamma^{p-1}, \ldots, \gamma,\ 1}_{p \text{ components}})$$

is a left eigenvector of the substitution matrix M associated with γ. On the other hand, according to (10), the unique left eigenvector $(\rho_0, \rho_1, \ldots, \rho_{m+p-1})$ of β such that $\sum_{i=0}^{m+p-1} \rho_i = 1$ satisfies that ρ_i is the frequency of letter i in u_β. Combining the two previous facts, we obtain for frequencies the following formula

$$\rho_i = \frac{\sigma_i}{\sum_{i=0}^{m-1} \beta^i (\beta^p - 1) + \sum_{i=0}^{p-1} \beta^i} = \frac{\sigma_i (\beta - 1)}{\beta^m (\beta^p - 1)}, \tag{24}$$

where

$$\begin{aligned} \sigma_i &= \beta^{m-1-i}(\beta^p - 1) && \text{for } 0 \leq i \leq m-1, \\ \sigma_i &= \beta^{m+p-1-i} && \text{for } m \leq i \leq m+p-1. \end{aligned}$$

Let $(\Delta_0, \Delta_1, \ldots, \Delta_{m+p-1})^T$ be the right eigenvector of M associated with β such that $\Delta_0 = 1$. Then it is easy to verify that Δ_i is the distance between consecutive β-integers which is coded by letter i in the infinite word u_β (see the formula for distances in (7)). Similarly as for simple Parry numbers, also for non-simple Parry numbers the following formula for distances holds and will be useful

$$\Delta_i = \beta^i - \sum_{j=1}^{i} t_j \beta^{i-j}, \quad i \in \{0, 1, \ldots, m+p-1\}. \tag{25}$$

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Theorem 3.9. *Let $p(x)$ be the Parry polynomial of the non-simple Parry number β. Then*

$$c_\beta := \lim_{n\to\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m(\beta^p-1)} p'(\beta).$$

Proof. Similarly as for simple Parry numbers, $\lim_{n\to\infty} \frac{b_n}{n}$ exists and we have

$$\lim_{n\to\infty} \frac{b_n}{n} = \rho_0\Delta_0 + \rho_1\Delta_1 + \cdots + \rho_{m+p-1}\Delta_{m+p-1}.$$

Applying (24) and (25), we obtain $\lim_{n\to\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m(\beta^p-1)}(A+B)$, where

$$A = \sum_{i=0}^{m-1} \beta^{m-1-i}(\beta^p-1)\left(\beta^i - \sum_{j=1}^{i} t_j\beta^{i-j}\right) \quad \text{and} \quad B = \sum_{i=m}^{m+p-1} \beta^{m+p-1-i}\left(\beta^i - \sum_{j=1}^{i} t_j\beta^{i-j}\right).$$

It is then straightforward to prove that

$$A = (\beta^p-1)\left(m\beta^{m-1} - \sum_{j=1}^{m-1} t_j(m-j)\beta^{m-j-1}\right),$$

$$B = p\beta^{m+p-1} + p\sum_{j=1}^{m-1} t_j\beta^{m+p-1-j} + \sum_{j=1}^{p-1}(p-j)t_{m+j}\beta^{p-j-1}$$

$$A+B = p'(\beta).$$

$\square$

Remark 3.10. *If we consider the infinite Rényi expansion of unity $d_\beta^*(1)$ instead of the "classical" Rényi expansion of unity $d_\beta(1)$, we have in the simple Parry case $d_\beta^*(1) = (t_1\ldots t_{m-1}(t_m-1))^\omega$. Thus the length l of preperiod is 0 and the length L of period is m. In the non-simple Parry case, we have $d_\beta^*(1) = d_\beta(1) = t_1 t_2 \ldots t_m(t_{m+1}\ldots t_{m+p})^\omega$. Hence the length l of preperiod is m and the length L of period is p. With this notation, the formulae for c_β from Theorems 3.3 and 3.9 may be rewritten for both simple and non-simple Parry numbers in a unique way as*

$$c_\beta = \frac{\beta-1}{\beta^l(\beta^L-1)} p'(\beta).$$

Corollary 3.11. *Let $\beta = \beta_1, \beta_2, \ldots, \beta_{m+p}$ be mutually different roots of the Parry polynomial $p(x)$ of a non-simple Parry number β. Then,*

$$c_\beta = \lim_{n\to\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m(\beta^p-1)} \prod_{i=2}^{m+p}(\beta-\beta_i).$$

Proof. Analogous as in Corollary 3.4. $\square$

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Remark 3.12. *If $p(x)$ is an irreducible polynomial, then β is an algebraic integer of order $m+p$ and $\beta_2,\ldots,\beta_{m+p}$ are algebraic conjugates of β, and hence mutually different.*

Let us now investigate the asymptotic behavior of $(b_n - c_\beta\, n)_{n\in\mathbb{N}}$. As we know already that the limit $\lim_{n\to\infty}\frac{b_n}{n}$ exists, we may rewrite it in terms of the subsequence (U_n)

$$\lim_{n\to\infty}\frac{b_n}{n} = \lim_{n\to\infty}\frac{b_{U_n}}{U_n} = \lim_{n\to\infty}\frac{\beta^n}{U_n}.$$

Under the assumption that all roots of $p(x)$ are mutually distinct, we will express U_n in an easier form. Since M is diagonalizable, there exists an invertible P such that PMP^{-1} is diagonal with $\left(PMP^{-1}\right)_{ii} = \beta_i$, $i \in \{1,\ldots,m+p\}$. Using (14), we may write

$$U_n = (1,\, 0,\ldots,0)P^{-1}\left(PMP^{-1}\right)^n P\,(1,1,\ldots,1)^T. \tag{26}$$

It follows from the Perron-Frobenius theorem that $\beta > |\beta_i|$, hence, the formula (26) leads to the following expression

$$\frac{1}{c_\beta} = \lim_{n\to\infty}\frac{U_n}{\beta^n} = (1,\, 0,\ldots,0)P^{-1}P_{e_1}P\,(1,1,\ldots,1)^T. \tag{27}$$

Now, let us turn our attention to the difference $b_n - c_\beta n$. Let $\langle b_n\rangle_\beta = a_{k-1}\ldots a_0\bullet$, thus $b_n = \sum_{i=0}^{k-1} a_i\beta^i$ and $n = \sum_{i=0}^{k-1} a_i U_i$. Employing (26) and (27), we obtain

$$\frac{1}{c_\beta}b_n - n = \sum_{i=0}^{k-1} a_i(1,0,\ldots,0)P^{-1}\left(\beta_1^i P_{e_1} - \left(PMP^{-1}\right)^i\right)P\,(1,1,\ldots,1)^T =$$

$$= (1,0,\ldots,0)P^{-1}ZP\,(1,1,\ldots,1)^T, \tag{28}$$

where Z is a diagonal matrix with $Z_{11} = 0$, $Z_{jj} = -z_j$ for $j \in \{2,\ldots,m+p\}$, and $z_j = \sum_{i=0}^{k-1} a_i\beta_j^i$. Since the coefficients of β-expansion satisfy $a_i \in \{0,\ldots,\lfloor\beta\rfloor\}$ and since for PV numbers β, it holds $|\beta_j| < 1$ for $j = 2,3,\ldots,m+p$, we have

$$|z_j| \le \sum_{i=0}^{k-1} |a_i||\beta_j^i| \le \frac{\beta}{1 - |\beta_j|}. \tag{29}$$

According to Remark 3.6 and since P does not depend on n, we have shown the following theorem.

Theorem 3.13. *Let β be a non-simple Parry number. If β is moreover a Pisot number and the Parry polynomial of β is its minimal polynomial, then $(b_n - c_\beta\, n)_{n\in\mathbb{N}}$ is a bounded sequence.*

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The explicit form of the matrix P reads

$$P = \begin{pmatrix} \beta^{m-1}(\beta^p-1) & \beta^{m-2}(\beta^p-1) & \cdots & (\beta^p-1) & \beta^{p-1} & \cdots & \beta & 1 \\ \beta_2^{m-1}(\beta_2^p-1) & \beta_2^{m-2}(\beta_2^p-1) & \cdots & (\beta_2^p-1) & \beta_2^{p-1} & \cdots & \beta_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ \beta_{m+p}^{m-1}(\beta_{m+p}^p-1) & \beta_{m+p}^{m-2}(\beta_{m+p}^p-1) & \cdots & (\beta_{m+p}^p-1) & \beta_{m+p}^{p-1} & \cdots & \beta_{m+p} & 1 \end{pmatrix}.$$

Hence,

$$P(1,1,\ldots,1)^T = \left(\frac{\beta^m(\beta^p-1)}{\beta-1}, \frac{\beta_2^m(\beta_2^p-1)}{\beta_2-1}, \ldots, \frac{\beta_{m+p}^m(\beta_{m+p}^p-1)}{\beta_{m+p}-1} \right). \tag{30}$$

In order to have for all $n \in \mathbb{N}$ an explicit formula for $\frac{1}{c_\beta}b_n - n$, it remains to determine $(1, 0, \ldots, 0)P^{-1}$, i.e., the first row of P^{-1}. By contrast to the simple Parry case, the matrix P is not in the Vandermonde's form. However, we notice that its determinant is equal to a Vandermonde determinant through a simple addition of columns. More precisely, we start with the addition of the last column to the m-th column, the last but one column to the $(m-1)$-st column and so forth. It is readily seen that this procedure leads after m steps to a Vandermonde matrix of order $m + p$ with the same determinant as P.

So $\det P = \prod_{i<k}(\beta_i - \beta_k)$. The expression of $(P^{-1})_{1j}$, $j \in \{1, \ldots, m+p\}$, is then given by

$$(P^{-1})_{1j} = \frac{\prod_{i<k,\, i,k\neq j}(\beta_i - \beta_k)}{\prod_{i<k}(\beta_i - \beta_k)} = \frac{(-1)^{j-1}}{\prod_{k\neq j}(\beta_j - \beta_k)} = \frac{(-1)^{j-1}}{p'(\beta_j)}. \tag{31}$$

Notice that since $p(x)$ does not have multiple roots, $p'(\beta_j) \neq 0$.

We obtain applying expressions (28), (30), and (31)

$$\frac{1}{c_\beta}b_n - n = \sum_{j=2}^{m+p} \frac{(-1)^j z_j}{p'(\beta_j)} \frac{\beta_j^m(1 - \beta_j^p)}{1 - \beta_j}.$$

In consequence, we may deduce an upper bound on $|b_n - c_\beta n|$

$$|b_n - c_\beta n| \leq 2c_\beta \beta \sum_{j=2}^{m} \frac{1}{(1 - |\beta_j|)^2} \frac{1}{|p'(\beta_j)|}. \tag{32}$$

Example 3.14. *Let us illustrate the previous results for the simplest non-simple Parry number β with Rényi expansion of unity $d_\beta(1) = 21^\omega$ and $p(x) = x^2 - 3x + 1$, i.e., $\beta = \tau^2 = \frac{3+\sqrt{5}}{2}$. The substitution matrix for the associated substitution $\varphi : 0 \to 001$, $1 \to 01$ is the square of the Fibonacci substitution matrix, i.e., $M_\varphi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Consequently, $(U_n)_{n\in\mathbb{N}}$ is just a subsequence of the Fibonacci sequence $(f_n)_{n\in\mathbb{N}}$ (defined in Example 3.8) given by $U_n = f_{2n}$.*

Applying Theorem 3.9, we get

$$c_\beta = \frac{\beta - 1}{\beta(\beta - 1)} p'(\beta) = \frac{p'(\beta)}{\beta} = \frac{2\beta - 3}{\beta} = 1 - \frac{1}{\beta^2},$$

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which is in correspondence with Proposition 1.1.

Let us denote the second root of $p(x)$ by β', $\beta' = \frac{3-\sqrt{5}}{2}$. Let $\langle b_n \rangle_\beta = a_{k-1} \ldots a_1 a_0 \bullet$, then $a_i \in \{0, 1, 2\}$ and

$$b_n - c_\beta\, n = c_\beta \left(\tfrac{1}{2\beta-3}, \tfrac{-1}{2\beta'-3} \right) \begin{pmatrix} 0 & 0 \\ 0 & -z_2 \end{pmatrix} \begin{pmatrix} \beta \\ \beta' \end{pmatrix} = \frac{-1}{\beta^2} \sum_{i=0}^{k-1} a_i (\beta')^i.$$

Since $0 < \beta' < 1$, the sequence $(b_n - c_\beta\, n)_{n \in \mathbb{N}}$ is bounded and we may easily determine an upper bound (taking into account that coefficients in β-expansions satisfy the Parry condition)

$$|b_n - c_\beta\, n| \le \frac{1}{\beta^2} \left(2 + \sum_{i=1}^{\infty} (\beta')^i \right) = \frac{1}{\beta^2} \left(2 + \frac{\beta'}{1-\beta'} \right) = \frac{1}{\beta^2} \left(2 + \frac{1}{\beta - 1} \right) = \frac{1}{\beta} \ ,$$

which is in correspondence with the estimate we get if we replace the fractional part $\left\{ \frac{n}{\beta} \right\}$ with 1 in Proposition 1.1.

Open questions:

It would be interesting to study the behaviour of β-integers even for non-Parry numbers β, i.e., when the Rényi expansion of unity of β is not eventually periodic. Among non-Parry numbers one may distinguish two cases:

• If the length of blocks of zero's in $d_\beta(1)$ is bounded, say by a length L, then the β-integers form a Delone set since the shortest distance between consecutive points is at least $\frac{1}{\beta^L}$ and the largest distance is 1.

• If $d_\beta(1)$ contains strings of zero's of unbounded length, then the set of distances (Δ_k) between consecutive β-integers have 0 as its accumulation point, see Equation (7). It means that in this case, the β-integers do not form a Delone set.

The first question to be answered in both cases is whether the limit b_n/n does exist for some of non-Parry numbers β.

Acknowledgements

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