Vzorové řešení LAB 30.5.2012

PRAXE (1.)

$$\left(\begin{array}{cccc|c} \beta & \beta & -1 & -1 & \beta \\ 1 & 1 & 0 & \beta & -\beta \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & \beta & -\beta \\ 0 & 0 & -1 & (-1-\beta^2) & \beta+\beta^2 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & \beta & -\beta \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -\beta^2 & \beta+\beta^2 \end{array}\right)$$

$\beta = 0$ 2b

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{mn. řeš.} \quad S = \left[\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right]_\lambda$$

$\beta \neq 0$ 2b

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & \beta & -\beta \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -\beta & 1+\beta \end{array}\right) \qquad S = \begin{pmatrix} 1 \\ 0 \\ \frac{1+\beta}{\beta} \\ \frac{-1-\beta}{\beta} \end{pmatrix} + \left[\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right]_\lambda$$

$-\beta + 1 + \beta = 1$

(2) a) a1) 1,5

$$\begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ -1 & 0 & 2 & 0 & -1 \end{vmatrix} = 1\cdot(-1)^{2+4} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ -1 & 0 & 2 & -1 \end{vmatrix} = 1(-1)^{2+2} \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} + (-1)(-1)^{1+2} \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} =$$

$$= -5 + 1 = -4$$

a2) 1,5

$$\begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & -2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 4 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = -4$$

b) $\det A^{-1} = \frac{1}{\det A} = -1/4$ 0,5 c) $\det\left(\frac{1}{2}A\right) = \left(\frac{1}{2}\right)^5 \cdot \det A = -\frac{1}{8}$ 0,5

(3) $p_A(\lambda) = \begin{vmatrix} 1-\lambda & 0 & -1 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = \lambda \begin{vmatrix} 1-\lambda & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3(1-\lambda) \quad \sigma(A) = \{0, 1\}$ 1,5

vl. vektory k 0: $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\vec{x_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ 0,5, $\vec{x_2} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ 0,5

$3 = \nu_a(0) \neq \nu_g(0) = 2$

vl. vekt. k 1: $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\vec{x_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ 0,5

1 A není diagonalizovatelná, protože $v_a(0) > v_g(0)$, proto A není podobná žádné diagonální matici, speciálně A není podobná B.

4. a) $$\vec{x} = \vec{x}_p + \vec{x}_{p\perp} = \alpha\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \beta\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \vec{x}_{p\perp} = \begin{pmatrix}\alpha+\beta+\gamma\\\alpha+\beta+\gamma\\\alpha+\beta\\\alpha\end{pmatrix} + \vec{x}_{p\perp}$$

2b (1b postup, 1b výsledek)

α, β, γ najdeme z podmínek, kt. vzniknou vynásobením $\vec{x}$ vektory $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$

$$\left\langle \vec{x} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}2\\0\\1\\0\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle = 3 = \left\langle \vec{x}_p + \vec{x}_{p\perp} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}\alpha+\beta+\gamma\\\alpha+\beta+\gamma\\\alpha+\beta\\\alpha\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle = 4\alpha+3\beta+2\gamma$$

$$\left\langle \vec{x} \,\middle|\, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right\rangle = 3 = 3\alpha+3\beta+2\gamma$$

$$\left\langle \vec{x} \,\middle|\, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\rangle = 2 = 2\alpha+2\beta+2\gamma$$

$$\left(\begin{array}{ccc|c}1&1&1&1\\3&3&2&3\\4&3&2&3\end{array}\right) \sim \left(\begin{array}{ccc|c}1&1&1&1\\0&0&-1&0\\0&-1&-2&-1\end{array}\right) \sim \left(\begin{array}{ccc|c}1&1&1&1\\0&1&2&1\\0&0&1&0\end{array}\right)$$

$\gamma = 0, \beta = 1, \alpha = 0$ $\quad \vec{x}_p = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}$ $\quad \vec{x}_{p\perp} = \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$

b) postup stejný, ale A-skalární součin

2b

$$\left\langle \vec{x} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle_A = \left\langle \begin{pmatrix}2&-1&0&0\\-1&1&0&0\\0&0&1&0\\0&0&0&2\end{pmatrix}\begin{pmatrix}2\\0\\1\\0\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}4\\-2\\1\\0\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle = 3 = \left\langle \begin{pmatrix}\alpha+\beta+\gamma\\\alpha+\beta+\gamma\\\alpha+\beta\\\alpha\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle_A =$$

$$\left\langle \begin{pmatrix}2&-1&0&0\\-1&1&0&0\\0&0&1&0\\0&0&0&2\end{pmatrix}\begin{pmatrix}\alpha+\beta+\gamma\\\alpha+\beta+\gamma\\\alpha+\beta\\\alpha\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}\alpha+\beta+\gamma\\0\\\alpha+\beta\\2\alpha\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle = 4\alpha+2\beta+\gamma$$

$$\left\langle \vec{x} \,\middle|\, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right\rangle_A = \left\langle \begin{pmatrix}4\\-2\\1\\0\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right\rangle = 3 = \left\langle \begin{pmatrix}\alpha+\beta+\gamma\\0\\\alpha+\beta\\2\alpha\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right\rangle = 2\alpha+2\beta+\gamma$$

$$\left\langle \vec{x} \,\middle|\, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\rangle_A = \left\langle \begin{pmatrix}4\\-2\\1\\0\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\rangle = 2 = \left\langle \begin{pmatrix}\alpha+\beta+\gamma\\0\\\alpha+\beta\\2\alpha\end{pmatrix} \,\middle|\, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\rangle = \alpha+\beta+\gamma$$

$$\left(\begin{array}{ccc|c}1&1&1&2\\2&2&1&3\\4&2&1&3\end{array}\right) \sim \left(\begin{array}{ccc|c}1&1&1&2\\0&0&-1&-1\\0&-2&-3&-5\end{array}\right) \sim \left(\begin{array}{ccc|c}1&1&1&2\\0&2&3&5\\0&0&1&1\end{array}\right)$$

$\gamma = 1, \beta = 1, \alpha = 0$ $\quad \vec{x}_p = \begin{pmatrix}2\\2\\1\\0\end{pmatrix}$

(5) $W_1: \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 \end{array}\right)$ $W_1 = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right]_\lambda$ rovina

$\dim W_1 = 2$

$W_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}\right]_\lambda$ $\dim W_2 = 1$ přímka 1b

$W_2 = \left[\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right]_\lambda$ 1b

$W_1 \cap W_2$: $\left.\begin{array}{l} 2(1+\lambda) + 2(-1+\lambda) = 0 \\ (1+\lambda) - (-1+\lambda) = 1 \end{array}\right\}$ N.Ř.

$W_1 \cap W_2 = \emptyset$ 1b

$Z(W_2) \not\subset Z(W_1) \Rightarrow W_1$ a W_2 jsou mimoběžné 1b

TEORIE (1) a) Nechť A je matice s prvky z T typu $m \times n$, $\vec{b} \in T^m$. Pak soustava $A \cdot \vec{x} = \vec{b}$:

1,5b 1) má řešení $\Leftrightarrow h(A) = h(A|\vec{b})$

2) označ-li $S_0 = \{\vec{x} \in T^n \mid A \cdot \vec{x} = \vec{o}\}$, pak $S_0 << T^n$ a $\dim S_0 = n - h(A)$

3) je-li $h(A) = h(A|\vec{b})$, pak mn. všech řešení $S = \{\vec{x} \in T^n \mid A \cdot \vec{x} = \vec{b}\}$ má tvar $\vec{a} + S_0$, kde $\vec{a}$ splňuje $A \cdot \vec{a} = \vec{b}$.

b) dk. 1): $A\vec{x} = \vec{b}$ má řešení $\Leftrightarrow \exists\, \alpha_1, \dots, \alpha_n \in T$ $A\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \vec{b} \Leftrightarrow \exists\, \alpha_1, \dots, \alpha_n \in T^n$

1,5b $\alpha_1 A_{\cdot 1} + \alpha_2 A_{\cdot 2} + \dots + \alpha_n A_{\cdot n} = \vec{b} \Leftrightarrow \vec{b} \in [A_{\cdot 1}, \dots, A_{\cdot n}]_\lambda \Leftrightarrow [A_{\cdot 1}, \dots, A_{\cdot n}]_\lambda =$

sloupce A

$= [A_{\cdot 1}, \dots, A_{\cdot n}, \vec{b}]_\lambda \Leftrightarrow \dim [A_{\cdot 1}, \dots, A_{\cdot n}]_\lambda = \dim [A_{\cdot 1}, \dots, A_{\cdot n}, \vec{b}]_\lambda \Leftrightarrow h(A) = h(A|\vec{b})$

(2) a) VĚTA 0,5 Nechť $\vec{x}, \vec{y} \in (V, \langle\cdot|\cdot\rangle)$. Pokud $\langle \vec{x}|\vec{y}\rangle = 0$, pak $\|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x}+\vec{y}\|^2$.

0,5 dk.: $\|\vec{x}+\vec{y}\|^2 = \langle \vec{x}+\vec{y} | \vec{x}+\vec{y} \rangle = \|\vec{x}\|^2 + \underbrace{\langle \vec{x}|\vec{y}\rangle}_{0} + \underbrace{\langle \vec{y}|\vec{x}\rangle}_{0} + \|\vec{y}\|^2$

0,5 b) je-li $T = \mathbb{R}$, tedy V vekt. pr. nad reálným tělesem, $\langle \vec{x}|\vec{y}\rangle = 0$

c) VĚTA: Nechť $\vec{x}, \vec{y} \in (V, \langle\cdot|\cdot\rangle)$, pak $\|\vec{x}\| + \|\vec{y}\| \geq \|\vec{x}+\vec{y}\|$.

1 Rovnost nastává $\Leftrightarrow \exists\, \alpha \geq 0$ $\vec{x} = \alpha\vec{y}$ nebo $\vec{y} = \alpha\vec{x}$.

d) Rovnost nenastává, protože $\vec{y} = -1000 \cdot \vec{x}$, což není nezá-

0,5 porný násobek.

(3) a) 1 Nechť $A \in \mathbb{C}^{n,n}$. A je unitární, pokud $A \cdot A^H = \mathbb{I}$. A je OG, pokud je navíc reálná, tedy $A \in \mathbb{R}^{n,n}$ a $A \cdot A^T = \mathbb{I}$.

$\overline{\left(A^T\right)}$

0,5 b) $\forall \lambda \in \sigma(A)$, kde A je OG, platí $|\lambda| = 1$. 0,5 d) vl. vekt. přísl. různým vl. číslům jsou OG

0,5 c) $|\det A| = 1$, je-li A OG, tedy $\det A = \pm 1$.

0,5 e) NE