Return words and recurrence function of a class of infinite words

Ľubomíra Balková

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Abstract

Many combinatorial and arithmetical properties have been studied for infinite words u_β associated with β-integers. Here, new results describing return words and recurrence function for a special case of u_β will be presented. The results can be applied to more general cases, but the description becomes then rather technical.

1 Introduction

Studying of factor complexity, palindromic complexity, arithmetics, balance property, return words, and recurrence function of infinite aperiodic words is an interesting combinatorial problem. Moreover, investigation of infinite words coding β-integers $\mathbb{Z}_\beta$, for β being a Pisot number, can be interpreted as investigation of one-dimensional quasicrystals. In this paper, new results concerning the return words and recurrence function will be presented. In general, only little is known about return words. Return word of a factor w of an infinite word u is any word which starts at some occurrence of w in u and ends just before the next occurrence of w. Vuillon [1] has shown that an infinite word over a bilateral alphabet is sturmian if and only if every of its factors has exactly two return words. Justin and Vuillon [2] proved that every factor of an Arnoux-Rauzy word of order m has m return words. Moreover, Vuillon [3] proved that every factor of an m-interval exchange coding word has also m return words. Return words in the Thue-Morse sequence and return words in the infinite words associated with simple Parry numbers have been studied in [4]. Even less is known about recurrence function which, for an infinite uniformly recurrent word u, determines the minimal length $R_u(n)$ such that if we take any subword of length $R_u(n)$ of u, we will find there all the factors of length n. Cassaigne [5] determined this function for sturmian words taking into account the continued fractions of their slope and in [6], he has given a general algorithm describing how to determine recurrence function if we know return words. This algorithm will constitute the headstone of our further results.

We will describe the cardinality of the set of return words for any factor w of the infinite word u_β associated with quadratic non-simple Parry (and, thus Pisot) number β and, as a consequence, we will be able to deduce the exact formula of recurrence function. These results will complete the list of properties already studied: the factor complexity has been studied in [7], the palindromic complexity is described in [8], the results on arithmetics and balances can be found in [9].

2 Preliminaries

First, let us introduce our "language" which will be used throughout this paper. An *alphabet* $\mathcal{A}$ is a finite set of symbols called *letters*. A concatenation of letters is a *word*. The *length* of a word w is denoted by $|w|$. The set $\mathcal{A}^*$ of all finite words (including the empty word ε) provided with the operation of concatenation is a free monoid. We will deal also with right-sided infinite words $u = u_0u_1u_2....$ A finite word w is called a *factor* of the word u (finite or infinite) if there exist a finite word $w^{(1)}$ and a word $w^{(2)}$ (finite or infinite) such that $u = w^{(1)}ww^{(2)}$. The word w is a *prefix* of u if $w^{(1)} = \varepsilon$ and it is a *suffix* of u if $w^{(2)} = \varepsilon$. A concatenation of k letters a (or words a) will be denoted by a^k. The *language* of u, $\mathcal{L}(u)$, is the set of all factors of a word u, and $\mathcal{L}_n(u)$ is the set of all factors of u of length n. We say that a letter $a \in \mathcal{A}$ is a *left extension* of a factor $w \in \mathcal{L}(u)$ if the factor aw belongs to the language $\mathcal{L}(u)$ and w is *left special* if w has at least two left extensions. Right extensions and right special factors are defined analogously. We call a factor $w \in \mathcal{L}(u)$ *bispecial* if it is left special and right special. A mapping φ on the free monoid $\mathcal{A}^*$ is called a *morphism* if $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in \mathcal{A}^*$. Obviously, for determining any morphism it suffices to give $\varphi(a)$ for all $a \in \mathcal{A}$. The action of a morphism can be naturally extended on right-sided infinite words by the prescription

$$\varphi(u_0u_1u_2...) := \varphi(u_0)\varphi(u_1)\varphi(u_2)...$$

A non-erasing morphism φ, for which there exists a letter $a \in \mathcal{A}$ such that $\varphi(a) = aw$ for some non-empty word $w \in \mathcal{A}^*$, is called a *substitution*. An infinite word u such that $\varphi(u) = u$ is called a *fixed point* of the substitution φ. Obviously, every substitution has at least one fixed point, namely

$$\lim_{n\to\infty} \varphi^n(a).$$

In all what follows, we will focus on the infinite word u_β being the only fixed point of the substitution φ given by

$$\varphi(0) = 0^a1,\ \varphi(1) = 0^b1, \quad a - 1 \geq b. \tag{1}$$

Let us only briefly mention that this infinite word codes the set of β-integers $\mathbb{Z}_\beta$, where β is a quadratic non-simple Parry number. All details can be found, for instance, in [8]. Since the main interest will be devoted to the study of return words and recurrence function, let us give the corresponding definitions.

Definition 1. *Let w be a factor of an infinite word $u = u_0u_1\dots$ (with $u_j \in \mathcal{A}$), $|w| = \ell$. An integer j is an occurrence of w in u if $u_ju_{j+1}\dots u_{j+\ell-1} = w$. Let j, k, $j < k$, be successive occurrences of w. Then $u_ju_{j+1}\dots u_{k-1}$ is a return word of w. The set of all return words of w is denoted by $Ret(w)$, i.e.,*

$$Ret(w) = \{u_ju_{j+1}\dots u_{k-1} \mid j, k \text{ being successive occurrences of } w\}.$$

An infinite word u is said to be *recurrent* if each of its factors appears infinitely many times in u, it is said to be *uniformly recurrent* if for every $n \in \mathbb{N}$ there exists $R(n) > 0$ such that any segment of u of length $\geq R(n)$ contains all the factors of $\mathcal{L}_n(u)$.

As we deal with the infinite word $u_\beta = \lim_{n\to\infty} \varphi(0)$, where $\varphi(0) = 0^a1, \varphi(1) = 0^b1, a-1 \geq b$, let us remind also what is known about uniform recurrence of words being fixed points of a substitution.

Definition 2. *A substitution φ over the alphabet $\mathcal{A} = \{a_1, a_2, \dots, a_k\}$ is called primitive if there exists $K \in \mathbb{N}$ such that for any letter $a_i \in \mathcal{A}$, the word $\varphi^K(a_i)$ contains all the letters of $\mathcal{A}$.*

Queffélec in [10] has shown that any fixed point of a primitive substitution φ is a uniformly recurrent word. Thus, since the infinite word u_β is a fixed point of a primitive substitution, u_β is uniformly recurrent. It is not difficult to see that the set of return words of w is finite for any factor $w \in \mathcal{L}(u)$ if u is a uniformly recurrent word.

Definition 3. *The recurrence function of an infinite word u is the function $R_u : \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by*

$$R_u(n) = \inf\big(\{N \in \mathbb{N} \mid \forall v \in \mathcal{L}_N(u), \mathcal{L}_n(v) = \mathcal{L}_n(u)\} \cup \{+\infty\}\big)$$

In other words, $R_u(n)$ is the smallest length such that if we take any segment of u of length $R_u(n)$, we will find there all the factors of u of length n.

Clearly, u is uniformly recurrent if and only if $R_u(n)$ is finite for every $n \in \mathbb{N}$. To get another expression for $R_u(n)$, convenient to work with, let us introduce some more terms.

Definition 4. *Let u be an infinite uniformly recurrent word.*

- *Let $w \in \mathcal{L}(u)$, then $l_u(w) = \max\{|v| \bigm| v \in Ret(w)\}$ is the maximal return time of w in u.*
- *For all $n \in \mathbb{N}$, we define $l_u(n) = \max\{l_u(w) \bigm| w \in \mathcal{L}_n(u)\}$.*

Once, we will have determined the lengths of return words, the following proposition from [5] will allow us to calculate the recurrence function $R_u(n)$.

Proposition 1. *For any infinite uniformly recurrent word u and for any $n \in \mathbb{N}$, one has*

$$R_u(n) = l_u(n) + n - 1.$$

3 Return words

The aim of this section is to determine return words of the infinite word u_β being the fixed point of the substitution φ introduced in (1). Vuillon in [1] has shown the following result on sturmian words.

Proposition 2. *Let u be an infinite word over a two letter alphabet. Then u is sturmian if and only if $\#Ret(w) = 2$ for every factor w of u.*

Let us mention that u_β is sturmian for $a - 1 = b$ (see e.g. [7]).

Example 1. *Let $u_\beta = 001001010010010100101....$ be the fixed point of the substitution $\varphi(0) = 001, \varphi(1) = 01$ (sturmian case). Let us show examples of return words:*

$$Ret(0) = \{0, 01\},$$

$$Ret(00) = \{001, 00101\},$$
$$Ret(001) = \{001, 00101\},$$
$$Ret(0010) = \{001, 00101\}.$$

Throughout this section, we will use analogical methods as those ones introduced in [4]. In order to study return words $Ret(w)$ of factors w of an infinite uniformly recurrent word u, it is possible to limit our considerations to bispecial factors. Namely, if a factor w is not right special, i.e., if it has a unique right extension $a \in \mathcal{A}$, then the sets of occurrences of w and wa coincide, and

$$Ret(w) = Ret(wa). \tag{2}$$

If a factor w has a unique left extension $b \in \mathcal{A}$, then $j \geq 1$ is an occurrence of w in the infinite word u if and only if $j - 1$ is an occurrence of bw. This statement does not hold for $j = 0$. Nevertheless, if u is a uniformly recurrent infinite word, then the set $Ret(w)$ of return words of w stays the same no matter whether we include the return word corresponding to the prefix w of u or not. Consequently, we have

$$Ret(bw) = bRet(w)b^{-1} = \{bvb^{-1} \mid v \in Ret(w)\}, \tag{3}$$

where bvb^{-1} means that the word v is prolonged to the left by the letter b and it is shortened from the right by erasing the letter b (which is always the suffix of v for $v \in Ret(w)$).

Observation 1. *Notice that Relations (2) and (3) imply that if $w \in \mathcal{L}(u)$ is not right special (left special), then the maximal return times from Definition 4 satisfy $l_u(w) = l_u(wa)$, where $a \in \mathcal{A}$ is the only right extension of w $\big(l_u(w) = l_u(bw)$, where $b \in \mathcal{A}$ is the only left extension of $w\big)$.*

For an aperiodic uniformly recurrent infinite word u, each factor w can be extended to the left and to the right to a bispecial factor. To describe the cardinality of $Ret(w)$, it suffices therefore to consider bispecial factors w.

Observation 2. *The only bipecial factors of u_β that do not contain the letter 1 are of the form 0^r, $r \le a-1$.*

Observation 3. *Every bispecial factor w of u_β containing at least one letter 1 has the prefix 0^b1 and the suffix 10^b. Consequently, there exists a bispecial factor v such that $w = 0^b1\varphi(v)0^b$. (The empty word ε is bispecial, too.)*

Let us summarize the previous observations to get the description of all the bispecial factors.

Corollary 1. *The set of all the bispecial factors of u_β is given by*

$$\{B_r^{(n)} \mid r \in \{0, \ldots, a-1\}, n \in \mathbb{N}\},$$

where $B_r^{(n+1)} = 0^b1\varphi(B_r^{(n)})0^b$ for all $n \in \mathbb{N}$, $B_0^{(1)} = \varepsilon$, and $B_r^{(1)} = 0^r$ for $r \in \{1, \ldots, a-1\}$.

In order to find return words of bispecial factors, let us remind a lemma from [4]. To formulate it, we need to remind as well a definition.

Definition 5. *Let w be a factor of a fixed point u of a substitution φ. We say that a word $v_0v_1\ldots v_{n-1}v_n \in \mathcal{L}_{n+1}(u)$ is an ancestor of w if*

- *w is a factor of $\varphi(v_0v_1\ldots v_{n-1}v_n)$,*
- *w is neither a factor of $\varphi(v_1\ldots v_{n-1}v_n)$ nor of $\varphi(v_0v_1\ldots v_{n-1})$.*

Clearly, any factor $\varphi(w)$ has at least one ancestor, namely the factor w.

Lemma 1. *Let an infinite word u be a fixed point of a substitution φ and w be a factor of u. If the only ancestor of $\varphi(w)$ is the factor w, then*

$$Ret(\varphi(w)) = \varphi(Ret(w)).$$

Proposition 3. *Let w be a non-empty factor of u_β, then $\varphi(w)$ has a unique ancestor, namely w.*

Proof. It is a direct consequence of the form of the substitution defined by (1) that any factor v having suffix 1 occurs only as suffix of $\varphi(w)$ for some factor $w \in \mathcal{L}(u_\beta)$. Moreover, any factor $\hat{v}$ preceded by 1 occurs only as prefix of $\varphi(\hat{w})$ for some $\hat{w} \in \mathcal{L}(u_\beta)$. The statement follows by injectivity of φ. □

Now, we are able to explain how to get the return words of all the bispecial factors.

Corollary 2. *Let $B_r^{(n)}$ be a bispecial factor of u_β. Then $\#Ret(B_r^{(n)}) = \#Ret(B_r^{(n-1)})$ for every $n \ge 2$ and $r \in \{0, \ldots, a-1\}$.*

Proof. It follows from Proposition 3 that $\#Ret(B_r^{(n-1)}) = \#Ret(\varphi(B_r^{(n-1)}))$. Then, Relations (2) and (3) say that $\#Ret(\varphi(B_r^{(n-1)})) = \#Ret(0^b1\varphi(B_r^{(n-1)})0^b) = \#Ret(B_r^{(n)})$. □

Let us sum up the results to get the main theorem about the cardinality of the set of return words of any factor of u_β.

Theorem 1. *Let w be a factor of u_β. Then $2 \le \#Ret(w) \le 3$.*

Proof. Using Corollary 2, it suffices to consider only bispecial factors of the form $0^r, r \le a-1$, and 0^b10^b to get all the possible cardinalities of the sets of return words of factors of u_β. It is not difficult to see that the return words of the simplest bispecial factors are the following ones.

1. Let $r \le b$, then $Ret(0^r) = \{0, 0^r1\}$.
2. Let $b < r \le a-1$, then $Ret(0^r) = \{0, 0^r1, 0^r10^b1\}$.
3. $Ret(\varepsilon) = \{0, 1\}$.

□

Observation 4. *From the proof of Theorem 1, we can notice that in the case $b = a-1$, $\#Ret(w) = 2$ for all the factors of u_β. Thanks to Proposition 2, we have another confirmation of the fact that u_β for $b = a-1$ is a sturmian word.*

4 How to compute the recurrence function

Let us show how to apply the knowledge of return words to describe the recurrence function (Definition 3) of the infinite word u_β. All the ideas of this section are taken from [6].

As you are surely expecting, we want to compute $l_u(n)$ in order to get the recurrence function $R_u(n)$ for every $n \in \mathbb{N}$. This task can be simplified using the notion of singular factors. A factor $w \in \mathcal{L}(u)$ is called *singular* if $|w| = 1$ or there exist a word $v \in \mathcal{L}(u)$ and letters $x, x', y, y' \in \mathcal{A}$ such that $w = xvy$, $x \neq x'$, $y \neq y'$, and $\{x'vy, xvy'\} \subset \mathcal{L}(u)$. Obviously, v is a bispecial factor.

Proposition 4. *Let u be a uniformly recurrent word and $n \geq 1$. If $l_u(n-1) < l_u(n)$, then there exists a singular factor w of length n such that $l_u(w) = l_u(n)$.*

A singular factor w is said to be *essential* if $l_u(w) = l_u(|w|) > l_u(|w|-1)$. It follows that to calculate $l_u(n)$, it is sufficient to consider singular, or, even, only essential singular factors.

Theorem 2. *Let u be a uniformly recurrent word and $n \geq 1$.*

$$l_u(n) = \max\{l_u(w) \mid |w| \leq n \text{ and } w \text{ singular}\} = \max\{l_u(w) \mid |w| \leq n \text{ and } w \text{ essential singular}\}.$$

Now, we are able to give an **algorithm for computing the recurrence function** of an infinite uniformly recurrent word u:

1. Determine bispecial factors.
2. Deduce the form of singular factors and compute their lengths.
3. For every singular factor, determine the associated return words and compute their lengths.
4. Compute the function $l_u(n)$ to get the recurrence function $R_u(n)$ for every $n \in \mathbb{N}$.

5 Computation of recurrence function of u_β

Let us apply the above described algorithm for computing the recurrence function of the infinite word u_β being the only fixed point of the substitution $\varphi(0) = 0^a1,\ \varphi(1) = 0^b1,\ a-1 \geq b$.

1. The bispecial factors of u_β are described in Corollary 1.
2. Let us show how the task of the description of singular factors can be simplified using the relation between a factor and its image.

   Observation 5. *Let $n \geq 2$ and $r \in \{0, \dots, a-1\}$. Then, $S_r^{(n)}(x,y) = xB_r^{(n)}y$ is a singular factor if and only if $S_r^{(n-1)}(x,y) = xB_r^{(n-1)}y$ is a singular factor.*

   If we describe all the simplest non-trivial singular factors, i.e., those ones of the form $S_r^{(1)}(x,y) = xB_r^{(1)}y,\ x,y \in \{0,1\}$, Observation 5 will allow us to get the set of all the singular factors of u_β.

   Proposition 5. *The set of all the singular factors of u_β is given by*

   $$\{0,1\} \cup \left\{S_r^{(n)}(x,y) \mid r \in \{0,\dots,a-1\}, n \in \mathbb{N}\right\},$$

   where $S_r^{(n)}(x,y) = xB_r^{(n)}y$, $n \in \mathbb{N}$, and all the simplest non-trivial singular factors are the following ones:

   $$S_r^{(1)}(0,0) = 00^r0, \quad r \leq a-2,$$
   $$S_b^{(1)}(1,0) = 10^b0,$$
   $$S_b^{(1)}(0,1) = 00^b1,$$
   $$S_b^{(1)}(1,1) = 10^b1,$$
   $$S_0^{(1)}(0,0) = 00.$$

To compute the length of singular factors, it is, of course, enough to compute the length of bispecial factors since $|S_r^{(n)}(x,y)| = |B_r^{(n)}| + 2$. For the lengths of bispecial factors, we have $|B_r^{(n)}| = |B_r^{(n)}|_0 + |B_r^{(n)}|_1$, where $|B_r^{(n)}|_0$ denotes the number of 0's in $B_r^{(n)}$ and $|B_r^{(n)}|_1$ denotes the number of 1's in $B_r^{(n)}$. Then $\begin{pmatrix} |B_r^{(n)}|_0 \\ |B_r^{(n)}|_1 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |B_r^{(n-1)}|_0 \\ |B_r^{(n-1)}|_1 \end{pmatrix}$ for all $n \geq 2$.

3. To describe the return words of singular factors, we can again make use of a relation between return words of a singular factor and return words of its image.

Proposition 6. *Let $n \geq 2$, $r \in \{0, \dots, a-1\}$. The following sets are equal*

$$\{|w| \bigm| w \in Ret(S_r^{(n)}(x,y))\} = \{|\varphi(v)| \bigm| v \in Ret(S_r^{(n-1)}(x,y))\}.$$

Proof. We will distinguish two situations.

(a) For $x = 0$ and $y \in \{0,1\}$, applying Observation 1 on return words of factors which are not right or left special, it follows that the set of lengths of return words of $S_r^{(n)}(x,y) = x0^b1\varphi(B_r^{(n-1)})0^by$ is the same as the one of $\varphi(x)\varphi(B_r^{(n-1)})\varphi(y)$. Thus, we get

$$\{|w| \bigm| w \in Ret(S_r^{(n)}(x,y))\} = \{|w| \bigm| w \in Ret(\varphi(S_r^{(n-1)}(x,y)))\}.$$

(b) The singular factor $S_r^{(n)}(1,1) = 10^b1\varphi(B_r^{(n-1)})0^b1$ can be extended to the left without changing the set of lengths of return words to $0^a10^b1\varphi(B_r^{(n-1)})0^b1 = \varphi(01)\varphi(B_r^{(n-1)})\varphi(1)$. Thus, we get

$$\{|w| \bigm| w \in Ret(S_r^{(n)}(1,1))\} = \{|w| \bigm| w \in Ret(\varphi(0S_r^{(n-1)}(1,1)))\}.$$

The statement follows then using Lemma 5 and Proposition 3, and, in the case of (b), also using Relation (3) for return words of $0S_r^{(n-1)}(1,1)$ and $S_r^{(n-1)}(1,1)$. $\square$

Now, it suffices to determine return words and their lengths for the simplest singular factors $S_r^{(1)}(x,y)$. Proposition 6 implies that the lengths of return words of all the other singular factors will be obtained by calculating the lengths of images of the simplest return words. Here are the return words of the simplest singular factors.

(a) For the trivial singular factors $0, 1$, one gets

$$Ret(0) = \{0, 01\} \text{ and } Ret(1) = \{10^a, 10^b\}. \tag{4}$$

(b) For $S_r^{(1)}(0,0) = 00^r0$, using the proof of Theorem 1, we have

$$Ret(00^r0) = \begin{cases} \{0^a1, 0^a10^b1\} & \text{if } r = a-2\ , \\ \{0, 0^{r+2}1, 0^{r+2}10^b1\} & \text{if } b-2 < r < a-2, \\ \{0, 0^{r+2}1\} & \text{if } 1 \leq r \leq b-2. \end{cases} \tag{5}$$

(c) For $S_b^{(1)}(1,0) = 10^b0$, one can easily find

$$Ret(10^b0) = \{10^a, 10^a10^b\}. \tag{6}$$

(d) For $S_b^{(1)}(0,1) = 00^b1$, one has

$$Ret(00^b1) = \{0^{b+1}10^{a-(b+1)}, 0^{b+1}10^b10^{a-(b+1)}\}. \tag{7}$$

(e) Since $S_b^{(1)}(1,1) = 10^b1 = 1\varphi(1)$, using Relation (3), we have

$$Ret(10^b1) = \{1v1^{-1} \mid v \in Ret(\varphi(1))\} = \{1\varphi(10^a)1^{-1}, 1\varphi(10^b)1^{-1}\}. \tag{8}$$

(f) For $S_0^{(1)}(0,0) = 00$, one has to consider more cases:

- Let $b = 1$ and $a = 2$ (a sturmian case), it holds $Ret(00) = \{001, 00101\}$.
- Let $b = 1$ and $a > 2$, then $Ret(00) = \{0, 00101\}$.
- Let $b \geq 2$, then $Ret(00) = \{0, 001\}$.

4. The last step is to compute $l_{u_\beta}(k) := l(k),\ k \in \mathbb{N}$. Before starting, let us exclude singular factors which are not essential. (Let us remind that a singular factor w is essential if $l(w) = l(|w|) > l(|w|-1)$.) Naturally, we will apply Proposition 6 to determine the maximal return times $l(w)$ of singular factor w.

(a) Since $|S_b^{(n)}(1,0)| = |S_b^{(n)}(0,1)| = |S_b^{(n)}(1,1)|$, but from Relations (6), (7), and (8), it is clear that for all $n \in \mathbb{N}$, one gets $l(S_b^{(n)}(1,0)) = l(S_b^{(n)}(0,1)) < l(S_b^{(n)}(1,1))$. Thus, $S_b^{(n)}(1,0)$ and $S_b^{(n)}(0,1)$ are not essential singular factors and one does not have to consider them in calculation of $l(k)$, $k \in \mathbb{N}$.

(b) Analogously, if $b \geq 2$, we have $|S_0^{(n+1)}(0,0)| > |S_b^{(n)}(1,1)|$, while $l(S_0^{(n+1)}(0,0)) < l(S_b^{(n)}(1,1))$ for all $n \in \mathbb{N}$. Hence, the singular factors $S_0^{(n+1)}(0,0)$ are not essential.

(c) Next, if $a - 2 \geq r \geq b$, we have $|S_r^{(n)}(0,0)| \geq |S_b^{(n)}(1,1)|$, nevertheless, $l(S_r^{(n)}(0,0)) < l(S_b^{(n)}(1,1))$ for all $n \in \mathbb{N}$. Therefore, $S_r^{(n)}(0,0)$, $r \geq b$, are not essential.

(d) If $1 \leq r \leq b-2$, then we obtain $|S_r^{(n+1)}(0,0)| > |S_b^{(n)}(1,1)|$, but $l(S_r^{(n+1)}(0,0)) < l(S_b^{(n)}(1,1))$ for all $n \in \mathbb{N}$, thus, the singular factors $S_r^{(n+1)}(0,0)$, $1 \leq r \leq b-2$, are not essential.

(e) The last remark is that for the trivial singular factor $w = 1$, we have $|w| < |S_r^{(1)}(0,0)|$, $1 \leq r \leq b-2$, but $l(S_r^{(1)}(0,0)) < l(w)$, hence, $S_r^{(1)}(0,0)$, $1 \leq r \leq b-2$, are not essential.

The previous facts imply that to calculate $l(k)$, we have to take into account only the trivial singular factors, the non-trivial singular factors of the form $S_b^{(n)}(1,1)$, and, eventually, $S_0^{(n)}(0,0)$ and $S_{b-1}^{(n)}(0,0)$. The formulae for $l(k)$, $k \in \mathbb{N}$, will split into more cases, according to the values taken by a and b given in the substitution (1).

(a) For $b \geq 2$, combining the previous facts and the description of the simplest singular factors, we obtain the following formula for $l(k), k \in \mathbb{N}$,

- If $2b + 1 \geq a$, then

$$\begin{aligned}
&l(1) = \cdots = l(b) = a + 1, \\
&l(b+1) = 2b + 3, \\
&l(k) = |\varphi^n(10^a)| && \text{for } |S_b^{(n)}(1,1)| \leq k < |S_{b-1}^{(n+1)}(0,0)|,\ n \in \mathbb{N}, \\
&l(k) = |\varphi^n(0^{b+1}10^b1)| && \text{for } |S_{b-1}^{(n+1)}(0,0)| \leq k < |S_b^{(n+1)}(1,1)|,\ n \in \mathbb{N}.
\end{aligned}$$

- If $2b + 1 < a$, then

$$\begin{aligned}
&l(1) = \cdots = l(b+1) = a + 1, \\
&l(k) = |\varphi^n(10^a)| && \text{for } |S_b^{(n)}(1,1)| \leq k < |S_b^{(n+1)}(1,1)|,\ n \in \mathbb{N}.
\end{aligned}$$

(b) For $b = 1$ and $2 \leq a \leq 3$, then it holds for $l(k)$, $k \in \mathbb{N}$,

$$\begin{aligned}
&l(1) = a + 1,\ l(2) = 5, \\
&l(k) = |\varphi^n(10^a)| && \text{for } |S_1^{(n)}(1,1)| \leq k < |S_0^{(n+1)}(0,0)|,\ n \in \mathbb{N}, \\
&l(k) = |\varphi^n(00101)| && \text{for } |S_0^{(n+1)}(0,0)| \leq k < |S_1^{(n+1)}(1,1)|,\ n \in \mathbb{N}.
\end{aligned}$$

$l(k)$	3	5	8	8	13	13	13	21	21	21	21	21	34	34	34	34	34	34	34	34
$R(k)$	3	6	10	11	17	18	19	28	29	30	31	32	46	47	48	49	50	51	52	53

Figure 1: The table of the first 20 values of $l(k)$ and $R(k)$ for the simplest case $a = 2$ and $b = 1$.

(c) For $b = 1$ and $a > 3$, we get

$$\begin{aligned} l(1) &= l(2) = a + 1, \\ l(k) &= |\varphi^n(10^a)| \qquad \text{for } |S_1^{(n)}(1,1)| \leq k < |S_1^{(n+1)}(1,1)|,\ n \in \mathbb{N}. \end{aligned}$$

Having calculated the formula for $l(k)$, $k \in \mathbb{N}$, we have also the recurrence function computed since $R(k) = l(k) + k - 1$.

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