Infinite words with finite defect

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Outline

1. Overview of finite defects
   - Basic definitions
   - Equivalent characterizations of words with finite defect
2. Classes P_{ret} and P
   - Class P_{ret}
   - Class P
   - P_{ret}, P and questions ...

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Defect of finite words

- Droubay, Justin, Pirillo 2001: any finite word w contains at most $|w|+1$ palindromes
- number of distinct palindromes in w
$=$ number of prefixes of w with a unioccurrent longest palindromic suffix (lps)
$=$ number of suffixes of w with a unioccurrent longest palindromic prefix

Definition

w is *rich* if w contains $|w|+1$ distinct palindromes

Definition (Brlek 2004)

defect of w $D(w) = |w| + 1 -$ number of distinct palindromes in w

- zero defect = richness
- $D(w) =$ number of prefixes of w with a non-unioccurrent lps

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Defect of infinite words

Definition

$\mathbf{u}$ infinite is *rich* if any of its factors (prefixes) w contains $|w| + 1$ palindromes

- w factor of $v \Rightarrow D(w) \leq D(v)$

Definition

defect of an infinite word $\mathbf{u}$

$D(\mathbf{u}) = \sup\{D(w) \mid w \text{ is a factor (prefix) of } \mathbf{u}\}$

- if $D(\mathbf{u}) < +\infty \Rightarrow$ infinite number of palindromes

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Rich infinite words

Theorem

Let $\mathbf{u}$ *be an infinite word with language closed under reversal. The following statements are equivalent:*

1. $\mathbf{u}$ *is rich,*
2. *complete return words of palindromic factors are palindromes,*
3. *any prefix of* $\mathbf{u}$ *has a unioccurrent lps,*
4. $\mathcal{P}(n+1) + \mathcal{P}(n) = \mathcal{C}(n+1) - \mathcal{C}(n) + 2 \quad \text{for all } n \in \mathbb{N}.$

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Oddity

Proposition

$D(\mathbf{u}) \geq$
$\#\{\{v, \overline{v}\} \mid v \neq \overline{v} \text{ \& } v \text{ or } \overline{v} \text{ complete return word of a palindrome}\}$

$\{v, \overline{v}\}$ is *oddity*

Example

$\mathbf{u} = (abcabcacbacb)^\omega$, $D(\mathbf{u}) = 4$, but number of oddities 3

Theorem

A uniformly recurrent word $\mathbf{u}$ *has* ∞ *oddities iff* $\mathbf{u}$ *has* ∞ *distinct palindromes and* $D(\mathbf{u}) = +\infty$.

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Equivalent characterizations of words with finite defect

Theorem

Let **u** *be a uniformly recurrent word containing* ∞ *palindromes. The following statements are equivalent:*

1. $D(\mathbf{u}) < +\infty$,
2. **u** *has a finite number of oddities,*
3. *there exists an integer* K *such that all complete return words of any palindrome of* **u** *of length at least* K *are palindromes,*
4. *there exists an integer* H *such that for any prefix* f *of* **u** *with* $|f| \geq H$ *lps of* f *is unioccurrent in* f,
5. *there exists an integer* N *such that*

$$\mathcal{P}(n) + \mathcal{P}(n+1) = \mathcal{C}(n+1) - \mathcal{C}(n) + 2 \text{ for all } n \geq N.$$

It coincides for $K = H = N = 0$ with definitions of rich words!

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1. **u** *is rich,*
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Class P_{ret}

Definition

We say that a morphism $\varphi : \mathcal{B}^* \mapsto \mathcal{A}^*$ is of class P_{ret} if there exists a palindrome $p \in \mathcal{A}^*$ such that

- *$\varphi(b)p$ is a palindrome for any $b \in \mathcal{B}$,*
- *$\varphi(b)p$ contains exactly 2 distinct occurrences of p, one as a prefix and one as a suffix, for any $b \in \mathcal{B}$,*
- *$\varphi(b) \neq \varphi(c)$ for all $b, c \in \mathcal{B}$, $b \neq c$.*

Example

Let $\mathcal{A} = \mathcal{B} = \{0, 1\})$ and $p = 010$.
The morphism $\varphi : 0 \to 01011, 1 \to 01$ is of class P_{ret}.

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Properties of morphisms of class P_{ret}

Let φ be of class P_{ret}, then

1. φ is injective,
2. $\overline{\varphi(x)p} = \varphi(\overline{x})p$ for any $x \in \mathcal{B}^*$,
3. $\varphi(s)p$ is a palindrome if and only if $s \in \mathcal{B}^*$ is a palindrome.

The class P_{ret} is closed under the composition of morphisms, i.e., for any $\varphi, \sigma \in P_{ret}$ we have $\varphi\sigma \in P_{ret}$ (if the composition is well defined).

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Relation with finite defects

Theorem

Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ be a uniformly recurrent word with finite defect. Then there exist a rich word $\mathbf{v} \in \mathcal{B}^{\mathbb{N}}$ and a morphism $\varphi : \mathcal{B}^* \mapsto \mathcal{A}^*$ of class P_{ret} such that

$$\mathbf{u} = \varphi(\mathbf{v}).$$

The word $\mathbf{v}$ is uniformly recurrent.

The converse in not true: there exist an infinite word $\mathbf{u}$ with finite defect and a morphism φ of class P_{ret} such that $D(\varphi(\mathbf{u})) = +\infty$.

Note: $D(\varphi(\mathbf{u}))$ depends on φ as well as on $\mathbf{u}$.

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Class P

Definition (Hof, Knill and Simon (1995))

A morphism φ is of class P if there exist a palindrome p and for every letter a there exists a palindrome q_a such that $\varphi(a) = pq_a$.

Example

Let $p = 00$ and $\varphi : 0 \to 001 \to 001001$ is of class P but not of class P_{ret}.

Definition (Glen et al., 2008)

A morphism φ of class P is ***special*** *if*

1. *all $\varphi(a)$ end with different letters,*
2. *whenever $\varphi(a)p$ occurs in some $\varphi(y_1 y_2 \ldots y_n)p$, then this occurrence is $\varphi(y_m)p$ for some m with $1 \leq m \leq n$.*

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Definition

A morphism σ is ***conjugate*** *to a morphism θ if there exists $w \in \mathcal{A}^*$ such that for all $a \in \mathcal{A}$ we have $\sigma(a)w = w\theta(a)$.*

Definition

A morphism ***is of class*** *P' if it is conjugate to a morphism of class P. A morphism of class P' is* ***special*** *if it is conjugate to a special morphism of class P.*

Theorem (Glen et al., 2008)

Suppose $\mathbf{v} = \sigma(\mathbf{u})$ where σ is a special morphism of class P'. Then

$$D(\mathbf{v}) = 0 \Leftrightarrow D(\mathbf{u}) = 0.$$

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P_{ret} vs. P

Proposition

If φ is a morphism of class P_{ret}, then φ is conjugate to a morphism of class P.

Proposition

Let $\mathbf{u}$ be a binary uniformly recurrent word such that $D(\mathbf{u})$ is finite. Let φ be a morphism of class P_{ret}. Then $D(\varphi(\mathbf{u}))$ is finite.

Thus, special P' is not the maximal class to preserve finite defect.

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Questions

1. Subclass of P_{ret} where the converse of Theorem 7 holds?
2. Class preserving rich words?
3. Class preserving words with finite defect?

Thank you.