Inverse Word Graphs

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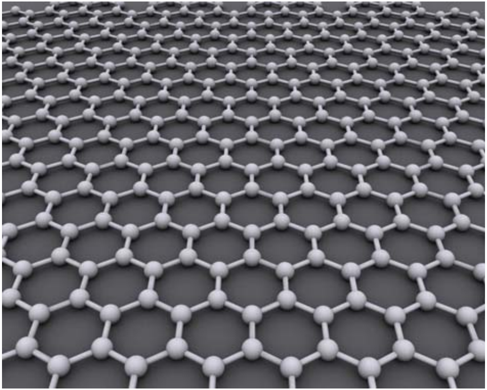
Symmetries $\Rightarrow$ Crystals

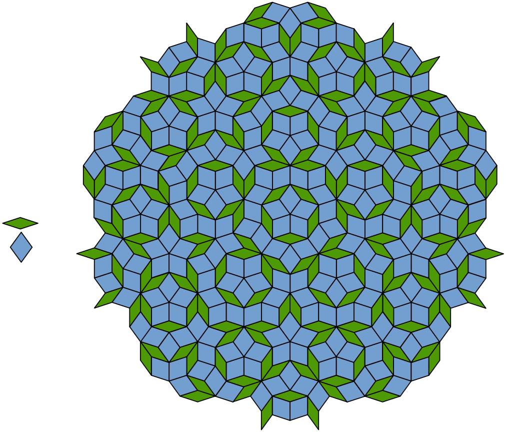
Example: Graphene crystal



Symmetries

Partial Symmetries ⇒ Quasi Crystals?

Example: Penrose tiling



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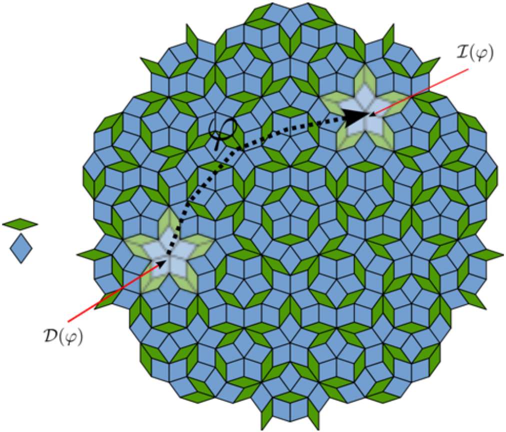

Inverse semigroups

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Every group can be embedded in the set of one to one transformations on a set.

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Inverse semigroups can be represented as partial symmetries.

Theorem (Vagner-Preston)

Every inverse semigroup can be embedded in the set of partial one to one transformations on a set.

Inverse semigroups historically

- Nineteenth century: Klein's Erlanger Programm of classifying geometries via the group of symmetries of the geometry (structure preserving bijections).

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- The algebraic framework of Pseudogroups: Vagner-Preston introducing Inverse Semigroups.

Inverse Semigroup

Group -

- associative binary operation
- identity
- inverses: $a \cdot a^{-1} = a^{-1} \cdot a = e$

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Inverse Semigroup

Group -

- associative binary operation
- identity
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permutations, symmetries, bijections,...

Semigroup - associative operation

concatenation of strings

Inverse Semigroup

- associative binary operation
- existence of inverses:
  $a \cdot a^{-1} \cdot a = a$
  $a^{-1} \cdot a \cdot a^{-1} = a^{-1}$

strings, paths in graphs, transition semigroups, partial transformations, do/undo proceses

Structural Questions

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A maximal subgroup of an inverse semigroup $(S, \cdot)$ is a subset G of S "centered around" an idempotent e of S and satisfying the property that $(G, \cdot)$ is in fact a group with e serving as its identity element.

Maximal Subgroups:

An element a is called an *idempotent* if it satisfies the property $a^2 = a$.

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In general, inverse semigroups can have many idempotents.

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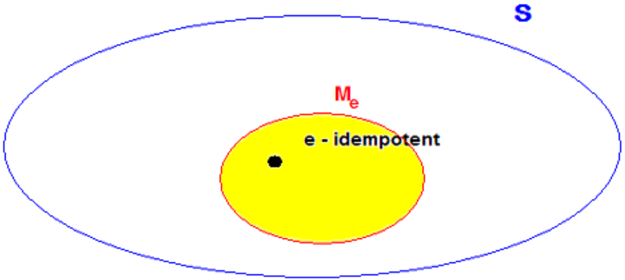

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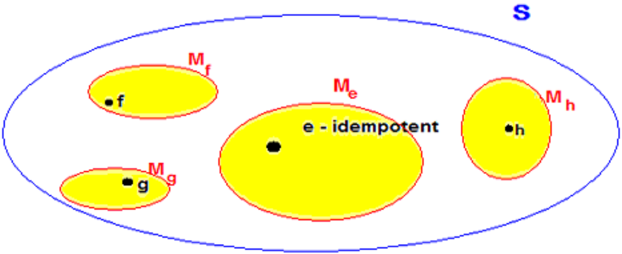

How do we present an algebraic structure?

Multiplication Table

Example from groups:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Multiplication Table

Example from groups:

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Multiplication Table

Example from groups: Klein 4-Group

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Presentations:

A presentation for an algebraic structure is "contracted" information about the structure allowing for a complete recovery of its original multiplication table.

For example,
$V_4 = \langle e, a, b, c | e^2 = e, ea = a, \ldots, cb = a, c^2 = e \rangle$

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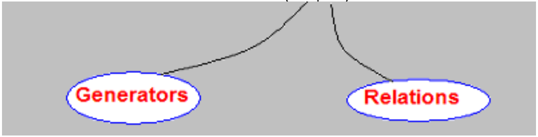
Every other multiplication in V_4 follows from this presentation and the fact that V_4 is a group.

$$V_4 = Gp\langle a, b | a^2 = b^2 = e, ab = ba\rangle$$

Formally, a presentation is a pair

$$A = \langle X | R \rangle$$



with the relations R being equations between expressions formed of the generators from X.

$A = Gp\langle X|R\rangle$
$A = Inv\langle X|R\rangle$
$A = InvM\langle X|R\rangle$

Presentations

We say $S = Inv\langle X|R\rangle$ (or S is presented by $\langle X|R\rangle$) if

$$S = (X \cup X^{-1})^{+}/\tau$$

where τ is the smallest congruence containing the relation R and Vagner's relations:

$$\{(xx^{-1}x, x), (xx^{-1}yy^{-1}, yy^{-1}xx^{-1}) : x, y \in X \cup X^{-1}\}$$

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Starting from a presentation, we would like to be able to answer *at least* some of the following questions:

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- given a product of generators, is this product equal to some other product of generators? "word problem"
- given two presentations, do they define the same structure? "isomorphism problem"

Decision problems

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Decision problems

All of the decision problems mentioned for the presentations are *undecidable* for the classes of groups and inverse semigroups.

Algorithmic problems in semigroups

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Definition (Word problem)

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Algorithmic problems in semigroups

- One of the most studied: the word problem for finitely presented (inverse) semigroups.
- It is a particular case of a more general problem in the framework of rewriting systems.

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Definition (Rewriting system)

$\langle X | R \rangle$ where X is a finite alphabet, $R \subseteq X^* \times X^*$ which is symmetric. We say $w_1 \to w_2$ if $w_1 = uxv, w_2 = uyv$ and $(x, y) \in R$, the transitive closure of such relation is denoted by $\xrightarrow{*}$, thus the word problem is reduced to ask whether or not, given $w, w' \in X^*$, $w \xrightarrow{*} w'$

Example: the free case in Groups

Consider the free group $FG(a, b) = Gp\langle a, b | \emptyset \rangle$,

$$w = aba^{-1}ab^{-1}a \qquad w' = aa$$

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can I find an algorithm to check weather or not $w\tau = w'\tau$?
Seen as a rewriting system:

$$aba^{-1}ab^{-1}a \rightarrow abb^{-1}a \rightarrow aa$$

so it is always decidable since relations R reduce the length and we have always a normal form...

Example for a Free Inverse Semigroup

Let $S = FIS(a,b) = Inv\langle a, b \mid \emptyset \rangle$ and $w = aa^{-1}b^2b^{-1}a^2a^{-1}$.

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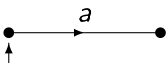

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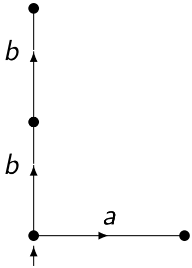

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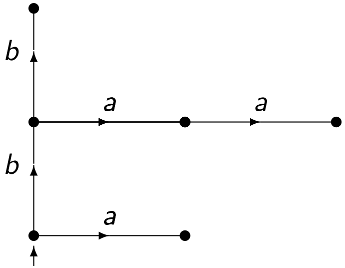

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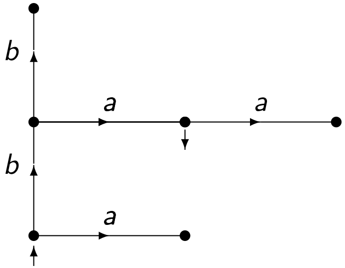

Example: the free case in Inverse Semigroups

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$$MT(u) = MT(v) \text{ and the roots are the same.}$$

We first build the Munn automaton of w, then of w' if they recognize the same language, then $w\tau = w'\tau$.

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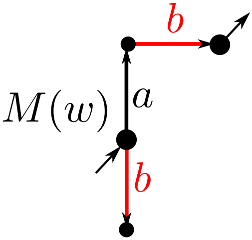

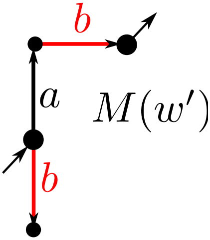

Inverse Word Graph

Let $X \neq \emptyset$ be a set (an alphabet).
An inverse word graph over X is a connected graph whose edges are labeled by the elements from X, and that satisfies the property that for each edge z the oppositely oriented edge $\overline{z}$ is labeled by the inverse of the label of z.

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- $H = Inv\langle Y | T \rangle = (Y \cup Y^{-1})^+/\tau$ the Schützenberger graphs $S\Gamma(Y, T, w)$ for $w \in (Y \cup Y^{-1})^+$ are the connected components of the Cayley graph of H containing $w\tau$.

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- $S\Gamma(Y, T, w)$ is the inverse word graph

Let $S = Inv\langle X|R\rangle$

Definition (Schützenberger graph)

Let w be a word in $(X \cup X^{-1})^{+}$. The Schützenberger graph of w relative to the presentation $Inv\langle X|R\rangle$ is the graph $S\Gamma(X, R, w\tau)$ whose vertices are the elements of the $\mathcal{R}$-class $\mathcal{R}_{w\tau}$ of $w\tau$ in S, and whose edges are of the form

$$\{(v_1, x, v_2) \mid v_1, v_2 \in \mathcal{R}_{w\tau} \text{ and } v_1(x\tau) = v_2\}.$$

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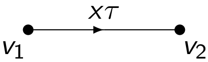

if $v_2 = v_1 \cdot x\tau$.

Schützenberger automata

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$$\mathcal{A}(Y, T, w) = (ww^{-1}\tau, S\Gamma(Y, T, w), w\tau)$$

- Many nice properties, one remarkable for the study of the word problem:
  $w\tau = w'\tau$ iff $L[\mathcal{A}(Y, T, w)] = L[\mathcal{A}(Y, T, w')]$.

General Inverse Semigroups

Theorem (Stephen, 1990)

Let w_1 and w_2 be two words in $(X \cup X^{-1})^+$. Then $w_1 = w_2$ in $S = Inv\langle X | R \rangle$ if and only if the Schützenberger graphs $S\Gamma(X, R, w_1\tau)$ and $S\Gamma(X, R, w_2\tau)$ corresponding to the words w_1 and w_2 and the inverse semigroup S are isomorphic and the isomorphism maps the beginning of w_1 to the beginning of w_2 and the end of w_1 to the end of w_2.

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Theorem (Stephen, 1990)

$G_e \cong Aut(S\Gamma(X, R, e))$

Applying Stephen's theorem assumes that we already know the Schützenberger graph for the given word and inverse semigroup.

General Inverse Semigroups

Applying Stephen's theorem assumes that we already know the Schützenberger graph for the given word and inverse semigroup.

In general, we do not know any effective procedure for constructing the Schützenberger graphs.

Stephen's iterative procedure.

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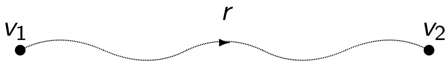

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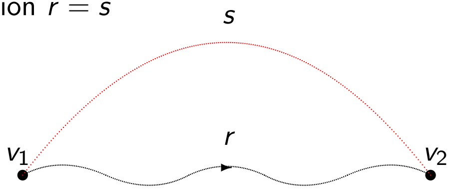

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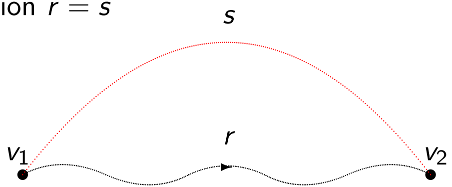
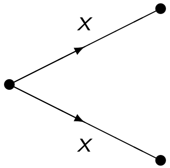


Elementary determination:

-edge folding



Stephen's iterative procedure.

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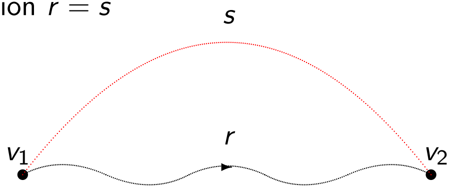

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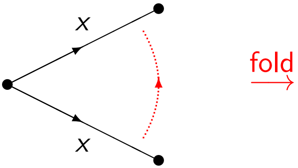

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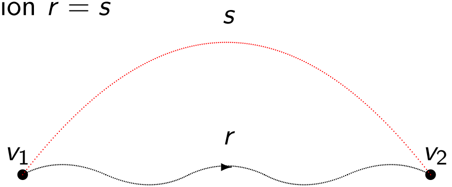

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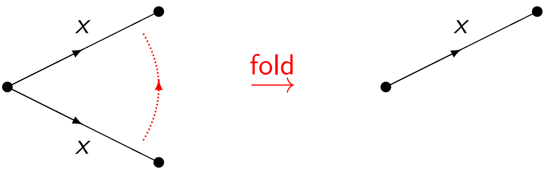

Schützenberger automata

In this way we get a directed system of inverse automata

$$\mathcal{A}_1 \to \mathcal{A}_2 \to \ldots \to \mathcal{A}_i \to \ldots$$

whose directed limit is the Schützenberger automata $\mathcal{A}(Y, T, w)$.

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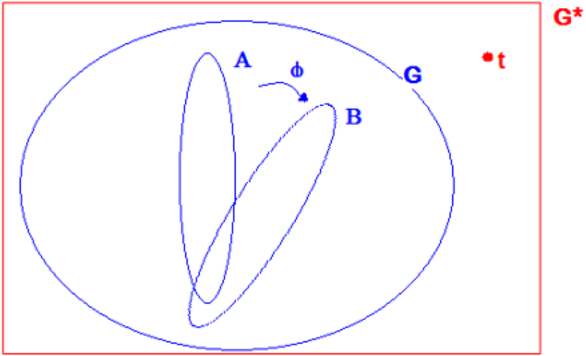
There are many product operations used successfully for both groups and semigroups – direct product, free product, amalgamated product.

Our focus will be on the product operation originally introduced for groups and called an HNN-extension.

HNN-extensions for groups

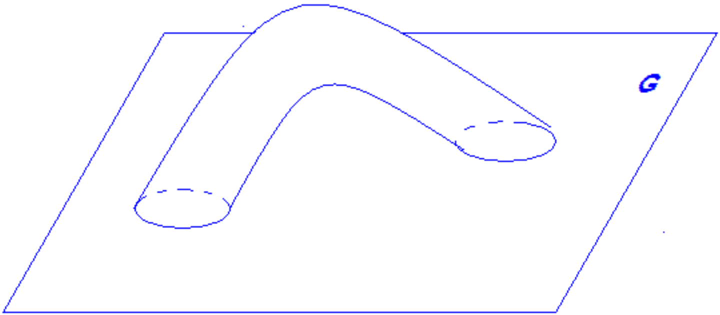
HigmanNeumannNeumann - extensions



$t^{-1}at = a\phi \quad \text{for} \quad \forall a \in A$

Handle

For example, the fundamental group of a surface with a handle is an HNN-extension of the fundamental group of the surface without the handle attached.



Definition of HNN-extensions for inverse semigroups

Definition (A.Yamamura)

Let $S = Inv\langle X \mid R\rangle$ be an inverse semigroup.
Let A, B be inverse subsemigroups of S,
$\varphi : A \longrightarrow B$ be an isomorphism

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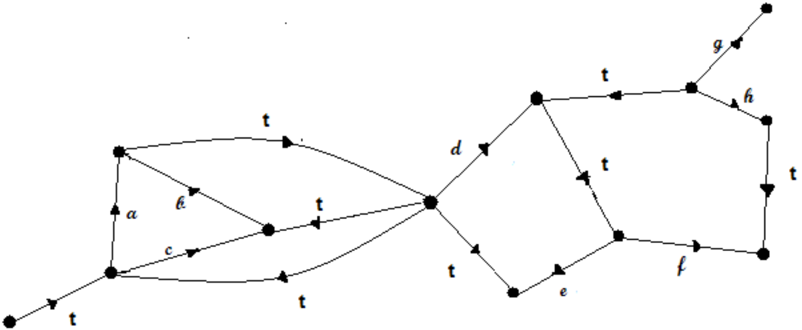
$$S \hookrightarrow S^*$$

HNN-extensions for groups

In what follows, we shall address the structural and decision questions concerning the HNN-extensions of inverse semigroups, $S^* = Inv\langle X, t \mid R \cup R_{HNN} \rangle$, via the use of the very visual and intuitive concept of a graph "constructed from a word in X according to the rules in $R \cup R_{HNN}$".

In the special case when $S = Inv\langle X, t | R \cup R_{HNN} \rangle$, a part of the word graph over $X \cup \{t\}$ may look something like this:



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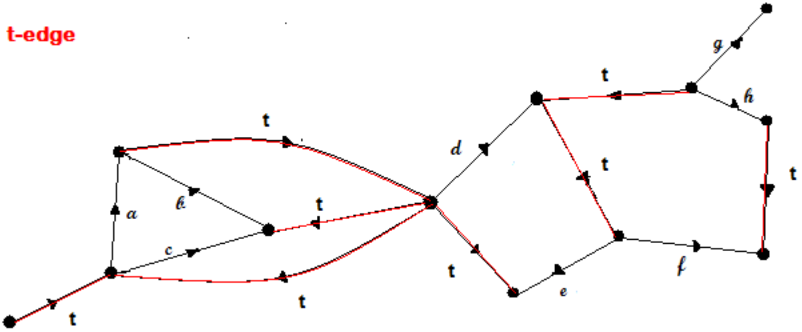

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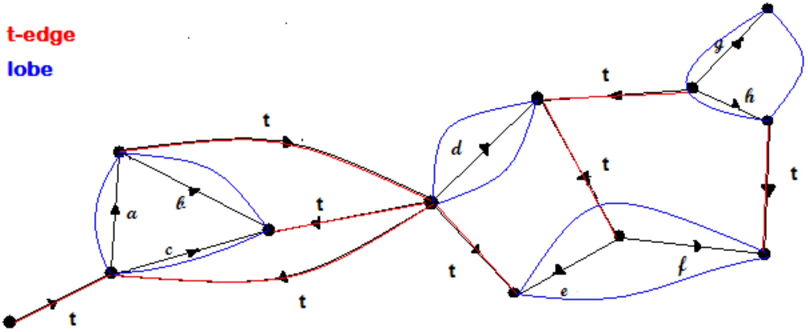
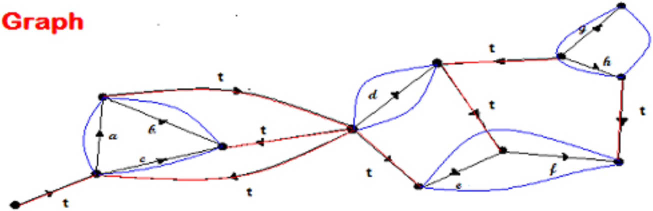


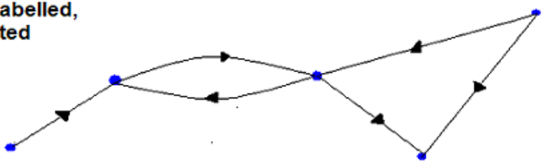
Graph



$$\Downarrow$$

Lobe Graph

non-labelled,
oriented



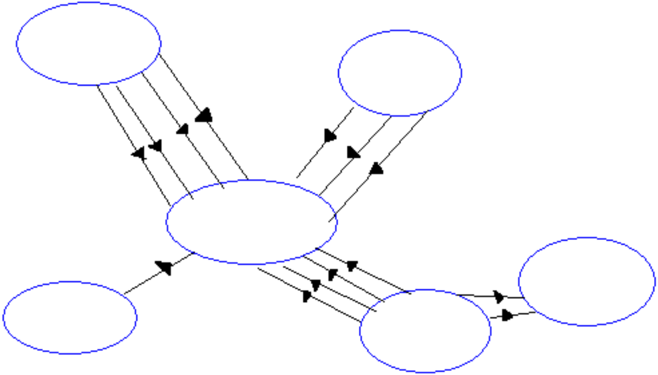
T(Γ)

The tree structure of lobe graphs

Theorem

The lobe graph $T(\Gamma)$ of a Schützenberger graph Γ relative to the presentation $Inv\langle X, t \mid R \cup R_{HNN} \rangle$ is an oriented tree.

The tree structure of lobe graphs



Characterization of the Schützenberger automata for HNN-extension.

Theorem

Let S^* be a lower bounded HNN-extension. The Schützenberger automata of S^* relative to the presentation $Inv\langle X \cup \{t\} | R \cup R_{HNN}\rangle$ are precisely the complete T-automata that possess a host.

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- Schützenberger graphs of HNN-extensions have tree like lobe structure and many other "nice" features – e.g., they contain a special subgraph with only finitely many lobes that contains the information for the whole graph.
- the tree like lobe structure of these graphs allows for the use of the Bass-Serre Theory of group actions on trees and graphs of groups.

Word Problem for HNN-extension.

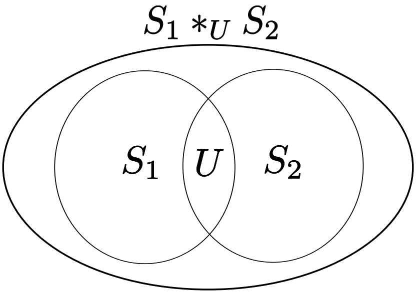
Theorem

The word problem is decidable for any HNN-extension of the form $S^* = [S; A, B; \varphi]$, where A and B are isomorphic finitely generated inverse subsemigroups of $FIS(X)$.

Amalgams of Inverse Semigroups

Amalgam is a 5-uple $[S_1, S_2; U, \omega_1, \omega_2]$ where S_1, S_2, U are inverse semigroups and $\omega_i : U \hookrightarrow S_i, i = 1, 2.$

Amalgams of Inverse Semigroups



If $S_1 = Inv\langle X_1 | R_1\rangle$, $S_2 = Inv\langle X_2 | R_2\rangle$ with $X_1 \cap X_2 = \emptyset$

$$S_1 *_U S_2 = Inv\langle X | R_1, R_2, R_w\rangle = Inv\langle X | R\rangle$$

where $X = X_1 \cup X_2$, $R_w = \{(\omega_1(u), \omega_2(u)) : u \in U\}$

Word problem for amalgams of (inverse)-semigroups

The word problem for amalgams of (inverse)-semigroups Given two (inverse)-semigroups S_1, S_2 which have decidable word problem and the embeddings $\omega_i : U \hookrightarrow S_i$ are computable, does $S_1 *_U S_2$ have decidable word problem?

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- Proof based on an ordered way to build Schützenberger automata

Theorem (Cherubini, Meakin, Piochi)

The word problem in $S_1 *_U S_2$ where S_1, S_2 are finite inverse semigroups is decidable.

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- Result in contrast with Sapir's results using Minsky machines.

Theorem (Sapir)

There are two finite semigroups for which the word problem in $S_1 *_U S_2$ is undecidable.

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- Group case is decidable.

Theorem

If S_1, S_2 are two groups which have decidable word problem and the embeddings $\omega_i : U \hookrightarrow S_i$ are computable, then $S_1 *_U S_2$ have decidable word problem.

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Theorem (R., Silva)

The word problem for $S_1 *_U S_2$ of inverse semigroups may be undecidable even if we assume S_1 and S_2 to have finite $\mathcal{R}$-classes and ω_1, ω_2 to be computable functions.